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# Necessary and Sufficient Condition for the Existence of Equilibrium in Finite Dimensional Asset Markets with Short-Selling and Preferences with Half-Lines

Thai Ha-Huy\*, Cuong Le Van<sup>†</sup>, Myrna Wooders<sup>‡</sup>

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## Abstract

We consider a pure exchange asset model with a finite number of agents and a finite number of states of nature where short sells are allowed. We present the definition of weak no-arbitrage price, a weaker notion of no-arbitrage price than the one of Werner, and prove that if the utility functions satisfy the maximal and closed gradients conditions we propose in this paper, then there exists an equivalence between existence of a general equilibrium and existence of a price which is weak no-arbitrage price for all the agents.

**Keywords:** asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, no-arbitrage condition.

**JEL Classification:** C62, D50, D81, D84, G1.

## 1 Introduction

In economic models of financial markets, where short-sales are allowed, the case of a finite number of states is well-treated in a huge literature. Since the consumption set is not any more compact, the literature focus on conditions which ensure the compactness of allocation set or of the utility set. These conditions are known as *no-arbitrage* conditions. We can classify them in three categories:

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- i) Conditions on prices, like Green [16], Grandmont [14], [15], Hammond [12] and Werner [22],
- ii) Conditions on net trade, like Hart [13], Page [18], Nielsen [17], Page and Wooders [19], Allouch [1], Page, Wooders and Monteiro [20],
- iii) Conditions on utility set, like Brown and Werner [5], Dana, Le Van, Magnien [8].

A natural question arises. Under which conditions there is an equivalence between these conditions. In [2], Allouch, Le Van and Page prove the equivalence between Hart's condition and No Unbounded Arbitrage of Page with the assumption that the utility functions have no half-line, i.e. there exists no trading direction in which the agent's utility is constant. These conditions imply existence of a general equilibrium. But the converse is not always true, i.e., the existence of equilibrium does not ensure these no-arbitrage conditions are satisfied. We can find in Ha-Huy and Le Van [11] an example of economy where these conditions fail but an equilibrium exists.

In this paper, we extend the idea of Dana and Le Van in [7] by using weak no-arbitrage price, a no-arbitrage price weaker than the one in Werner [22], or in [2]. Following [7], we use the derivatives of utility functions as weak no-arbitrage prices. Under the conditions of closed gradient (condition **C**), which is similar of the one of Chichilnisky [6], or maximal condition (condition **M**) for the utility functions, we can establish the equivalence between existence of a price which is weak no-arbitrage for every agent and existence of general equilibrium. Moreover, these conditions are equivalent to the compactness of the individually rational utility set. We emphasize that in our paper existence of half-lines i.e. trading directions in which the agent's utility is constant is not excluded.

The paper is organized as follows. In Section 2, we present the model with the definitions of equilibrium, individually rational attainable allocations set, individually rational utility set, useful vectors and useless vectors. In Section 3, we review some no-arbitrage conditions in the literature. In particular we define weak no-arbitrage prices. Section 4 links usual no-arbitrage conditions and existence of equilibrium. In Section 5, we introduce **C, M** conditions. In particular we show that a separable utility satisfies these conditions. In Section 6, we introduce the assumption that the utility functions, if they do not satisfy No Half-line condition then they satisfy either **C** or **M**. We then prove equivalence between the existence of a price which is weak no-arbitrage for every agent and existence of a general equilibrium. Moreover, these conditions are equivalent to the compactness of the individually rational utility set. Section 7 is the appendix where we put most of the proofs.

## 2 The model

We have an exchange economy  $\mathcal{E}$  with  $m$  agents. Each agent is characterized by a consumption set  $X^i = \mathbb{R}^S$ , an endowment  $e^i$  and a utility function  $U^i : \mathbb{R}^S \rightarrow \mathbb{R}^S$ . We suppose that  $\sup_{x \in \mathbb{R}^S} U(x) = +\infty$ .

For the sake of simplicity, we suppose that utility functions are concave, strictly increasing and differentiable. In the general case of concave functions, the sub differential of  $U^i$  exists. The results and economic intuitions do not change, but the computations become tedious.

We first define an equilibrium of this economy.

**Definition 1** *An equilibrium is a list  $((x^{*i})_{i=1, \dots, m}, p^*)$  such that  $x^{*i} \in X^i$  for every  $i$  and  $p^* \in \mathbb{R}_+^S \setminus \{0\}$  and*

- (a) *For any  $i$ ,  $U^i(x) > U^i(x^{*i}) \Rightarrow p^* \cdot x > p^* \cdot e^i$*
- (b)  $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$ .

**Definition 2** *A quasi-equilibrium is is a list  $((x^{*i})_{i=1, \dots, m}, p^*)$  such that  $x^{*i} \in X^i$  for every  $i$  and  $p^* \in \mathbb{R}_+^S \setminus \{0\}$  and*

- (a) *For any  $i$ ,  $U^i(x) > U^i(x^{*i}) \Rightarrow p^* \cdot x \geq p^* \cdot e^i$*
- (b)  $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m e^i$ .

Since short-sales are allowed, from Geistdorfer-Florenzano [10], actually any quasi-equilibrium is an equilibrium.

**Definition 3** *1. The individually rational attainable allocations set  $\mathcal{A}$  is defined by*

$$\mathcal{A} = \{(x^i) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } U^i(x^i) \geq U^i(e^i) \text{ for all } i\}.$$

*2. The individually rational utility set  $\mathcal{U}$  is defined by*

$$\mathcal{U} = \{(v^1, v^2, \dots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \text{ such that } U^i(e^i) \leq v^i \leq U^i(x^i) \text{ for all } i\}.$$

**Definition 4** *i) The vector  $w$  is called useful vector of agent  $i$  if for any  $x \in \mathbb{R}^S$ , for any  $\lambda \geq 0$  we have  $U^i(x + \lambda w) \geq U^i(x)$ .*

*ii) The vector  $w$  is called useless vector of agent  $i$  if for any  $x \in \mathbb{R}$ , for any  $\lambda \in \mathbb{R}$  we have  $U^i(x + \lambda w) = U^i(x)$ .*

*iii) We say that  $w \in \mathbb{R}^S$  is a half-line direction for agent  $i$  if there exists  $x \in \mathbb{R}^S$  such that  $U^i(x + \lambda w) = U^i(x)$ ,  $\forall \lambda \geq 0$ .*

Denote by  $R^i$  the set of useful vectors.  $L_i$  the set of useless vectors. Following Rockafellar [21],  $R^i$  and  $L^i$  are closed. By the very definition, the set of useless vectors of agent  $i$  is the biggest linear subspace included in  $R^i$ :

$$L_i = R^i \cap (-R^i).$$

We have that  $R^i$  is the positive dual cone of  $P^i$ , i.e.  $R^i = -(P^i)^o$ . Observe that  $R^i$  has no empty interior since  $\mathbb{R}_+^S \subseteq R^i$ .

### 3 No-arbitrage conditions in the literature

We will review some well-known no-arbitrage conditions in the literature.

1. Hart [13] proposed the Weak No Market Arbitrage (*WNMA*) condition:

**Definition 5** *The economy satisfies WNMA if  $(w^1, w^2, \dots, w^m) \in R^1 \times R^2 \times \dots \times R^m$  satisfies  $\sum_{i=1}^m w^i = 0$  then  $w^i \in L_i$  for every  $i$ .*

2. Page [18] proposed the No Unbounded Arbitrage (*NUBA*) condition:

**Definition 6** *The economy satisfies NUBA if  $(w^1, w^2, \dots, w^m) \in R^1 \times R^2 \times \dots \times R^m$  satisfies  $\sum_{i=1}^m w^i = 0$  then  $w^i = 0$  for every  $i$ .*

3. We present the definition of *no-arbitrage prices* proposed by Werner [22].

**Definition 7** *The vector  $p \in \mathbb{R}^S$  is a no-arbitrage price for agent  $i$  if for any  $w \in R^i \setminus L_i$  we have  $p \cdot w > 0$ , and for  $w \in L_i$ ,  $p \cdot w = 0$ .*

Denote by  $S^i$  the set of no-arbitrage prices of agent  $i$ . It is a cone. The *no-arbitrage condition* is  $\bigcap_i^m S^i \neq \emptyset$ .

