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AN ALGEBRAIC FORMULATION OF THE LOCALITY PRINCIPLE IN RENORMALISATION

PIERRE CLAVIER, LI GUO, SYLVIE PAYCHA, AND BIN ZHANG

ABSTRACT. We study the mathematical structure underlying the concept of locality which lies at the heart of classical and quantum field theory, and develop a machinery used to preserve locality during the renormalisation procedure. Viewing renormalisation in the framework of Connes and Kreimer as the algebraic Birkhoff factorisation of characters on a Hopf algebra with values in a Rota-Baxter algebra, we build locality variants of these algebraic structures, leading to a locality variant of the algebraic Birkhoff factorisation. This provides an algebraic formulation of the conservation of locality while renormalising. As an application in the context of the Euler-Maclaurin formula on cones, we renormalise the exponential generating function which sums over the lattice points in convex cones. For a suitable multivariate regularisation, renormalisation from the algebraic Birkhoff factorisation amounts to composition by a projection onto holomorphic multivariate functions.

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1. INTRODUCTION

1.1. Locality in quantum field theory. Locality is a widespread notion in mathematics and physics. In physics, the principle of locality states that an object is only directly influenced by its immediate surroundings. A theory which includes the principle of locality is said to be a “local theory”. Various notions of locality are also used in analysis (local operators, local Dirichlet forms), geometry (geometric localisation, locality in index theory), algebra (localised rings) and number theory (local fields).

In classical field theory, for a classical action $\mathcal{A}(f) = \mathcal{B}(f, f)$ given by a bilinear form $\mathcal{B} : C_c^\infty(U, \mathbb{C}^k) \times C_c^\infty(U, \mathbb{C}^k) \rightarrow \mathbb{C}$ on an open subset $U \subset \mathbb{R}^n$, the locality principle translates to

$$\text{for any } f_1, f_2 \in C_c^\infty(U, \mathbb{C}^k) \text{ if } \text{Supp}(f_1) \cap \text{Supp}(f_2) = \emptyset, \text{ then } \mathcal{B}(f_1, f_2) = 0.$$

We interpret the binary relation \top defined by

$$(1) \quad f_1 \top f_2 \iff \text{Supp}(f_1) \cap \text{Supp}(f_2) = \emptyset$$

on pairs $\{f_1, f_2\}$ as an independence relation on $C_c^\infty(U, \mathbb{C}^k)$.

In quantum field theory, the locality principle governs the construction of consistent subtraction (of divergences) algorithms that preserve locality during the renormalisation process. Subtracting divergences may be interpreted as resulting from the addition to the effective action of new properly chosen terms (known as counterterms) that are local polynomials in the fields and their derivatives. A systematic algorithm to subtract divergent momentum space integrals while preserving the fundamental postulates of relativistic quantum field theory including locality was proposed by Bogoliubov, Parasiuk, Hepp and Zimmermann (abbreviated BPHZ renormalisation, and based on the so-called forest formula) [6, 17, 24]. More recently, Connes and Kreimer [8] gave an interpretation of this algorithm by means of a coproduct which enables to build – using dimensional regularisation – a renormalised map via its algebraic Birkhoff factorisation, regarded as an algebra homomorphism from the Hopf algebra of Feynman graphs to the Rota-Baxter algebra of meromorphic functions in one variable. Alternatively, following Speer [20], one can use analytic regularisation, which gives rise to a map on graphs with values in multivariate meromorphic functions.¹ In that case, locality is reflected in the fact that this map preserves locality (what we call a locality map in Definition 2.4).

Separation of supports, which in QFT reflects independence of events, also arises in the early algebraic study of locality, in terms of locality ideals, initiated by the work of H.-J. Borchers [2, 23]. Here a locality ideal is defined to be the two-sided ideal generated by commutators of test functions with space-like separated supports. Its importance comes from the fact that quantum fields satisfying the requirement of local commutativity can be regarded as Hilbert space representations of the tensor algebra annihilating the locality ideal. See [3, 4] for the recent progresses initiated by R. E. Borchers.

1.2. Locality in algebraic Birkhoff factorisation. In this paper we take an algebraic approach to investigate how locality is preserved in the renormalisation process, and choose to work in the framework of algebraic Birkhoff factorisation à la Connes and

¹In the Epstein-Glaser formalism, an analytic regularisation à la Speer yields Feynman amplitudes obeying amongst other axioms, a factorisation condition reflecting the locality principle [9, Theorem 10.1].

Kreimer [8]. Our starting point is to view locality as a symmetric binary relation comprising all pairs of independent events as in Eq. (1). Our main task is to explore the structures which are compatible with and preserve the locality, thus providing a mathematical formulation of the locality principle.

To make our point more precise, let us briefly recall the algebraic Birkhoff factorisation in the approach of Connes and Kreimer.

Theorem 1.1. (Algebraic Birkhoff factorisation, Hopf algebra version) *Let H be a connected Hopf algebra and let (A, P) be a commutative Rota-Baxter algebra of weight -1 with an idempotent Rota-Baxter operator P . Any algebra homomorphism $\phi : H \rightarrow A$ factors uniquely as the convolution product*

$$(2) \quad \phi = \phi_-^{*(-1)} \star \phi_+$$

of algebra homomorphisms $\phi_- : H \rightarrow K + P(A)$ and $\phi_+ : H \rightarrow K + (\text{Id} - P)(A)$.

As an instance of physics applications, H is the Connes-Kreimer Hopf algebra of Feynman diagrams, A is the Rota-Baxter algebra $\mathbb{C}[\varepsilon^{-1}, \varepsilon]$ of Laurent series and ϕ is the regularisation map sending a Feynman diagram to the (dimensional) regularisation of the corresponding Feynman integral with Laurent expansion in A .

We can reformulate the theorem as follows: for a multiplicative regularisation map ϕ , the renormalised map ϕ_+ is also multiplicative. Thus renormalisation preserves multiplicativity, a property which is the driving thread underlying the Hopf algebra method introduced by Connes and Kreimer. Also, it is essential in the applications of the algebraic Birkhoff factorisation in mathematics, specifically when renormalising multiple zeta values [10, 14, 19] while preserving their shuffle product and quasi-shuffle product.

In a recent study [15] of the renormalisation of conical zeta values and Euler-Maclaurin formula on lattice cones, the algebraic Birkhoff factorisation was generalised to weaken the Hopf algebra condition of H to a connected coalgebra as well as the Rota-Baxter algebra condition of A to allow for an algebra with a decomposition $A = A_1 \oplus A_2$ where only A_1 is required to be a subalgebra of A .

Theorem 1.2. (Algebraic Birkhoff factorisation, coalgebra version) [15, Theorem 4.4] *Let H be a connected coalgebra with coaugmentation $H_0 = K \oplus J$ and let A be a commutative (unital) algebra with an idempotent linear operator P on A such that $\ker P$ is a subalgebra of A . Any linear map $\phi : H \rightarrow A$ with $\phi(J) = 1_A$ factors uniquely as the convolution product*

$$(3) \quad \phi = \phi_-^{*(-1)} \star \phi_+$$

of linear maps $\phi_- : H \rightarrow K + P(A)$ and $\phi_+ : H \rightarrow K + (\text{Id} - P)(A)$ with $\phi_{\pm}(J) = 1_A$.

As our motivation and first application of this generalised algebraic Birkhoff factorisation, H is taken to be the connected coalgebra $\mathbb{Q}\mathcal{C}$, the linear span over \mathbb{Q} of the set \mathcal{C} of lattice cones, with the transverse coproduct, A is the algebra $\mathcal{M}_{\mathbb{Q}}$ of multivariate meromorphic functions with linear poles and rational coefficients, equipped with the direct sum $\mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}^+} \oplus \mathcal{M}_{\mathbb{Q}^-}$ where $\mathcal{M}_{\mathbb{Q}^+}$ is the algebra of holomorphic functions and $\mathcal{M}_{\mathbb{Q}^-}$ is the space of polar germs. See Section 2.2 for details. The linear map ϕ is

$$(4) \quad S : \mathbb{Q}\mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q}} \quad \text{with} \quad S(C, \Lambda_C)(z) = \sum_{n \in C^\circ \cap \Lambda_C} e^{(n, z)}$$

which corresponds to the exponential generating function summing over the lattice points in a lattice cone (C, Λ_C) .

Then a natural locality issue is whenever $S(C, \Lambda_C)$ and $S(D, \Lambda_D)$ are orthogonal under the locality relation on $\mathcal{M}_{\mathbb{Q}}$ induced from the \mathbb{Q} -Euclidean space, to ask whether $S_+(C, \Lambda_C)$ and $S_+(D, \Lambda_D)$ also orthogonal. Another version of the question is, if (C, Λ_C) and (D, Λ_D) are orthogonal, whether $S_+(C, \Lambda_C)$ and $S_+(D, \Lambda_D)$ are orthogonal. We reformulate this first question relative to the locality principle in renormalisation in more general terms as follows:

Problem 1.3. (Locality Conservation Principle in Renormalisation) *Consider a connected coalgebra H and a commutative algebra A with a linear map P as in Theorem 1.2. Let $\top_H \subseteq H \times H$ and $\top_A \subseteq A \times A$ be relations on H and A respectively. Let $\phi : H \rightarrow A$ be a linear map compatible with the relations in the sense that $(\phi \times \phi)(\top_H) \subseteq \top_A$. Determine the conditions under which the renormalised map ϕ_+ is also compatible with the relations.*

The first main goal of this paper is to provide a solution to this problem as a consequence of the locality generalisation (Theorem 4.10) of the algebraic Birkhoff factorisation in Theorem 1.2.

Note that the lack of multiplication in the coalgebra H means that ϕ and ϕ_+ are only linear maps, leaving out the other algebraic structures. However, as we will see next, it is precisely the interaction of the binary relations with all the existing and potential algebraic structures involved, that makes the locality principle work. This interaction of binary relations with algebraic structures leads us to partially defined binary operations, which we dubbed locality structures, throughout the whole hierarchy of algebraic structures from locality set up to locality algebra and locality coalgebra, then further to locality Rota-Baxter algebra and locality Hopf algebra.

1.3. Locality and partially defined operations. In mathematics one often encounters multiplications which are meaningful only partially even if they might be everywhere defined. This phenomenon was long known in number theory where certain functions are multiplicative only for coprime positive integers, for instance Euler's totient function $\phi(n)$ counting the number of integers modulo $n \geq 1$ which are relatively prime to n and the Ramanujan tau function $\tau(n)$ in modular forms. In fact, such phenomena have become so prevalent that such a restricted multiplicative function is simply called a multiplicative function in number theory [1]. Such operations with restrictions can be viewed in the general framework of partial algebras in universal algebra [12] (see Footnote 2).

An example of special importance for us is that of lattice cones. Even though the product given by the Minkowski sum is defined for any two convex cones, and can be extended to any two lattice cones, compatibilities with either the coalgebra structure or the regularisation maps $\phi : H \rightarrow A$, such as the exponential sum, can be expected only when the cones are orthogonal. See later sections for details on this (Propositions 3.7 and 5.2) and other instances.

This leads to another natural question: whether, in the absence of a fully defined multiplicativity that is preserved by renormalisation, as in the classical algebraic Birkhoff factorisation in Theorem 1.1, one can hope for a partially defined multiplication preserved by renormalisation. So we propose the following

Problem 1.4. (Locality Product Conservation Principle in Renormalisation)

Consider a connected bialgebra H and a unital commutative algebra A with a linear map P as in Theorem 1.1. Let $\top_H \subseteq H \times H$ and $\top_A \subseteq A \times A$ be relations on H and A respectively for which a partial multiplication $m_H : \top_H \rightarrow H$ and $m_A : \top_A \rightarrow A$ are defined. Let $\phi : H \rightarrow A$ be partially multiplicative in the sense that $\phi(m_H(u, v)) = m_A(\phi(u), \phi(v))$ for $(u, v) \in \top_H$. Determine a condition under which the renormalised map ϕ_+ is also partially multiplicative.

At this point it is worthwhile to observe the mutual effects of the interplay between locality relations and the partial algebraic structures mentioned above. In one direction it allows us to pass the locality of ϕ represented by the relations onto the corresponding renormalised map ϕ_+ , thus giving a solution of the Locality Conservation Principle in Problem 1.3. This is achieved in Theorem 4.10. In the other direction this interplay allows us to transmit the partial multiplicativity of ϕ onto ϕ_+ , thus giving a solution of the Locality Product Conservation Principle in Problem 1.4. This is achieved in Theorem 5.8.

