



**HAL**  
open science

# On the relation between perfect powers and tetration frozen digits

Marco Ripà

► **To cite this version:**

Marco Ripà. On the relation between perfect powers and tetration frozen digits. 2023. hal-04131657v1

**HAL Id: hal-04131657**

**<https://hal.science/hal-04131657v1>**

Preprint submitted on 29 Jun 2023 (v1), last revised 20 Jul 2024 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - ShareAlike 4.0 International License

# On the relation between perfect powers and tetration frozen digits

Marco Ripà

World Intelligence Network

Rome, Italy

e-mail: marco.ripa@mensa.it

**Abstract:** This paper provides a link between integer exponentiation and integer tetration, since it is devoted to introduce some peculiar sets of perfect powers characterized by any given value of their constant congruence speed, revealing a fascinating relation between the degree of every perfect power belonging to any congruence class modulo 20 and the number of digits frozen by these special tetration bases, in radix-10, for any unit increment of the hyperexponent. In particular, given any positive integer  $c$ , we constructively prove the existence of infinitely many  $c$ -th perfect powers having a constant congruence speed of  $c$ .

**Keywords:** Perfect power, Tetration, Power tower, Congruence speed, Convergence.

**2020 Mathematics Subject Classification:** 11A07, 11A15.

## 1 Introduction

Since 2020 [12], by assuming the standard decimal numeral system (radix-10), we have shown that the integer tetration (also known as hyper-4)  ${}^b a := \begin{cases} a & \text{if } b = 1 \\ a^{({}^{b-1} a)} & \text{if } b \geq 2 \end{cases}$  has a unique, mesmerizing, property [4] involving the number of new frozen rightmost digits for any unit increment of its hyperexponent,  $b \in \mathbb{Z}^+$  [13, 14]. Indeed, this value does not depend on  $b$  when  $b$  is sufficiently large and the tetration base,  $a \in \mathbb{Z}^+$ , is not a multiple of 10.

We refer to the mentioned property as the constancy of the *congruence speed* of tetration (see Definition 2.1).

From Reference [13], we know that any positive integer  $a > 1$ , which is not a multiple of 10, is characterized by a finite, strictly positive, integer value of its own constant congruence speed (the map of the constant congruence speed of every  $a$  is fully described in [13, 14]).

The aim of the present paper is to constructively prove the existence of infinitely many perfect powers with any given positive constant congruence speed.

A noteworthy result, which follows from Theorem 3.4 as a corollary, is the existence, for any given positive integer  $c$ , of infinitely many  $c$ -th perfect powers (i.e., an integer  $a > 1$  is a  $c$ -th perfect power if there are integers  $\tilde{a}$  and  $c$  such that  $a = \tilde{a}^c$ , so we have *perfect squares* if  $c = 2$ , *perfect cubes* if  $c = 3$ , and so forth) having a constant congruence speed of  $c$ .

## 2 Preliminary investigations with the automorphic numbers

In order to present the results in a compact way, let us properly define the constant congruence speed of tetration as already done in Reference [14], p. 442, Definitions 1.1&1.2.

**Definition 2.1.** *Let  $n \in \mathbb{N}_0$  and assume that  $a \in \mathbb{N} - \{0, 1\}$  is not a multiple of 10. Then, given  ${}^{b-1}a \equiv {}^b a \pmod{10^n} \wedge {}^{b-1}a \not\equiv {}^b a \pmod{10^{n+1}}, \forall b \in \mathbb{Z}^+, V(a, b)$  returns the non-negative integer such that  ${}^b a \equiv {}^{b+1}a \pmod{10^{n+V(a,b)}} \wedge {}^b a \not\equiv {}^{b+1}a \pmod{10^{n+V(a,b)+1}}$ , and we define  $V(a, b)$  as the “congruence speed” of the base  $a$  at the given height of its hyperexponent  $b$ . Furthermore, let  $\bar{b} := \min_b \{b \in \mathbb{Z}^+ : V(a, b) = V(a, b+k), \forall k \in \mathbb{Z}^+\}$  so that we define as “constant congruence speed” of  $a$  the positive integer  $V(a) := V(a, \bar{b})$ .*

We know that a sufficient but not necessary condition for having  $V(a) = V(a, \bar{b})$  is to set  $\bar{b} := a + 1$  (for a tighter bound on  $\bar{b} := \bar{b}(a)$ , holding for any  $a \not\equiv 0 \pmod{10}$ , see [14], p. 450, line 6).