We have identity between no-arbitrage condition of Werner and WNMA of Hart:

**Proposition 1**  $\bigcap_i^m S^i \neq \emptyset \Leftrightarrow \text{WNMA}$

For a proof see e.g. Allouch, Le Van and Page [2].

4. In [20], Page, Wooders and Monteiro introduced the notion of *Inconsequential arbitrage*.

**Definition 8** *The economy satisfies Inconsequential arbitrage condition if for any  $(w_1, w_2, \dots, w_m)$  with  $w_i \in R_i$  for all  $i$  and  $\sum_{i=1}^m w^i = 0$  and  $(w^1, w^2, \dots, w^m)$  is the limit of  $\lambda_n(x^1(n), x^2(n), \dots, x^m(n))$  with  $(x^1(n), x^2(n), \dots, x^m(n)) \in \mathcal{A}$  and  $\lambda_n$  converges to zero when  $n$  tends to infinity, there exists  $\epsilon > 0$  such that for  $n$  sufficiently big we have  $U^i(x^i(n) - \epsilon w^i) \geq U^i(x^i(n))$ .*

In [7], Dana and Le Van propose to use the derivative of utility function as no-arbitrage price. They introduce *weak no-arbitrage prices*.

**Definition 9** *A vector  $p$  is a weak no-arbitrage price for agent  $i$  if there exists  $\lambda > 0$  and  $x^i \in \mathbb{R}^S$  such that  $p = \lambda U^{i'}(x^i)$ .*

Let  $P^i$  denote the set of weak no-arbitrage prices for the agent  $i$ . Their no-arbitrage condition is  $\bigcap_i^m P^i \neq \emptyset$

We have  $S^i \subset \text{conv}P^i \subset \bar{S}^i$  where  $\text{conv}P^i$  denotes the convex hull of  $P^i$  and  $\bar{S}^i$  is the closure of  $S^i$ . Actually we have  $S^i = \text{ri}(\text{clconv}P^i)$  (the relative interior of the closure of convex hull of  $P^i$ ).

The next Lemma gives a property of useful vectors.

**Lemma 1** *The vector  $w$  is useful a vector for agent  $i$  if, and only if, for all  $p \in P^i$ , we have:  $p \cdot w \geq 0$ .*

**Proof:** Appendix. ■

## 4 No-arbitrage condition and existence of equilibrium

Existence of equilibrium can be derived from the following result.

**Theorem 1** *The compactness of  $\mathcal{U}$  implies the existence of equilibrium.*

**Proof:** For the existence of a quasi-equilibrium, see e.g. Brown and Werner [5] or Dana, Le Van and Magnien [9]. Since short-sales are allowed, in our model quasi-equilibrium is also equilibrium. See Geistdorfer-Florenzano [10]. ■

**Theorem 2**  *$NUBA \Rightarrow WNMA \Rightarrow Inconsequential\ Arbitrage \Rightarrow Compactness\ of\ \mathcal{U} \Rightarrow Existence\ of\ equilibrium$ .*

For a proof see e.g. Allouch, Le Van and Page [2].

It is well known that, with the assumptions we state until now, if an equilibrium exists, we are not sure either that the equilibrium price belongs to the set  $S = \bigcap_i S^i$ , or the set  $\mathcal{U}$  is compact.

However we have with the weak no-arbitrage prices the following result.

### Proposition 2

*If an equilibrium exists then the equilibrium price is a weak no-arbitrage price for any agent.*

$$\text{Hence } \bigcap_{i=1}^m P^i \neq \emptyset.$$

**Proof:** Denote by  $(p^*, (x^{*i})_i)$  the equilibrium. The equilibrium allocation  $(x^{*i})$  solves the problems:

$$\begin{aligned} & \max U^i(x^i) \\ \text{s.t. } & p^* \cdot x^i = p^* \cdot e^i. \end{aligned}$$

From Theorem V.3.1, page 91, in Arrow-Hurwicz-Uzawa (1958) [3], for any  $i$ , there exists  $\zeta_i > 0$  s.t.

$$U^i(x^{*i}) - \zeta_i p^* \cdot x^{*i} \geq U^i(x^i) - \zeta_i p^* \cdot x^i$$

for any  $x^i \in \mathbb{R}^S$ . Hence  $U^{i'}(x^{*i}) = \zeta_i p^*, \forall i$ . Let  $\lambda_i = \frac{1}{\zeta_i}$ , we have  $p^* = \lambda_i U^{i'}(x^{*i}) \in P^i$  for all  $i$ . Hence  $\bigcap_i P^i \neq \emptyset$ . ■

**Proposition 3** *Suppose that  $\bigcap_i P^i \neq \emptyset$ . Then  $\mathcal{U}$  is bounded.*

**Proof:** Take any  $\bar{p} \in \bigcap_i P^i$ . For each  $i$ , there exists  $\lambda_i > 0$ ,  $\bar{x}^i \in \mathbb{R}^S$  such that  $\bar{p} = \lambda_i U^{i'}(\bar{x}^i)$ . For  $(x^1, x^2, \dots, x^m) \in \mathcal{A}$ , we have:

$$\begin{aligned} \sum_{i=1}^m \lambda_i [U^i(x^i) - U^i(\bar{x}^i)] & \leq \sum_{i=1}^m \bar{p} \cdot (\bar{x}^i - x^i) \\ & = \bar{p} \cdot \left( \sum_{i=1}^m \bar{x}^i - \bar{e} \right) \end{aligned}$$

then for all  $i$ :

$$\lambda_i U^i(e^i) \leq \lambda_i U^i(x^i) \leq \bar{p} \cdot \left( \sum_{i=1}^m \bar{x}^i - \bar{e} \right) - \sum_{j \neq i} \lambda_j U^j(e^j) + \sum_{j \neq i} \lambda_j U^j(\bar{x}^j).$$

■

When any agent has no half-line direction, we have

**Theorem 3** *Assume no agent has half-line direction. Then the following claims are equivalent*

- $\bigcap_i S^i \neq \emptyset$ ,
- *NUBA holds,*
- *$\mathcal{A}$  is compact,*
- *WNMA holds,*
- *$\mathcal{U}$  is compact,*

- *Inconsequential Arbitrage holds,*
- *Existence of equilibrium.*

For a proof see Allouch, Le Van and Page [2].

The interesting case is when at least one agent has half line directions. Our aim is to prove now

$$\bigcap_{i=1}^m P^i \neq \emptyset \text{ implies an equilibrium exists,}$$

even one agent has half line directions. In this case, we will have the equivalence

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \text{Existence of an equilibrium.}$$

To obtain this result we require the utility functions satisfy some more properties (*Closed Gradient Condition* or *Maximality Condition*).

In [11], we give an example of economy where the no-arbitrage condition, NUBA, WNMA are not satisfied. However this economy has an equilibrium because there exists a common weak no-arbitrage price.

## 5 The closed gradient condition and the maximality condition

In this section, if there is no confusion, we will use  $U(x)$  instead of  $U^i(x)$ , the set  $R$  will denote the set of useful vectors associated with  $U$ .

We recall that  $w \in \mathbb{R}^S$  is a *half-line direction* if there exists  $x \in \mathbb{R}^S$  such that  $U(x + \lambda w) = U(x), \forall \lambda \geq 0$ . Let  $\mathbf{HL}$  denote the set of half-line directions.

