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# INNER AMENABILITY OF THE SUBGROUPS OF $\text{PL}^+(I)$ .

HAJER HMILI AND ISABELLE LIOUSSE

ABSTRACT. We give a short proof of the fact that all subgroups of  $\text{PL}^+(I)$ , the group of orientation preserving piecewise homeomorphisms of the unit interval, are inner amenable in the sense of Effros.

## 1. INTRODUCTION

The notion of inner amenability was introduced by E. Effros [Eff75] for countable groups and it is a weakening of that of amenability: one considers the action of a group  $G$  on itself by conjugation rather than by left multiplication. Examples, characterizations and related properties to inner amenability may be found in [BdlH86] as well as the mention of some possible generalisations to uncountable groups. In [CSS01], Ceccherini-Silberstein and Scarabotti were interested in inner amenability for some large groups and following [CSS01], precise definitions are given in Section 2.

On the other hand, the group of orientation preserving piecewise linear homeomorphisms of the unit interval  $I$  has attracted considerable attention, in particular since it contains many algebraically interesting groups:

The Thompson group  $F$  can be regarded as the group of piecewise linear homeomorphisms of the unit interval whose break points are dyadic points and slopes are powers of 2. More generally, given  $p \in \mathbb{N}$ ,  $p \geq 2$ , Brown ([Bro87]) defined  $F_p$  to be the set of piecewise linear homeomorphisms of the unit interval whose break points are  $p$ -adic points and slopes are powers of  $p$ .

An important question is whether  $F$  is amenable especially as  $\text{PL}^+(I)$  could also be amenable. It is well known that  $\text{PL}^+(I)$  shares many properties with amenable groups: In [BS85], Brin and Squier proved that  $\text{PL}^+(I)$  contains no free non abelian subgroups and a theorem of Calegari establishes the vanishing stable commutator length in all subgroups of  $\text{PL}^+(I)$  ([Cal07]).

In [Jol97] and [Jol98], Jolissaint proved that  $F$  is inner amenable and, in a different way, Ceccherini-Silberstein and Scarabotti ([CSS01]) proved that the group of orientation preserving piecewise linear homeomorphisms of the real line and some of its subgroups including  $F$  are inner amenable. Later, Picioroaga ([Pic06]) proved that the Brown-Thompson groups  $F_p$  are inner amenable.

Using the method of Ceccherini-Silberstein and Scarabotti ([CSS01]) and Calegari's arguments for its vanishing theorem, we prove

**Theorem.** *Every subgroup of  $\text{PL}^+(I)$  is inner amenable.*

Recently, in [TD20] (Theorem 20 and section 7E), Tucker-Drob proved that countable subgroups of the Monod group  $H(\mathbb{R})$  of piecewise projective homeomorphisms of the real line are stable in the sense of Jones-Schmidt and then inner amenable (see [JS87]). Here, restricting to subgroups of piecewise affine homeomorphisms of the unit

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interval, we provide a direct proof of inner amenability that does not require groups to be countable. However, since every group is the direct union of its finitely generated subgroups and inner amenability is stable by taking direct union, Tucker-Drob's result implies our theorem.

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## 2. BACKGROUND

### 2.1. Inner amenability.

**Definition 1.** ([Eff75])

A group  $G$  is **inner amenable** (as a discrete group) if there exists an inner-invariant mean, that is a map  $\mu : \mathcal{P}(G \setminus \{e\}) \rightarrow [0, 1]$  such that

- $\mu(A \sqcup B) = \mu(A) + \mu(B)$ , for all  $A, B \subset G \setminus \{e\}$  disjoint,
- $\mu(gAg^{-1}) = \mu(A)$  for all  $A, B \subset G \setminus \{e\}$ ,  $g \in G$  and
- $\mu(G \setminus \{e\}) = 1$ .