**Lemma 2.2.** *Let  $a \in \mathbb{N} : a \not\equiv 0 \pmod{10}$ . Then,  $\forall t \in \mathbb{N}_0, \exists^\infty c \in \mathbb{Z}^+ : V(a^c) = t$ .*

*Proof.* Disregarding the special case  $t = 0$ , this proof immediately follows from Definition 2.1.

For any integer  $a > 1$  which is not a multiple of 10, the constant congruence speed of the tetration  ${}^b a$  is well-defined and it is the same for any  $b \in \{a + 1, a + 2, a + 3, \dots\}$ . Thus, by the last line of Equation (2) in Reference [13], it is sufficient to consider  $\hat{a} := 10^t - 1$  so that  $V(\hat{a}) = t$  is true for any given positive integer  $t$  and then we can easily complete the proof by observing that  $V(1) = 0$  is stated in Definition 1.3 of Reference [14].

Trivially,  $V(\hat{a}, b) = V(\hat{a}, b + 1) = V(\hat{a}, b + 2) = \dots$  is certainly true for every integer  $b \geq \hat{a} + 1$  and, by assuming that  $t \in \mathbb{Z}^+$ , for all the aforementioned values of  $b$ , we have that  $\hat{a} = 10^t - 1 \Rightarrow V(\hat{a}, b) = t$  (while from  $a = 1 \Rightarrow t = 0$  it follows that  $V(1^c) = 0$  for any non-negative integer  $c$ ).

Consequently, let  $t \in \mathbb{Z}^+$ , assume  $b \in \{10^t, 10^t + 1, 10^t + 2, \dots\}$ , and then  $V((10^t - 1)^c, b) = V((10^t - 1)^c) = t$  is true for any  $c \in \{b^{-1}(10^t - 1), {}^b(10^t - 1), {}^{b+1}(10^t - 1), \dots\}$  so that the proof of Lemma 2.2 is complete.  $\square$

Thus, Lemma 2.2 shows the existence of infinitely many  $c$ -th powers of  $a : a \equiv \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \pmod{10}$  characterized by any given (arbitrarily large) non-negative constant congruence speed.

**Remark 2.1.** *We note that, in radix-10, there exist only 3 positive 1-automorphic numbers [16] and they are congruent modulo 100 to 1, 25, and 76. Thus, the corresponding three integers found by considering the two rightmost digits of the analogous solutions of the fundamental decadic equation  $y^5 = y$ , by [13], describe 1-automorphic numbers (e.g.,  $\alpha_{76} \mapsto a_{76} := 76$  since  $76^2 \equiv 76 \pmod{10^2}$ ) and we know that, in radix-10, there are only 4 decadic solutions, including  $\alpha_{00} := \dots 000000$ , of the equation  $y^2 = y$  [2, 7, 8]). Consequently, by looking at lines 4, 5, and 7 of Equation (16) [14], we can see that the recurrences described by Equations (1) to (3) hold for every  $c \in \mathbb{Z}^+$ .*

$$\tilde{a} \equiv 6 \pmod{10} \Rightarrow V(\tilde{a}) \begin{cases} = V(\tilde{a}^c) & \text{iff } c \equiv \{1, 2, 3, 4\} \pmod{5} \\ \leq V(\tilde{a}^c) & \text{iff } c \equiv 0 \pmod{5} \end{cases} . \quad (1)$$

$$\tilde{a} \equiv 5 \pmod{20} \Rightarrow V(\tilde{a}) \begin{cases} = V(\tilde{a}^c) & \text{iff } c \equiv 1 \pmod{2} \\ \leq V(\tilde{a}^c) & \text{iff } c \equiv 0 \pmod{2} \end{cases} . \quad (2)$$

$$\tilde{a} \equiv 1 \pmod{20} \Rightarrow V(\tilde{a}) \begin{cases} = V(\tilde{a}^c) & \text{iff } c \equiv \{1, 2, 3, 4\} \pmod{5} \\ \leq V(\tilde{a}^c) & \text{iff } c \equiv 0 \pmod{5} \end{cases} . \quad (3)$$

The investigation of this observation (with specific reference to Equation (2)) leads us to the following theorem.