Of particular interest to us is the exponential generating sum $S : \mathbb{QC} \rightarrow \mathcal{M}_{\mathbb{Q}}$ mentioned above. In spite of the fact that the space \mathbb{QC} of lattice cones is a genuine algebra when equipped with the extended Minkowski product, the product is not compatible with the transverse coproduct, so we do not have a bialgebra. Likewise, in the decomposition $\mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q},+} \oplus \mathcal{M}_{\mathbb{Q},-}$, even though the summand $\mathcal{M}_{\mathbb{Q},+}$, the space of holomorphic germs, is a subalgebra, the summand $\mathcal{M}_{\mathbb{Q},-}$, the space of polar germs, is not. Hence this decomposition does not give a Rota-Baxter algebra. Furthermore, the linear map S does not send the Minkowski product on \mathbb{QC} to the function multiplication in $\mathcal{M}_{\mathbb{Q}}$. However if one considers only orthogonal pairs of lattice cones and suitable orthogonal relation of meromorphic germs, all these structures can be recovered in the form of locality structures. In fact \mathbb{QC} is not only a locality bialgebra, it is a connected locality Hopf algebra. Moreover, $\mathcal{M}_{\mathbb{Q},-}$ is not only a locality subalgebra, it is a locality ideal, showing that the projection $\pi_+ : \mathcal{M}_{\mathbb{Q}} \rightarrow \mathcal{M}_{\mathbb{Q},+}$ onto $\mathcal{M}_{\mathbb{Q},+}$ along $\mathcal{M}_{\mathbb{Q},-}$ is a locality algebra homomorphism. Consequently, the full algebraic Birkhoff factorisation can be recovered on the locality level, from which it then follows that the renormalised map ϕ_+ is a locality algebra homomorphism. Further the locality ideal property of $\mathcal{M}_{\mathbb{Q},-}$ implies that its convolution inverse ϕ_+^{*-1} is ϕ composed with the projection of π_+ (see Eq. (37)). This applies whenever the regularisation map ϕ is a locality algebra homomorphism and takes values in $\mathcal{M}_{\mathbb{Q}}$. In this situation a recursive formula for ϕ_- in terms of the projections π_+ and $\text{Id} - \pi_+$ is given in (29), which is reminiscent of the forest formula of the renormalisation of Feynman graphs in the context of Quantum Field Theory.

1.4. Outline of the paper. We next give a summary of the locality construction as an outline of the paper.

We consider vector spaces H and A , each equipped with a suitable symmetric binary relation, and a linear map $\phi : H \rightarrow A$ preserving the relations. In order to pass this property of ϕ onto the corresponding renormalised map ϕ_+ via the algebraic Birkhoff factorisation for a suitably enriched H , A and ϕ , we need to make the relations compatible with all the algebraic structures involved in the algebraic Birkhoff factorisation, including a Hopf algebra or bialgebra structure on H , a Rota-Baxter algebra or algebra structure on A , and the corresponding structures on ϕ .

Throughout the paper we use the space of convex lattice cones, the space of meromorphic functions and the exponential generating sum in Eq. (4) between the two spaces as

the primary examples of the algebraic constructions, and of our main theorems on locality. Further applications will be given in subsequent work, such as locality of branched zeta values [7]. The locality relation relates to Weinstein’s “congeniality” condition [21, 22] in a selective category, which in turn is motivated by the drive to build a quantising functor from the category of canonical relations between symplectic manifolds to a category of quantum morphisms. We hope to explore these relations in future work.

Thus we begin in Section 2 with the general concepts of a locality set and locality map, emphasising examples on lattice cones and meromorphic functions, while mentioning several other examples in passing. In Section 3, we equip a locality set with a compatible associative multiplication to give a locality semigroup, locality monoid and locality group. Then through the intermediate structure of a locality vector space, we obtain a locality algebra and further a locality Rota-Baxter algebra. In Section 4, we consider the coalgebraic aspect of the construction which begins with the preliminary but subtle notion of locality tensor product. With it, we introduce the concepts of locality coalgebra, the convolution product for maps from a locality coalgebra to a locality algebra. At this point we can give our first main result, Theorem 4.10, addressing the Locality Conservation Principle in Problem 1.3. Finally in Section 5, we bring the locality algebra and locality coalgebra together to form a locality bialgebra and further a locality Hopf algebra under an extra connectedness condition. Then we prove our second main result, Theorem 5.8, addressing the Locality Product Conservation Principle in Problem 1.4. Both results are applied to the example of the exponential generating sum $S : \mathbb{QC} \rightarrow \mathcal{M}_{\mathbb{Q}}$, showing that the orthogonality property and the partial multiplicativity on orthogonal pairs of convex lattice cones are indeed preserved by the renormalised version of S .

Notations. Unless otherwise specified, all algebras are taken to be unitary commutative over a field K , and linear maps and tensor products are taken over K . A nonunitary algebra means one which does not necessarily have a unit. The same applies to coalgebras.

2. LOCALITY FOR SETS AND MAPS

2.1. Concepts of locality sets and locality maps. We begin with a set with an independent relation.

Definition 2.1. (i) An **locality set** is a couple (X, \top) where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X , referred to as the **locality relation** (or **independence relation**) of the locality set. So for $x_1, x_2 \in X$, denote $x_1 \top x_2$ if $(x_1, x_2) \in \top$. When the underlying set X needs to be emphasised, we use the notation $X \times_{\top} X$ or \top_X for \top .

(ii) For any subset U of a locality set (X, \top) , let

$$U^{\top} := \{x \in X \mid (U, x) \subseteq X \times_{\top} X\}$$

be the **polar subset** of U .

Remark 2.2. (i) Thus a locality set is simply a set with a binary symmetric relation: we use the term locality set to be consistent with the derived terminology to be introduced later for various algebraic structures built on top of the locality set.

(ii) The binary relation \top plays two roles in our study which are related and yet complementary. On the one hand, it serves as a condition under which two elements are related in various ways (independent, orthogonal, etc.), hence the symmetry requirement. As noted in the introduction, the locality relation is intended to

encode the notion of independence of events in physics (thus the alternative name independence relation); on the other hand, it assigns the subset of $X \times X$ as the domain for partial binary operations in the context of universal algebra [12]. Strictly speaking, the symmetric condition is not required for the latter purpose even though in most of our applications, the algebras are commutative and hence the domain is symmetric. As the two roles are so closely related, we will only deal with symmetric relations unless otherwise needed.

From a locality set one can derive other locality sets as follows.

Lemma 2.3. *Let (X, \top) be a locality set.*

- (i) *For a subset X' of X , denote $\top' := (X' \times X') \cap \top$. Then the pair (X', \top') is a locality set, called a **sub-locality set** of (X, \top) ;*
- (ii) *For subsets A and B of (X, \top) , denote $A \top^{\mathcal{P}} B$, or simply $A \top B$ should the context be clear, if $A \times B \subseteq \top$. Then $\top^{\mathcal{P}}$ equips the power set $\mathcal{P}(X)$ of X with a locality set structure;*
- (iii) *Combining the above two items, any subset Y of $\mathcal{P}(X)$ with the restriction of $\top^{\mathcal{P}}$ defines a locality set $(Y, (Y \times Y) \cap \top^{\mathcal{P}})$.*

As a very simple yet fundamental example which underlies the more sophisticated examples to be discussed below, the orthogonality relation \perp^Q between vectors or subsets in an Euclidean vector space (V, Q) equips V or the set of subsets of V with the structure of a locality set.

Definition 2.4. A **locality map** from a locality set (X, \top_X) to a locality set (Y, \top_Y) is a map $\phi : X \rightarrow Y$ such that $(\phi \times \phi)(\top_X) \subseteq \top_Y$. More generally, maps $\phi, \psi : (X, \top_X) \rightarrow (Y, \top_Y)$ are called **independent** and denoted $\phi \top \psi$ if $(\phi \times \psi)(\top_X) \subseteq \top_Y$. To be specific, if $\phi \neq \psi$, ϕ is called independent of ψ .

Example 2.5. Any orthogonal map between two Euclidean vector spaces $(V_i, Q_i), i = 1, 2$, is a locality map between the locality spaces (V_i, \perp^{Q_i}) .

Remark 2.6. (i) The identity map on a locality set (X, \top) is trivially a locality map. Also, the composition of two locality maps is still a locality map. Thus locality sets and locality maps form a category.

- (ii) A map independent of the identity is a locality map. Indeed let (Ω, \top_Ω) be a locality set and $\phi : \Omega \mapsto \Omega$ be a map such that $\phi \top \text{Id}_\Omega$. Then for any $(x, y) \in \top_\Omega$ we have

$$\phi(x) \top_\Omega \text{Id}_\Omega(y) \Rightarrow y \top_\Omega \phi(x) \Rightarrow \phi(y) \top_\Omega \text{Id}_\Omega(\phi(x)) \Rightarrow \phi(x) \top_\Omega \phi(y).$$

Note that here we need the symmetric condition.

2.2. Examples of locality sets and locality maps.

2.2.1. *Convex lattice cones and meromorphic functions.* We now give some background for the main examples which serve as both the motivation and prototype of the theoretical structures in this paper. See [15, 16] for details.

Our first example of locality sets is given by convex polyhedral lattice cones. Consider the filtered rational Euclidean lattice space

$$\left(\mathbb{R}^\infty = \bigcup_{\geq 1} \mathbb{R}^k, \mathbb{Z}^\infty = \bigcup_{\geq 1} \mathbb{Z}^k, Q = (Q_k(\cdot, \cdot))_{k \geq 1} \right),$$

where

$$Q_k(\cdot, \cdot) : \mathbb{R}^k \otimes \mathbb{R}^k \rightarrow \mathbb{R}, \quad k \geq 1,$$

is an inner product in \mathbb{R}^k such that $Q_{k+1}|_{\mathbb{R}^k \otimes \mathbb{R}^k} = Q_k$ and $Q_k(\mathbb{Z}^k \otimes \mathbb{Z}^k) \subset \mathbb{Q}$. A lattice cone is a pair (C, Λ_C) where C is a polyhedral cone in some \mathbb{R}^k , that is,

$$C = \langle u_1, \dots, u_m \rangle := \left\{ \sum_{i=1}^m c_i u_i \mid c_i \in \mathbb{R}_{\geq 0}, 1 \leq i \leq m \right\}$$

for some $u_1, \dots, u_m \in \mathbb{Q}^k$, and Λ_C is a lattice in the linear subspace spanned by C . Let \mathcal{C}_k be the set of lattice cones in \mathbb{R}^k and

$$\mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}_k$$

be the set of lattice cones in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$. Let $\mathbb{Q}\mathcal{C}_k$ and $\mathbb{Q}\mathcal{C}$ be the linear spans of \mathcal{C}_k and \mathcal{C} over \mathbb{Q} .

In $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$, we write \perp^Q for the corresponding orthogonality relation.

Definition 2.7. We call two lattice cones (C, Λ_C) and (D, Λ_D) **orthogonal** (with respect to Q), if $Q(u, v) = 0$ for all $u \in \Lambda_C, v \in \Lambda_D$. Then we write $(C, \Lambda_C) \perp^Q (D, \Lambda_D)$.

Multivariate meromorphic functions provide another fundamental motivation for the forthcoming algebraic setup. Again in $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$, let $\mathcal{M}_{\mathbb{Q}}((\mathbb{R}^k)^* \otimes \mathbb{C})$ be the space of meromorphic germs at 0 with linear poles and rational coefficients [15, 16] and let

$$(5) \quad \mathcal{M}_{\mathbb{Q}} := \bigcup_{k \geq 1} \mathcal{M}_{\mathbb{Q}}((\mathbb{R}^k)^* \otimes \mathbb{C}).$$

An element of $\mathcal{M}_{\mathbb{Q}}$ can be written as a sum of a holomorphic germ and elements the form

$$(6) \quad \frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \dots L_n^{s_n}}, \quad s_1, \dots, s_n \in \mathbb{Z}_{>0},$$

where h is a holomorphic germ with rational coefficients in linear forms $\ell_1, \dots, \ell_m \in \mathbb{Q}^k$, and L_1, \dots, L_n are linearly independent linear forms in \mathbb{Q}^k , $\ell_i \perp^Q L_j$ for all $i \in \{1, \dots, m\}$ for all $j \in \{1, \dots, n\}$, which is called a **polar germ**.

Definition 2.8. Two meromorphic germs with rational coefficients f and f' are **Q-orthogonal** which we denote by $f \perp^Q f'$ if there exist linear functions $L_1, \dots, L_m \in \mathbb{Q}^k$ and $L'_1, \dots, L'_n \in \mathbb{Q}^k$ satisfying $Q(L_i, L'_j) = 0$ for $i = 1, \dots, m, j = 1, \dots, n$, and meromorphic germs $g \in \mathcal{M}_{\mathbb{Q}}(\mathbb{R}^m \otimes \mathbb{C})$ and $g' \in \mathcal{M}_{\mathbb{Q}}(\mathbb{R}^n \otimes \mathbb{C})$, such that $f = g(L_1, \dots, L_m)$, $f' = g'(L'_1, \dots, L'_n)$. Let $(\mathcal{M}_{\mathbb{Q}}, \perp^Q)$ denote the resulting locality set.

We next give examples of locality maps. Let (C, Λ_C) be a strongly convex lattice cone in \mathbb{R}^k with interior C° . For z in the dual cone

$$C^- := \{z \in (\mathbb{R}^k)^* \mid \langle x, z \rangle < 0, \forall x \in C\},$$

we define its **exponential generating function** to be the sum

$$S(C, \Lambda_C)(z) := \sum_{n \in C^\circ \cap \Lambda_C} e^{\langle n, z \rangle}.$$

We also define its **exponential integral** $I(C, \Lambda_C)$ to be the integral

$$I(C, \Lambda_C)(z) := \int_C e^{\langle x, z \rangle} d\Lambda_x,$$

where $d\Lambda_x$ is the volume form induced by generators of Λ_C such that the polytope generated by a basis of Λ_C has volume 1.