According to Tarski (Theorem 16.12 (ii) of [Tar49]), one has

**Proposition.** *A group  $G$  is inner amenable if and only if there does not exist inner-paradoxical decompositions of  $G \setminus \{e\}$  that is non-empty subsets  $A_1, \dots, A_k ; B_1, \dots, B_l$  of  $G \setminus \{e\}$  and elements  $\alpha_1, \dots, \alpha_k ; \beta_1, \dots, \beta_l$  in  $G$  not all trivial such that:*

- (1)  $G \setminus \{e\} = (A_1 \sqcup \dots \sqcup A_k) \sqcup (B_1 \sqcup \dots \sqcup B_l)$ ,
- (2)  $G \setminus \{e\} = \alpha_1 A_1 \alpha_1^{-1} \sqcup \dots \sqcup \alpha_k A_k \alpha_k^{-1}$  and
- (3)  $G \setminus \{e\} = \beta_1 B_1 \beta_1^{-1} \sqcup \dots \sqcup \beta_l B_l \beta_l^{-1}$ .

**Remark.** Note that Effros definition requires that  $\mu \neq \delta_e$  unlike other notions of inner amenability. Namely, Paterson (see [Pat88], 2.35) defined a locally compact group  $G$  to be "inner amenable" if there exists a conjugation-invariant mean  $m \in L^\infty(G)^*$  and with this definition every discrete group would be inner amenable.

**Proposition 1.** (Corollary 2 (iv) of [BdlH86])

*Let  $G$  be a group, if  $N$  is an inner amenable normal subgroup of  $G$  and  $K = G/N$  is amenable then  $G$  is inner amenable.*

In particular, a group  $G$  is inner amenable provided that the same holds for its commutator subgroup  $[G, G]$  (i.e. the group generated by the commutators  $[f, g] = fgf^{-1}g^{-1}$ ,  $f, g \in G$ ), since  $G/[G, G]$  is abelian so amenable.

**2.2. Piecewise linear homeomorphisms.** Let  $\mathcal{I} = [a, b]$  be a closed interval.

**Definition 2.**

• An orientation preserving homeomorphism  $f$  of  $\mathcal{I}$  is **piecewise linear** if there exist finitely many points  $a = a_1 < a_2 < \dots < a_p < a_{m+1} = b$  called **break points** such that for any  $i = 1, \dots, m$ ,

$$f_{|[a_i, a_{i+1})}(x) = \lambda_i x + \beta_i, \quad \lambda_i \in \mathbb{R}^{+*}, \beta_i \in \mathbb{R}.$$

- The group  $\text{PL}^+(\mathcal{I})$  is the set of all piecewise linear homeomorphisms of  $\mathcal{I}$ .
- For  $f \in \text{PL}^+(\mathcal{I})$ , we set  $\text{Fix } f := \{x \in \mathcal{I} \mid f(x) = x\}$  and given a subgroup  $G$  of  $\text{PL}^+(\mathcal{I})$  we denote by  $\text{Fix } \mathbf{G}$  the set of common fixed points of all elements of  $G$  that is the set  $\{x \in \mathcal{I} \mid \forall g \in G, g(x) = x\}$ .

- The **support** of  $f$  [resp.  $\Gamma$ ] is the set  $\mathcal{I} \setminus \text{Fix } f$  [resp.  $\mathcal{I} \setminus \text{Fix } \Gamma$ ].
- Given a closed subinterval  $J$  of  $\mathcal{I}$ , we define the group  $\mathbf{S}_J$  as the set of all piecewise linear homeomorphisms of  $\mathcal{I}$  whose support is included in  $J$  and given a group  $G < \text{PL}^+(\mathcal{I})$ , we set  $\mathbf{S}_J(\mathbf{G}) := \mathbf{G} \cap \mathbf{S}_J$ .

**Remark 1.** For any element of  $\text{PL}^+(\mathcal{I})$ , the set  $\text{Fix } f$  is a union of finitely many points and intervals and so the same remains true for  $\text{Fix } \Gamma$  provided that  $\Gamma$  is a finitely generated subgroup of  $\text{PL}^+(\mathcal{I})$ .

In this last case, the support set  $\mathcal{I} \setminus \text{Fix } \Gamma$  of  $\Gamma$  is a finite union of open intervals  $I_i$  that are  $\Gamma$ -invariant and it makes sense to define for all  $i$ , the set  $\Gamma_{|\bar{I}_i} := \{g_{|\bar{I}_i}, g \in \Gamma\}$  which is a finitely generated subgroup of  $\text{PL}^+(\bar{I}_i)$ .