Let  $\sigma_\alpha$  denote an indifference surface,  $\sigma_\alpha = \{x \mid U(x) = \alpha\}$  with  $\alpha \in \mathbb{R}$ .

**Proposition 4** a)  $\mathbf{HL} \subset \partial R$ .

b) Assume that for any  $\alpha \in \mathbb{R}$ , the set  $\left\{ \frac{U'(x)}{\|U'(x)\|} \right\}_{x \in \sigma_\alpha}$  is closed, then  $\mathbf{HL} = \partial R$ .

**Proof:** See Appendix. ■

Given an affine space  $H$  of  $\mathbb{R}^S$  such that  $\sup_{x \in H} U(x) = +\infty$ , define the function  $U_H : H \rightarrow \mathbb{R}$  as the restriction of  $U$  on  $H$ . Define  $U'_H$  the projection of gradient of  $U$  on  $H$ . We set the following assumptions.

**C** (*Closed gradient condition*): For any  $\alpha \in \mathbb{R}$ , any  $H$  affine space such that  $\sup_H U(x) = +\infty$ , define  $\sigma_\alpha^H$  as the indifference surface of  $U$  on  $H$ . Then the



map  $x \in \sigma_\alpha^H \rightarrow U'_H(x)/\|U'_H(x)\|$  is closed.

**M** (*Maximality*) condition: For all affine space  $H$  of  $\mathbb{R}^S$ , only one of the two following cases holds:

- i)  $\sup_H U(x) = +\infty$ .
- ii)  $\max_H U(x)$  exists.

**Remark 1** *C* condition is borrowed from Chichilnisky [6].

**Proposition 5** *Let  $U$  satisfy C condition. Let  $w \in \partial R$ . Then for all  $x$ , the function  $\lambda \mapsto U(x + \lambda w)$  has a maximum.*

**Proof:** See Appendix. ■

The next Proposition 6 gives us an example of utility function which satisfies **C** and **M** conditions. This is the case of an utility function satisfying being "linear at far".

**Proposition 6** *We consider the case where the utility function is defined as  $U(x) = \sum_{s=1}^m \pi_s u(x_s)$ , with  $u : \mathbb{R} \rightarrow \mathbb{R}$  is a concave, strictly increasing, differentiable function,  $\pi$  is a probability measure on  $\{1, \dots, S\}$ , with  $\pi_s > 0, \forall s$ . Define  $a = u'(+\infty)$ ,  $b = u'(-\infty)$ . Suppose that  $0 < a \leq b < +\infty$  and there exists  $z > 0$  such that  $u'(x) = a, \forall x > z$  and  $u'(x) = b, \forall x < -z$ . Then the utility function  $U$  satisfies **C** and **M**.*

**Proof:** See Appendix. ■

**Definition 10** *Given a concave function  $U$ , and  $w \in R$ . Define  $V[U] : \mathbb{R}^S \times R \rightarrow \bar{\mathbb{R}}$ :*

$$V[U](x, w) = \sup_{\lambda \in \mathbb{R}} U(x + \lambda w) = \lim_{\lambda \rightarrow +\infty} U(x + \lambda w).$$

We will present here some properties of the function  $V[U]$ . From now on, since there exists no confusion, we will write  $V(x, w)$  instead of  $V[U](x, w)$ . The proofs of lemmas 3, 4, 5 and 6 are given in Appendix.

**Lemma 2** *The vector  $w$  is a useless vector of  $V(\cdot, w)$ .*

**Proof:** Let  $\mu \in \mathbb{R}$ . We have

$$\begin{aligned} V(x + \mu w, w) &= \lim_{\lambda \rightarrow +\infty} U(x + \mu w + \lambda w) \\ &= \lim_{\lambda \rightarrow +\infty} U(x + (\mu + \lambda)w) \\ &= \lim_{\zeta \rightarrow +\infty} U(x + \zeta w) \\ &= V(x, w). \end{aligned}$$

■

**Lemma 3** a)  $V$  is concave.

b) If there exists  $\bar{x}$  such that  $V(\bar{x}, w) = +\infty$  then  $V(x, w) = +\infty, \forall x \in \mathbb{R}^S$ .

c) If  $V(x, w) < +\infty \forall x \in \mathbb{R}^S$ ,  $V(\cdot, w)$  is continuous on  $\mathbb{R}^S$ .

**Lemma 4** If  $w \in \text{int}R$  then  $V(x, w) = +\infty$  for all  $x$ .

**Remark 2** Let  $\partial R$  denote the boundary of  $R$ . From Lemma 4, the necessary condition to have  $V(x, w) < +\infty$  is  $w \in \partial R$ .

The set of useful vector of  $U$  is included in the set of useful vectors of  $V$ .

**Lemma 5** Given  $w \in \partial R$  such that  $V(x, w) \in \mathbb{R}$ , a useful vector of  $U$  is a useful vector of  $V(\cdot, w)$ .

Lemma 6 gives us the derivative of the function  $V$  at the points  $x$  satisfying  $V(x, w) = \max_{\lambda} U(x + \lambda w)$ .

**Lemma 6** Given  $w \in \partial R, x \in \mathbb{R}^S$ . Suppose that there exists  $\bar{\lambda} \geq 0$  such that  $U(x + \lambda w) = U(x + \bar{\lambda} w)$  for all  $\lambda \geq \bar{\lambda}$ . Then  $V$  is differentiable at  $x$  and  $V'_x(x, w) = U'(x + \bar{\lambda} w)$ .

**Proposition 7** Suppose that  $U$  satisfies either No Half-line condition, either **C** or **M**. Take a useful vector of  $U$  such that there exists  $\bar{x} \in \mathbb{R}^S$  satisfying  $\sup_{\lambda} U(\bar{x} + \lambda w) < +\infty$ . Then for any  $x \in \mathbb{R}^S$ , there exists  $\max_{\lambda \in \mathbb{R}} U(x + \lambda w)$ . The function  $V(x, w) = \max_{\lambda} U(x + \lambda w)$  also satisfies also the same property as  $U$ .

**Proof:** Appendix. ■

## 6 The main results

In this section, we prove the equivalence between the existence of general equilibrium and the existence of common weak no-arbitrage price under **C** and **M**.

Let  $\mathcal{E} = \{(U^i, e^i, X^i)_i\}$  be the initial economy, with  $X^i = \mathbb{R}^S$  for any  $i$ .

Denote

$$W = \{(w^1, w^2, \dots, w^m) \in R^1 \times R^2 \times \dots \times R^m \text{ such that } \sum_{i=1}^m w^i = 0\}.$$

The set  $W$  is a cone and will be called the cone of mutual opportunities of arbitrage. Fix a vector  $w = (w^1, w^2, \dots, w^m) \in W$ , as in Definition 10, define  $V^i(x^i, w^i) = V[U^i](x^i, w^i)$ . We define  $\mathcal{E}^V$  the economy with  $m$  agents where each agent  $i$  has a utility function defined by  $V^i$ , endowment  $e^i$  and consumption set  $X^i = \mathbb{R}^S$ . The attainable allocation set  $\mathcal{A}^V$  is defined as:

$$\mathcal{A}^V = \{(x^1, x^2, \dots, x^m) \in (\mathbb{R}^S)^m \mid \sum_{i=1}^m x^i = \sum_{i=1}^m e^i \text{ and } V^i(x^i, w^i) \geq U^i(e^i)\}.$$

The attainable utility set  $\mathcal{U}^V$  is defined as:

$$\mathcal{U}^V = \{(v^1, v^2, \dots, v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A}^V : U^i(e^i) \leq v^i \leq V^i(x^i, w^i), \forall i\}.$$

We have the result:

**Lemma 7** *Suppose that for all  $(x^1, x^2, \dots, x^m) \in \mathcal{A}$ , there exists  $\max_{\lambda \geq 0} U^i(x^i + \lambda w^i)$ ,  $\forall i$ . Then  $\mathcal{U}^V = \mathcal{U}$ .*

**Proof:** Evidently, since  $U^i(x^i) \leq V^i(x^i, w^i)$ , we have  $\mathcal{U} \subset \mathcal{U}^V$ .

Take  $(v^1, v^2, \dots, v^m) \in \mathcal{U}^V$ . There exists  $(x^1, x^2, \dots, x^m) \in \mathcal{U}^V$  such that  $U^i(e^i) \leq v^i \leq V^i(x^i, w^i)$ ,  $\forall i$ . By assumption, there exists  $\lambda \geq 0$  big enough such that  $V^i(x^i, w^i) = U^i(x^i + \lambda w^i)$ . Observe that  $(x^1 + \lambda w^1, x^2 + \lambda w^2, \dots, x^m + \lambda w^m) \in \mathcal{A}$ . This implies  $(v^1, v^2, \dots, v^m) \in \mathcal{U}$ . ■

**Theorem 4** *Consider economy  $\mathcal{E}$ . Suppose that for all  $i$ , either  $U^i$  satisfies No Half-line condition, or it satisfies **C**, or it satisfies **M**. We have the equivalent between existence of weak no-arbitrage price and equilibrium:*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{there exists an equilibrium.}$$

**Proof:** Appendix. ■

**Remark 3** *The condition  $\bigcap_i P^i \neq \emptyset$  is weaker than  $\bigcap_i S^i \neq \emptyset$ . In [11], we have an example in which  $\bigcap_{i=1}^m S^i = \emptyset$ , but  $\bigcap_i P^i \neq \emptyset$ . In this example, an equilibrium exists.*

**Corollary 1** *Suppose the utility functions are separable, i.e for any agent  $i$ ,*

$$U^i(x) = \sum_{s=1}^S \pi_s^i u^i(x_s)$$

*with  $\pi_s^i > 0$  for all  $i$ , for all  $s$ . We have*

$$\bigcap_{i=1}^m P^i \neq \emptyset \Leftrightarrow \mathcal{U} \text{ is compact} \Leftrightarrow \text{an equilibrium exists.}$$

**Proof:** For each agent  $i$ , if  $u^{i'}(z) < u^{i'}(-\infty)$  for any  $z$ , or  $u^{i'}(z) > u^{i'}(+\infty)$  for any  $z$ , then the function  $U^i$  has no half-line. For this result, see [7]. In the other case,  $U^i$  satisfies **C** and **M** conditions. The corollary is hence a direct consequence of Proposition 6 and Theorem 4.

Another proof of the corollary can be found in [11]. ■

## 7 Appendix

### 7.1 Proof of Lemma 1

( $\Rightarrow$ ): Let  $p = \mu U^{i'}(x)$  for some  $\mu > 0$  some  $x \in \mathbb{R}^S$ . From the concavity of  $U$  we have:

$$0 \geq \mu U^i(x) - \mu U^i(x + \lambda w) \geq \mu U^{i'}(x) \cdot (-\lambda w) = p \cdot (-\lambda w) \text{ for all } \lambda \geq 0.$$

Hence  $\mu U^{i'}(x) \cdot w \geq 0$ .

( $\Leftarrow$ ): Observe that for all  $\lambda \geq 0$ , all  $\mu > 0$  :

$$\mu U^i(x + \lambda w) - \mu U^i(x) \geq \mu U^{i'}(x + \lambda w) \cdot (\lambda w) \geq 0.$$

Hence  $U^i(x + \lambda w) \geq U^i(x)$ ,  $\forall x$ . ■

### 7.2 Proof of Lemma 3

- (a) Consider two couples  $(x, w), (x', w')$  with  $w \in R, w' \in R$  and  $\theta \in [0, 1]$ . We have

$$U(\theta x + (1 - \theta)x' + \lambda(\theta w + (1 - \theta)w')) \geq \theta U(x + \lambda w) + (1 - \theta)U(x' + \lambda w')$$

By taking the limits

$$\begin{aligned} & \lim_{\lambda \rightarrow +\infty} U(\theta x + (1 - \theta)x' + \lambda(\theta w + (1 - \theta)w')) \\ & \geq \lim_{\lambda \rightarrow +\infty} [\theta U(x + \lambda w) + (1 - \theta)U(x' + \lambda w')] \\ & = \theta \lim_{\lambda \rightarrow +\infty} U(x + \lambda w) + (1 - \theta) \lim_{\lambda \rightarrow +\infty} U(x' + \lambda w'). \end{aligned}$$

Equivalently

$$V(\theta x + (1 - \theta)x', \theta w + (1 - \theta)w') \geq \theta V(x, w) + (1 - \theta)V(x', w').$$

- (b) Let  $x \in \mathbb{R}^S$ . There exist  $\theta \in (0, 1)$  and  $y \in \mathbb{R}^S$  such that  $x = \theta\bar{x} + (1 - \theta)y$ . We use the concavity of  $U$ :

$$\begin{aligned} \text{Let } \lambda \in \mathbb{R}_+ : U(x + \lambda w) &= U(\theta(\bar{x} + \lambda w) + (1 - \theta)(y + \lambda w)) \\ &\geq \theta U(\bar{x} + \lambda w) + (1 - \theta)U(y + \lambda w) \\ &\geq \theta U(\bar{x} + \lambda w) + (1 - \theta)U(y). \end{aligned}$$

Take the limits:

$$\begin{aligned} V(x, w) = \lim_{\lambda \rightarrow +\infty} U(x + \lambda w) &\geq \theta \lim_{\lambda \rightarrow +\infty} U(\bar{x} + \lambda w) + (1 - \theta)U(y) \\ &= \theta V(\bar{x}, w) + (1 - \theta)U(y) = +\infty. \end{aligned}$$

- (c) The function  $V(\cdot, w)$  is concave, real valued over  $\mathbb{R}^S$  is therefore continuous everywhere on  $\mathbb{R}^S$ .

### 7.3 Proof of Lemma 4

Take  $\bar{w} = (1, 1, \dots)$ . Since  $U$  is strictly increasing,  $\mathbb{R}_+^S \subset R$ . Therefore,  $\bar{w} \in \text{int}R$  since  $\bar{w} \in \text{int}\mathbb{R}_+^S$ . Let  $x$  be given and let  $\{y^n\}$  be a sequence which satisfies  $\lim_n U(y^n) = \sup_{z \in \mathbb{R}^S} U(z) = +\infty$ . For any  $n$  there exists  $\lambda_n > 0$  such that  $x_s + \lambda_n \bar{w}_s > y_s^n$  for any  $s = 1, \dots, S$ . That implies  $U(x + \lambda_n \bar{w}) > U(y^n)$  since  $U$  is strictly increasing. Hence  $V(x, \bar{w}) = \sup_{\lambda \in \mathbb{R}} U(x + \lambda \bar{w}) = +\infty$ .

Now let  $w \in \text{int}R$ . There exist  $\theta \in (0, 1)$ ,  $r \in R$  such that  $w = \theta\bar{w} + (1 - \theta)r$ . Let  $\{\lambda_n\}_n$  be an increasing sequence of positive numbers which converge to  $+\infty$ . We have  $x + \lambda_n w = \theta(x + \lambda_n \bar{w}) + (1 - \theta)(x + \lambda_n r)$  and hence

$$U(x + \lambda_n w) \geq \theta U(x + \lambda_n \bar{w}) + (1 - \theta)U(x + \lambda_n r) \geq \theta U(x + \lambda_n \bar{w}) + (1 - \theta)U(x).$$

Since  $\lim_n U(x + \lambda_n \bar{w}) = +\infty$  we have also  $\lim_n U(x + \lambda_n w) = +\infty$  and  $V(x, w) = +\infty$ .