**Theorem 2.3.**  $\forall c \in \mathbb{Z}^+ \exists^\infty a \equiv 5 \pmod{20} : (\sqrt[c]{a} \in \mathbb{Z}^+ \wedge V(\sqrt[c]{a}) = t \wedge V(a) \geq t, \forall t \in \mathbb{N} - \{0, 1\})$ . Symmetrically,  $\forall t \in \mathbb{N} - \{0, 1\} \exists^\infty a \equiv 5 \pmod{20} : (\sqrt[c]{a} \in \mathbb{Z}^+ \wedge V(\sqrt[c]{a}) = t \wedge V(a) \geq t, \forall c \in \mathbb{Z}^+)$ .

*Proof.* Let us (constructively) prove first the last statement of Theorem 2.3, since it simply follows from the constancy of the congruence speed as it has been shown in Reference [13], Section 2.1 (our strategy here is also reminiscent of the proof of Theorem 2 in [13], Section 4).

Consider the rightmost  $t \in \mathbb{N} - \{0, 1\}$  digits of the decadic integer  $\alpha_{25} := \{5^{2^n}\}_\infty$ , say  $x_t x_{t-1} \dots 2_5$ , and then juxtapose to the left the  $(t+1)$ -th digit plus 1 if  $x_{t+1} \leq 8$  or the  $(t+1)$ -th digit minus 1 if  $x_{t+1} = 9$ . So, let  $\tilde{x}_{t+1} := \begin{cases} x_{t+1} + 1 & \text{if } x_{t+1} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \\ x_{t+1} - 1 & \text{if } x_{t+1} \in \{9\} \end{cases}$ .

Thus, the base  $\tilde{a} := \tilde{x}_{t+1} x_t x_{t-1} \dots 2_5$  is characterized by a constant congruence speed of  $t$  (i.e.,  $V(\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5) = V(\tilde{a}) = t$  for any  $t \in \mathbb{N} - \{0, 1\}$ ). This property follows from Reference [14], Equation (16), line 5 (i.e.,  $2^t \mid \tilde{x}_{t+1} x_t x_{t-1} \dots 2_5 \wedge 2^{t+1} \nmid \tilde{x}_{t+1} x_t x_{t-1} \dots 2_5$ , for any  $t \geq 2$ ). Since (as discussed in Remark 2.1)  $\alpha_{25} \mapsto a_{25} := 25$  and  $25^2 \equiv 25 \pmod{10^2}$ , from Hensel's lemma [6] (see also [3, 15]), we have that  $(\tilde{a} \equiv \alpha_{25} \pmod{10^t} \wedge \tilde{a} \not\equiv \alpha_{25} \pmod{10^{t+1}}) \Rightarrow \tilde{a}^c \equiv \alpha_{25} \pmod{10^t}$ , for any given  $c \in \mathbb{Z}^+$  (in general, we cannot assert that  $(\tilde{a} \equiv \alpha_{25} \pmod{10^t} \wedge \tilde{a} \not\equiv \alpha_{25} \pmod{10^{t+1}})$  implies  $\tilde{a}^c \not\equiv \alpha_{25} \pmod{10^{t+1}}$  for the given pair  $(t, c)$ ).