**Remark 2.** Let  $G$  be a subgroup of  $\text{PL}^+(\mathcal{I})$ . By the chain rule, the commutator subgroup  $[G, G]$  of  $G$  consists of piecewise linear homeomorphisms that fix a neighborhood of any point  $a$  in  $\text{Fix } G$ . Obviously, these neighborhoods depend on the considering map. However, when taking a finite number of such maps, it is possible to find common neighborhoods by intersecting the finitely many neighborhoods related to these maps.

Noting that elements with non full support are particularly interesting because of their large centralizers, we make

**Definition 3.** Two subsets  $A_1$  and  $A_2$  of a group  $G$  **commute** if any  $\alpha \in A_1$  commutes with any  $\alpha' \in A_2$ .

It is easy to check that if  $J_1$  and  $J_2$  are two disjoint subintervals of  $\mathcal{I}$  then the groups  $\mathbf{S}_{J_1}$  and  $\mathbf{S}_{J_2}$  commute.

### 3. PROOF.

The key property in order to apply the method of [CSS01] is provided by a claim proved in [Cal07] (proof of Theorem A case 2), we reformulate it as

**Proposition 2.** Let  $\mathcal{I}$  be a closed interval,  $\Gamma$  be a finitely generated subgroup of  $\text{PL}^+(\mathcal{I})$  and  $I_i = (a_i, b_i)$ ,  $i \in \{1, \dots, n\}$ , be the connected components of  $\mathcal{I} \setminus \text{Fix } \Gamma$ .

Then, for all set  $J = \sqcup J_i$ , where  $J_i \subset I_i$  is a closed interval, there exists  $\phi \in \Gamma$  such that  $J$  and  $\phi(J)$  are disjoint.

And for the sake of completeness, we detail its proof.

**Claim 1.** Let  $c \in \mathcal{I}$  then  $\delta(c) := \sup_{g \in \Gamma} g(c) \in \text{Fix } \Gamma$ .

Indeed, let  $\gamma \in \Gamma$ , by definition, for all  $g \in \Gamma$ , we have  $\delta(c) \geq g(c)$ . As  $\gamma$  is orientation preserving, it holds that  $\gamma(\delta(c)) \geq \gamma \circ g(c)$ . Therefore passing to the supremum over  $g \in \Gamma$ , we get  $\gamma(\delta(c)) \geq \sup_{g \in \Gamma} \gamma \circ g(c) = \sup_{k \in \Gamma} k(c) = \delta(c)$ . Changing  $\gamma$  for  $\gamma^{-1}$  leads to the opposite inequality and finally to  $\gamma(\delta(c)) = \delta(c)$  for all  $\gamma \in \Gamma$ .

**Claim 2.** Let  $J = [c, d] \subset I_i$  for some  $i \in \{1, \dots, n\}$  then there exists  $\phi \in \Gamma$  such that  $\phi(J) \cap J = \emptyset$  and  $\sup J < \inf \phi(J)$ .

As the maps in consideration are orientation preserving, it suffices to find  $\phi \in \Gamma$  such that  $\phi(c) > d$  and this is equivalent to prove that  $\delta(c) = \sup_{g \in \Gamma} g(c) > d$ .

As  $I_i$  is  $\Gamma$ -invariant, we get that  $g(J) \subset I_i$  for all  $g \in \Gamma$  so  $a_i \leq g(c) \leq b_i$  and then  $a_i \leq \delta(c) \leq b_i$ . Therefore, according to Claim 1,  $\delta(c) \in \text{Fix } \Gamma \cap \bar{I}_i$  so  $\delta(c) \in \{a_i, b_i\}$ . As

$c \notin \text{Fix } \Gamma$  there exists  $g \in \Gamma$  such that  $g(c) \neq c$ , eventually replacing  $g$  by its inverse we can assume that  $g(c) > c$ , it follows that  $\delta(c) > c$ . Finally, we get that  $\delta(c) = b_i > d$ .

