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On the relation between perfect powers and tetration frozen digits

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Abstract

This paper provides a link between integer exponentiation and integer tetration since it is devoted to introducing some peculiar sets of perfect powers characterized by any given value of their constant congruence speed, revealing a fascinating relation between the degree of every perfect power belonging to any congruence class modulo 20 and the number of digits frozen by these special tetration bases, in radix-10, for any unit increment of the hyperexponent. In particular, given any positive integer c , we constructively prove the existence of infinitely many c -th perfect powers having a constant congruence speed of c .

Keywords: Perfect power, Tetration, Power tower, Congruence speed, Convergence.
MSC 2020: Primary 11A07, 11A15.

1. Introduction

In recent years, by assuming the standard decimal numeral system (radix-10), we have shown that the integer tetration ${}^b a := \begin{cases} a & \text{if } b = 1 \\ a^{(b-1)a} & \text{if } b \geq 2 \end{cases}$ has a unique property [4] involving the number of new frozen rightmost digits for any unit increment of its hyperexponent, $b \in \mathbb{Z}^+$ [11, 12]. Indeed, this value no longer depends on b as b becomes sufficiently large (see the sequence [A372490](#) in the On-Line Encyclopedia of Integer Sequences [7]) and the tetration base, $a \in \mathbb{Z}^+$, is not a multiple of 10.

We refer to the mentioned property as the constancy of the *congruence speed* of tetration (see Definition 1 and also the comments of the OEIS sequence [A317905](#)).

From Reference [11], we know that each positive integer $a \geq 2$, which is not a multiple of 10, is characterized by a finite, strictly positive, integer value of its constant congruence speed (the map of the constant congruence speed of every a as above is provided by References [11, 12]).

Then, the present paper aims to constructively prove the existence of infinitely many perfect powers having any given positive constant congruence speed.

A pleasant result, which follows from Theorem 3 as a corollary, is the existence, for any given positive integer c , of infinitely many c -th perfect powers (i.e., an integer $a > 1$ is a c -th perfect power if there are integers \tilde{a} and c such that $a = \tilde{a}^c$, so we have *perfect squares* if $c = 2$, *perfect cubes* if $c = 3$, and so forth) having a constant congruence speed of c .

2. Preliminary investigations with the automorphic numbers

In order to present the results compactly, let us properly define the constant congruence speed of tetration as already done in Reference [12], p. 442, Definitions 1.1 and 1.2.

Definition 1. Let $n \in \mathbb{N}_0$ and assume that $a \in \mathbb{N} - \{0, 1\}$ is not a multiple of 10. Then, given $b^{-1}a \equiv {}^b a \pmod{10^n} \wedge b^{-1}a \not\equiv {}^b a \pmod{10^{n+1}}$, for all $b \in \mathbb{Z}^+$, $V(a, b)$ returns the non-negative integer such that ${}^b a \equiv {}^{b+1}a \pmod{10^{n+V(a,b)}} \wedge {}^b a \not\equiv {}^{b+1}a \pmod{10^{n+V(a,b)+1}}$, and we define $V(a, b)$ as the *congruence speed* of the base a at the given height of its hyperexponent b .

Furthermore, let $\bar{b} := \min_b \{b \in \mathbb{Z}^+ : V(a, b) = V(a, b+k) \text{ for all } k \in \mathbb{Z}^+\}$ so that we define as *constant congruence speed* of a the positive integer $V(a) := V(a, \bar{b})$.

We know that a sufficient but not necessary condition for having $V(a) = V(a, \bar{b})$ is to set $\bar{b} := a + 1$ (for a tighter bound on $\bar{b} := \bar{b}(a)$, holding for any $a \not\equiv 0 \pmod{10}$, see Reference [12], p. 450, line 6).

Lemma 1. Let $a \in \mathbb{N}$ be such that $a \not\equiv 0 \pmod{10}$. Then, for all $t \in \mathbb{N}_0$, there exist infinitely many $c \in \mathbb{Z}^+$ such that $V(a^c) = t$.

Proof. Disregarding the special case $t = 0$, this proof immediately follows from Definition 2.1.

For any integer $a > 1$ which is not a multiple of 10, the constant congruence speed of the tetration ${}^b a$ is well-defined and it is the same for any $b \in \{a + 1, a + 2, a + 3, \dots\}$. Thus, by the last line of Equation (2) in Reference [11], it is sufficient to consider $\hat{a} := 10^t - 1$ so that $V(\hat{a}) = t$ is true for any given positive integer t , and then we can easily complete the proof by observing that $V(1) = 0$ is stated in Definition 1.3 of Reference [12].