Consequently, by simply taking  $a := (\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5)^c$  (as  $c$  is free to run over the positive integers), we have proven the existence, for any given  $t \in \mathbb{N} - \{0, 1\}$ , of infinitely many tetration bases  $a \equiv 5 \pmod{20}$  such that  $V(a) \geq t$  holds for all the elements of the aforementioned set, a set that contains  $\aleph_0$  distinct perfect powers originated from the string  $\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5$  (since  $\tilde{a}$  is a positive integer by definition, then  $a = \tilde{a}^c \Rightarrow \sqrt[c]{a} \in \mathbb{Z}^+$ ). Hence,  $V(\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5) = t \Rightarrow V((\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5)^c) \geq t$  by observing that  $V((\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5)^c) = V(\dots x_t x_{t-1} \dots 2_5)$ , and trivially  $\sqrt[c]{(\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5)^c} \in \mathbb{Z}^+ \forall c \in \mathbb{Z}^+$  (we point out that, for any  $t \geq 2$  and as long as  $c$  is a positive integer,  $(\tilde{x}_{t+1} x_t x_{t-1} \dots 2_5)^c \equiv x_t x_{t-1} \dots 2_5 \pmod{10^t}$  holds by construction [16]).

Now, let us prove the first statement of Theorem 2.3 and complete the proof.

For this purpose, it is sufficient to note that  $V(\tilde{x}_{t+1}x_t x_{t-1} \dots 2.5) = V(10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2.5)$  is true for any positive integer  $k$ . So, we can take the  $c$ -th power of every integer of the form  $10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2.5$  in order to get  $k$  distinct sets of cardinality  $\aleph_0$  each, whose elements, by construction, always satisfy the first statement of the theorem (we have already shown that, for any given  $c \in \mathbb{Z}^+$ ,  $V(10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2.5) = t \Rightarrow V(10^{k+t} + \tilde{x}_{t+1}x_t x_{t-1} \dots 2.5)^c \geq t$  holds for every  $t \in \mathbb{N} - \{0, 1\}$ ).

Therefore, both the statements of Theorem 2.3 have been shown to be true and this concludes the proof.  $\square$

### 3 Main result

From here on, let us indicate the  $p$ -adic valuation [11] of any tetration base  $a$  as  $\nu_p(a)$ , for any prime number  $p$ .

Then, we need the following lemma in order to prove the existence, for any  $\tilde{a} \in \mathbb{N} - \{0, 1\} : (\tilde{a} \not\equiv 0 \pmod{10} \wedge V(\tilde{a}) = t)$ , of infinitely many  $c$ -th powers of  $\tilde{a}$  having a constant congruence speed of  $t, t + 1, t + 2, t + 3$ , and so forth. Furthermore, for any given positive integer  $c$ , Lemma 3.1 shows the existence of infinitely many tetration bases of the form  $(10^{k+t} + 10^t + 1)^c$  (where  $k \in \mathbb{Z}^+$ , as usual) characterized by any positive value,  $t$ , of their constant congruence speed.

**Lemma 3.1.**  $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$  is true for any  $t, k, c \in \mathbb{Z}^+$ .

*Proof.* We prove Lemma 3.1 by induction on  $c$ : “ $\forall c \in \mathbb{Z}^+, \nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$ , where  $t, k \in \mathbb{Z}^+$ ”.

Let us start with the base case, so  $c = 1$ . By [10], we know that the  $p$ -adic valuation is a valuation and then  $c = 1 \Rightarrow \nu_5((10^{k+t} + 10^t + 1)^c - 1) = \nu_5(10^{k+t} + 10^t) = \nu_5(10^t \cdot (10^k + 1)) = \nu_5(10^t) + \nu_5(10^k + 1)$ . Since  $10^k + 1 \equiv 1 \pmod{5}$  for any positive integer  $k$ , it follows that  $\nu_5(10^k + 1) = 0$ . Hence,  $\nu_5(10^t) + \nu_5(10^k + 1) = \nu_5(10^t) = \nu_5(2^t \cdot 5^t) = \nu_5(5^t) = t$ .

Thus,  $c = 1 \Rightarrow \nu_5((10^{k+t} + 10^t + 1)^c - 1) = t$ .

