UNIFORM SIMPLICITY FOR SUBGROUPS OF PIECEWISE CONTINUOUS BIJECTIONS OF THE UNIT INTERVAL

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ABSTRACT. Let $I = [0, 1)$ and $\mathcal{PC}(I)$ [resp. $\mathcal{PC}^+(I)$] be the quotient group of the group of all piecewise continuous [resp. piecewise continuous and orientation preserving] bijections of I by its normal subgroup consisting in elements with finite support (i.e. that are trivial except at possibly finitely many points). Arnoux's thesis states that $\mathcal{PC}^+(I)$ and certain groups of interval exchanges are simple, and the proofs of these results are the purpose of the Appendix. We prove the simplicity of the group $\mathcal{A}^+(I)$ of locally orientation preserving, piecewise continuous, piecewise affine maps of the unit interval. These results can be improved. Indeed, a group G is uniformly simple if there exists a positive integer N such that for any $f, \phi \in G \setminus \{\mathrm{Id}\}$, the element ϕ can be written as a product of at most N conjugates of f or f^{-1}.

We provide conditions which guarantee that a subgroup G of $\mathcal{PC}(I)$ is uniformly simple. As corollaries, we obtain that $\mathcal{PC}(I)$, $\mathcal{PC}^+(I)$, $\mathrm{PL}^+(\mathbb{S}^1)$, $\mathcal{A}(I)$, $\mathcal{A}^+(I)$ and some Thompson like groups included the Thompson group T are uniformly simple.

1. INTRODUCTION

The algebraic study of groups consisting in continuous transformations of a topological space was initiated by Schreier and Ulam in 1934 ([SU34]) and the question of the simplicity of such groups was raised.

Definition 1.1.

- A group G is **simple** if any normal subgroup of G is either trivial or equal to G.
- A group G is **perfect** if G coincides with $G' = [G, G]$ the normal subgroup generated by its commutators $[a, b] = aba^{-1}b^{-1}$ with $a, b \in G$.

Remark 1.2. In particular, a simple group G is perfect.

In [UvN47], Ulam and Von Neumann proved that the identity component in the group of homeomorphisms of the circle or the 2-sphere is a simple group. In the seventies lots of (smooth) transformation groups were studied by Epstein, Herman, Thurston, Mather, Banyaga, and proved to be simple (see the books [Ban97] or [Bou08]).

In [Ula64], Ulam explained that [UvN47] establishes a sharper theorem: *"for every f and ϕ non-trivial and isotopic to identity homeomorphisms of the circle or the 2-sphere, there exists a fixed number N of conjugates of f or f^{-1} whose product is ϕ"*. This number does not exceed 23 and Ulam raised the question of finding the optimal bound. The issue was taken up again in updated versions of the Scottish book (see [Mau15], Problem 29) in relation with Nunnally's work ([Nul66]) which states that N is less than 3 for certain groups of homeomorphisms. This leads to the following

Definition 1.3. Let N be a positive integer.

A perfect group G is $\boldsymbol{N-}$**uniformly perfect** if any product of commutators in G can be written as a product of at most N commutators in G.

A non-trivial group G is $\boldsymbol{N-}$**uniformly simple** if for any pair $\{f, \phi\}$ of non-trivial elements of G, one can express ϕ as a product of at most N conjugates of f or f^{-1} in G.

Note that uniform simplicity implies simplicity. In this context, Nunnally's work establishes the 3-simplicity for certain groups of homeomorphisms. But Nunnally's techniques fail when requiring groups to preserve additional structures (e.g. smooth, PL or area). Tsuboi ([Tsu09]) showed the uniform simplicity of the identity component $\mathrm{Diff}^r(M^n)_0$ of the group of C^r-diffeomorphisms ($1 \leq r \leq \infty$, $r \neq n+1$) of a compact connected n-dimensional manifold M^n with handle decomposition without handles of index $\frac{n}{2}$. As a corollary and under the same assumption on r, he obtained that $\mathrm{Diff}^r(\mathbb{S}^n)_0$ *is* 12*-uniformly simple.*

Remark 1.4. Uniform simplicity is related to conjugacy-invariant lengths on G, that is $L : G \to \mathbb{R}^+$ such that $L(gh) \leq L(g) + L(h)$, $\quad L(g^{-1}) = L(g) = L(hgh^{-1})$ and $\quad L(g) = 0$ iff $g = \mathrm{Id}$. Namely, if G is N-uniformly simple then for every conjugacy-invariant length L on G and for any pair $\{f, \phi\}$ of non-trivial elements of G, one has $L(\phi) \leq NL(f)$.