Trivially, $V(\hat{a}, b) = V(\hat{a}, b+1) = V(\hat{a}, b+2) = \dots$ is certainly true for every integer $b \geq \hat{a}+1$ and, by assuming that $t \in \mathbb{Z}^+$, for all the aforementioned values of b , we have that $\hat{a} = 10^t - 1$ implies $V(\hat{a}, b) = t$ (while from $a = 1 \Rightarrow t = 0$ it follows that $V(1^c) = 0$ for any non-negative integer c). Consequently, let $t \in \mathbb{Z}^+$, assume $b \in \{10^t, 10^t + 1, 10^t + 2, \dots\}$, and then $V((10^t - 1)^c, b) = V((10^t - 1)^c) = t$ is true for any $c \in \{b^{-1}(10^t - 1), {}^b(10^t - 1), {}^{b+1}(10^t - 1), \dots\}$ so that the proof of Lemma 1 is complete. \square

Thus, Lemma 1 shows the existence of infinitely many c -th powers of the tetration base $a : a \equiv \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \pmod{10}$ that are characterized by any given (arbitrarily large) non-negative constant congruence speed.

Remark 1. We note that, in radix-10, there exist only three positive 1-automorphic numbers and they are congruent modulo 100 to 1, 25, and 76 (respectively). Thus, the corresponding three integers found by considering the two rightmost digits of the analogous solutions of the fundamental 10-adic (decadic) equation $y^5 = y$, by Equation (2) of Reference [11] (see also the OEIS sequences [A018247](#) and [A018248](#)), describe 1-automorphic numbers [14] (e.g., $\alpha_{76} \mapsto a_{76} := 76$ since $76^2 \equiv 76 \pmod{10^2}$ and we know that, in radix-10, there are only four 10-adic solutions, including $\alpha_{00} := \dots 000000$, to the equation $y^2 = y$ [2]). Consequently, by looking at lines 4, 5, and 7 of Equation (16) in Reference [12], we can see that the recurrences described by Equations (1) to (3) hold for every $c \in \mathbb{Z}^+$.

$$\tilde{a} \equiv 6 \pmod{10} \Rightarrow V(\tilde{a}) \begin{cases} = V(\tilde{a}^c) & \text{iff } c \equiv \{1, 2, 3, 4\} \pmod{5} \\ \leq V(\tilde{a}^c) & \text{iff } c \equiv 0 \pmod{5} \end{cases} . \quad (1)$$

$$\tilde{a} \equiv 5 \pmod{20} \Rightarrow V(\tilde{a}) \begin{cases} = V(\tilde{a}^c) & \text{iff } c \equiv 1 \pmod{2} \\ \leq V(\tilde{a}^c) & \text{iff } c \equiv 0 \pmod{2} \end{cases} . \quad (2)$$

$$\tilde{a} \equiv 1 \pmod{20} \Rightarrow V(\tilde{a}) \begin{cases} = V(\tilde{a}^c) & \text{iff } c \equiv \{1, 2, 3, 4\} \pmod{5} \\ \leq V(\tilde{a}^c) & \text{iff } c \equiv 0 \pmod{5} \end{cases} . \quad (3)$$

The investigation of this observation (with specific reference to Equation (2)) leads us to the following theorem.

Theorem 1. *For each $c \in \mathbb{Z}^+$, there exist infinitely many $a : a \equiv 5 \pmod{20}$ such that $\sqrt[t]{a} \in \mathbb{Z}^+ \wedge V(\sqrt[t]{a}) = t \wedge V(a) \geq t$ holds for all $t \in \mathbb{N} - \{0, 1\}$. Symmetrically, for each $t \in \mathbb{N} - \{0, 1\}$, there exist infinitely many $a : a \equiv 5 \pmod{20}$ such that $\sqrt[t]{a} \in \mathbb{Z}^+ \wedge V(\sqrt[t]{a}) = t \wedge V(a) \geq t$ holds for all $c \in \mathbb{Z}^+$.*

Proof. Let us (constructively) prove first the last statement of Theorem 1 since it simply follows from the constancy of the congruence speed as it has been shown in Reference [11], Section 2.1.