These assignments extend by subdivisions to maps:

$$(7) \quad S, I : \mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q}}.$$

Proposition 2.9. *For lattice cones (C, Λ_C) and (D, Λ_D) , if $(C, \Lambda_C) \perp^{\mathcal{Q}} (D, \Lambda_D)$, then $S(C, \Lambda_C)(z) \perp^{\mathcal{Q}} S(D, \Lambda_D)(z)$ and $I(C, \Lambda_C)(z) \perp^{\mathcal{Q}} I(D, \Lambda_D)(z)$ in the sense of Definition 2.8, that is, the exponential integral and exponential generating function maps I and S are locality maps.*

Proof. For any subdivision $\{(C_i, \Lambda_{C_i})\}$ of (C, Λ_C) , since $\Lambda_{C_i} = \Lambda_C$, we know $(C_i, \Lambda_{C_i}) \perp^{\mathcal{Q}} (D, \Lambda_D)$. Because any lattice cone can be subdivided into smooth lattice cones, we can reduce the proof to smooth lattice cones.

For a smooth lattice cone $(C, \Lambda_C) = (\langle u_1, \dots, u_n \rangle, \sum_{i=1}^n \mathbb{Z}u_i)$, we have

$$S(C, \Lambda_C)(z) = \prod_{i=1}^n \frac{e^{\langle u_i, z \rangle}}{1 - e^{\langle u_i, z \rangle}}; \quad I(C, \Lambda_C)(z) = \prod_{i=1}^k \frac{1}{\langle u_i, z \rangle}.$$

So $S(C, \Lambda_C)(z)$ and $S(D, \Lambda_D)(z)$ (resp. $I(C, \Lambda_C)(z)$ and $I(D, \Lambda_D)(z)$) are meromorphic germs in perpendicular linear functions. Therefore $S(C, \Lambda_C)(z) \perp^{\mathcal{Q}} S(D, \Lambda_D)(z)$ and $I(C, \Lambda_C)(z) \perp^{\mathcal{Q}} I(D, \Lambda_D)(z)$. \square

2.2.2. Other examples. There are many other examples of locality sets. To save space, we only briefly list some of them and refer the reader to the references for further details.

A large number of examples come from disjointness of subsets noted in Lemma 2.3.

- (i) On the one hand, locality structures can be built on functions or distributions by requiring the disjointness of (adequately chosen) supports of such maps, such as their ordinary supports, their singular supports or wavefront sets (see [5] and references therein).
- (ii) On the other hand, locality structures on maps can be built by requiring disjointness of their image sets. This is the case for the decorating maps from the vertices of graphs or trees to a decorating set, which yields locality sets of labelled graphs and trees [11].
- (iii) When the set S is equipped with a linear structure, we can replace disjointness by trivial intersections, transversality or linear independence. If S further has an inner product, then the relation can be chosen to be the one of orthogonality.

Further examples of locality sets include independence of events in probability and coprimeness of natural numbers as discussed in the introduction.

3. BUILDING UP LOCALITY FROM SEMIGROUPS TO ROTA-BAXTER ALGEBRAS

In this section, we equip a locality set with various algebraic structures, from that of a semigroup to that of a Rota-Baxter algebra.

3.1. Locality semigroups. For a locality set (X, T) and an integer $k \geq 2$, denote

$$(8) \quad X^{\top k} := \underbrace{X \times_{\top} \dots \times_{\top} X}_{k \text{ factors}} := \{(x_1, \dots, x_k) \in X^k \mid (x_i, x_j) \in \top, 1 \leq i \neq j \leq k\}.$$

Definition 3.1. (i) An **locality semigroup**² is a locality set (G, \top) together with a product law defined on \top :

$$m_G : G \times_{\top} G \longrightarrow G$$

for which the product is compatible with the locality relation on G , namely

$$(9) \quad \text{for all } U \subseteq G, \quad m_G((U^{\top} \times U^{\top}) \cap \top) \subset U^{\top}$$

and such that the following **locality associativity property** holds:

$$(10) \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) \text{ for all } (x, y, z) \in G \times_{\top} G \times_{\top} G.$$

Note that, because of the condition (9), both sides of Eq. (10) are well-defined for any triple in the given subset.

- (ii) An locality semigroup is **commutative** if $m_G(x, y) = m_G(y, x)$ for $(x, y) \in \top$, noting that both sides of the equations are defined since \top is symmetric.
- (iii) An **locality monoid** is a locality semigroup (G, \top, m_G) together with a **unit element** $1_G \in G$ given by the defining property

$$\{1_G\}^{\top} = G \quad \text{and} \quad m_G(x, 1_G) = m_G(1_G, x) = x \quad \text{for all } x \in G.$$

We denote the locality monoid by $(G, \top, m_G, 1_G)$.

- (iv) An **locality group** is a locality monoid $(G, \top, m_G, 1_G)$ equipped with a morphism $\iota : G \rightarrow G$ of locality sets, called the **inverse map**, such that $(\iota(g), g) \in \top$ and $m_G(\iota(g), g) = m_G(g, \iota(g)) = 1_G$ for any $g \in G$.
- (v) A **sub-locality semigroup** of a locality semigroup (G, \top, m_G) is a locality semigroup $(G', \top', m_{G'})$ with $G' \subseteq G$, $\top' = (G' \times G') \cap \top$ and $m_{G'} = m_G|_{\top'}$, that is, for $x, y \in G'$ and $(x, y) \in \top$, $m_G(x, y)$ is in G' . A **sub-locality monoid** of a locality monoid is a sub-locality semigroup of the corresponding locality semigroup which share the same unit. A **sub-locality group** of a locality group is a sub-locality monoid of the corresponding locality monoid which is also a locality group.

For notational convenience, we usually abbreviate $m_G(x, y)$ by $x \cdot y$ or simply xy .

Remark 3.2. One easily checks that on a locality monoid $(G, \top, m_G, 1_G)$ if (x_1, x_2, y_1, y_2) is in $G^{\top 4}$, then $(x_1 x_2, y_1, y_2)$ and $(x_1, x_2, y_1 y_2)$ are in $G^{\top 3}$ and hence $(x_1 x_2, y_1 y_2) \in \top$.

As a simple counter example of locality semigroup, we have

Counterexample 3.3. The set G of linear subspaces of \mathbb{R}^2 is a locality set with respect to the following relation \top_G on linear subspaces of G : $U, V \subseteq \mathbb{R}^2$ are called transverse if they intersect trivially, namely if $U \cap V = \{0\}$. The set G equipped with linear sums $+$ is a monoid. But the corresponding $(G, \top_G, +)$ is not a locality monoid. Indeed, for the standard basis $\{e_1, e_2\}$ of \mathbb{R}^2 , the subspaces $\mathbb{R}e_1$ and $\mathbb{R}e_2$ both intersect $\mathbb{R}(e_1 + e_2)$ trivially, but $\mathbb{R}e_1 + \mathbb{R}e_2$ does not.

Example 3.4. The locality set $(\mathcal{M}_{\mathbb{Q}}, \perp^{\mathbb{Q}})$ in Definition 2.8, equipped with the restricted multiplication $m_{\mathbb{Q}} : \mathcal{M}_{\mathbb{Q}} \times_{\perp^{\mathbb{Q}}} \mathcal{M}_{\mathbb{Q}} \mapsto \mathcal{M}_{\mathbb{Q}}$, is a locality monoid.

²As a special case of partial algebras [12], the terminology ‘‘partial semigroup’’ is used for a set equipped with a partial associative product defined only for certain pairs of elements in the set. The condition for a locality semigroup is more restrictive than that of a partial semigroup in that the former requires that the pairs for which the partial product is defined stem from a symmetric relation and that the partial product should be compatible with the locality relation in the sense of Eq. (9).

Definition 3.5. Let (X, \top_X, \cdot_X) and (Y, \top_Y, \cdot_Y) (resp. $(X, \top_X, \cdot_X, 1_X)$ and $(Y, \top_Y, \cdot_Y, 1_Y)$) be locality semigroups (resp. monoids). A map $\phi : X \rightarrow Y$ is called a **locality semigroup** (resp. **locality monoid**) **homomorphism**, if it

- (i) is a locality map;
- (ii) is **locality multiplicative**: for $(a, b) \in \top_X$ we have $\phi(a \cdot_X b) = \phi(a) \cdot_Y \phi(b)$,
- (iii) (resp. preserves the unit $\phi(1_X) = 1_Y$.)

Example 3.6. Classical examples of locality monoid homomorphisms are given by **multiplicative functions** in number theory. Here a function $f : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 1}$ is multiplicative means $f(1) = 1$ and $f(mn) = f(m)f(n)$ if m and n are coprime. This means precisely that f is a locality monoid homomorphism from the locality monoid $(\mathbb{Z}_{\geq 1}, \top_{\text{cop}})$ where \top_{cop} is the coprime relation, to the locality monoid $(\mathbb{Z}_{\geq 1}, \top_{\text{full}})$, where \top_{full} is the full relation $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

Now let us take a closer look at the set \mathcal{C} of lattice cones. For convex cones $C := \langle u_1, \dots, u_m \rangle$ and $D := \langle v_1, \dots, v_n \rangle$ spanned by u_1, \dots, u_m and v_1, \dots, v_n respectively, their **Minkowski product** (usually called Minkowski sum) is the convex cone

$$C \cdot D := \langle u_1, \dots, u_m, v_1, \dots, v_n \rangle.$$

This product can be extended to a product in \mathcal{C} :

$$(11) \quad (C, \Lambda_C) \cdot (D, \Lambda_D) := (C \cdot D, \Lambda_C + \Lambda_D),$$

where $\Lambda_C + \Lambda_D$ is the abelian group generated by Λ_C and Λ_D in \mathbb{Q}^∞ . This product endows a monoid structure on \mathcal{C} with unit $(\{0\}, \{0\})$, which also restricts to a locality monoid structure on $(\mathcal{C}, \perp^{\mathcal{Q}})$.

Even though \mathcal{C} and $\mathcal{M}_{\mathbb{Q}}$ have their natural multiplications defined on the full spaces, the importance of the locality structures on \mathcal{C} and $\mathcal{M}_{\mathbb{Q}}$ becomes evident when studying the multiplicative property of the maps I and S from \mathcal{C} to $\mathcal{M}_{\mathbb{Q}}$ introduced in Section 2.2. Because of the idempotency $(C, \Lambda_C) \cdot (C, \Lambda_C) = (C, \Lambda_C)$ for $(C, \Lambda_C) \in \mathcal{C}$, the multiplicativity $I((C, \Lambda_C) \cdot (C, \Lambda_C))(z) = I(C, \Lambda_C)(z)I(C, \Lambda_C)(z)$ or $S((C, \Lambda_C) \cdot (C, \Lambda_C))(z) = S(C, \Lambda_C)(z)S(C, \Lambda_C)(z)$ does not hold in general since that would force the integral or the sum to be 0 or 1, which can not be the case for example by taking $(C, \Lambda_C) = (\langle e_1 \rangle, \mathbb{Z}e_1)$. But the multiplicativity can be recovered in the context of locality monoids, as follows.

Proposition 3.7. *If $(C, \Lambda_C) \perp^{\mathcal{Q}} (D, \Lambda_D)$, then*

$$S((C, \Lambda_C) \cdot (D, \Lambda_D)) = S(C, \Lambda_C)S(D, \Lambda_D), \quad I((C, \Lambda_C) \cdot (D, \Lambda_D)) = I(C, \Lambda_C)I(D, \Lambda_D).$$

Thus the locality maps I and S are locality semigroup homomorphisms from $(\mathcal{C}, \perp^{\mathcal{Q}})$ to $(\mathcal{M}_{\mathbb{Q}}, \perp^{\mathcal{Q}})$.

Proof. For $(C, \Lambda_C) \perp^{\mathcal{Q}} (D, \Lambda_D)$, if both are smooth, let $C := \langle u_1, \dots, u_m \rangle$ with $\Lambda_C = \oplus \mathbb{Z}u_i$, and $D := \langle v_1, \dots, v_n \rangle$ with $\Lambda_D = \oplus \mathbb{Z}v_j$. Since $Q(u_i, v_j) = 0$, we have $\Lambda_C + \Lambda_D = (\oplus_i \mathbb{Z}u_i) \oplus (\oplus_j \mathbb{Z}v_j)$. Thus $(C, \Lambda_C) \cdot (D, \Lambda_D)$ is smooth. By a direct calculation, we obtain

$$S((C, \Lambda_C) \cdot (D, \Lambda_D)) = S(C, \Lambda_C)S(D, \Lambda_D), \quad I((C, \Lambda_C) \cdot (D, \Lambda_D)) = I(C, \Lambda_C)I(D, \Lambda_D).$$

The general case follows by a subdivision of lattice cones into smooth lattice cones. \square

Moreover, the locality monoid $(\mathcal{C}, \perp^{\mathcal{Q}})$ has a natural grading which does not extend to the monoid \mathcal{C} . For $n \geq 0$, let \mathcal{C}_n denote the subset of \mathcal{C} consisting of lattice cones of

dimension n . Then for orthogonal lattice cones (C, Λ_C) and (D, Λ_D) , we have $\dim((C, \Lambda_C) \cdot (D, \Lambda_D)) = \dim(C, \Lambda_C) + \dim(D, \Lambda_D)$. Hence,

$$(12) \quad \mathcal{C} = \sqcup_{n \geq 0} \mathcal{C}_n, \quad m_{\mathcal{C}}((\mathcal{C}_m \times \mathcal{C}_n) \cap \perp^{\mathcal{Q}}) \subseteq \mathcal{C}_{m+n} \quad \text{for all } m, n \geq 0,$$

which we take to be the defining conditions for \mathcal{C} to be a **graded locality monoid**.