Given a group G and a non-empty subset S of $G \setminus \{\mathrm{Id}\}$ which is closed under inversion and conjugation, if G is N-uniformly simple then any $g \in G$ can be expressed as a product of at most N elements of S. In particular, S can be the set consisting of involutions, finite order elements, commutators or reversible maps. Recall that $g \in G$ is said to be **reversible** if g is conjugate in G to its inverse. In the O'Farell and Short survey on reversibility ([OS15] p.35), the authors raised the related questions: "*Given G a group, does there exist a positive integer n [resp. m] such that G coincides with $\mathcal{I}^n = \{\ h_1 \ \cdots \ h_n$ with $h_i^2 = \mathrm{Id}\ \}$ [resp. with $\mathcal{R}^m = \{\ r_1 \ \cdots \ r_m$ with r_i reversible $\}$] ?*" Clearly, for uniformly simple groups containing involutions both questions have a positive answer.

In this paper, we do not further assume that transformations are continuous and we focus on dimension one. The groups we are interested in are described by the following

Definition 1.5. Let $I = [0, 1)$ be the unit interval.

- A **piecewise continuous bijection of** I is a bijection f of I that is continuous outside a finite subset of I called **discontinuity set** and denoted by $\mathbf{Disc}(f)$. The **support** of f is the set $\mathbf{Supp}(f) = \{x \in I \mid f(x) \neq x\}$.
- Let $\widehat{\mathcal{PC}}(I)$ be the group of piecewise continuous bijections of I. We denote by $\mathcal{PC}(I)$ the quotient group of $\widehat{\mathcal{PC}}(I)$ by its normal subgroup consisting of elements with finite support and the subgroup of $\mathcal{PC}(I)$ consisting of classes of piecewise increasing elements is referred as $\mathcal{PC}^+(I)$.

By taking the unique right continuous representative for all f in $\mathcal{PC}^+(I)$, the group $\mathcal{PC}^+(I)$ can be identified with the group of right continuous and piecewise increasing bijections of I. But such a representative may not exist for some elements of $\mathcal{PC}(I)$.

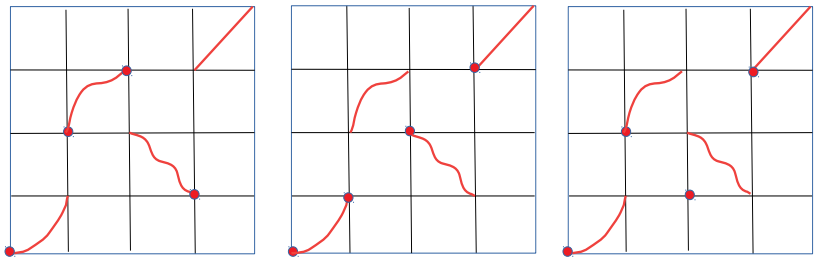
Definition 1.6. Let $f \in \mathcal{PC}(I)$. We say that a representative of f is **good** if it minimizes the number of discontinuity points among the elements of the class f.

Note that this minimizing condition does not guarantee uniqueness, but all the good representatives of a given element of $\mathcal{PC}(I)$ have the same discontinuity point set, the same image of the discontinuity point set and they coincide on their common continuity set. However, it is possible to require more properties in order to exhibit "canonical" representatives.

More precisely, let $f \in \mathcal{PC}(I)$ and $\tilde{f}$ be a good representative of f with discontinuity points a_i where $0 = a_1 < \cdots < a_n < 1$. We consider σ the finitely supported bijection which sends $\tilde{f}(a_i)$ to b_j the left endpoint of $\tilde{f}((a_i, a_{i+1}))$ with the convention that $a_{n+1} = 1$. Note that σ is well-defined since the set of all $\tilde{f}(a_i)$ is equal to the set of all b_j.

The map $\sigma\tilde{f}$ is a good representative of f and it satisfies $\sigma\tilde{f}([a_i, a_{i+1})) = [b_j, b_{j+1})$, with the convention that $b_{n+1} = 1$. Clearly $\sigma\tilde{f}$ is the unique good representative of f that has this property. Then we give the following

Definition 1.7. Let $f \in \mathcal{PC}(I)$. We define the **best** representative $\widehat{\boldsymbol{f}}$ of f to be the unique good representative of f such that $\hat{f}([a_i, a_{i+1}))$ is a right-open and left-closed interval, where a_i, $1 \leq i \leq n$ are the discontinuity points of $\hat{f}$ and $a_{n+1} = 1$.



Remark 1.8.