Going back to the proof of Proposition 2, we argue by induction on the number  $n$  of the connected components of  $I \setminus \text{Fix } \Gamma$  for showing that:

*If  $\mathcal{I}$  is a closed interval,  $n \in \mathbb{N}^*$ ,  $\Gamma$  is a subgroup of  $PL^+(\mathcal{I})$  such that  $\mathcal{I} \setminus \text{Fix } \Gamma$  is a finite union of  $n$  pairwise disjoint open intervals  $I_1, \dots, I_n$  and  $J_i = [c_i, d_i] \subset I_i$  for all  $i$ , then there exists  $\phi \in \Gamma$  such that  $\phi(J_i)$  is disjoint from  $J_i$  for all  $i$ .*

- If  $n = 1$  then  $\text{Fix } \Gamma = \partial\mathcal{I}$ ,  $J = [c, d]$  and we directly conclude by Claim 2
- Let  $n \in \mathbb{N}$ ,  $n \geq 2$  and assume that  $\mathcal{I} \setminus \text{Fix } \Gamma$  has  $n + 1$  connected components, namely  $I_1 = (a_1, b_1)$ , ...,  $I_i = (a_i, b_i)$ , ...,  $I_{n+1} = (a_{n+1}, b_{n+1})$ .

Applying Claim 2 to  $J_1$  and the group  $\Gamma|_{\overline{I_1}}$ , we deduce that there exists  $\phi_1 \in \Gamma$  such that  $\phi_1(J_1)$  is disjoint from  $J_1$  and  $\sup J_1 < \inf \phi_1(J_1)$ .

Now, we apply the induction hypothesis to the group  $\Gamma|_{[b_1, 1]}$  and the sets  $J'_i \subset I_i$ ,  $i > 1$ , where  $J'_i$  is the smallest closed interval containing both  $J_i$  and  $\phi_1^{-1}(J_i)$ . We obtain an element  $\phi_2 \in \Gamma$  such that  $\phi_2(J'_i)$  is disjoint from  $J'_i$ , for all  $i > 1$ . After replacing  $\phi_2$  by  $\phi_2^{-1}$ , if necessary, we may also assume that  $\inf J_1 \leq \phi_2(\inf J_1)$ .

Finally, let  $\phi = \phi_1 \circ \phi_2$ , we have:

$$\inf J_1 \leq \phi_2(\inf J_1) \quad , \quad \sup J_1 < \inf \phi_1(J_1).$$

Then, maps being orientation preserving, we get:

$$\sup J_1 < \phi_1(\inf J_1) \leq \phi_1 \circ \phi_2(\inf J_1) = \inf \phi(J_1).$$

This implies that  $\phi(J_1) \cap J_1 = \emptyset$ .

Moreover for all  $i > 1$ , it holds that:

$$\phi(J_i) \cap J_i = (\phi_1 \circ \phi_2)(J_i) \cap J_i = \phi_1 \left( \underbrace{\phi_2(J_i)}_{\subset J'_i} \cap \underbrace{\phi_1^{-1}(J_i)}_{\subset J'_i} \right) \subset \phi_1(\phi_2(J'_i) \cap J'_i) = \emptyset$$

This ends the induction argument.

In addition, since the  $I_i$ 's are the connected components of  $\mathcal{I} \setminus \text{Fix } \Gamma$ , they are  $\Gamma$ -invariant and pairwise disjoint. Hence, for every  $i \in \{1, \dots, n\}$ , the intervals  $J_i$  and  $\phi(J_i)$  are included in  $I_i$  and therefore are disjoint from all  $J_j$  and  $\phi(J_j)$  with  $j \neq i$ . This completes the proof of Proposition 2.  $\square$

Now, we prove the theorem by developing the strategy proposed in Remark 1 of [CSS01] and using Propositions 1 and 2.

Let  $G$  be a subgroup of  $PL^+(I)$ , according to Proposition 1,  $G$  is inner amenable provided that its commutator subgroup  $[G, G]$  is.

We argue by contradiction supposing that the group  $[G, G]$  is not inner amenable. Then there exist non-empty subsets  $A_1, \dots, A_k$ ;  $B_1, \dots, B_l$  of  $[G, G] \setminus \{e\}$  and elements  $\alpha_1, \dots, \alpha_k$ ;  $\beta_1, \dots, \beta_l$  of  $[G, G]$  not all trivial such that:

- (1)  $[G, G] \setminus \{e\} = (A_1 \sqcup \dots \sqcup A_k) \sqcup (B_1 \sqcup \dots \sqcup B_l)$ ,
- (2)  $[G, G] \setminus \{e\} = \alpha_1 A_1 \alpha_1^{-1} \sqcup \dots \sqcup \alpha_k A_k \alpha_k^{-1}$  and
- (3)  $[G, G] \setminus \{e\} = \beta_1 B_1 \beta_1^{-1} \sqcup \dots \sqcup \beta_l B_l \beta_l^{-1}$ .