3.2. Locality vector spaces. We now consider locality relations on vector spaces.

Definition 3.8. An **locality vector space** is a vector space V equipped with a locality relation \top which is compatible with the linear structure on V in the sense that, for any subset X of V , X^\top is a linear subspace of V .

Remark 3.9. For a locality vector space (V, \top) , since V^\top is a linear subspace of V , we have $\{0\} \times V \subset \top$, or equivalently $0 \in V^\top$. Note that there is no locality restrictions for the vector space structure (addition and scalar product) on V , that is, the addition and scalar product are everywhere defined.

Example 3.10. The vector space $\mathcal{M}_{\mathbb{Q}}$ equipped with the relation $\perp^{\mathcal{Q}}$ in Definition 2.8 is a locality vector space $(\mathcal{M}_{\mathbb{Q}}, \perp^{\mathcal{Q}})$.

Remark 3.11. Clearly, constant germs are orthogonal of any germs, namely $\mathbb{R} \subseteq \mathcal{M}_{\mathbb{Q}}^{\perp^{\mathcal{Q}}}$. In fact, the converse is also true. Thus $\mathcal{M}_{\mathbb{Q}}^{\perp^{\mathcal{Q}}} = \mathbb{R}$.

From a locality set (X, \top) we can build a locality vector space (KX, \top) from the vector space generated by X whose defining relation (denoted by the same symbol \top) is the linear extension of that on X . More precisely for $u, v \in KX$, $(u, v) \in \top$ if the basis elements from X appearing in u are related via \top to the basis elements appearing in v . Thus

$$KX \times_{\top_{KX}} KX = \bigcup_{U, V \subseteq X, (U, V) \subseteq \top} KU \times KV.$$

Example 3.12. The locality set \mathcal{C} of lattice cones with the orthogonal relation in Definition 2.7 gives rise to the corresponding locality vector space $\mathbb{Q}\mathcal{C}$. Likewise, the locality set of labelled rooted trees described in Section 2.2.2 gives rise to the corresponding locality vector space generated by the set.

Definition 3.13. Let (U, \top_U) and (V, \top_V) be locality vector spaces, a linear map $\phi : (U, \top_U) \rightarrow (V, \top_V)$ is called a **locality linear map** if it is a locality map.

Example 3.14. The locality maps given by the exponential integral $I : \mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q}}$ and the exponential generating sum $S : \mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q}}$ in Proposition 2.9 extend to locality linear maps from $\mathbb{Q}\mathcal{C}$ to $\mathcal{M}_{\mathbb{Q}}$.

Here are further useful properties of locality linear maps.

Proposition 3.15. *Let $(U, \top_U), (V, \top_V)$ be locality vector spaces and $\phi, \psi : (U, \top_U) \rightarrow (V, \top_V)$ be independent locality linear maps. Any two linear combinations of ϕ and ψ are also independent. In particular, any linear combination $\lambda\phi + \mu\psi$ with $\lambda, \mu \in K$ is a locality linear map.*

Proof. Let u_1, u_2 be in \top_U . Since ϕ and ψ are independent locality linear maps, we have $\{\phi(u_1), \psi(u_1)\} \top_V \{\phi(u_2), \psi(u_2)\}$ and hence $(\lambda\phi(u_1) + \mu\psi(u_1)) \top_V (\lambda\phi(u_2) + \mu\psi(u_2))$. \square

3.3. Locality algebras. We begin with a preliminary notion. Let V and W be vector spaces and let $\top := V \times_{\top} W \subseteq V \times W$. A map $f : V \times_{\top} W \rightarrow U$ to a vector space U is called **locality bilinear** if

$$\begin{aligned} f(v_1 + v_2, w_1) &= f(v_1, w_1) + f(v_2, w_1), & f(v_1, w_1 + w_2) &= f(v_1, w_1) + f(v_1, w_2), \\ f(kv_1, w_1) &= kf(v_1, w_1), & f(v_1, kw_1) &= kf(v_1, w_1) \end{aligned}$$

for all $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $k \in K$ such that all the pairs arising in the above expressions are in $V \times_{\top} W$.

Definition 3.16. (i) A **nonunitary locality algebra** over K is a locality vector space (A, \top) over K together with a locality bilinear map

$$m_A : A \times_{\top} A \rightarrow A$$

such that (A, \top, m_A) is a locality semi-group in the sense of Definition 3.1.(i).

- (ii) An **locality algebra** is a nonunitary locality algebra (A, \top, m_A) together with a **unit** $1_A : K \rightarrow A$ in the sense that $(A, \top, m_A, 1_A)$ is a locality monoid defined in Definition 3.1. (iii). We shall omit explicitly mentioning the unit 1_A and the product m_A unless this generates an ambiguity.
- (iii) A linear subspace B of a locality algebra (A, \top, m_A) is called a **sub-locality algebra** of A if B is a sub-locality semigroup of A in the sense of Definition 3.1.(v).
- (iv) A sub-locality algebra I of a locality commutative algebra (A, \top, m_A) is called a **locality ideal** of A if for any $b \in I$ we have $b^{\top} \cdot b \subseteq I$ for all $b^{\top} \in \{b\}^{\top}$.
- (v) An locality-linear map $f : (A, \top_A, \cdot_A) \rightarrow (B, \top_B, \cdot_B)$ between two (non necessarily unital) locality algebras is called a **locality algebra homomorphism** if

$$(13) \quad f(u \cdot_A v) = f(u) \cdot_B f(v) \quad \text{for all } (u, v) \in \top_A.$$

- (vi) A locality algebra A with a linear grading $A = \bigoplus_{n \geq 0} A_n$ is called a locality graded algebra if $m_A((A_m \times A_n) \cap \top_A) \subseteq A_{m+n}$ for all $m, n \in \mathbb{Z}$.

It is easy to check that a locality linear map $f : (A, \top_A, \cdot_A) \rightarrow (B, \top_B, \cdot_B)$ between two locality algebras is a locality algebra homomorphism if and only if $\ker f$ is a locality ideal of A , by the same argument as the one for the corresponding result on an algebra homomorphism.

Remark 3.17. (i) For a locality algebra (A, \top) we have $\{0, 1_A\} \subset A^{\top}$ since $0 \in A^{\top}$ by Remark 3.9.

- (ii) If $A \times_{\top} A$ is $A \times A$ in a locality monoid and locality algebra, we recover the usual notions of monoid and algebra.

The locality space \mathbb{QC} , with the multiplication obtained from the linear extension of the locality monoid structure on \mathcal{C} by the Minkowski product in Eq. (11), is a locality commutative algebra. By Eq. (12), we in fact have

Lemma 3.18. *With the grading induced from $\mathcal{C} = \sqcup_{n \geq 0} \mathcal{C}_n$ in Eq. (12), $\mathbb{QC} = \bigoplus_{n \geq 0} \mathbb{QC}_n$ is a graded locality algebra.*

Another important locality algebra for our purpose is the space $(\mathcal{M}_{\mathbb{Q}}, \perp^{\mathbb{Q}})$ with $\perp^{\mathbb{Q}}$ as in Definition 2.8 and the pointwise multiplication. Further by Proposition 3.7, the linear maps

$$(14) \quad S, I : \mathbb{QC} \rightarrow \mathcal{M}_{\mathbb{Q}}$$

linearly extended from those in Eq. (7), are locality algebra homomorphisms.

We can say more about the locality algebra $\mathcal{M}_{\mathbb{Q}}$. By [16, Corollary 4.7], there is a direct sum decomposition

$$(15) \quad \mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q},+} \oplus \mathcal{M}_{\mathbb{Q},-}^{\mathbb{Q}},$$

where $\mathcal{M}_{\mathbb{Q},+}$ is the subspace of holomorphic functions and $\mathcal{M}_{\mathbb{Q},-}^{\mathbb{Q}}$ is the subspace spanned by polar germs defined by Eq. (6). Then we have the following result from [16, Corollary 4.18] reformulated in the terminology of locality structures.

Proposition 3.19. *In the decomposition in Eq. (15), the space $\mathcal{M}_{\mathbb{Q},+}$ is a subalgebra and a locality subalgebra of $\mathcal{M}_{\mathbb{Q}}$. The space $\mathcal{M}_{\mathbb{Q},-}^{\mathbb{Q}}$ is not a subalgebra but a locality subalgebra, in fact a locality ideal of $\mathcal{M}_{\mathbb{Q}}$. Consequently, the projection $\pi_+^{\mathbb{Q}} : \mathcal{M}_{\mathbb{Q}} \rightarrow \mathcal{M}_{\mathbb{Q},+}$ is a locality algebra homomorphism.*

In contrast to the multivariate case, the space $\mathcal{M}_{\mathbb{Q},-}^{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C}) = \epsilon^{-1}\mathbb{C}[\epsilon^{-1}]$ is a subalgebra in the space $\mathcal{M}_{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C})$ of meromorphic functions in one variable. This is a major difference between our multivariate setup and the usual single variate framework used for renormalisation purposes. We circumvent the difficulty in relaxing ordinary multiplicativity to a multiplicativity allowed only on independent elements. In fact, $\mathcal{M}_{\mathbb{Q},-}^{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C})$ is a locality ideal of $\mathcal{M}_{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C})$ under the restriction of independence relation since the locality relation $\perp^{\mathbb{Q}}$ restricted to $\mathcal{M}_{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C})$ is simply $(\mathbb{C} \times \mathcal{M}_{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C})) \cup (\mathcal{M}_{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C}) \times \mathbb{C})$. Thus, the locality algebra homomorphism $\pi_+^{\mathbb{Q}}$ restricts to a mere linear map on $\mathcal{M}_{\mathbb{Q}}(\mathbb{R}^* \otimes \mathbb{C})$ with no additional multiplicativity property.

3.4. Locality Rota-Baxter algebras and projection maps. The reader is referred to [13] for background on Rota-Baxter algebras.

Definition 3.20. A linear operator $P : A \rightarrow A$ on a locality algebra (A, \top) over a field K is called a **locality Rota-Baxter operator of weight $\lambda \in K$** , or simply a **Rota-Baxter operator**, if it is a locality map, independent of Id_A , and satisfies the following **locality Rota-Baxter relation**:

$$(16) \quad P(a)P(b) = P(P(a)b) + P(aP(b)) + \lambda P(ab) \quad \text{for all } (a, b) \in \top.$$

We call the triple (A, \top, P) a **locality Rota-Baxter algebra**.

Remark 3.21. (i) The right hand side of Eq. (16) is well defined due to the condition that P is independent of the identity.

(ii) As in the classical setup [13, Proposition 1.1.12], if P is a locality Rota-Baxter operator of weight λ , then $-\lambda - P$ is also a locality Rota-Baxter of weight λ .

An important class of locality Rota-Baxter algebras arises from idempotent operators.

Proposition 3.22. *Let (A, \top, m_A) be a locality algebra. Let $P : A \rightarrow A$ be a locality linear idempotent operator in which case there is a linear decomposition $A = A_1 \oplus A_2$ with $A_1 = \ker(\text{Id} - P)$ and $A_2 = \ker(P)$ so that P is the projection onto A_1 along A_2 . The following statements are equivalent:*

- (i) P or $\text{Id} - P$ is a locality Rota-Baxter operator of weight -1 ;
- (ii) A_1 and A_2 are locality subalgebras of A , and P and $\text{Id} - P$ are independent locality maps.

If one of the conditions holds, then P is a locality multiplicative map if and only if A_2 is a locality ideal of A .

Proof. We write $\pi_1 = P$ and $\pi_2 = \text{Id} - P$.

((i) \implies (ii)) It follows from the locality Rota-Baxter identity (16) that $A_1 = P(A)$ is a sub-locality algebra of A . Since $\text{Id} - P$ is again an idempotent locality Rota-Baxter operator, $A_2 = (\text{Id} - P)(A)$ is also a sub-locality algebra of A . Finally, P and $\text{Id} - P$ are independent locality maps as a consequence of Definition 3.20.

((ii) \implies (i)) Since π_1 and $\pi_2 = \text{Id} - \pi_1$ are locality Rota-Baxter operators of weight -1 at the same time in view of Remark 3.21, we only need to verify that π_1 is a locality Rota-Baxter operator of weight -1 :

$$(17) \quad \pi_1(a) \pi_1(b) + \pi_1(ab) = \pi_1(\pi_1(a)b) + \pi_1(a\pi_1(b)) \quad \text{for all } (a, b) \in \top.$$

Write $a = a_1 + a_2$ and $b = b_1 + b_2$. Since the projections $\pi_i, i = 1, 2$, are independent locality maps, it follows that $\{a_1, a_2\} \top \{b_1, b_2\}$. Thus every term in

$$ab = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$$

is well defined, with $a_1b_1 \in A_1$ and $a_2b_2 \in A_2$. Then the left hand side of Eq. (17) becomes

$$a_1b_1 + \pi_1(a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2) = 2a_1b_1 + \pi_1(a_1b_2) + \pi_1(a_2b_1).$$

The right hand side of Eq. (17) becomes

$$\pi_1(a_1b) + \pi_1(ab_1) = \pi_1(a_1b_1 + a_1b_2) + \pi_1(a_1b_1 + a_2b_1) = \pi_1(a_1b_2) + \pi_1(a_2b_1) + 2a_1b_1,$$

as needed. Then the last statement follows from the remark after Definition 3.16. \square

As an immediate consequence of Proposition 3.19 and 3.22, we obtain

Corollary 3.23. *The projection $\pi_+^{\mathcal{Q}}$ onto $\mathcal{M}_{\mathbb{Q},+}$ along $\mathcal{M}_{\mathbb{Q},-}^{\mathcal{Q}}$ is locality multiplicative and $(\mathcal{M}_{\mathbb{Q}}, \pi_{\pm}^{\mathcal{Q}})$ is a locality Rota-Baxter algebra.*

4. LOCALITY FOR COALGEBRAS AND THE LOCALITY CONSERVATION PRINCIPLE

We introduce the concept of a locality coalgebra which involves a suitable locality tensor product. Between a locality coalgebra and a locality algebra, we consider locality convolution and locality convolution inverse when the locality coalgebra is connected. We then prove our Locality Conservation Principle, showing that a locality (independent) relation is preserved by the renormalisation procedure à la Connes and Kreimer i.e., carried out by means of algebraic Birkhoff factorisation in the coalgebra-algebraic context [15].