- If $f \in \mathcal{PC}^+(I)$ then $\widehat{f}$ is the right-continuous representative of f. More generally, for $f \in \mathcal{PC}(I)$, $\widehat{f}$ is the good representative of f that is right continuous at the left endpoints of the continuity intervals where $\widehat{f}$ is orientation preserving and for the continuity intervals where $\widehat{f}$ is orientation reversing, $\widehat{f}$ sends their left endpoints to the left endpoints of their images.
- Note that the map $f \mapsto \widehat{f}$ is not a morphism (i.e. there exist f and g such that $\widehat{f \circ g} \neq \widehat{f} \circ \widehat{g}$).

Since the maps we deal with, are only piecewise continuous, the interval $[0, 1)$ can be identified with the unit circle $\mathbb{S}^1$ and it is equivalent to consider a piecewise continuous bijection as a map $\mathbb{S}^1 \to \mathbb{S}^1$ (see [Cor21]). We refer as "continuous versions" of a subgroup G of $\mathcal{PC}(I)$ the subgroups of G consisting in classes of continuous elements of either the interval or the circle. The continuous versions of $\mathcal{PC}^+(I)$ are $\mathrm{Homeo}^+(I)$ and $\mathrm{Homeo}^+(\mathbb{S}^1)$ and their simplicity was shown by Epstein ([Eps70]).

Arnoux ([Arn81b]) proved that $\mathcal{PC}^+(I)$ and certain groups of interval exchanges, defined below, are simple. Unfortunately, these works are unpublished and we express our gratitude to P. Arnoux for reproducing and joining them as an appendix.

Definition 1.9. Let $f \in \widehat{\mathcal{PC}}(I)$.

• The map f is an **affine interval exchange transformation (AIET)** if there exists a finite subdivision $0 = a_1 < a_2 < \cdots < a_p < a_{p+1} = 1$ of $[0, 1]$ such that for any $i = 1, \cdots, p$

$$f_{|[a_i, a_{i+1})}(x) = \lambda_i x + \beta_i, \qquad \lambda_i \in \mathbb{R}^{+*},\ \beta_i \in \mathbb{R}.$$

We define the group $\boldsymbol{\mathcal{A}^+(I)}$ to be the set of all AIET.

• The map f is an **affine interval exchange transformation with flips (FAIET)** if there is a finite subdivision $0 = a_1 < a_2 < \cdots < a_p < a_{p+1} = 1$ of $[0, 1)$ such that for any $i = 1, \cdots, p$

$$f_{|(a_i, a_{i+1})}(x) = \lambda_i x + \beta_i, \qquad \lambda_i \in \mathbb{R},\ \lambda_i \neq 0,\ \beta_i \in \mathbb{R}.$$

The numbers λ_i are called the **slopes of** f and their set is denoted by $\boldsymbol{\Lambda(f)}$.

We denote by $\widehat{\boldsymbol{\mathcal{A}(I)}}$ the group of all FAIET and we define $\boldsymbol{\mathcal{A}(I)}$ to be the group of all classes of FAIET.

• An **interval exchange transformation (IET)** is $f \in \mathcal{A}^+(I)$ with $\Lambda(f) = \{1\}$. We define the group $\boldsymbol{\mathcal{G}^+(I)}$ to be the set of all IET.

• An **interval exchange transformation with flips (FIET)** is $f \in \widehat{\mathcal{A}(I)}$ with $\Lambda(f) \subset \{1, -1\}$. We define the group $\boldsymbol{\mathcal{G}(I)}$ to be the set of all classes of FIET.

The continuous versions of $\mathcal{A}^+(I)$ are $\mathrm{PL}^+(I)$ and $\mathrm{PL}^+(\mathbb{S}^1)$, the groups of piecewise affine homeomorphisms (commonly referred as **PL-homeomorphisms**) of the unit interval and the circle respectively.

For interval exchange transformations, Arnoux ([Arn81b], [Arn81a]) and Sah ([Sah81]) established that $[\mathcal{G}^+(I), \mathcal{G}^+(I)]$ *is a simple group* and an unpublished part of [Arn81b] (III Proposition 1.4) showed that $\mathcal{G}(I)$ *is simple.* In [Eps70], Epstein proved that $\mathrm{PL}^+(\mathbb{S}^1)$ *and* $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ *are simple.* In Section 6, we prove

Theorem 1. *The group $\mathcal{A}^+(I)$ is simple.*

It was not proven in [Arn81b] that $\mathcal{A}^+(I)$ is simple, however the tools of [Arn81b] provide a different proof which is detailed in the appendix. Recently, Lacourte ([Lac22] Theorem 1.4) proved that $\mathcal{PC}(I)$ *and* $\mathcal{A}(I)$ *are simple.*

Note that groups of piecewise affine bijections are particularly known because of the popularity of Thompson's groups and their generalizations.