### 7.4 Proof of Lemma 5

Indeed, take  $r \in R$ . For all  $\lambda, \mu \geq 0$  we have:

$$U(x + \mu r + \lambda w) \geq U(x + \lambda w).$$

This implies  $\sup_\lambda U(x + \mu r + \lambda w) \geq \sup_\lambda U(x + \lambda w)$ . ■

### 7.5 Proof of Lemma 6

Take  $p \in \partial V(x, w)$ . We have  $p \cdot w = 0$ . Indeed, observe that  $w$  is an useless vector of  $V(\cdot, w)$ , then for all  $\lambda \in \mathbb{R}$ , by the concavity of  $V$  we have:

$$0 = V(x + \lambda w) - V(x, w) \leq \lambda p \cdot w.$$

Hence  $p \cdot w = 0$ .

By assumption,  $U(x + \lambda w) = U(x + \bar{\lambda}w) \forall \lambda \geq \bar{\lambda}$ , so  $V(x, w) = U(x + \bar{\lambda}w)$ . For all  $y \in \mathbb{R}^S$  we have:

$$U(y) - U(x + \bar{\lambda}w) \leq V(y, w) - V(x, w) \leq p \cdot (y - x) = p \cdot (y - (x + \bar{\lambda}w)).$$

Then  $p = U'(x + \bar{\lambda}w)$ .

Take  $p = U'(x + \bar{\lambda}w)$ . Observe that  $p \cdot w = 0$ . We have:

$$\begin{aligned} V(y, w) - V(x, w) &= \lim_{\lambda \rightarrow +\infty} [U(y + \lambda w) - U(x + \bar{\lambda}w)] \\ &\leq \lim_{\lambda \rightarrow +\infty} p \cdot (y - x + \lambda w - \bar{\lambda}w) \\ &= \lim_{\lambda \rightarrow +\infty} p \cdot (y - x) = p \cdot (y - x). \end{aligned}$$

Then  $p \in \partial V(x, w)$ . ■

## 7.6 Proof of Proposition 4

(a) Suppose that  $w$  is **HL**, by using Lemma 4, we have  $w \in \partial R$ .

(b) With  $\alpha \in \mathbb{R}$ , let  $\sigma_\alpha = \{x \mid U(x) = \alpha\}$ .

We will prove that if  $w \in \partial R$ , then there exists  $\hat{x} \in \sigma_\alpha$  such that  $U(\hat{x} + \lambda w) = U(\hat{x}) = \alpha$  for all  $\lambda \geq 0$ . If  $w = 0$ , obviously, the claim is true. We assume now that  $w \in \partial R \setminus \{0\}$ .

Take some  $x \in \sigma_\alpha$ . If  $U(x + \lambda w) = U(x) \forall \lambda \geq 0$ , the claim is true. Assume that there exists  $\mu > 0$  such that  $U(x + \mu w) > U(x)$ . If  $\mu > 1$ , the concavity of  $U$  implies that  $U(x + w) > U(x)$ . If  $\mu \leq 1$ , then  $U(x + w) \geq U(x + \mu w) > U(x)$ . This implies that  $U(x + w) > U(x)$ . From the continuity of  $U$ , there exists  $\rho > 0$  such that for all  $y \in B(x + w, \rho)$ , a ball center  $x + w$  and radius  $\rho$ , we have  $U(y) > U(x)$ . Since  $w \in \partial R$ , we can take a sequence  $w^n$  converging to  $w$  such that  $x + w^n \in B(x + w, \rho)$  and  $w^n \notin R$ . We have  $U(x + w^n) > U(x)$ . Therefore, for any  $n$ , there exists  $\lambda^n > 1$  such that  $U(x + \lambda^n w^n) = U(x)$  and  $U(x + \lambda w^n) < U(x)$  for all  $\lambda > \lambda^n$ .

Define  $x^n := x + \lambda^n w^n$ . We claim that  $\lambda^n \rightarrow +\infty$ . Indeed, in the contrary,  $\lambda^n \rightarrow \bar{\lambda} \geq 1$ . We have  $U(x + \bar{\lambda}w) \geq U(x + w) > U(x)$ : a contradiction because  $U(x + \bar{\lambda}w) = \lim_n U(x^n) = U(x)$ .

Since  $\lambda^n \rightarrow \infty$ , we have  $\|x^n\| \rightarrow \infty$ .

By the concavity of  $U$  we have:

$$0 = U(x^n) - U(x) \geq U'(x^n) \cdot (x^n - x).$$

This implies:

$$\frac{U'(x^n)}{\|U'(x^n)\|} \cdot \frac{x^n}{\|x^n\|} \leq \frac{U'(x^n)}{\|U'(x^n)\|} \cdot \frac{x}{\|x^n\|}.$$

Observe that when  $n \rightarrow \infty$ ,  $\|x^n\| \rightarrow +\infty$ , and  $\frac{x^n}{\|x^n\|} \rightarrow w$ . From the closeness of  $\left\{ \frac{U'(x)}{\|U'(x)\|} \right\}_{x \in \sigma_\alpha}$  we have:

$$\frac{U'(x^n)}{\|U'(x^n)\|} \rightarrow \frac{U'(\hat{x})}{\|U'(\hat{x})\|}$$

where  $\hat{x} \in \sigma_\alpha$ . We obtain that:

$$\frac{U'(\hat{x})}{\|U'(\hat{x})\|} \cdot w \leq 0.$$

So  $U'(\hat{x}) \cdot w = 0$  and that implies  $U(\hat{x} + \lambda w) = U(\hat{x}) \forall \lambda \geq 0$ . ■

## 7.7 Proof of Proposition 5

From Proposition 4, there exist  $\hat{x}$  such that  $U(\hat{x} + \lambda w) = U(\hat{x})$ ,  $\forall \lambda \geq 0$ . Hence  $V(\hat{x}, w) < +\infty$ . By Lemma 3, we have  $V(x, w) < +\infty \forall x \in \mathbb{R}^S$ , or  $\lim_{\lambda \rightarrow +\infty} U(x + \lambda w) < +\infty$ .

Suppose that there exists  $x$  such that  $U(x + \lambda w)$  has no maximum.

Take any  $p \in \partial V(x, w)$ . Take any  $z$  in interior of  $R$ . Let  $\Pi$  denote the plane spanned by  $(w, z)$  and  $\Pi_x = \Pi + \{x\}$ . We will prove that  $p$  is orthogonal to  $\Pi$  and then orthogonal to  $z$ .

By the way of choosing  $z$  and Lemma 3, we have  $\sup_\lambda U(x + \lambda z) = +\infty$ , so  $\sup_{\Pi_x} U(x) = +\infty$ . Let  $\alpha := \lim_{\lambda \rightarrow \infty} U(x + \lambda w) = V(x, w)$ . Since  $U(x + \lambda w) < \alpha$  and  $U(x + \lambda z) \rightarrow +\infty$  when  $\lambda \rightarrow \infty$ , there exists  $y \in \Pi_x$  which satisfies  $U(y) = \alpha$ . In other words,  $C_\alpha := \Pi_x \cap \sigma_\alpha \neq \emptyset$ .

By using the same argument as in the proof of Proposition 4 and the closed gradient condition for the affine space  $\Pi_x$ , we can prove that there exists  $\hat{x} \in C_\alpha$  such that

$$U(\hat{x}) = U(\hat{x} + \lambda w) = \alpha, \forall \lambda \geq 0.$$

We have  $V(x, w) = V(\hat{x}, w) = \alpha$ , so

$$0 = V(x, w) - V(\hat{x}, w) \geq p \cdot (x - \hat{x}).$$

We obtained  $p \cdot (x - \hat{x}) \leq 0$ .