4.1. Locality tensor product. We first give a locality version of the tensor product by considering a relative generalisation of the locality relation. As noted before, vector spaces and tensor products are taken over the base field K unless otherwise specified.

Let V and W be vector spaces and let $\top \subseteq V \times W$. For $X \subseteq V$ and $Y \subseteq W$, denote

$$X^{\top} := \{w \in W \mid X \top w\}, \quad {}^{\top}Y := \{v \in V \mid v \top Y\}.$$

A **relative locality vector space** is a triple (V, W, \top) where V and W are vector spaces and $\top := V \times_{\top} W$ is a subset of $V \times W$ such that for any sets $X \subseteq V$ and $Y \subseteq W$, the sets X^{\top} and ${}^{\top}Y$ are linear subspaces of V and W respectively.

Example 4.1. Given vector spaces V and W , any subspaces $V_1 \subset V$ and $W_1 \subset W$ give rise to a relative locality vector space (V, W, \top) with $\top = (V_1 \times W_1) \cup (\{0\} \times W) \cup (V \times \{0\})$.

Given two vector spaces (resp. a relative locality vector space (V, W, \top)), define I_{bilin} (resp. $I_{\top, \text{bilin}}$) to be the subspace of $K(V \times W)$ (resp. $K(V \times_{\top} W)$) spanned by the bilinear relations

$$(18) \quad \begin{aligned} & (v_1 + v_2, w_1) - (v_1, w_1) - (v_2, w_1), \quad (v_1, w_1 + w_2) - (v_1, w_1) - (v_1, w_2), \\ & (kv_1, w_1) - (v_1, kw_1), \quad (kv_1, w_1) - k(v_1, w_1) \end{aligned}$$

for all $v_1, v_2 \in V, w_1, w_2 \in W, k \in K$ (resp. such that the pairs in Eq. (18) are in \top).

Counterexample 4.2. Take $\{e_1, e_2, e_3, e_4\}$ to be the canonical orthonormal basis of $V := \mathbb{R}^4$ and

$$\begin{aligned} V \times_{\top} V &= ((V \times \{0\}) \cup (\{0\} \times V)) \cup (K(e_1 + e_2) \times K e_3) \\ &\cup (K(e_1 + 2e_2) \times K e_4) \cup (K e_1 \times K(e_3 + e_4)) \cup (K e_2 \times K(e_3 + 2e_4)). \end{aligned}$$

Then

$$(-e_1 - e_2, e_3) + (-e_1 - 2e_2, e_4) + (e_1, e_3 + e_4) + (e_2, e_3 + 2e_4)$$

is an element of $K(V \times_{\top} W) \cap I_{\text{bilin}}$ as can easily be seen using the defining relations for I_{bilin} , but it is not in $I_{\top, \text{bilin}}$.

So in general

$$K(V \times_{\top} W) \cap I_{\text{bilin}} \not\supseteq I_{\top, \text{bilin}},$$

and thus

$$h : K(V \times_{\top} W) / I_{\top, \text{bilin}} \rightarrow V \otimes W$$

is not injective. Therefore

$$K(V \times_{\top} W) / I_{\top, \text{bilin}}$$

is not an appropriate candidate for a locality tensor product, since the image of a locality coproduct Δ should lie in $V \otimes W$. Instead, we set

$$(19) \quad V \otimes_{\top} W := \text{im } h \subset V \otimes W,$$

which is also the image of the composition map $K(V \times_{\top} W) \rightarrow K(V \times W) \rightarrow V \otimes W$.

4.2. Locality coalgebras. We recall that a coalgebra (C, Δ) over a field K is **counital** if there is a map $\varepsilon : C \rightarrow K$ such that $(\varepsilon \otimes \text{Id}_C)\Delta = (\text{Id}_C \otimes \varepsilon)\Delta = \text{Id}_C$. It is **$(\mathbb{Z}_{\geq 0}$ -)graded** if

$$C = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} C_n \quad \text{and} \quad \Delta(C_n) \subseteq \bigoplus_{p+q=n} C_p \otimes C_q, \quad \bigoplus_{n \geq 1} C_n \subseteq \ker \varepsilon.$$

Thus $C = C_0 + \ker \varepsilon$. Moreover a graded coalgebra is called **connected** if $C = C_0 \oplus \ker \varepsilon$. Consequently, ε restricts to a linear bijection $\varepsilon : C_0 \cong K$ and $\ker \varepsilon = \bigoplus_{n \geq 1} C_n$.

For the sake of simplicity, we shall drop the explicit mention of the grading and simply call such a coalgebra a connected coalgebra.

The following definition is dual to that of a locality algebra. As in Eq. (19), let

$$C^{\otimes_{\top} n} = \underbrace{C \otimes_{\top} \cdots \otimes_{\top} C}_{n \text{ factors}}$$

denote the image of $KC^{\top n}$ in $C^{\otimes n}$.

Definition 4.3. Let (C, \top) be a locality vector space and let $\Delta : C \rightarrow C \otimes C$ be a linear map. (C, \top, Δ) is a **locality noncounital coalgebra** if it satisfies the following two conditions

(i) for any $U \subset C$ (compare with Eq. (9))

$$(20) \quad \Delta(U^\top) \subset U^\top \otimes_{\top} U^\top.$$

In particular, $\Delta(C) \subseteq C \otimes_{\top} C$;

(ii) the following **coassociativity** holds:

$$(\text{Id}_C \otimes \Delta) \Delta = (\Delta \otimes \text{Id}_C) \Delta.$$

- If in addition, there is a **counit**, namely a linear map $\varepsilon : C \rightarrow K$ such that $(\text{Id}_C \otimes \varepsilon) \Delta = (\varepsilon \otimes \text{Id}_C) \Delta = \text{Id}_C$, then $(C, \top, \Delta, \varepsilon)$ is called a **locality coalgebra**.
- A **connected locality coalgebra** is a locality coalgebra (C, \top, Δ) with a grading $C = \bigoplus_{n \geq 0} C_n$ such that, for any $U \subseteq C$,

$$(21) \quad \Delta(C_n \cap U^\top) \subseteq \bigoplus_{p+q=n} (C_p \cap U^\top) \otimes_{\top} (C_q \cap U^\top), \quad \bigoplus_{n \geq 1} C_n = \ker \varepsilon.$$

We denote by J the unique element of C_0 with $\varepsilon(J) = 1_K$, giving $C_0 = K J$.

Remark 4.4. We note that whereas the conditions for a locality algebra are weaker than those for an algebra, the conditions for a locality coalgebra are stronger than those for a coalgebra. In particular, a locality coalgebra is a coalgebra and a connected locality coalgebra is a connected coalgebra.

A natural example of locality coalgebra is given by the coalgebra $\mathbb{Q}\mathcal{C}$ of lattice cones in [15]. Let (C, Λ_C) be a lattice cone, (F, Λ_F) a face of (C, Λ_C) , which we denote by $(F, \Lambda_F) < (C, \Lambda_C)$. The transverse cone $t((C, \Lambda_C), (F, \Lambda_F))$ is the orthogonal projection of (C, Λ_C) to the orthogonal subspace of the subspace spanned by F . Then the coproduct $\Delta(C, \Lambda_C)$ of (C, Λ_C) is defined by

$$(22) \quad \Delta((C, \Lambda_C)) = \sum_{F < C} t((C, \Lambda_C), (F, \Lambda_F)) \otimes (F, \Lambda_F).$$

Since $t((C, \Lambda_C), (F, \Lambda_F)) \perp^{\mathcal{Q}} (F, \Lambda_F)$ by definition, the quadruple $(\mathbb{Q}\mathcal{C}, \perp^{\mathcal{Q}}, \Delta, \varepsilon)$ is a locality coalgebra with the locality counit given by the linear extension of the map

$$\varepsilon : \mathcal{C} \rightarrow \mathbb{Q}, (C, \Lambda_C) \mapsto \begin{cases} 1, & (C, \Lambda_C) = (\{0\}, \{0\}), \\ 0, & (C, \Lambda_C) \neq (\{0\}, \{0\}). \end{cases}$$

Further the connectedness conditions in Eq. (21) are satisfied. Thus we have proved

Lemma 4.5. $(\mathbb{Q}\mathcal{C}, \perp^{\mathcal{Q}}, \Delta, \varepsilon)$ with the grading $\mathbb{Q}\mathcal{C} = \bigoplus_{n \geq 0} \mathbb{Q}\mathcal{C}_n$ from Eq. (12) is a connected locality coalgebra.

Let us list a few useful general properties of locality coalgebras.

Lemma 4.6. Let (C, \top, Δ) be a locality coalgebra.

(i) For any $n \geq 2$ and $0 \leq i \leq n$,

$$(23) \quad \text{Id}_C^{\otimes i} \otimes \Delta \otimes \text{Id}_C^{\otimes (n-i-1)} : C^{\otimes_{\top} n} \rightarrow C^{\otimes_{\top} (n+1)}$$

- (ii) We have $(\Delta \otimes \Delta)(C \otimes_{\top} C) \subseteq C^{\otimes_{\top} 4}$;
- (iii) We have $(\Delta \times \Delta)(C \times_{\top} C) \subseteq (C \otimes_{\top} C) \times_{\top} (C \otimes_{\top} C)$;

(iv) For any locality linear map ϕ independent of Id , $n \geq 2$ and $0 \leq i \leq n$, we have $\text{Id}^{\otimes i} \otimes \phi \otimes \text{Id}^{\otimes(n-i-1)} : C^{\otimes \top n} \rightarrow C^{\otimes \top n}$.

Proof. (i) Let $n \geq 2$ and $1 \leq i \leq n$ be given. By the definition of $C^{\otimes \top n}$, any of its elements is a sum of pure tensors $c_1 \otimes \cdots \otimes c_n$ with $(c_1, \dots, c_n) \in C^{\top n}$. Let $U = \{c_j \mid j \neq i+1\}$. Then $c_{i+1} \in U^\top$, so there exist $(d_1, e_1), \dots, (d_k, e_k) \in U^\top \times_{\top} U^\top$, such that

$$\Delta(c_{i+1}) = \sum_{\ell} d_{\ell} \otimes e_{\ell}.$$

Now

$$(\text{Id}_C^{\otimes i} \otimes \Delta \otimes \text{Id}_C^{\otimes(n-i-1)})(c_1 \otimes \cdots \otimes c_n) = \sum_{\ell} c_1 \otimes \cdots \otimes c_i \otimes d_{\ell} \otimes e_{\ell} \otimes c_{i+2} \otimes \cdots \otimes c_n,$$

and $c_1 \otimes \cdots \otimes c_i \otimes d_{\ell} \otimes e_{\ell} \otimes c_{i+2} \otimes \cdots \otimes c_n \in C^{\otimes \top(n+1)}$.

(ii) Since $(\Delta \otimes \Delta) = (\Delta \otimes \text{Id})(\text{Id} \otimes \Delta)$, from Eq. (23) we obtain

$$\begin{aligned} (\Delta \otimes \Delta)(C \otimes_{\top} C) &= (\Delta \otimes \text{Id})(\text{Id} \otimes \Delta)(C \otimes_{\top} C) \\ &\subseteq (\Delta \otimes \text{Id})(C \otimes_{\top} C \otimes_{\top} C) \\ &\subseteq C \otimes_{\top} C \otimes_{\top} C \otimes_{\top} C. \end{aligned}$$

(iii) Let $(c_1, c_2) \in C \times_{\top} C$. Then $c_2 \in \{c_1\}^\top$. So by Eq. (20), $\Delta(c_2) = \sum_{(c_2)} c_{2,(1)} \otimes c_{2,(2)}$ with $c_{2,(1)} \top c_{2,(1)}$ and $\{c_{2,(1)}, c_{2,(2)}\} \subseteq \{c_1\}^\top$. Thus $c_1 \in \{c_{2,(1)}, c_{2,(2)}\}^\top$. By Eq. (20) again, $\Delta(c_1) = \sum_{(c_1)} c_{1,(1)} \otimes c_{1,(2)}$ with $c_{1,(1)} \top c_{1,(2)}$ and $\{c_{1,(1)}, c_{1,(2)}\} \subseteq \{c_{2,(1)}, c_{2,(2)}\}^\top$. This shows that $(c_{1,(1)}, c_{1,(2)}, c_{2,(1)}, c_{2,(2)})$ is in $C^{\top 4}$ and hence $((c_{1,(1)} \otimes c_{1,(2)}), (c_{2,(1)} \otimes c_{2,(2)}))$ is in $(C \otimes_{\top} C) \times_{\top} (C \otimes_{\top} C)$.