Definition 1.10. Let $\Lambda \subset \mathbb{R}^{+*}$ be a multiplicative subgroup and $A \subset \mathbb{R}$ be an additive subgroup which is closed under multiplication by Λ and such that $1 \in A$.

• The **Bieri-Strebel groups** are:
  – $V_{\Lambda,A}$ the subgroup of $\mathcal{A}^+(I)$ consisting of elements with slopes in Λ, discontinuity points and their images in A,
  – $T_{\Lambda,A}$ the intersection subgroup of $\mathrm{PL}^+(\mathbb{S}^1)$ with $V_{\Lambda,A}$ and
  – $F_{\Lambda,A}$ the intersection subgroup of $\mathrm{PL}^+(I)$ with $V_{\Lambda,A}$.

• In the case that $A = \mathbb{Z}[1/m]$ and $\Lambda = \langle m \rangle$ with $m \in \mathbb{N}^*$, we get the **Higman-Thompson group** V_m and the **Brown-Thompson groups** T_m and F_m.

• Let $1 < n_1 < \cdots < n_p$ be p integers generating a rank p free abelian multiplicative subgroup $\Lambda = \langle n_1, n_2, ..., n_p \rangle \subset \mathbb{Q}^{+*}$. The **Stein-Thompson groups** are $T_{\Lambda,A}$ and $F_{\Lambda,A}$ with $A = \mathbb{Z}[\Lambda]$. They are denoted by $T_{\{n_1,n_2,...,n_p\}}$ and $F_{\{n_1,n_2,...,n_p\}}$.

It was shown by Thompson that T_2 *and* V_2 *are simple* (see e.g. [CFP96]). Generalizing a result of Brown ([Bro87]), Stein ([Ste92]) proved that $T_{\{n_1,n_2,...,n_p\}}$ *and* $F_{\{n_1,n_2,...,n_p\}}$ *are finitely presented and* $T''_{\{n_1,n_2,...,n_p\}}$ *is simple.* In Section 7, we prove

Theorem 2. *The Stein-Thompson groups $T_{\{n_1,n_2,...,n_p\}}$ with $n_2 = n_1^{2k} - 1$ are simple.*

From now on, we focus on uniform simplicity. Burago and Ivanov in [BI08] implicitly show that the group $[\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ is uniformly simple.

Cornulier communicated to us that $[\mathcal{G}^+(I), \mathcal{G}^+(I)]$ and $\mathcal{G}(I)$ are not uniformly simple. Indeed, if the support of an IET or FIET f has length less than $\frac{1}{N}$ then any product of N conjugates of f or f^{-1} can not have full support. However, in [GL22], we prove that $\mathcal{G}(I)$ is 6-perfect.

Before stating our main result, we give necessary related notions.

Definition 1.11. Let $a \in [0,1)$.

- Let $\delta > 0$, we set $\boldsymbol{V_\delta(0)} = [0,\delta) \cup (1-\delta, 1)$ and $\boldsymbol{V_\delta(a)} = (a-\delta, a+\delta)$, for δ small enough.
- When identifying $[0,1)$ with $\mathbb{S}^1$, an arc contained in $\mathbb{S}^1 \setminus V_\delta(a)$ for some positive δ is referred as ***a*-proper interval**.
- Let $g \in \widehat{\mathcal{PC}}(I)$, the **fixed-point-set** of g is the set $\mathbf{Fix(g)} = \{x \in I \mid g(x) = x\}$.
- We denote $\boldsymbol{B\widehat{\mathcal{PC}}(I)_a} = \{g \in \widehat{\mathcal{PC}}(I) \mid \exists \delta > 0 : \ V_\delta(a) \subset \mathrm{Fix}(g)\}$ and we define $\boldsymbol{B\mathcal{PC}(I)_a}$ to be the image of $B\widehat{\mathcal{PC}}(I)_a$ in $\mathcal{PC}(I)$ by the quotient morphism.

Let G be a subgroup of $\mathcal{PC}(I)$.

- We set $\boldsymbol{BG_a} = G \cap B\mathcal{PC}(I)_a$.
- The **regular *G*-orbit** of a is the set $\boldsymbol{G_{reg}(a)}$ consisting of points $x \in I$ for which there exists $g \in G$ such that $\widehat{g}$ is continuous at a and $\widehat{g}(a) = x$, with the convention that $\widehat{g}$ is continuous at 0 if $\lim\limits_{x\to 0+} \widehat{g}(x) = \lim\limits_{x\to 1-} \widehat{g}(x)$.

Remark 1.12. Note that $f \in B\mathcal{PC}(I)_a$ if and only if its best representative $\widehat{f} \in B\widehat{\mathcal{PC}}(I)_a$.