Consider the rightmost $t \in \mathbb{N} - \{0, 1\}$ digits of the 10-adic integer $\alpha_{25} := \{5^{2^n}\}_\infty$, say $x_t x_{t-1} \dots \dots 2_5$, and then juxtapose to the left the $(t+1)$ -th digit plus 1 if $x_{t+1} \leq 8$ or the $(t+1)$ -th digit minus 1 if $x_{t+1} = 9$.

$$\text{So, let } \tilde{x}_{t+1} := \begin{cases} x_{t+1} + 1 & \text{if } x_{t+1} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \\ x_{t+1} - 1 & \text{if } x_{t+1} \in \{9\} \end{cases} .$$

Thus, by using the symbol “ $_$ ” to indicate the juxtaposition of consecutive digits (e.g., $3_6_1 = 36_1 = 3_61 = 361$), the base $\tilde{a} := \tilde{x}_{t+1} x_t x_{t-1} \dots \dots 2_5$ is characterized by a constant congruence speed of t (i.e., $V(\tilde{x}_{t+1} x_t x_{t-1} \dots \dots 2_5) = V(\tilde{a}) = t$ for any $t \in \mathbb{N} - \{0, 1\}$). This property follows from Reference [12], Equation (16), line 5 (i.e., $2^t \mid \tilde{x}_{t+1} x_t x_{t-1} \dots \dots 2_5 \wedge 2^{t+1} \nmid \tilde{x}_{t+1} x_t x_{t-1} \dots \dots 2_5$, for any $t \geq 2$). Since (as discussed in Remark 1) $\alpha_{25} \mapsto a_{25} := 25$ and $25^2 \equiv 25 \pmod{10^2}$, from Hensel’s lemma [6] (see also [3, 13]), we have that if $\tilde{a} \equiv \alpha_{25} \pmod{10^t} \wedge \tilde{a} \not\equiv \alpha_{25} \pmod{10^{t+1}}$, then $\tilde{a}^c \equiv \alpha_{25} \pmod{10^t}$ for any given $c \in \mathbb{Z}^+$ (in

general, we cannot assert that $(\tilde{a} \equiv \alpha_{25} \pmod{10^t} \wedge \tilde{a} \not\equiv \alpha_{25} \pmod{10^{t+1}})$ implies $\tilde{a}^c \not\equiv \alpha_{25} \pmod{10^{t+1}}$ for the given pair (t, c) .

Consequently, by simply taking $a := (\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)^c$ (as c is free to run over the positive integers), we have proven the existence, for any given $t \in \mathbb{N} - \{0, 1\}$, of infinitely many tetration bases $a \equiv 5 \pmod{20}$ such that $V(a) \geq t$ holds for all the elements of the aforementioned set, a set that contains infinitely many distinct perfect powers originated from the string $\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5$ (since \tilde{a} is a positive integer by definition, then $a = \tilde{a}^c$ implies that $\sqrt[c]{a} \in \mathbb{Z}^+$). Hence, $V(\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5) = t$ implies $V((\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)^c) \geq t$ by observing that $V((\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)^c) = V(\dots x_t x_{t-1} \dots 2_5)$, and trivially $\sqrt[c]{(\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)^c} \in \mathbb{Z}^+$ for all $c \in \mathbb{Z}^+$ (we point out that, for any $t \geq 2$ and as long as c is a positive integer, $(\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)^c \equiv x_t x_{t-1} \dots 2_5 \pmod{10^t}$ holds by construction [14]).

Now, let us prove the first statement of Theorem 1 and complete the proof.

For this purpose, it is sufficient to note that $V(\tilde{x}_{t+1}x_t x_{t-1} \dots 2_5) = V(10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)$ is true for any positive integer k . So, we can take the c -th power of every integer of the form $10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2_5$ to get k distinct sets of cardinality \aleph_0 each, whose elements, by construction, always satisfy the first statement of the theorem (we have already shown that, for any given $c \in \mathbb{Z}^+$, if $V(10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2_5) = t$, then $V(10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2_5)^c \geq t$ holds for every $t \in \mathbb{N} - \{0, 1\}$).

Therefore, both the statements of Theorem 1 have been shown to be true, and this concludes the proof. \square

3. Main result

From here on, let us indicate the p -adic valuation [10] of any tetration base a as $\nu_p(a)$, for any prime number p .

Then, we need the following lemma to prove the existence, for any $\tilde{a} \in \mathbb{N} - \{0, 1\}$ such that $\tilde{a} \not\equiv 0 \pmod{10} \wedge V(\tilde{a}) = t$, of infinitely many c -th powers of \tilde{a} having a constant congruence speed of $t, t + 1, t + 2, t + 3$, and so forth. Furthermore, for any given positive integer c , Lemma 2 shows the existence of infinitely many tetration bases of the form $(10^{k+t} + 10^t + 1)^c$ (where $k \in \mathbb{Z}^+$, as usual) characterized by any positive value, t , of their constant congruence speed.