(iv) Again any element of $C^{\otimes \top n}$ is a sum of pure tensors $c_1 \otimes \cdots \otimes c_n$ with $(c_1, \dots, c_n) \in C^{\top n}$. Thus $(c_1, \dots, c_{i-1}, \phi(c_i), c_{i+1}, \dots, c_n)$ is in $C^{\top n}$. This is what we want since $(\text{Id}^{\otimes i} \otimes \phi \otimes \text{Id}^{\otimes(n-i-1)})(c_1 \otimes \cdots \otimes c_n) = c_1 \otimes \cdots \otimes c_{i-1} \otimes \phi(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_n$. \square

Lemma 4.7. Let $(C = \bigoplus_{n \geq 0} C_n, \top, \Delta)$ be a connected locality coalgebra. Define the **reduced coproduct** $\tilde{\Delta}(c) := \Delta(c) - J \otimes c - c \otimes J$. Recursively define

$$(24) \quad \tilde{\Delta}^{(1)} = \tilde{\Delta}, \quad \tilde{\Delta}^{(k)} := (\text{Id} \otimes \tilde{\Delta}^{(k-1)})\tilde{\Delta}, \quad k \geq 2.$$

(i) For $c \in \bigoplus_{n \geq 1} C_n$, $\tilde{\Delta}(c) = \sum_{(c)} c' \otimes c''$ with $\deg(c'), \deg(c'') > 0$ and $(c', c'') \in C \times_{\top} C$;

(ii) If in addition $c \in U^\top$ for some $U \subset C$, then the above pairs (c', c'') are in $U^\top \times_{\top} U^\top$;

(iii) $\tilde{\Delta}^{(k)}(x)$ is in $C^{\otimes \top(k+1)}$ for all $x \in C, k \in \mathbb{N}$;

(iv) $\tilde{\Delta}^{(k)}(C_n) = \{0\}$ for all $k \geq n$.

Proof. We only need to prove (ii) since then (i) is the special case when $U = \{0\}$.

Let $c \in C_n \cap U^\top$. By Eq. (21), we can write

$$\Delta(c) = y \otimes J + J \otimes z + \sum_{(c)} c' \otimes c''$$

with $y, z \in C_n, c', c'' \in U^\top$ and each $c' \otimes c'' \in C_p \otimes_{\top} C_q, p+q=n, p, q \geq 1$. Then by the same argument for a connected coalgebra [13, Theorem 2.3.3], we obtain $y = z = x$. This proves (ii).

Then (iii) follows from an easy induction on k by the locality property of Δ ; while the proof of (iv) is similar to the case without a locality structure [18, Proposition II.2.1]. \square

4.3. Locality of the convolution product. We show that the locality (independence) of linear maps are preserved under the convolution product.

Lemma 4.8. *Let (C, \top_C) be a locality coalgebra with counit $\varepsilon_C : C \rightarrow K$. Let (A, \top_A) be a locality algebra with unit $u_A : K \rightarrow A$. The map $e := u_A \varepsilon_C : C \rightarrow A$ is independent to any linear map $\phi : C \rightarrow A$. In particular, the map e is a locality linear map.*

Proof. This is because $\text{im } e = K \cdot 1_A \subset A^{\top_A}$ as we can see from Remark 3.17. \square

Now we give a general result.

Proposition 4.9. *Let (C, \top_C, Δ) be a locality coalgebra and let (A, \top_A, m_A) be a locality algebra. Let $\mathcal{L} := L\text{-Hom}(C, A)$ be the space of locality linear maps. Define*

$$\top_{\mathcal{L}} := \{(\phi, \psi) \in \mathcal{L} \times \mathcal{L} \mid (\phi \times \psi)(C \times_{\top} C) \subseteq A \times_{\top} A\}.$$

(i) For $(\phi, \psi) \in \top_{\mathcal{L}}$, define the **convolution product** of ϕ and ψ by

$$(25) \quad \phi \star \psi : C \xrightarrow{\Delta_C} C \otimes_{\top} C \xrightarrow{\phi \otimes \psi} A \otimes_{\top} A \xrightarrow{m_A} A.$$

Then $\phi \star \psi$ is a locality linear map and the triple $(\mathcal{L}, \top_{\mathcal{L}}, \star)$ is a locality algebra.

(ii) If moreover C is connected then

$$\mathcal{GL} := \{\phi \in \mathcal{L} \mid \phi(J) = 1_A\}$$

is a locality group for the convolution product.

(iii) Under this same assumption, we have

$$(\phi, \psi) \in \top_{\mathcal{L}} \cap (\mathcal{GL} \times \mathcal{GL}) \implies (\phi^{\star k}, \psi^{\star l}) \in \top_{\mathcal{L}} \cap (\mathcal{GL} \times \mathcal{GL}) \quad \text{for all } k, l \in \mathbb{Z}.$$

Proof. (i) For $(\phi, \psi) \in \top_{\mathcal{L}}$, since $(\phi \times \psi)(C \times_{\top} C) \subseteq A \times_{\top} A$, we have

$$(\phi \otimes \psi)(C \otimes_{\top} C) \subseteq A \otimes_{\top} A.$$

Hence the composition in Eq. (25) is well-defined, giving a well-defined convolution product.

We next verify that $\phi \star \psi$ is a locality linear map. For $c_1 \top_C c_2$, by Lemma 4.6.(iii), there are finitely many $(d_i, e_i), (f_j, g_j) \in C \times_{\top} C$ with $(d_i, e_i, f_j, g_j) \in C^{\top^4}$, such that

$$\Delta(c_1) = \sum_i d_i \otimes e_i \quad \text{and} \quad \Delta(c_2) = \sum_j f_j \otimes g_j.$$

Then

$$\phi \star \psi(c_1) = \sum_i \phi(d_i)\psi(e_i) \quad \text{and} \quad \phi \star \psi(c_2) = \sum_j \phi(f_j)\psi(g_j).$$

From $(\phi, \psi) \in \top_{\mathcal{L}}$, we obtain $(\phi(d_i), \psi(e_i), \phi(f_j), \psi(g_j)) \in A^{\top^4}$. So

$$\left(\sum_i \phi(d_i)\psi(e_i), \sum_j \phi(f_j)\psi(g_j) \right)$$

is in $A \times_{\top} A$ and thus $(\phi \star \psi(c_1)) \top_A (\phi \star \psi(c_2))$.

Thus we are left to verify the axioms for a locality semigroup: the closeness of $U^{\top_{\mathcal{L}}}$ under the convolution product for every $U \subseteq \mathcal{L}$ and the associativity.

Let ψ and χ be independent locality linear maps in $U^{\top\mathcal{L}}$ and let ϕ be in U . Then ϕ, ψ, χ are pairwise independent. Therefore

$$\phi \times \psi \times \chi : C^{\top 3} \longrightarrow A^{\top 3}$$

is well defined. For $(c_1, c_2) \in C^{\top 2}$, that is $c_1 \in \{c_2\}^\top$, there exist $(d_1, e_1), \dots, (d_k, e_k) \in \{c_2\}^\top \times_{\top} \{c_2\}^\top$, such that

$$\Delta(c_1) = \sum_i d_i \otimes e_i,$$

with $(d_i, e_i, c_2) \in C^{\top 3}$. Then $(\psi(d_i), \chi(e_i), \phi(c_2)) \in A^{\top 3}$ and hence $(\psi(d_i)\chi(e_i))\top_A\phi(c_2)$. So we have $(\psi \star \chi)(c_1)\top_A\phi(c_2)$, which means $\psi \star \chi$ is in $\phi^{\top\mathcal{L}}$. Thus $\psi \star \chi \in U^{\top\mathcal{L}}$. This verifies the first axiom. The associativity of \star follows from the associativity of m and coassociativity of Δ as in the classical case.

(ii) We now assume that C is a connected locality coalgebra. For a locality linear map $\phi : C \rightarrow A$, we now prove by induction on the degree of c_1 that the map

$$(26) \quad \phi^{\star(-1)}(c_1) = \begin{cases} 1_A, & c_1 = J, \\ -\phi(c_1) - \sum_{(c_1)} \phi(c'_1)\phi^{\star(-1)}(c''_1), & c_1 \in \ker \varepsilon, \end{cases}$$

is well defined, and that $c_1 \top_C c$ implies $\phi^{\star(-1)}(c_1)\top_A\phi(c)$.

This is trivial for degree 0 since $\phi^{\star(-1)}(J) = 1_A$. Assume for any $c_1 \in C$ of degree $\leq n$, $\phi^{\star(-1)}(c_1)$ is well defined, and for c with $c_1 \top_C c$, $\phi^{\star(-1)}(c_1)\top_A\phi(c)$ holds.

Now for any c_1 of degree $n+1 \geq 1$ with $c_1 \top_C c$, by Lemma 4.7.(ii), we have

$$\Delta(c_1) = c_1 \otimes J + J \otimes c_1 + \sum_{(c_1)} c'_1 \otimes c''_1 \quad \text{with } (c'_1, c''_1, c) \in C^{\top 3}.$$

By the induction hypothesis, $\phi^{\star(-1)}(c''_1)$ is well defined, such that $\phi^{\star(-1)}(c''_1)\top_A\phi(c)$ and $\phi^{\star(-1)}(c'_1)\top_A\phi(c'_1)$. Since ϕ is a locality linear map, we also have $\phi(c_1)\top_A\phi(c)$ and $\phi(c'_1)\top_A\phi(c)$. Thus $\phi(c'_1)\phi^{\star(-1)}(c''_1)$ is well defined and $(\phi(c'_1)\phi^{\star(-1)}(c''_1))\top_A\phi(c)$. So, $\phi^{\star(-1)}(c_1)$ is well defined and $\phi^{\star(-1)}(c_1)\top_A\phi(c)$, which means $\phi \top_{\mathcal{L}} \phi^{\star(-1)}$.

Again by induction on the degree of c_1 , we now prove that $\phi^{\star(-1)}$ is a locality linear map by checking

$$(27) \quad \phi^{\star(-1)}(c_1)\top_A\phi^{\star(-1)}(c_2) \quad \text{for all } c_2 \in C, c_1 \top_C c_2,$$

a fact which is obvious at degree 0. Assume that, for a given $n \geq 0$ and any c_1 of degree $\leq n$ the equation holds. Consider c_1 of degree $n+1 \geq 1$. Since $c_1 \top_C c_2$, we can choose

$$\Delta(c_1) = c_1 \otimes J + J \otimes c_1 + \sum_{(c_1)} c'_1 \otimes c''_1$$

such that $\{c_1, c'_1, c''_1\} \top_C c_2$. From this we have $\{\phi(c_1), \phi(c'_1), \phi^{\star(-1)}(c''_1)\} \top_A \phi^{\star(-1)}(c_2)$. So Eq. (26) gives $\phi^{\star(-1)}(c_1)\top_A\phi^{\star(-1)}(c_2)$. Therefore, we conclude that \mathcal{GL} is a locality group with unit $u_{A \in C}$ by Example 4.8.

(iii) Similar inductions show that $\phi \top_{\mathcal{L}} \psi$ implies $\phi \top_{\mathcal{L}} \psi^{\star(-1)}$ and $\phi^{\star(-1)} \top_{\mathcal{L}} \psi^{\star(-1)}$, so that $\phi^{\star k} \top_{\mathcal{L}} \psi^{\star l}$ for any $k, l \in \mathbb{Z}$. \square

4.4. The Locality Conservation Principle. We now put together all the locality structures we have obtained so far to provide an answer to the question addressed by the Locality Conservation Principle proposed in Problem 1.3. It is formulated as the preservation of locality by the algebraic Birkhoff factorisation in the coalgebra context in [15, Theorem 4.4]. See Section 5.3 for the formulation of the Locality Product Conservation Principle built on the algebraic Birkhoff factorisation in the Hopf algebra context originated from Connes-Kreimer [8].

Theorem 4.10. (Algebraic Birkhoff factorisation, locality coalgebra version) *Let (C, \top_C, Δ) be a connected locality coalgebra, $C = \bigoplus_{n \geq 0} C_n$, $C_0 = KJ$. Let (A, \top_A, \cdot) be a commutative locality algebra with decomposition $A = A_1 \oplus A_2$ as a vector space satisfying the following*

(Basic Assumption) *the linear projections π_i onto A_i along $A_i, \{\hat{i}\} := [2] \setminus \{i\}, i = 1, 2$, are independent locality linear maps and 1_A is in A_1 .*

Let

$$\phi : (C, \top_C) \longrightarrow (A, \top_A)$$

be a locality linear map such that $\phi(J) = 1_A$. Then there are unique independent locality linear maps $\phi_i : C \rightarrow K + A_i$ with $\phi_i(J) = 1_A$ and $\phi_i(\ker \varepsilon) \subseteq A_i, i = 1, 2$, such that

$$(28) \quad \phi = \phi_1^{\star(-1)} \star \phi_2.$$

The map $(\phi_1)^{\star(-1)}$ is a locality linear map and $\phi_1 \top_C \{\phi, \phi_2\}, \phi_1^{\star(-1)} \top_C \{\phi_1, \phi_2\}$.

- (i) If in addition to the Basic Assumption, A_1 is a sub-locality algebra of A , then $\phi_1^{\star(-1)} : C \rightarrow K + A_1$.
- (ii) If in addition to the Basic Assumption and Item (i), A_2 is a locality ideal of A , then $\phi_1^{\star(-1)} = \pi_1 \phi$ and ϕ_2 is recursively given by

$$(29) \quad \phi_2(J) = 1_A, \quad \phi_2(c) = (\pi_2 \phi)(c) - \sum_{(c)} (\pi_1 \phi)(c') \phi_2(c'') \text{ for all } c \in \ker \varepsilon,$$

with $\tilde{\Delta}(c) = \sum_{(c)} c' \otimes c''$ the reduced coproduct defined in Lemma 4.7.