Now we introduce the conditions that will guarantee that a perfect subgroup of $\mathcal{PC}(I)$ is uniformly simple.

Definition 1.13. Let $a \in I = [0,1)$ and G be a subgroup of $\mathcal{PC}(I)$.

- We say that G is ***a*-LBS** (***a*-L**ocally **B**oundedly **S**upported) if for every $g \in G$ and every a-proper interval J such that $\widehat{g}$ is continuous on J and $\widehat{g}(J)$ is a-proper, there exists $g_a \in BG_a$ such that $\widehat{g}|_J = \widehat{g_a}|_J$.
- Let J be a subinterval of I, we say that G is **(a, J)-proximal** if for every a-proper interval K there exists $k \in G$ such that $\widehat{k}(K) \subset J$.
- We say that G is ***a*-proximal** if for every subinterval J of I, the group G is (a, J)-proximal.
- We say that G is ***NCI*** (**N**on **C**ommuting **I**nvolution) if for any involution $i \in G$, there exists $h \in G$ such that i and hih^{-1} do not commute, it means that $i \circ (hih^{-1})$ is not an involution.

Remark 1.14.

- If G is infinite and simple then G is NCI. This follows by contradiction as the simplicity of G implies that G coincides with its normal subgroup generated by $\{hih^{-1}, h \in G\}$ which is abelian, but abelian simple groups are cyclic of prime order. More formally:
- For an arbitrary group, NCI means that G has no normal subgroup of exponent 2, therefore non-abelian simple groups are NCI.
- If G is perfect and non-trivial then G contains elements that are not involutions. This follows from the fact that a group in which $g^2 = 1$ for all g is abelian.

Inspired by ideas of Dennis-Vaserstein ([DV89]) and Burago-Ivanov ([BI08]), we obtain the following results on uniform simplicity.

Theorem 3. *Let $a \in I$ and G be a perfect a-proximal subgroup of $B\mathcal{PC}(I)_a$.*

- *If G does not contain any involution then G is 8-uniformly simple.*
- *If G has the NCI property then G is 16-uniformly simple.*

Theorem 4. *Let $a \in I$ and G be an a-LBS subgroup of $\mathcal{PC}(I)$ such that*

(1) the regular G-orbit of a is infinite and
(2) the subgroup BG_a is perfect and a-proximal.

- *If G does not contain any involution then G is 12-uniformly simple.*
- *If G has the NCI property then G is 24-uniformly simple.*

The hypotheses of Theorem 3 are closely related to the ones of Theorems 1.1 and 5.1 of Gal and Gismatullin in [GG17]. However, their theorems that concern either boundedly supported order preserving actions or full group actions on a Cantor set do not apply directly to all subgroups of $\mathcal{PC}(I)$. The proof of [GG17] and our proof use the idea of f-commutator of Burago-Ivanov. Here, for proving uniform perfectness, we add ideas of Dennis-Vaserstein: this is explained in Section 2.

Consequences of Theorem 4 are

Corollary 1. *The groups $\mathcal{PC}(I)$ and $\mathcal{PC}^+(I)$ are uniformly simple.*

Corollary 2. *The groups $\mathrm{PL}^+(\mathbb{S}^1)$, $\mathcal{A}(I)$ and $\mathcal{A}^+(I)$ are uniformly simple.*

Theorems 3 and 4 apply to certain Thompson like groups. They imply that the commutator subgroups of the Brown-Thompson groups F_n and the Higman-Thompson groups V_n are uniformly simple. This was proved in [GG17] with smaller bounds. Moreover, Theorem 4 applies to some Stein-Thompson groups $T_{\{n_1,n_2,\dots,n_p\}}$, in particular to the Thompson group T_2. The uniform simplicity of these groups cannot be obtained by Gal and Gismatullin's results.

Corollary 3. *The Thompson group T_2, the Stein-Thompson groups $T_{\{n_1,n_2,\dots,n_p\}}$ with $n_2 = n_1^{2k} - 1$ and in particular, $T_{\{2,3,\cdots\}}$, are uniformly simple.*

Remark 1.15.
Theorem 4 does not apply to subgroups of $\mathrm{Homeo}^+(I)$, since its Hypothesis (1) implies that $a \neq 0$ and the a-proximality excludes the possibility that 0 might be a global fix point.

In addition, a simple subgroup G of $\mathrm{Homeo}^+(I)$ that contains an f having support in some $[c,d]$ with $0 < c < d < 1$ is a subgroup of $B\mathrm{Homeo}^+(I)_0$ and might satisfy the hypotheses of Theorem 3. Indeed, let G be a simple group and $f \in G$ having support in some $[c,d]$ with $0 < c < d < 1$, then any $g \in G$ belongs to the normal closure of $\langle f \rangle$, that is g is the product of p conjugates $k_i f^{\pm 1} k_i^{-1}$ of f or f^{-1}. Therefore g has support in $[\min_i k_i(c), \max_i k_i(d)]$ so $g \in B\mathrm{Homeo}^+(I)_0$.