By Lemma 6, we have  $V'_x(\hat{x}, w) = U'(\hat{x})$ , and then  $V'_x(\hat{x}, w) \cdot w = 0$ .

Denote  $q = V'(\hat{x}, w)$ . Let  $p_1, q_1$  respectively the orthogonal projections of  $p$  and  $q$  on  $\Pi$ . Since  $p \cdot w = q \cdot w = 0$ , then  $p_1$  and  $q_1$  are orthogonal to  $w$ , and so  $p_1 = \mu q_1$  with  $\mu \in \mathbb{R}$ . Since  $z \in \text{int}(R)$ ,  $z$  is also a useful vector of  $V(\cdot, w)$ , and  $V'(x, w) \cdot z \geq 0$  which implies  $p_1 \cdot z \geq 0$ . We have also  $U'(\hat{x}) \cdot z > 0 \Rightarrow q_1 \cdot z > 0$ . These inequalities imply  $\mu \geq 0$ .

Observe that:

$$U(\hat{x}) - U(x + \lambda w) \geq U'(\hat{x}) \cdot (\hat{x} - x) - \lambda U'(\hat{x}) \cdot w = U'(\hat{x}) \cdot (\hat{x} - x) = q_1 \cdot (\hat{x} - x).$$

Let  $\lambda \rightarrow +\infty$ . We obtain  $q_1 \cdot (\hat{x} - x) \leq 0$ . And then  $p_1 \cdot (\hat{x} - x) \leq 0$ . Remember that  $p \cdot (x - \hat{x}) \leq 0$  or  $p_1 \cdot (x - \hat{x}) \leq 0$ . Then, we have  $p_1 \cdot (x - \hat{x}) = 0$ .

By the assumption of the non existence of a maximum of  $U(x + \lambda w)$ ,  $\hat{x}$  does not belong to the line  $\{x + \lambda w\}$ , so  $w$  and  $x - \hat{x}$  are linearly independent and orthogonal to  $p_1$ , so  $p$  is orthogonal to  $\Pi$ , or  $p \cdot z = 0$ .

Since  $z$  is arbitrarily chosen,  $p$  is orthogonal to all vectors who belongs to  $\text{int}R$ , and hence to all vectors in  $\mathbb{R}_+^S$ . This implies  $p = 0$ . Then  $V(x, w) \geq V(y, w) \geq U(y)$  for all  $y$ , or  $\sup_y U(y) < +\infty$ : a contradiction. ■

## 7.8 Proof of Proposition 6

Before beginning the proof, for each vector  $w \in \mathbb{R}$ , define  $S_+(w) = \{s \text{ such that } w_s > 0\}$ ,  $S_-(w) = \{s \text{ such that } w_s < 0\}$ .

(a) Fix an affine subspace  $H$ . Denote by  $\sigma_\alpha^H$  the set  $\{x \in H \text{ such that } U(x) = \alpha\}$ . Suppose that  $p$  is a limit point of  $\{\frac{U'(x)}{\|U'(x)\|}\}_{\sigma_\alpha^H}$ . Since  $U'(x) \cdot w \geq 0$  we have  $p \cdot w \geq 0$ . For each  $n$ , define

$$A_n = \{x \in \sigma_\alpha^H \text{ such that } \|p - \frac{U'(x)}{\|U'(x)\|}\| \leq \frac{1}{n}\}.$$

Denote by  $x_n$  the element of  $A_n$  which has the smallest norm:  $\|x_n\| \leq \|x\|$  for all  $x \in A_n$ . We prove that the set  $\{x_n\}$  is bounded.

Suppose the contrary,  $\lim_n \|x_n\| = \infty$ . Without loss of generality, we can suppose that  $\lim_n \frac{x_n}{\|x_n\|} = w$ . Observe that  $w$  is parallel to  $H$ . Since  $U(x_n) = \alpha$  for all  $n$ , the vector  $w$  is a useful vector. We have also  $p \cdot w = 0$ . Indeed, denote by  $\mathbf{0}$  the vector  $(0, 0, \dots, 0)$ . We have  $U(x_n) - U(\mathbf{0}) \geq U'(x_n) \cdot x_n$ . Hence

$$\frac{\alpha - U(\mathbf{0})}{\|x_n\|} \geq U'(x_n) \cdot \frac{x_n}{\|x_n\|}.$$

Let  $n$  converges to infinity, the LHS converges to 0. Observe that since  $0 < a \leq b < +\infty$ , we have  $\{U'(x)\}$  is bounded. Hence without loss of generality, assume that  $U'(x_n)$  converges to a vector  $q$ , with  $\frac{q}{\|q\|} = p$ . The inequality above implies that  $p \cdot w \leq 0$ , that implies  $p \cdot w = q \cdot w = 0$ .

Dana and Le Van in [7] prove that for all  $x$  we have

$$U'(x) \cdot w \geq a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s \geq 0.$$

Since  $\lim_n U'(x_n) \cdot w = 0$ , we have  $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s = 0$ .



Fix  $0 < \epsilon < 1$ . Take  $N$  big enough such that for all  $n \geq N$  we have  $x_{n,s} - \epsilon w_s > z$  for  $s \in S_+(w)$  and  $x_{n,s} - \epsilon w_s < -z$  for  $s \in S_-(w)$ . This implies  $\|x_n - \epsilon w\| < \|x_n\|$ . We will prove that  $U(x_n - \epsilon w) = \alpha$ ,  $\forall n \geq N$ . Indeed,

$$\begin{aligned}
U(x_n - \epsilon w) - U(x_n) &\geq -\epsilon U'(x_n - \epsilon w) \cdot w \\
&= -\epsilon \sum_{s=1}^S \pi_s u'(x_{n,s} - \epsilon w_s) w_s \\
&= -\epsilon \sum_{s \in S_+(w)} \pi_s u'(x_{n,s} - \epsilon w_s) w_s - \epsilon \sum_{s \in S_-(w)} \pi_s u'(x_{n,s} - \epsilon w_s) w_s \\
&= -\epsilon a \sum_{s \in S_+(w)} \pi_s w_s - \epsilon b \sum_{s \in S_-(w)} \pi_s w_s \\
&= -\epsilon [a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s] \\
&= 0.
\end{aligned}$$

Hence  $U(x_n - \epsilon w) \geq U(x_n) \geq U(x_n - \epsilon w)$ . This implies  $U(x_n - \epsilon w) = \alpha$ .

Observe that for  $s \in S_+ \cup S_-$ , we have  $u'(x_{n,s} - \epsilon w_s) = u'(x_{n,s})$ , which is equal to  $a$  if  $s \in S_+(w)$ , and to  $b$  if  $s \in S_-$ , so  $U'(x_n - \epsilon w) = U'(x_n)$ . Since  $w$  is parallel to  $H$ ,  $x_n - \epsilon w \in H$ . We have proved above  $\|x_n - \epsilon w\| < \|x_n\|$ . This is a contradiction to the definition of  $x_n$ .

So, the set  $\{x_n\}$  is bounded. Take a limit point  $x$  of this set, we have  $x \in H$ ,  $U(x) = \alpha$ , and  $\frac{U'(x)}{\|U'(x)\|} = p$ . We have proved that **C** condition holds.