In order to prove the inductive step, we assume that  $\nu_5((10^{t+k} + 10^t + 1)^c - 1) = t + \nu_5(c)$  is true for  $c = n$  and we show that it is also true for  $c = n + 1$ . For this purpose, we note that  $\nu_5((10^{t+k} + 10^t + 1)^{n+1} - 1) = \nu_5((10^{t+k} + 10^t + 1)^{n+1} - (1^{n+1}))$ , so we can invoke the Lifting The Exponent lemma (see [9] and [5], Lemma 2.6) for odd primes, stating that for any integers  $x, y$ , a positive integer  $n + 1$ , and a prime number  $p$  such that  $p \nmid x \wedge p \nmid y$ , if  $p$  divides  $x - y$ , then  $v_p(x^{n+1} - y^{n+1}) = v_p(x - y) + v_p(n + 1)$ . Thus, by observing that  $p := 5$  is an odd prime satisfying all the conditions above for  $x := 10^{k+t} + 10^t + 1 \wedge y := 1$  (since  $10^{k+t} + 10^t$  is a multiple of 5),  $\nu_5((10^{k+t} + 10^t + 1)^{n+1} - 1^{n+1}) = \nu_5(10^{k+t} + 10^t) + \nu_5(n + 1)$  and then  $c = n + 1 \Rightarrow \nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$  is finally proven.

Therefore, we have shown that  $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$  is true for any  $t, c, k \in \mathbb{Z}^+$  and the proof is complete.  $\square$

**Theorem 3.2.** For any  $t, k, c \in \mathbb{Z}^+$ ,  $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$ .

*Proof.* Theorem 3.2 easily follows from Lemma 3.1. By [14], Equation (16), line 7, we have that  $V((10^{k+t} + 10^t + 1)^c) = \nu_5((10^{k+t} + 10^t + 1)^c - 1)$  and, since Lemma 3.1 asserts that  $\nu_5((10^{k+t} + 10^t + 1)^c - 1) = t + \nu_5(c)$ , it follows that  $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$  for any positive integers  $t, k$ , and  $c$ .  $\square$

**Remark 3.1.** Let the tetration base  $a : a \equiv 6 \pmod{10}$  be given. Then, by looking at the two rightmost digits of the corresponding decadic solution,  $\alpha_{76} = \dots 7109376$  (see Remark 2.1), and applying the usual strategy (already described in the proof of Theorem 2.3), we find the sequence  $a_n := 10^{n+1} + 86$ ,  $n \in \mathbb{Z}^+$  (defining also the set  $\{186, 1086, 10086, 100086, \dots\}$ ). Now, we can obviously create an infinite set consisting of the  $c$ -th powers of each aforementioned term, a set whose elements are all characterized by a unit constant congruence speed as long as  $c$  is not a multiple of 5. Thus,  $5 \nmid c \Rightarrow V(186^c) = V(1086^c) = V(10086^c) = V(100086^c) = \dots = 1$ .

The above is just another example of the  $\tilde{x}_{t+1}$  idea, introduced in the proof of Theorem 2.3, shown by taking into account  $t = 1$  and the solution  $\alpha_{76} := 1 - \{5^{2^n}\}_\infty$  of the equation  $y^2 = y$  in the commutative ring of decadic integers (as we know, the other three solutions are  $\alpha_{00} := 0$ ,  $\alpha_{01} := 1$ , and  $\alpha_{25} := \{5^{2^n}\}_\infty$  [13]).

**Corollary 3.3.** Let  $t \in \mathbb{Z}^+$  and assume that  $V(\tilde{a}) = t$ . Then,  $\exists^\infty c \in \mathbb{Z}^+ : V(\tilde{a}^c) = t + h$ , for any non-negative integer  $h$ .

*Proof.* Let  $k \in \mathbb{Z}^+$ . If  $t > 1$ , it is sufficient to observe that, by Theorem 3.2,  $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$ . So, let  $\tilde{a} := 10^{k+t} + 10^t + 1$  and then, for any  $h \in \mathbb{N}_0$ ,  $\tilde{a}^c := \tilde{a}^{2^{k-1} \cdot 5^h}$  identifies an infinite set of valid tetration bases (since  $\nu_5(2^{k-1} \cdot 5^h) = h$  is true for any positive integer  $k$  and, consequently, the constant congruence speed of  $\tilde{a}^{2^{k-1} \cdot 5^h}$  does not depend on  $k$ ).