Lemma 2. *For every triad $t, k, c \in \mathbb{Z}^+$, $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$.*

Proof. We prove Lemma 2 by induction on c : “For every positive integer c , $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$, where $t, k \in \mathbb{Z}^+$ ”.

Let us start with the base case, so $c = 1$. It is well known that the p -adic valuation is a valuation [9] and consequently $c = 1$ implies $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = \nu_5(10^{k+t} + 10^t) = \nu_5(10^t \cdot (10^k + 1)) = \nu_5(10^t) + \nu_5(10^k + 1)$. Since $10^k + 1 \equiv 1 \pmod{5}$ for any positive integer k , it follows that $\nu_5(10^k + 1) = 0$. Hence, $\nu_5(10^t) + \nu_5(10^k + 1) = \nu_5(10^t) = \nu_5(2^t \cdot 5^t) = \nu_5(5^t) = t$. Thus, if $c = 1$, then $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t$.

To prove the inductive step, we assume that $\nu_5((10^{t+k} + 10^t + 1)^c - 1) = t + \nu_5(c)$ is true for $c = n$ and we show that it is also true for $c = n + 1$.

For this purpose, we note that $\nu_5((10^{t+k} + 10^t + 1)^{n+1} - 1) = \nu_5((10^{t+k} + 10^t + 1)^{n+1} - (1^{n+1}))$, so we can invoke the Lifting The Exponent lemma (see [8] and [5], Lemma 2.6) for odd primes, stating that for any integers x, y , a positive integer $n + 1$, and a prime number p such that $p \nmid x \wedge p \nmid y$, if p divides $x - y$, then $\nu_p(x^{n+1} - y^{n+1}) = \nu_p(x - y) + \nu_p(n + 1)$. Thus, by observing that $p := 5$ is an odd prime satisfying all the conditions above for $x := 10^{k+t} + 10^t + 1 \wedge y := 1$ (since $10^{k+t} + 10^t$ is a multiple of 5), $\nu_5((10^{k+t} + 10^t + 1)^{n+1} - 1^{n+1}) = \nu_5(10^{k+t} + 10^t) + \nu_5(n + 1)$ and then $c = n + 1 \Rightarrow \nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$ is finally proven.

Therefore, we have shown that $\nu_5((10^{t+k} + 10^t + 1)^c - 1) = t + \nu_5(c)$ is true for every triad of positive integers t, c, k and the proof is complete. \square

Theorem 2. *For any given triad $t, k, c \in \mathbb{Z}^+$, $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$.*

Proof. Theorem 2 easily follows from Lemma 2. By [12], Equation (16), line 7, we have that $V((10^{k+t} + 10^t + 1)^c) = \nu_5((10^{k+t} + 10^t + 1)^c - 1)$ and, since Lemma 2 asserts that $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$, it follows that $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$ for any positive integers t, k , and c . \square

Remark 2. Let the tetration base $a : a \equiv 6 \pmod{10}$ be given. Then, by looking at the two rightmost digits of the corresponding 10-adic solution, $\alpha_{76} = \dots 7109376$ (see Remark 1), and applying the usual strategy (already described in the proof of Theorem 1), we find the sequence $a_n := 10^{n+1} + 86$, $n \in \mathbb{Z}^+$ (defining also the set $\{186, 1086, 10086, 100086, \dots\}$). Now, we can create an infinite set consisting of the c -th powers of each aforementioned term, a set whose elements are all characterized by a unit constant congruence speed as long as c is not a multiple of 5. Thus, $5 \nmid c$ implies $V(186^c) = V(1086^c) = V(10086^c) = V(100086^c) = \dots = 1$.

The above is just another example of the \tilde{x}_{t+1} idea, introduced in the proof of Theorem 1, shown by taking into account $t = 1$ and the solution $\alpha_{76} := 1 - \{5^{2^n}\}_\infty$ of the equation $y^2 = y$ in the commutative ring of 10-adic integers (as we know, the other three solutions are $\alpha_{00} := 0$, $\alpha_{01} := 1$, and $\alpha_{25} := \{5^{2^n}\}_\infty$ [11]).

Corollary 1. *Let $t \in \mathbb{Z}^+$ and assume that $V(\tilde{a}) = t$. Then, for any non-negative integer h , there exist infinitely many $c \in \mathbb{Z}^+$ such that $V(\tilde{a}^c) = t + h$.*

Proof. Let $k \in \mathbb{Z}^+$. If $t > 1$, it is sufficient to observe that, by Theorem 2, $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$. So, let $\tilde{a} := 10^{k+t} + 10^t + 1$ and then, for any $h \in \mathbb{N}_0$, $\tilde{a}^c := \tilde{a}^{2^{k-1} \cdot 5^h}$ identifies an infinite set of valid tetration bases (since $\nu_5(2^{k-1} \cdot 5^h) = h$ is true for any positive integer k and, consequently, the constant congruence speed of $\tilde{a}^{2^{k-1} \cdot 5^h}$ does not depend on k).