In conclusion, Theorems 3 and 4 do not apply to simple subgroups of $\mathrm{Homeo}^+(I)$ whose elements have dense supports.

Finally, going back to the O'Farell and Short questions mentioned above, if G is one of the groups considered in Corollaries 1, 2 and 3, then there exists a finite positive integer n such that $G = \mathcal{I}^n = \mathcal{R}^n$.

Acknowledgements. We express our gratitude to P. Arnoux for writing the appendix. We thank E. Ghys for communicating us, a long time ago, how Proposition 1 of [DV89] allows to conclude that certain groups of homeomorphisms are uniformly perfect. His argument is reproduced in Section 2. We thank O. Lacourte for for several stimulating discussions during this work. We gratefully acknowledge Yves Cornulier for bringing the question of uniform simplicity to our attention and for fruitful further exchanges. We are deeply indebted to the anonymous referees whose comments and suggestions substantially improved this work, in

particular for indicating some related contents and for a more elegant construction of the full support involution in Section 6.3.

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2. Uniform perfectness

2.1. Uniform perfectness for subgroups of $B\mathcal{PC}(I)_a$.

Definition 2.1.

• Two subsets S_1 and S_2 of a group G **centralize each other** or are **commuting** if any $\alpha \in S_1$ commutes with any $\alpha' \in S_2$.

• Given J a subset of I, we denote $\widehat{\mathcal{PC}}(J) = \{g \in \widehat{\mathcal{PC}}(I) : \mathrm{Supp}(g) \subset J\}$ and we define $\mathcal{PC}(J)$ to be the image of $\widehat{\mathcal{PC}}(J)$ in $\mathcal{PC}(I)$ by the quotient morphism.

In this section, inspired by the proof of Proposition 1 of Dennis and Vaserstein ([DV89]), we establish

Proposition 2.2. *Let $a \in I$ and G be a subgroup of $B\mathcal{PC}(I)_a$. Suppose that there exist $f \in G$ and a subinterval $J \subset (0,1)$ such that J, $\widehat{f}(J)$ and $\widehat{f}^{-1}(J)$ are pairwise disjoint and G is (a,J)-proximal then any element of $[G,G]$ is the product of 2 commutators in G.*

Proof. As Dennis and Vaserstein have noted, it suffices to prove that any product of 3 commutators is the product of 2 commutators.

Let $g = \gamma_1\gamma_2\gamma_3$ with $\gamma_i = [a_i, b_i]$. By definition of $B\mathcal{PC}(I)_a$, there exists an a-proper interval K which contains the support of all $\widehat{a_i}, \widehat{b_i}$ and by (a,J)-proximality, there exists k in G such that $\widehat{k}$ sends K into J. Thereby, conjugating by k, we can suppose that the supports of $\widehat{a_i}, \widehat{b_i}$ are included in J. For $\gamma, h \in G$, we denote by $C_h(\gamma) = h\gamma h^{-1}$.

Note that $\widehat{\mathcal{PC}}(J)$, $\widehat{\mathcal{PC}}(\widehat{f}(J))$ and $\widehat{\mathcal{PC}}(\widehat{f}^{-1}(J))$ are pairwise commuting subgroups of $\widehat{\mathcal{PC}}(I)$ since J, $\widehat{f}(J)$ and $\widehat{f}^{-1}(J)$ are pairwise disjoint. Thus, for any $w_0, w_1, w_2 \in \langle a_i, b_i\rangle$, it holds that w_0, $C_f(w_1)$ and $C_{f^{-1}}(w_2)$ are pairwise commuting. Therefore

$$g = \gamma_1\gamma_2\gamma_3 = \gamma_1 \;\; C_f(\gamma_2)\; C_{f^{-1}}(\gamma_3) \;\; C_{f^{-1}}(\gamma_3^{-1})\; C_f(\gamma_2^{-1}) \;\; \gamma_2\gamma_3 = C_1C_2, \text{ where}$$

• $C_1 = \gamma_1\; C_f(\gamma_2)\; C_{f^{-1}}(\gamma_3)$ is a commutator. As the supports of a_i and b_i are disjoint from the supports of $C_f(a_i)$ and $C_f(b_i)$ and all these supports are disjoint from those of $C_{f^{-1}}(a_i)$ and $C_{f^{-1}}(b_i)$, we get that the product of these commutators is a commutator.