For the remaining case,  $t = 1$ , we note that Remark 3.1 gives us a valid set of solutions (i.e., any tetration base of the form  $(10^{k+1} + 86)^{2^{k-1} \cdot 5^h}$  does the job, since  $V(10^{k+1} + 86) = 1$  for every positive integer  $k$ ).  $\square$

**Theorem 3.4.** Let the integer  $a > 1$  not be a multiple of 10. Then,  $\forall c \in \mathbb{Z}^+, \exists^\infty a : (\sqrt[c]{a} \in \mathbb{Z}^+ \wedge V(a) = V(\sqrt[c]{a}) = t, \forall t \in \mathbb{Z}^+ : t > \nu_5(c))$ .

*Proof.* Since Theorem 3.2 states that  $V((10^{k+t} + 10^t + 1)^c) = t + \nu_5(c)$  for any  $k \in \mathbb{Z}^+$ , it follows that every tetration base of the form  $(10^{k+t-\nu_5(c)} + 10^{t-\nu_5(c)} + 1)^c$  is characterized by a constant congruence speed of  $(t - \nu_5(c)) + \nu_5(c)$ , as long as  $t \geq 1 + \nu_5(c)$ . Although this is enough to constructively prove the theorem, we are free to simplify the generic form of the above by observing that  $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = \nu_5((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c - 1^c) = \nu_5(10^{k+t} + 10^{t-\nu_5(c)} + \nu(c) = \min\{\nu_5(10^{k+t}), \nu_5(10^{t-\nu_5(c)})\} + \nu(c)$  [1] (since  $k > -\nu_5(c) \Rightarrow \nu_5(10^{k+t}) > \nu_5(10^{t-\nu_5(c)})$ ) and then  $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = \nu_5(10^{t-\nu_5(c)} + \nu(c) = t - \nu_5(c) + \nu_5(c) = t$ .

Therefore,  $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = t$  is true for any  $c, k \in \mathbb{Z}^+$  and  $t > \nu_5(c)$ , so the proof is complete.  $\square$

Now, let  $t = c$  and observe that  $\nu_5(c) \leq c - 1$  holds for any positive integer  $c$ .

Then, the proof of Theorem 3.4 shows the existence of a very special set of tetration bases that are  $c$ -th powers of an integer and whose constant congruence speed is  $c$ , a set including

all the bases of the form  $(10^{c+k} + 10^{c-\nu_5(c)} + 1)^c, k \in \mathbb{Z}^+$  (e.g.,  $(c = 250 \wedge k = 10000) \Rightarrow V(10^{10250} + 10^{247} + 1)^{250} = 250$ ).

Therefore, we have finally proven Corollary 3.5 as a special case of Theorem 3.4.

**Corollary 3.5.** *Let  $a \in \mathbb{N} - \{0, 1\}$  not be a multiple of 10. Then,  $\forall c \in \mathbb{Z}^+, \exists^\infty a : (\sqrt[c]{a} \in \mathbb{Z}^+ \wedge V(a) = V(\sqrt[c]{a}) = c)$ .*

Since, for any given  $c \in \mathbb{Z}^+$ , the set of all the integers of the form  $10^{c+k} + 10^{c-\nu_5(c)} + 1$  contains infinitely many elements that are not divisible by themselves (as  $digit\_sum(10^{c+k} + 10^{c-\nu_5(c)} + 1) = 3, 3 \mid 10^{c+k} + 10^{c-\nu_5(c)} + 1$  and  $3^2 \nmid 10^{c+k} + 10^{c-\nu_5(c)} + 1$ , implying that none of these numbers can be a perfect power of degree greater than 1), there exist infinitely many perfect powers of degree  $c = 1, 2, 3, 4, \dots$  having a constant congruence speed of  $c$ , and it is sufficient to consider all the tetration bases of the form  $(10^{c+k} + 10^{c-\nu_5(c)} + 1)^c$ .

## 4 Conclusion

In the previous section, we have shown the existence of infinitely many perfect powers of degree  $c$  with a constant congruence speed of  $\nu_5(c) + 1, \nu_5(c) + 2, \nu_5(c) + 3$ , and so forth.