If $\psi : (C, \top_C) \longrightarrow (A, \top_A)$ is also a locality linear map independent of ϕ with $\psi(J) = 1_A$, then ϕ_i and ψ_j are independent for $i, j = 1, 2$.

Remark 4.11. Theorem 4.10 provides an answer to Problem 1.3: the locality of the renormalised map follows from that of the initial map under the assumption of the theorem. See also Remark 4.12 and its subsequent example.

Proof. The proof of the uniqueness of the maps ϕ_i is the same as the proof [15, Theorem 4.4] for the case of a trivial locality relation i.e., when $\top = C \times C$.

Let $n \geq 1$ and $c \in C_n$. Since C is a connected locality coalgebra, we can write

$$\Delta(c) = J \otimes c + c \otimes J + \sum_{(c)} c' \otimes c''$$

with $\deg(c'), \deg(c'') > 0$ and $c' \top_C c''$.

We first prove by induction on the degree n of c that the map given by

$$(30) \quad \phi_1(c) = \begin{cases} 1_A, & c = J, \\ -\pi_1 \left(\phi(c) + \sum_{(c)} \phi_1(c') \phi(c'') \right), & c \in C_n, n > 0, \end{cases}$$

is well-defined, and for any $d \in C$ with $d \top_C c$, there is

$$\phi_1(c) \top_A \phi(d),$$

which clearly hold for c of degree 0.

Assume that these hold true for c of degree $\leq n$. Then for c of degree $n + 1$, c' is of degree $\leq n$, so $\phi_1(c')$ is defined and $\phi_1(c') \top_A \phi(c'')$. Therefore $\phi_1(c') \phi(c'')$ makes sense, and $\phi_1(c)$ is well-defined.

Further for any $d \in C$ with $c \top_C d$, we can take $\{c', c''\} \top_C d$. Since ϕ is a locality map, we obtain $\phi(c) \top_A \phi(d)$ and $\phi(c'') \top_A \phi(d)$. Also the induction hypothesis gives $\phi_1(c') \top_A \phi(d)$. Thus $(\phi_1(c') \phi(c'')) \top_A \phi(d)$. Therefore,

$$\left(\phi(c) + \sum_{(c)} \phi_1(c') \phi(c'') \right) \top_A \phi(d).$$

Now since π_1 is a locality map and π_1 and π_2 are independent, π_1 and $\text{Id}_A = \pi_1 + \pi_2$ are independent. Thus $\phi_1(c) \top_A \phi(d)$. Therefore we have proved that ϕ_1 is well-defined and $\phi_1 \top_{\mathcal{L}} \phi$.

Now for any $c \top_C d$, we have $\phi(c) \top_A \phi_1(d)$. By a similar induction on the degree of c , we obtain $\phi_1(c) \top_A \phi_1(d)$, so ϕ_1 is a locality linear map. Therefore, the map

$$(31) \quad \phi_2(c) := \begin{cases} 1_A, & c = J, \\ (\text{Id}_A - \pi_1)(\phi(c) + \sum_{(c)} \phi_1(c') \phi(c'')), & c \in C_n, n > 0, \end{cases}$$

is well-defined.

Notice that for $c \in C_n, n > 0$, Eq. (31) means

$$(32) \quad \phi_2(c) = \phi(c) + \phi_1(c) + \sum_{(c)} \phi_1(c') \phi(c'').$$

With the condition on J , this in turn reads $\phi_2 = \phi_1 \star \phi$ and hence $\phi = \phi_1^{\star(-1)} \star \phi_2$. By Proposition 4.9, $\phi_1^{\star(-1)}$ is a locality linear map. From Eq. (32), we easily obtain $\phi_1 \top_{\mathcal{L}} \phi_2$. By Eq. (26), an easy induction on the degree of c shows $\phi_1^{\star(-1)} \top_{\mathcal{L}} \{\phi_1, \phi_2\}$.

A similar induction shows that if $\psi : (C, \top_C) \rightarrow (A, \top_A)$ is also a locality map, independent of ϕ with $\psi(J) = 1_A$, then ϕ_i and ψ_j are independent for $i, j = 1, 2$, proving the last statement of the theorem.

Now to prove (i), letting A_1 be a sub-locality algebra, Eq. (26) and a simple induction on $n \geq 0$ show that $\phi_1^{\star(-1)}(c) \in K + A_1$ for any $c \in C_n$.

To prove (ii), suppose A_1 is a sub-locality algebra and A_2 is a locality ideal. We prove by induction on $n \geq 0$ that

$$(33) \quad \phi_1^{\star(-1)}(c) = (\pi_1 \phi)(c) \quad \text{for all } c \in C_n.$$

Notice that $\phi(J) = \phi_1(J) = 1_A$ implies $(\phi_1 \star (\pi_1 \phi))(J) = 1_A$, so Eq. (33) holds for $n = 0$ since $C_0 = KJ$. Assuming that Eq. (33) holds for any c in C of degree $\leq n$, we prove that Eq. (33) holds for any element $c \in C_{n+1}$.

We write

$$\Delta(c) = c \otimes J + J \otimes c + \sum_{(c)} c' \otimes c''$$

with $c' \top_C c''$ of degree $\leq n$. By the definition of ϕ_1 ,

$$\phi_1(c) = -\pi_1(\phi(c) + \sum_{(c)} \phi_1(c') \phi(c''))$$

and $\phi_1(c') \top_A \phi(c'')$. By the assumption on π_i , we have $\phi_1(c') \top_A \{\pi_1\phi(c''), (\pi_2\phi)(c'')\}$. Then

$$\begin{aligned} \phi_1(c) &= -\pi_1\left(\phi(c) - \sum_{(c)} \phi_1(c')((\pi_1\phi)(c'') + (\pi_2\phi)(c''))\right) \\ &= -(\pi_1\phi)(c) - \sum_{(c)} \phi_1(c')(\pi_1\phi)(c''). \end{aligned}$$

Consequently,

$$(\phi_1 \star (\pi_1\phi))(c) = \phi_1(c) + \pi_1\phi(c) + \sum_{(c)} \phi_1(c') \pi_1\phi(c'') = 0.$$

We conclude that $\phi_1 \star (\pi_1\phi) = e$, leading to $\phi_1^{\star(-1)} = \pi_1\phi$ since $\phi_1^{\star(-1)}$ has been shown to exist. The locality algebraic Birkhoff factorization $\phi = \phi_1^{\star(-1)} \star \phi_2$ then yields for $c \in \ker(\varepsilon)$:

$$\pi_1(\phi(c)) + \phi_2(c) + \sum_{(c)} \pi_1(\phi(c'))\phi_2(c'') = \phi(c)$$

Using $\pi_2 = \text{Id} - \pi_1$ gives the recursive expression (29) for ϕ_2 in terms of π_1 and π_2 . \square

Remark 4.12. That the locality of ϕ implies that of ϕ_2 and ϕ_1 can be summarised under the motto “renormalisation preserves locality” in analogy with quantum field theory.

We illustrate this motto by applying Theorem 4.10 to the locality linear map $S : \mathbb{Q}\mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q}}$ that we have been taken as the main example throughout this paper. The map is denoted by S° in [15] where it is shown, by the algebraic Birkhoff factorisation for connected coalgebras [15, Theorem 4.4], that both

$$S_1^{\star(-1)} : \mathbb{Q}\mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q},+} \text{ and } S_2 : \mathbb{Q}\mathcal{C} \rightarrow \mathcal{M}_{\mathbb{Q},-}$$

are linear maps sending the trivial lattice cone $(\{0\}, \{0\})$ to 1 in $\mathcal{M}_{\mathbb{Q}}$. Applying Theorem 4.10, we further learn that both $S_1^{\star(-1)}$ and S_2 are locality linear maps. This means that for cones (C, Λ_C) and (D, Λ_D) with $(C, \Lambda_C) \perp^{\mathcal{Q}} (D, \Lambda_D)$, we also have

$$(34) \quad S_1^{\star(-1)}(C, \Lambda_C) \perp^{\mathcal{Q}} S_1^{\star(-1)}(D, \Lambda_D), \quad S_2(C, \Lambda_C) \perp^{\mathcal{Q}} S_2(D, \Lambda_D).$$

5. LOCALITY FOR HOPF ALGEBRAS AND THE LOCALITY PRODUCT CONSERVATION PRINCIPLE

In this section, we combine a locality algebra and a locality coalgebra to give a locality bialgebra, and then a locality Hopf algebra under an additional connectedness condition. Together with the locality Rota-Baxter algebra structure, we obtain the locality version of the algebraic Birkhoff factorisation originally given by Connes and Kreimer. This provides an answer to the question addressed in the Locality Product Conservation Principle in Problem 1.4. An application is given to the renormalisation of the exponential generating function of lattice cones.

5.1. Locality bialgebras and locality Hopf algebras.

Definition 5.1. (i) An **locality bialgebra** is a sextuple $(B, \top, m, u, \Delta, \varepsilon)$ consisting of a locality algebra (B, m, u, \top) and a locality coalgebra $(B, \Delta, \top, \varepsilon)$ that are locality compatible in the sense that Δ and ε are locality algebra homomorphisms.

(ii) A locality bialgebra B is called **connected** if there is a $\mathbb{Z}_{\geq 0}$ -grading $B = \bigoplus_{n \geq 0} B_n$ with respect to which B is both a locality graded algebra in the sense of Definition 3.16 and a connected locality coalgebra in the sense of Definition 4.3. Then $J = 1_B$.

Let us go back to the space \mathbb{QC} of lattice cones with the Minkowski product and the coproduct Δ defined in Eq. (22). We observe that the idempotence $(C, \Lambda_C) \cdot (C, \Lambda_C) = (C, \Lambda_C)$ hinders the compatibility between the product and the coproduct. For example, taking $(C, \Lambda_C) = (\langle e_1 \rangle, \mathbb{Z}e_1)$, then $\Delta(C \cdot C) = \Delta(C) \cdot \Delta(C)$ does not hold. However, the following result shows that this compatibility can be recovered in the context of locality bialgebras.

Proposition 5.2. $(\mathbb{QC}, \perp^{\mathcal{Q}}, \cdot, u, \Delta, \varepsilon)$ is a connected locality bialgebra.

Proof. We first verify the compatibility of the locality coalgebra structure and the locality algebra structure. To show that Δ is a locality algebra homomorphism, we note that for $(C, \Lambda_C) \perp^{\mathcal{Q}} (D, \Lambda_D)$, the faces of $(C, \Lambda_C) \cdot (D, \Lambda_D)$ are of the form $(F_1, \Lambda_{F_1}) \cdot (F_2, \Lambda_{F_2})$ with F_1 a face of C and F_2 a face of D , and

$$t((C, \Lambda_C) \cdot (D, \Lambda_D), (F_1, \Lambda_{F_1}) \cdot (F_2, \Lambda_{F_2})) = t((C, \Lambda_C), (F_1, \Lambda_{F_1})) \cdot t((D, \Lambda_D), (F_2, \Lambda_{F_2})).$$

So by definition, we have the desired equation:

$$(35) \quad \Delta((C, \Lambda_C) \cdot (D, \Lambda_D)) = \Delta(C, \Lambda_C) \cdot \Delta(D, \Lambda_D).$$

The counit $\varepsilon : \mathbb{QC} \rightarrow \mathbb{Q}$ is evidently an algebra homomorphism and hence a locality algebra homomorphism.

Finally, we have checked that \mathbb{QC} is both a graded locality algebra (Lemma 3.18) and a connected locality coalgebra (Lemma 4.5). This completes the proof. \square

Definition 5.3. A **locality Hopf algebra** is a locality bialgebra $(B, \top, m, \Delta, u, \varepsilon)$ with an antipode, defined to be a linear map $S : B \rightarrow B$ such that S and Id_B are mutually independent (in the sense of Definition 3.13) and

$$S \star \text{Id} = \text{Id} \star S = u\varepsilon.$$

The usual proof (see e.g. [13, 18]) for the existence of the antipode on connected bialgebras extends to locality bialgebras as follows. For $k \geq 1$, denote $m_1 = m$ and $m_k = m(\text{Id}_B \otimes m_{k-1})$.

Lemma 5.4. Let $(B, \top, m, u, \Delta, \varepsilon)$ be a connected locality bialgebra, $\tilde{\Delta}^{\otimes k}$ as in Eq. (24) and $\alpha : B \rightarrow B$ a locality linear map with $\alpha(1_B) = 0$. Then

- (i) $\alpha^{\star k} = m_{k-1} \alpha^{\otimes k} \tilde{\Delta}^{(k-1)}$ for all $k \geq 2$;
- (ii) $\alpha^{\star k}(B_n) = \{0\}$ for all $k \geq n + 1$.