For the remaining case, $t = 1$, we note that Remark 2 gives us a valid set of solutions (i.e., any tetration base of the form $(10^{k+1} + 86)^{2^{k-1} \cdot 5^h}$ does the job, since $V(10^{k+1} + 86) = 1$ for every positive integer k). \square

Theorem 3. *Let the integer $a > 1$ not be a multiple of 10. Then, for all $c \in \mathbb{Z}^+$, there exist infinitely many a such that $\sqrt[c]{a} \in \mathbb{Z}^+ \wedge V(a) = V(\sqrt[c]{a}) = t$ holds for every positive integer $t > \nu_5(c)$.*

Proof. Since Theorem 2 states that $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$ for any $k \in \mathbb{Z}^+$, it follows that every tetration base of the form $(10^{k+t-\nu_5(c)} + 10^{t-\nu_5(c)} + 1)^c$ is characterized by a constant congruence speed of $(t - \nu_5(c)) + \nu_5(c)$, as long as $t \geq 1 + \nu_5(c)$. Although this is enough to constructively prove the theorem, we are free to simplify the generic form of the above by observing that $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = \nu_5((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c - 1^c) = \nu_5(10^{k+t} + 10^{t-\nu_5(c)} + \nu(c) = \min\{\nu_5(10^{k+t}), \nu_5(10^{t-\nu_5(c)})\} + \nu(c)$ [1] (since $k > -\nu_5(c)$ implies that $\nu_5(10^{k+t}) > \nu_5(10^{t-\nu_5(c)})$), and then it follows that $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = \nu_5(10^{t-\nu_5(c)} + \nu(c) = t - \nu_5(c) + \nu_5(c) = t$.

Therefore, $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = t$ is true for any $c, k \in \mathbb{Z}^+$ and $t > \nu_5(c)$, so the proof is complete. \square

Now, let $t = c$ and observe that $\nu_5(c) < c$ holds for any positive integer c .

Then, the proof of Theorem 3 shows the existence of a very special set of tetration bases that are c -th powers of an integer and whose constant congruence speed is c , a set including all the bases of the form $(10^{c+k} + 10^{c-\nu_5(c)} + 1)^c, k \in \mathbb{Z}^+$ (e.g., $(c = 250 \wedge k = 10000)$ implies $V(10^{10250} + 10^{247} + 1)^{250} = 250$).

Therefore, we have finally proven Corollary 2 as a special case of Theorem 3.

Corollary 2. *Let $a \in \mathbb{N} - \{0, 1\}$ not be a multiple of 10. Then, for every $c \in \mathbb{Z}^+$, there exist infinitely many a such that $\sqrt[c]{a} \in \mathbb{Z}^+ \wedge V(a) = V(\sqrt[c]{a}) = c$.*

Since, for any given $c \in \mathbb{Z}^+$, the set of all the integers of the form $10^{c+k} + 10^{c-\nu_5(c)} + 1$ contains infinitely many elements that are not divisible by themselves (as $digit_sum(10^{c+k} + 10^{c-\nu_5(c)} + 1) = 3, 3 \mid 10^{c+k} + 10^{c-\nu_5(c)} + 1$ and $3^2 \nmid 10^{c+k} + 10^{c-\nu_5(c)} + 1$, implying that none of these numbers can be a perfect power of degree greater than 1), there exist infinitely many perfect powers of degree $c = 1, 2, 3, 4, \dots$ having a constant congruence speed of c , and it is sufficient to consider all the tetration bases of the form $(10^{c+k} + 10^{c-\nu_5(c)} + 1)^c$.

4. Conclusion

In the previous section, we have shown the existence of infinitely many perfect powers of degree c with a constant congruence speed of $\nu_5(c) + 1, \nu_5(c) + 2, \nu_5(c) + 3$, and so forth.

Moreover, for any given positive integer c , we have constructed an infinite set of c -th perfect powers that are also characterized by a constant congruence speed of c , providing the general equation $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = t$ ($k \in \mathbb{Z}^+, t \in \mathbb{Z}^+ : t > \nu_5(c)$) so that if $t = c$, then $V((10^{k+c} + 10^{c-\nu_5(c)} + 1)^c) = c$ is always true.

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(Concerned with sequences [A018247](#), [A018248](#), [A317905](#), and [A372490](#).)