• $C_2 = C_{f^{-1}}(\gamma_3^{-1})\; C_f(\gamma_2^{-1})\; \gamma_2\gamma_3 = C_{f^{-1}}(\gamma_3^{-1})\gamma_2 \;\; C_f(\gamma_2^{-1})\gamma_3$.

Noticing that $C_f(\gamma_2^{-1})\gamma_3 = C_f\left(\gamma_2^{-1}\; C_{f^{-1}}(\gamma_3)\right) = C_f\left((C_{f^{-1}}(\gamma_3^{-1})\gamma_2)^{-1}\right)$, we conclude that C_2 is a commutator as a product of an element by a conjugate of its inverse. □

2.2. Uniform perfectness for subgroups of $\mathcal{PC}(I)$.

In this section, we prove a lemma that will make the link between the uniform perfectness of $G < \mathcal{PC}(I)$ and the one of its subgroups BG_a.

Lemma 2.3.

If $G < \mathcal{PC}(I)$ is 0-LBS then for any $g \in G$ and $a \in I \setminus \left(\mathrm{Disc}(\widehat{g}) \cup \{0, \widehat{g}^{-1}(0)\}\right)$, there exist $g_0 \in BG_0$ and $g_a \in BG_a$ such that $g = g_0g_a$.

Proof. Let $g \in G$. As $a \neq 0$, $\widehat{g}(a) \neq 0$ and $a \notin \mathrm{Disc}(\widehat{g})$, there exists $\delta > 0$ such that $\widehat{g}$ is continuous on $V_\delta(a)$, where $V_\delta(a)$ and $\widehat{g}(V_\delta(a))$ are 0-proper intervals.

Since G is 0-LBS, there exists $g_0 \in BG_0$ such that $\widehat{g}|_{V_\delta(a)} = \widehat{g_0}|_{V_\delta(a)}$, thereby $(\widehat{g_0}^{-1} \circ \widehat{g})|_{V_\delta(a)} = \mathrm{Id}|_{V_\delta(a)}$ and then $g_a := g_0^{-1} \circ g \in BG_a$. □

3. The Burago and Ivanov method (adapted from Lemma 3.6 and 3.8 of [BI08])

Definition 3.1. Let G be a group and $f \in G$. An **f-commutator** is an element of the form $[\tilde{f}, h]$ for some $h \in G$ and some $\tilde{f}$ conjugate to f or f^{-1}.

Remark 3.2. Any conjugate of an f-commutator is an f-commutator. All elements of the form $[h, f]$ and $[h, f^{-1}]$ are f-commutators. Any f-commutator is product of 2 conjugates of f or f^{-1}.

Proposition 3.3. *Let G be a group of bijections of a space X. Let $f \in G$ and $\Omega \subset X$ such that Ω, $f(\Omega)$ and $f^2(\Omega)$ are pairwise disjoint. Let g_1, g_2 and k be elements of G such that $k(\mathrm{Supp}(g_1) \cup \mathrm{Supp}(g_2)) \subset \Omega$ then $[g_1, g_2]$ is a product of two f-commutators.*

Proof. Let us recall that $C_f(w) = fwf^{-1}$. We first prove the following

Lemma 3.4. *If $\mathrm{Supp}(g_i) \subset \Omega$ for $i = 1, 2$ then*

$$(\star) \qquad g_1 g_2 \; C_f(g_1^{-1}) \; C_{f^2}(g_2^{-1}) = [\; C_f(g_2) \; g_1 g_2 \; , \; f \;]$$

is an f-commutator.

$$\text{Indeed,} \qquad [\; C_f(g_2) \; g_1 g_2 \; , \; f \;] \;=\; C_f(g_2) \; g_1 g_2 \; f \quad (g_1 g_2)^{-1} \; C_f(g_2^{-1}) \; f^{-1}$$

$$= \; C_f(g_2) \; g_1 g_2 \; C_f((g_1 g_2)^{-1}) \; C_{f^2}(g_2^{-1}).$$

Since $g_1 g_2$ and $C_f(g_2)$ have disjoint supports, they commute and we get

$$= \; g_1 g_2 \; C_f(g_1^{-1}) \; C_{f^2}(g_2^{-1}).$$

Remark 3.5. Writing $(\star)$ for g_1^{-1} and g_2^{-1}, we get that $g_1^{-1} g_2^{-1} \; C_f(g_1) \; C_{f^2}(g_2)$ is also an f-commutator.