For any of the  $\alpha_i, \beta_j$ , we fix a writing as product of commutators and we consider the finitely generated subgroup  $\Gamma$  of  $\text{PL}^+(I)$  generated by all the elements of  $G$  involved in these writings so that all the  $\alpha_i, \beta_j$  belong to  $[\Gamma, \Gamma]$ .

Then, by Remarks 1 and 2, all the  $\alpha_i, \beta_j$  have support included in a common finite union  $J$  of closed intervals  $J_i \subset I \setminus \text{Fix}(\Gamma)$ , in particular  $\alpha_i, \beta_j \in \mathbf{S}_J([G, G])$ .

Therefore, according to Proposition 2, there exists  $\phi \in \Gamma$  such that  $J$  and  $\phi(J)$  are disjoint. As a consequence, the groups  $\mathbf{S}_J$  and  $\mathbf{S}_{\phi(J)}$  commute.

Thus for all  $i = 1, \dots, k$  and all  $j = 1, \dots, l$ , we have

$$(\alpha_i A_i \alpha_i^{-1}) \cap \mathbf{S}_{\phi(J)} = A_i \cap \mathbf{S}_{\phi(J)} \quad \text{and} \quad (\beta_j B_j \beta_j^{-1}) \cap \mathbf{S}_{\phi(J)} = B_j \cap \mathbf{S}_{\phi(J)}$$

Indeed, for any  $i = 1, \dots, k$  :

$$(\alpha_i A_i \alpha_i^{-1}) \cap \mathbf{S}_{\phi(J)} = (\alpha_i A_i \alpha_i^{-1}) \cap (\alpha_i \mathbf{S}_{\phi(J)} \alpha_i^{-1}) = \alpha_i (A_i \cap \mathbf{S}_{\phi(J)}) \alpha_i^{-1} = A_i \cap \mathbf{S}_{\phi(J)},$$

since  $\alpha_i \in \mathbf{S}_J$  commutes with  $\mathbf{S}_{\phi(J)}$  and  $A_i \cap \mathbf{S}_{\phi(J)}$ .

Similarly, we get the analogous property for  $j = 1, \dots, l$ .

Therefore, intersecting the subsets in (1), (2) and (3) with  $\mathbf{S}_{\phi(J)}$  leads to:

- (1')  $\mathbf{S}_{\phi(J)}([G, G]) \setminus \{e\} = ((A_1 \sqcup \dots \sqcup A_k) \cap \mathbf{S}_{\phi(J)}) \sqcup ((B_1 \sqcup \dots \sqcup B_l) \cap \mathbf{S}_{\phi(J)})$ ,
- (2')  $\mathbf{S}_{\phi(J)}([G, G]) \setminus \{e\} = (\alpha_1 A_1 \alpha_1^{-1} \sqcup \dots \sqcup \alpha_k A_k \alpha_k^{-1}) \cap \mathbf{S}_{\phi(J)} = (A_1 \sqcup \dots \sqcup A_k) \cap \mathbf{S}_{\phi(J)}$ ,
- (3')  $\mathbf{S}_{\phi(J)}([G, G]) \setminus \{e\} = (\beta_1 B_1 \beta_1^{-1} \sqcup \dots \sqcup \beta_l B_l \beta_l^{-1}) \cap \mathbf{S}_{\phi(J)} = (B_1 \sqcup \dots \sqcup B_l) \cap \mathbf{S}_{\phi(J)}$ .

A contradiction with the fact that  $\mathbf{S}_{\phi(J)}([G, G]) \setminus \{e\}$  is non-empty (it contains the maps  $\phi \circ \alpha_i \circ \phi^{-1}$  and  $\phi \circ \beta_i \circ \phi^{-1}$ ).

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