(b) Fix an affine space  $H$  such that  $\sup_H U(x) = a < +\infty$ . For each  $n$ , define

$$A_n = \{x \in H \text{ such that } U(x) \geq a - \frac{1}{n}\}.$$

Take  $x_n \in A_n$  such that  $\|x_n\| \leq \|x\|$  for all  $x \in A_n$ . We prove that the set  $\{x_n\}$  is bounded. Suppose the contrary. We can suppose that  $\lim_n \frac{x_n}{\|x_n\|} = w$ . Since  $\{U(x_n)\}$  is bounded below,  $w$  is a useful vector. We will prove that  $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s = 0$ . Indeed, suppose that  $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s > 0$ . Fix  $x \in H$ . For  $\lambda > 0$  big enough we have  $x_s + \lambda w_s > z$ ,  $\forall s \in S_+(w)$ , and  $x_s + \lambda w_s < -z$ ,  $\forall s \in S_-$ . Hence

$$\begin{aligned}
U(x + \lambda w) - U(x) &\geq \lambda U'(x + \lambda w) \cdot w \\
&= \lambda \left[ a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s \right].
\end{aligned}$$

Let  $\lambda$  converges to infinity, the RHS tends to  $+\infty$ . Observe that since  $w$  is parallel to  $H$ ,  $x + \lambda w \in H$  for all  $\lambda$ . This implies  $\sup_H U(x) = +\infty$ , a contradiction.

We have  $a \sum_{s \in S_+(w)} \pi_s w_s + b \sum_{s \in S_-(w)} \pi_s w_s = 0$ . By using the same arguments in the part (a), we have  $U(x_n - \epsilon w) \geq U(x_n) \geq a - \frac{1}{n}$ , and  $\|x_n - \epsilon w\| < \|x_n\|$ , a contradiction. Hence the set  $\{x_n\}$  is bounded. Since  $\lim_n U(x_n) = a$ , we have that  $\max_H U(x)$  exists. We have proved that **M** condition holds. ■

## 7.9 Proof of Proposition 7

Obviously, we must consider only the case  $U$  satisfies either **C**, either **M**.

Firstly, suppose that  $U$  satisfies **C**.

By Proposition 5, for all  $x \in \mathbb{R}^S$ , there exists  $\lambda \geq 0$  big enough such that  $V'(x, w) = U'(x + \lambda w)$ .

Suppose that  $H$  is an affine space of  $\mathbb{R}^S$ , and  $\sup_{x \in H} V(x, w) = +\infty$ . We consider two cases:

1.  $H$  is parallel to  $w$ :  $\forall x \in H, x + w \in H$ .
2.  $H$  is not parallel to  $w$ .

Denote  $\sigma_\alpha^{V,H} := \{x \in H \mid V(x, w) = \alpha\}$  and consider the sequence  $\{x_n\}_n \subset \sigma_\alpha^{w,H}$  such that  $\lim_n \frac{V'_H(x_n, w)}{\|V'_H(x_n, w)\|} = p$ . For each  $n$ , there exists  $\lambda_n \geq 0$  such that  $V(x_n, w) = U(x_n + \lambda_n w)$ , or  $x_n + \lambda_n w \in \sigma_\alpha$ , and using Lemma 6,  $\lim_n \frac{U'_H(x_n + \lambda_n w)}{\|U'_H(x_n + \lambda_n w)\|} = p$ .

For any affine subspace  $F$ ,  $\pi_F$  denotes the projection on  $F$ .

Consider the first case. In this case,  $x_n + \lambda_n w \in H$  for all  $n$ , and  $\sup_{x \in H} U(x) = \sup_{x \in H} V(x, w) = +\infty$ . By applying the closed gradients condition for the affine space  $H$ , there exists  $\hat{x} \in \sigma_\alpha^{V,H}$  such that  $\frac{U'_H(\hat{x})}{\|U'_H(\hat{x})\|} = p$ . Observe that  $U'(x_n + \lambda_n w) \cdot w = 0 \Rightarrow \pi_H[U'(x_n + \lambda_n w)] \cdot w = 0 \Rightarrow \pi_H[U'(\hat{x})] \cdot w = 0 \Rightarrow U'(\hat{x}) \cdot w = 0$  and so  $V(\hat{x}, w) = U(\hat{x}) = \alpha$ , or  $U(\hat{x} + \lambda w) = U(\hat{x}) = \alpha$  for all  $\lambda \geq 0$ , and  $V'(\hat{x}, w) = U'(\hat{x})$ . So we have

$$\frac{V'_H(\hat{x}, w)}{\|V'_H(\hat{x}, w)\|} = \frac{\pi_H(V'(\hat{x}, w))}{\|\pi_H(V'(\hat{x}, w))\|} = \frac{\pi_H(U'(\hat{x}))}{\|\pi_H(U'(\hat{x}))\|} = \frac{U'_H(\hat{x})}{\|U'_H(\hat{x})\|} = p.$$

Consider the second case. Let  $G := H + \{\lambda w\}_{\lambda \in \mathbb{R}}$ . Then  $G$  is parallel to  $w$ . Observe that  $\sup_{x \in G} U(x) = \sup_{x \in H} V(x, w) = +\infty$ .

Suppose that  $\frac{V'_H(x_n, w)}{\|V'_H(x_n, w)\|} = p$  with  $x_n \in H$  and  $V(x_n, w) = \alpha$  for all  $n$ .

Without loss of generality, we can suppose that  $\frac{U'_G(x_n + \lambda_n w)}{\|U'_G(x_n + \lambda_n w)\|} \rightarrow \bar{p}$ . Observe that  $H \subset G$ , so  $\pi_H(\pi_G(q)) = \pi_H(q)$  for all vector  $q \in \mathbb{R}^S$ .

Firstly, we prove that  $\bar{p}$  is not orthogonal to  $H$ . Using the same argument as the first case, applying closed gradient condition for the affine space  $G$ , we know that there exists  $\hat{x} \in G$  such that

$$\frac{U'_G(\hat{x})}{\|U'_G(\hat{x})\|} = \bar{p} \text{ and } U'(\hat{x}) \cdot w = 0$$

or we can write

$$\frac{V'_G(\hat{x}, w)}{\|V'_G(\hat{x}, w)\|} = \bar{p}.$$

$H$  is not parallel to  $w$ , so there exists  $\lambda \in \mathbb{R}$  satisfies  $\hat{x} + \lambda w \in H$ . Denote  $\bar{x} = \hat{x} + \lambda w$ . Observe that  $V(\bar{x}, w) = V(\hat{x}, w)$  and  $V'(\bar{x}, w) = V'(\hat{x}, w)$ .

If  $\bar{p}$  is orthogonal to  $H$  then  $V'(\bar{x}, w)$  is orthogonal to  $H$ , then for all  $y \in H$  we have:

$$V(y, w) - V(\bar{x}, w) \leq V'(\bar{x}, w) \cdot (y - \bar{x}) = 0.$$

Thus  $V(\bar{x}, w) = \sup_{y \in H} V(y, w) = +\infty$ : a contradiction.

Since  $\bar{p}$  is not orthogonal to  $H$ , we have  $\pi_H(\bar{p}) \neq 0$ .

Secondly, we prove that  $\frac{\pi_H(\bar{p})}{\|\pi_H(\bar{p})\|} = p$ . Indeed, for all  $n$  we have

$$\frac{\pi_H(\pi_G(U'(x_n + \lambda_n w)))}{\|\pi_H(\pi_G(U'(x_n + \lambda_n w)))\|} = \frac{\pi_H(\frac{\pi_G(U'(x_n + \lambda_n w))}{\|\pi_G(U'(x_n + \lambda_n w))\|})}{\|\pi_H(\frac{\pi_G(U'(x_n + \lambda_n w))}{\|\pi_G(U'(x_n + \lambda_n w))\|})\|}.$$

And we let  $n \rightarrow \infty$ , the left hand side tends to  $p$ , and the right hand side tends to  $\frac{\pi_H(\bar{p})}{\|\pi_H(\bar{p})\|}$ . Hence,  $\frac{\pi_H(\bar{p})}{\|\pi_H(\bar{p})\|} = p$ .