Proof. (i) We proceed by induction on $k \geq 2$ and first observe that $\alpha^{*k}(1_B) = 0$ for every $k \geq 1$ as can easily be shown by induction. The result holds for $k = 2$ since $\alpha(1_B) = 0$. Assuming it holds at degree k we write

$$\begin{aligned} \alpha^{*k+1}(x) &= m(\alpha \otimes \alpha^{*k})\Delta(x) \\ &= m(\alpha \otimes \alpha^{*k})\tilde{\Delta}(x) \quad (\text{since } \alpha^{*k}(1_B) = \alpha(1_B) = 0) \\ &= m(\alpha \otimes (m_{k-1}\alpha^{\otimes k}\tilde{\Delta}^{(k-1)}))\tilde{\Delta}(x) \\ &= m(\text{Id}_B \otimes m_{k-1})(\alpha \otimes \alpha^{\otimes k})(\text{Id}_B \otimes \tilde{\Delta}^{(k-1)})\tilde{\Delta}(x) \\ &= m_k\alpha^{\otimes(k+1)}\tilde{\Delta}^{(k)}(x) \end{aligned}$$

where we have used the locality property of α and the fact that m is associative.

(ii) is a direct consequence of (i) and Lemma 4.7.(iv). \square

Proposition 5.5. *Let $(B, \top, m, u, \Delta, \varepsilon)$ be a connected locality bialgebra. There is a linear map $S : B \rightarrow B$ with the properties of the antipode stated above. It is given by*

$$S = \sum_{k=0}^{\infty} (u\varepsilon - \text{Id})^{*k}.$$

Proof. The map $\alpha : B \rightarrow B$ defined by $\alpha = \text{Id} - u\varepsilon$ is locality linear, and $\alpha(1_B) = 0$. The von Neumann series $S = \sum_{k=0}^{\infty} (-1)^k \alpha^{*k}$ which is locally finite by Lemma 5.4.(ii) and hence well-defined, gives the inverse of the identity for the convolution product. \square

As an immediate consequence of Propositions 5.2 and 5.5, we have

Corollary 5.6. *The locality bialgebra $(\mathbb{QC}, \perp^Q, \cdot, u, \Delta, \varepsilon)$ is a locality Hopf algebra.*

5.2. Locality of the convolution of locality algebra homomorphisms.

Proposition 5.7. *Let $(B, \top_B, m, u, \Delta, \varepsilon)$ be a locality bialgebra. Let (A, \top_A, \cdot) be a locality commutative algebra. Let*

$$\phi, \psi : (B, \top_B) \longrightarrow (A, \top_A)$$

be independent locality linear maps.

- (i) *If ϕ and ψ are locality multiplicative then so is their convolution product $\phi \star \psi$.*
- (ii) *Assume further that B is connected. If ϕ is a homomorphism of locality algebras, then so is its convolution inverse $\phi^{*(-1)}$. So the set \mathcal{G} of homomorphisms of locality algebras from (B, \top_B) to (A, \top_A) is a locality group with respect to the independent relation of locality linear maps.*

Proof. (i) For $(c, d) \in \top_B$, by the proof of Lemma 4.6.(iii), we can write

$$\Delta(c) = \sum_i c_{i1} \otimes c_{i2}, \quad \Delta(d) = \sum_j d_{j1} \otimes d_{j2}$$

with $(c_{i1}, c_{i2}, d_{j1}, d_{j2}) \in B^{\top^4}$. Then

$$\Delta(cd) = (m \otimes m)\tau_{23}(\Delta \otimes \Delta)(c \otimes d) = \sum_{i,j} c_{i1}d_{j1} \otimes c_{i2}d_{j2}.$$

So

$$\begin{aligned}
(\phi \star \psi)(cd) &= \sum_{i,j} \phi(c_{i1}d_{j1}) \psi(c_{i2}d_{j2}) \\
&= \sum_{i,j} \phi(c_{i1}) \phi(d_{j1}) \psi(c_{i2}) \psi(d_{j2}) \\
&= \sum_i \phi(c_{i1}) \psi(c_{i2}) \sum_j \phi(d_{j1}) \psi(d_{j2}) \\
&= (\phi \star \psi)(c) (\phi \star \psi)(d).
\end{aligned}$$

(ii) Now we use induction on the sum of degrees of c and d , $c \top d$ to prove

$$\phi^{\star(-1)}(c)\phi^{\star(-1)}(d) = \phi^{\star(-1)}(cd),$$

which is true if the sum of degrees is 0.

In general, by Lemma 4.7.(ii), we write

$$\Delta(c) = c \otimes J + J \otimes c + \sum_{(c)} c' \otimes c'', \quad \Delta(d) = d \otimes J + J \otimes d + \sum_{(d)} d' \otimes d''$$

with $(c', c'', d', d'') \in B^{\top 4}$. So by $\Delta(cd) = \Delta(c)\Delta(d)$, we obtain

$$\begin{aligned}
\Delta(cd) &= cd \otimes J + J \otimes cd + c \otimes d + d \otimes c + \sum_{(d)} cd' \otimes d'' + \sum_{(d)} d' \otimes cd'' \\
&\quad + \sum_{(c)} c'd \otimes c'' + \sum_{(c)} c' \otimes c''d + \sum_{(c)(d)} c'd' \otimes c''d''.
\end{aligned}$$

By Eq. (26) we obtain

$$\begin{aligned}
\phi^{\star(-1)}(cd) &= -\phi(cd) - \phi(c)\phi^{\star(-1)}(d) - \phi(d)\phi^{\star(-1)}(c) \\
&\quad - \sum_{(d)} \phi(cd')\phi^{\star(-1)}(d'') - \sum_{(d)} \phi(d')\phi^{\star(-1)}(cd'') \\
&\quad - \sum_{(c)} \phi(c'd)\phi^{\star(-1)}(c'') - \sum_{(c)} \phi(c')\phi^{\star(-1)}(c''d) \\
&\quad - \sum_{(c)(d)} \phi(c'd')\phi^{\star(-1)}(c''d'').
\end{aligned}$$

By Eq. (26) applied to c and d , the locality multiplicativity of ϕ , the commutativity of A and induction hypothesis, we have

$$\begin{aligned}
\phi^{\star(-1)}(cd) &= \phi(c)\phi(d) + \sum_{(d)} \phi(c)\phi(d')\phi^{\star(-1)}(d'') + \sum_{(c)} \phi(c')\phi(d)\phi^{\star(-1)}(c'') \\
&\quad + \sum_{(c)(d)} \phi(d')\phi^{\star(-1)}(d'')\phi(c')\phi^{\star(-1)}(c'') + \sum_{(c)(d)} \phi(c')\phi^{\star(-1)}(c'')\phi(d')\phi^{\star(-1)}(d'') \\
&\quad - \sum_{(c)(d)} \phi(c'd')\phi^{\star(-1)}(c''d'') \\
&= \phi(c)\phi(d) + \sum_{(d)} \phi(c)\phi(d')\phi^{\star(-1)}(d'') + \sum_{(c)} \phi(c')\phi(d)\phi^{\star(-1)}(c'')
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{(c)(d)} \phi(c')\phi^{*(-1)}(c'')\phi(d')\phi^{*(-1)}(d'') \\
 = & \left(\phi(c) + \sum_{(c)} \phi(c')\phi^{*(-1)}(c'')\right)\left(\phi(d) + \sum_{(d)} \phi(d')\phi^{*(-1)}(d'')\right) \\
 = & \phi^{*(-1)}(c)\phi^{*(-1)}(d).
 \end{aligned}$$

This completes the induction. \square

5.3. The Locality Product Conservation Principle. Now we give the locality of the algebraic Birkhoff factorisation in the Hopf algebra context [8].

Theorem 5.8. (Algebraic Birkhoff factorisation, locality Hopf algebra version) *Let (H, \top_H) be a locality connected Hopf algebra, $H = \bigoplus_{n \geq 0} H_n$, $H_0 = Ke$. Let (A, \top_A, \cdot, P) be a commutative locality Rota-Baxter algebra of weight -1 with P idempotent. Denote $A_1 = P(A)$ and $A_2 = (\text{Id} - P)(A)$. Let*

$$\phi : (H, \top_H) \longrightarrow (A, \top_A)$$

be a locality algebra homomorphism. Then there are unique independent locality algebra homomorphisms $\phi_i : H \rightarrow K + A_i$ with $\phi_i(\ker \varepsilon) \subseteq A_i$, $i = 1, 2$, such that

$$(36) \quad \phi = \phi_1^{*(-1)} \star \phi_2.$$

The map $\phi_1^{(-1)}$ is also a locality algebra homomorphism and $\phi_1 \top_{\mathcal{L}} \phi$, $\phi_1 \top_{\mathcal{L}} \phi_2$.*

If in addition A_2 is a locality ideal of A , then $\phi_1^{(-1)} = \pi_1 \phi$ and ϕ_2 is recursively given by*

$$\phi_2(1_H) = 1_A, \quad \phi_2(c) = (\pi_2 \phi)(c) - \sum_{(c)} (\pi_1 \phi)(c')\phi_2(c'') \text{ for all } c \in \ker \varepsilon.$$

with $\tilde{\Delta}(c) = \sum_{(c)} c' \otimes c''$ the reduced coproduct defined in Lemma 4.7.

Remark 5.9. Theorem 5.8 provides an answer to Problem 1.4: the locality multiplicativity of the renormalised map follows from that of the initial map under the assumption of the theorem.

Proof. All the statements follows from Theorem 4.10 except the claims that ϕ_i , $i = 1, 2$ and $\phi_1^{*(-1)}$ are locality algebra homomorphisms.

For $c \top_H d$, by Lemma 4.7.(ii), we can write

$$\Delta(c) = c \otimes 1_H + 1_H \otimes c + \sum_{(c)} c' \otimes c'', \quad \Delta(d) = d \otimes 1_H + 1_H \otimes d + \sum_{(d)} d' \otimes d''$$

with $(c', c'', d', d'') \in H^{\top 4}$. Using the locality Rota-Baxter property of P , by a similar argument as in the non-locality case [13, Theorem 2.4.3], we can prove that ϕ_1 and ϕ_2 are homomorphisms of locality algebras. By Proposition 5.7, $\phi_1^{*(-1)}$ is a homomorphism of locality algebras. \square

Applying Theorem 5.8 to $(A, \top_A, \cdot) = (\mathcal{M}_{\mathbb{Q}}, \perp^{\mathcal{Q}}, \cdot)$ yields the following result.

Corollary 5.10. *Let (H, \top_H) be a connected locality Hopf algebra. Let*

$$\phi : (H, \top_H) \longrightarrow (\mathcal{M}_{\mathbb{Q}}, \perp^{\mathbb{Q}})$$

be a locality linear map such that $\phi(1_H) = 1_{\mathcal{M}_{\mathbb{Q}}}$. Let $\phi = (\phi_1^{\mathbb{Q}})^{\star(-1)} \star \phi_2^{\mathbb{Q}}$ be the algebraic Birkhoff factorisation in Eq. (36) with $\phi_1^{\mathbb{Q}}(1_H) = \phi_2^{\mathbb{Q}}(1_H) = 1_{\mathcal{M}_{\mathbb{Q}}}$. Then

- (i) $\pi_1^{\mathbb{Q}}\phi$ is a locality linear map;
 - (ii) $(\phi_1^{\mathbb{Q}})^{\star(-1)} = \pi_1^{\mathbb{Q}}\phi$ so that
- (37)
$$\phi = (\pi_1^{\mathbb{Q}}\phi) \star \phi_2^{\mathbb{Q}};$$
- (iii) the maps $\phi_1^{\mathbb{Q}}, \phi_2^{\mathbb{Q}}$ are locality linear maps and $\phi_1^{\mathbb{Q}}\top_{\mathcal{L}}\phi, \pi_1^{\mathbb{Q}}\phi\top_{\mathcal{L}}\phi_2^{\mathbb{Q}};$
 - (iv) assuming in addition that ϕ is a locality algebra homomorphism, then the maps $\pi_1^{\mathbb{Q}}\phi, \phi_1^{\mathbb{Q}}$ and $\phi_2^{\mathbb{Q}}$ are locality algebra homomorphisms.

Proof. The proof is straightforward; let us nevertheless mention that $\pi_1^{\mathbb{Q}}\phi\top_{\mathcal{L}}\phi_2^{\mathbb{Q}}$ follows from $\phi_1\top_{\mathcal{L}}\phi_2$ in Eq. (36) combined with the fact that the convolution inverse preserves locality. \square

We end the paper by applying Corollary 5.10 to the locality algebra homomorphism $S : \mathbb{QC} \rightarrow \mathcal{M}_{\mathbb{Q}}$. The classical algebraic Birkhoff factorisation does not apply while Theorem 4.10 does, telling us that

$$S_1^{\star(-1)} : \mathbb{QC} \rightarrow \mathcal{M}_{\mathbb{Q},+} \text{ and } S_2 : \mathbb{QC} \rightarrow \mathcal{M}_{\mathbb{Q},-}$$

are locality linear maps. From Corollary 5.10, we conclude that the two maps are also locality algebra homomorphisms and thus are multiplicative for orthogonal pairs of lattice cones. Noting further [15] that $S_2 = I$, the exponential integral and $S_1^{\star(-1)} = \mu$, the interpolation factor in the Euler-Maclaurin formula $S = \mu \star I$, we obtain the following consequence of Theorem 5.8, Proposition 3.19 and Corollary 5.10.

Corollary 5.11. *For any orthogonal pair of lattice cones (C, Λ_C) and (D, Λ_D) , we have*

$$\mu((C, \Lambda_C) \cdot (D, \Lambda_D)) = \mu(C, \Lambda_C)\mu(D, \Lambda_D), \quad I((C, \Lambda_C) \cdot (D, \Lambda_D)) = I(C, \Lambda_C)I(D, \Lambda_D).$$

Further, $\mu = \pi_+^{\mathbb{Q}}S$.

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