Moreover, for any given positive integer  $c$ , we have constructed an infinite set of  $c$ -th perfect powers that are also characterized by a constant congruence speed of  $c$ , providing the general equation  $V((10^{k+t} + 10^{t-\nu_5(c)} + 1)^c) = t$  ( $k \in \mathbb{Z}^+, t \in \mathbb{Z}^+ : t > \nu_5(c)$ ) so that if  $t = c$ , then  $V((10^{k+c} + 10^{c-\nu_5(c)} + 1)^c) = c$  is always true.

## Acknowledgements

The author would like to thank his friend Aldo Roberto Pessolano for having shared with him first thoughts about the outcome of the computer search performed on the smallest integers of the form  $a^c$ , congruent to 5 modulo 20, and characterized by a constant congruence speed of  $c$ .

## References

- [1] Cassels, J. W. S. (1986). *Local fields*. London Mathematical Society Student Texts 3, Cambridge University Press.
- [2] deGuerre, V. and Fairbairn, R. A. (1968). Automorphic numbers. *Journal of Recreational Mathematics*, 1(3), 173-179.
- [3] Germain, J. (2009). On the Equation  $a^x \equiv x \pmod{b}$ . *Integers*, 9(6), 629–638.
- [4] Googology Wiki contributors (2023). *Tetration*. In *Hyper operators*. Googology Wiki | Fandom (Updated on May 28 2023). Accessed: May 30 2023. Available online at: <https://googology.wikia.org/wiki/Tetration>.

- [5] Heuberger, C. and Mazzoli, M. (2017). Elliptic curves with isomorphic groups of points over finite field extensions. *Journal of Number Theory*, 181, 89–98.
- [6] Lewis, R. Y. (2019). A formal proof of Hensel’s lemma over the  $p$ -adic integers. *Proceedings of the 8th ACM SIGPLAN International Conference on Certified Programs and Proofs*, 15–26. Available online at: <https://robertylewis.com/padics/padics.pdf>.
- [7] OEIS Foundation Inc. (2023), *The 10-adic integer  $x = \dots 8212890625$  satisfying  $x^2 = x$* , Entry A018247 in the On-Line Encyclopedia of Integer Sequences. Accessed: Jun. 15 2023, Available online at: <http://oeis.org/A018247>.
- [8] OEIS Foundation Inc. (2023), *The 10-adic integer  $x = \dots 1787109376$  satisfies  $x^2 = x$* , Entry A018248 in the On-Line Encyclopedia of Integer Sequences. Accessed: Jun. 15 2023, Available online at: <https://oeis.org/A018248>.
- [9] Parvardi, A. H. (2011). Lifting The Exponent Lemma (LTE) - *Version 6*. Accessed: Jun. 10 2023. Available online at: <https://citeseerx.ist.psu.edu/doc/10.1.1.221.5543>.
- [10] ProofWiki Contributors (2023). *P-adic Valuation is Valuation*, ProofWiki (Updated on 16 Sep. 2022). Accessed: May 30 2023. Available online at: [https://proofwiki.org/wiki/P-adic\\_Valuation\\_is\\_Valuation](https://proofwiki.org/wiki/P-adic_Valuation_is_Valuation).
- [11] Quick, L. (2020).  $p$ -adic Absolute Values. *The University of Chicago Mathematics*, REU 2020. Accessed: May 30 2023. Available online at: <https://math.uchicago.edu/~may/REU2020/REUPapers/Quick.pdf>.
- [12] Ripà, M. (2020). On the constant congruence speed of tetration. *Notes on Number Theory and Discrete Mathematics*, 26(3), 245–260.
- [13] Ripà, M. (2021). The congruence speed formula. *Notes on Number Theory and Discrete Mathematics*, 27(4), 43–61.
- [14] Ripà, M. and Onnis, L. (2022). Number of stable digits of any integer tetration. *Notes on Number Theory and Discrete Mathematics*, 28(3), 441–457.
- [15] Urroz, J. J. and Yebra, J. L. A. (2009). On the Equation  $a^x \equiv x \pmod{b^n}$ . *Journal of Integer Sequences*, 8(8), 1–8.
- [16] Weisstein, E. W. (2023). MathWorld - A Wolfram Web Resource. *Automorphic Number*. Accessed: Jun 16 2023. Available online at: <https://mathworld.wolfram.com/AutomorphicNumber.html>.