Finally we have

$$\frac{V'_H(\bar{x}, w)}{\|V'_H(\bar{x}, w)\|} = \frac{\pi_H(V'(\hat{x}, w))}{\|\pi_H(V'(\hat{x}, w))\|} = \frac{\pi_H(\pi_G(V'(\hat{x}, w)))}{\|\pi_H(\pi_G(V'(\hat{x}, w)))\|} = \frac{\pi_H(\frac{\pi_G(V'(\hat{x}, w))}{\|\pi_G(V'(\hat{x}, w))\|})}{\|\pi_H(\frac{\pi_G(V'(\hat{x}, w))}{\|\pi_G(V'(\hat{x}, w))\|})\|}$$

$\Rightarrow$

$$\frac{V'_H(\bar{x}, w)}{\|V'_H(\bar{x}, w)\|} = \frac{\pi_H(\bar{p})}{\|\pi_H(\bar{p})\|} = p.$$

$V(\cdot, w)$  satisfies **C** condition.

Now consider the case  $U$  satisfies **M**. Observe that from the property of condition **M**, if  $\sup_{\lambda} U(x + \lambda w) < +\infty$  for some  $x$ , then there exists  $\bar{\lambda}$  such that  $U(x + \bar{\lambda} w) = \sup_{\lambda} U(x + \lambda w) = V(x, w)$ .

Suppose that  $\sup_H V(x, w) < \infty$ . Let  $G = H + \{\lambda w\}_{\lambda \in \mathbb{R}}$ . It is an affine subspace. We can verify easily that  $\sup_G U(x) = \sup_H V(x, w) < +\infty$ , so there exists  $\bar{x} \in G$  such that  $U(\bar{x}) = \sup_G U(x)$ . Observe that  $V(\bar{x}, w) = \sup_{\lambda} U(\bar{x} + \lambda w) \geq U(\bar{x}) = \sup_G U(x) = \sup_H V(x, w)$ , so  $V(\bar{x}, w) = \sup_H V(x, w)$ . Since  $\bar{x} \in H + \{\lambda w\}_{\lambda}$ , there exists  $\hat{x} \in H$ ,  $\lambda \in \mathbb{R}$  such that  $\bar{x} = \hat{x} + \lambda w$ . We have  $V(\hat{x}, w) = \sup_H V(x, w)$ . ■

## 7.10 Proof of Theorem 4

Suppose there exists an equilibrium. By Proposition 2, we have  $\bigcap_i P^i \neq \emptyset$ .

We prove the converse. Suppose that  $\bigcap_i P^i \neq \emptyset$ . We will prove that  $\mathcal{U}$  is compact, and this implies the existence of equilibrium. Take any  $\bar{p} \in \bigcap_i P^i$ . There exists  $\lambda_i > 0$ ,  $\bar{x}_i \in \mathbb{R}^S$  such that  $\bar{p} = \lambda_i U^{i'}(\bar{x}_i)$ ,  $\forall i$ .

Suppose that  $(w^1, w^2, \dots, w^m) \in W$ . From the Lemma 1, we know that for all  $i$ ,  $\bar{p} \cdot w \geq 0$ , so we have:

$$0 = \bar{p} \cdot \sum_{i=1}^m w^i \geq 0.$$

This implies  $\bar{p} \cdot w^i = 0$  for all  $i$ .

Define  $L(W) := W \cap (-W)$ , the biggest sublinear space included in  $W$ . If  $W$  is a linear subspace, then the  $WNMA$  condition is satisfied. By Theorem 2,  $\mathcal{U}$  is compact and there exists an equilibrium.

Suppose that  $W \neq L(W)$ , or there exists  $w = (w^1, w^2, \dots, w^m) \in W$  such that  $-w \notin W$ , i.e.  $w \notin L(W)$ . Denote  $U_1^i(x^i) = V[U^i](x^i, w^i)$  as in Definition 10

$$U_1^i(x^i) = \sup_{\lambda \in \mathbb{R}} U^i(x^i + \lambda w^i).$$

We have  $p \cdot w^i = 0$ . Indeed,  $p \cdot w^i \geq 0$  for all  $i$  and  $p \cdot \sum_{i=1}^m w^i = 0$ .

This implies  $U^{i'}(\bar{x}^i) \cdot w^i = 0$ , so  $U^i(\bar{x}^i + \lambda w^i) = U^i(\bar{x}^i)$ ,  $\forall \lambda \geq 0$ . By Proposition 7,

$$U_1^i(x^i) = \max_{\lambda} U^i(x^i + \lambda w^i),$$

and it has the same property as  $U^i$ . The new utility function  $U_1^i$  satisfies either No Half-line condition, either **C**, either **M**.

Now we define  $\mathcal{E}_1$  the economy which is characterized by  $\{(U_1^i, e^i, X^i)\}$ . As above, denote by  $W_1$  the cone of mutual opportunities of arbitrage.

Observe that any useful vector of  $U^i$  is also a useful vector of  $U_1^i$ . So,  $W \subset W_1$ . From the very definition of  $V[U^i]$ , we have that, for all  $i$ ,  $w^i$  is a useless vector of  $U_1^i$ . Thus,  $(w^1, w^2, \dots, w^m) \in L(W_1)$ . Hence we have  $\dim L(W_1) \geq \dim L(W) + 1$ .

We will prove that the utility set  $\mathcal{U}_1$  equals  $\mathcal{U}$ . Recall that  $w^i$  is a half-line. If  $U^i$  satisfies no half-line condition, then  $w^i = 0$ , and evidently  $\max_{\lambda} U^i(x^i + \lambda w^i)$  exists. If  $U^i$  does not satisfy no half-line condition, then  $U^i$  satisfies **C** condition. By Proposition 5,  $\max_{\lambda} U^i(x^i + \lambda w^i)$  exists. By applying Lemma 7, we have  $\mathcal{U}_1 = \mathcal{U}$ .

If we denote by  $P_1^i$ , the set of weak no-arbitrage prices of agent  $i$ , we have  $\bar{p} \in P_1^i$  for all  $i$ . Indeed,  $\bar{p} \cdot w^i = 0$ , this implies  $U^i(\bar{x}^i) = V^i(\bar{x}^i, w^i)$ . Hence by Lemma 6,  $\bar{p} = \lambda_i V_x^{i'}(\bar{x}^i, w^i)$ .

The economy  $\mathcal{E}_1$  satisfies  $\bigcap_i P_1^i \neq \emptyset$ . Its utility functions satisfy either no half-line condition, or **C**, or **M**. If  $W_1$  is linear, using the *WNMA* condition, we have the compactness of  $\mathcal{U}_1$ . In the contrary case, take  $\tilde{w} = (\tilde{w}^1, \tilde{w}^2, \dots, \tilde{w}^m) \in W_1$  such that  $-\tilde{w} \notin W_1$ . We define  $U_2^i(x^i) = V[U_1^i](x^i, \tilde{w}^i)$ .

By induction and the same arguments, suppose that we have constructed the economy  $\mathcal{E}_t$ , with  $t \geq 1$ . If  $W_t$  is not a linear space, we can construct a new economy  $\mathcal{E}_{t+1}$ , with  $\dim L(W_{t+1}) \geq \dim L(W_t) + 1$ . Our economies have the same utility sets  $\mathcal{U}_t = \mathcal{U}, \forall t$ .

For all  $t, W_t \subset \mathbb{R}^{S \times m}$ . So, we have to stop at some step  $T$ . In economy  $\mathcal{E}_T$ ,  $W_T$  is a linear space, and hence satisfies *WNMA* condition. By Theorem 2, we have  $\mathcal{U}_T$  is compact. This implies  $\mathcal{U}$  is compact and our initial economy  $\mathcal{E}$  has an equilibrium. ■

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