We turn now on to the proof of Proposition 3.3. Let $g_1, g_2 \in G$ and $k \in G$ such that $k(\cup \mathrm{Supp}(g_i)) \subset \Omega$. The commutator $[g_1, g_2]$ can be written as $C_{k^{-1}}([C_k(g_1), C_k(g_2)])$, with $C_k(g_i)$ of support in Ω. So by Remark 3.2, we can suppose w.l.o.g that the g_i have support in Ω and we have:

$$[g_1, g_2] \;=\; g_1 g_2 \quad C_f(g_1^{-1}) \; C_{f^2}(g_2^{-1}) \quad C_{f^2}(g_2) \; C_f(g_1) \quad g_1^{-1} g_2^{-1}.$$

Since $g_1^{-1} g_2^{-1}$, $C_f(g_1)$ and $C_{f^2}(g_2)$ have pairwise disjoint supports, they all commute and we get

$$[g_1, g_2] \;=\; g_1 g_2 \; C_f(g_1^{-1}) \; C_{f^2}(g_2^{-1}) \quad g_1^{-1} g_2^{-1} \; C_f(g_1) \; C_{f^2}(g_2)$$

Finally, by Lemma 3.4 and Remark 3.5, $[g_1, g_2]$ is a product of two f-commutators. □

4. Uniform simplicity, proof of Theorems 3 and 4

We first show two lemmas which ensure that Propositions 2.2 and 3.3 will apply.

Lemma 4.1. *Let $f \in \mathcal{PC}(I)$ such that $f^2 \neq \mathrm{Id}$, then there exists $J = J_{\widehat{f}} \subset (0,1)$ such that J, $\widehat{f}(J)$ and $\widehat{f}^{\,2}(J)$ are pairwise disjoint subintervals.*

Indeed, as $f^2 \neq \mathrm{Id}$ in $\mathcal{PC}(I)$, the support of $\widehat{f}^{\,2}$ contains an interval and there exists $a \in \mathrm{Supp}(\widehat{f}^{\,2})$ a continuity point of both $\widehat{f}$ and $\widehat{f}^{\,2}$. The required statement follows from a standard argument of continuity.

The proof of the next lemma follows from the definition and we leave it to the reader.

Lemma 4.2. *Let $a \in I$ and G be an a-proximal subgroup of $B\mathcal{PC}(I)_a$. Then for all $g, h \in G$ and any subinterval J there exists $k \in G$ such that $\widehat{k}(\mathrm{Supp}(\widehat{g}) \cup \mathrm{Supp}(\widehat{h})) \subset J$.*

4.1. Proof of Theorem 3.

We recall

Theorem 3. *Let $a \in I$ and G be a perfect a-proximal subgroup of $B\mathcal{PC}(I)_a$.*

- *If G does not contain any involution then G is 8-uniformly simple.*
- *If G has the NCI property then G is 16-uniformly simple.*

Let $G < B\mathcal{PC}(I)_a$ and $f, \phi \in G \setminus \{\mathrm{Id}\}$.

If f is not an involution, by Lemma 4.1, there exists $J \subset (0,1)$ such that J, $\widehat{f}(J)$ and $\widehat{f}^{\,2}(J)$ are pairwise disjoint intervals.

Since G is perfect, $\phi \in [G, G]$. Thus, G being a-proximal, Proposition 2.2 (changing J for $\widehat{f}(J)$) implies that ϕ is a product of 2 commutators $[g_i, h_i]$, $i = 1, 2$.

In addition, by Lemma 4.2, the interval J and the maps $\widehat{g_i}$ and $\widehat{h_i}$ satisfy the hypotheses of Proposition 3.3, hence their commutator is a product of two $\widehat{f}$-commutators and applying the quotient morphism, each $[g_i, h_i]$ is a product of two f-commutators. Then ϕ is a product of 4 f-commutators. As any f-commutator is a product of 2 conjugates of f or f^{-1}, we finally get that ϕ is a product of 8 conjugates of f or f^{-1}.

If f is an involution, the NCI property implies that there exists $h \in G$ such that $F = f \circ (hfh^{-1})$ is not an involution. Applying the previous case to F we get that ϕ is the product of 8 conjugates of F or F^{-1} that is a product of 16 conjugates of $f = f^{-1}$.

4.2. Proof of Theorem 4.

Let G be a subgroup of $\mathcal{PC}(I)$. We begin by proving

Lemma 4.3. *If BG_0 is perfect and 0-proximal then for every $a \in I$ and $g \in G$ such that $a = \widehat{g}(0)$ and $\widehat{g}$ is continuous at 0, it holds that $BG_a = gBG_0g^{-1}$ is perfect and a-proximal.*

The continuity of $\widehat{g}$ at 0 implies that $BG_a = gBG_0g^{-1}$ and then, from the fact that perfect subgroups are taken to perfect subgroups under conjugation, BG_a is perfect.

Let J be a subinterval of I, J' be a subinterval of $\widehat{g}^{-1}(J)$ and K_a be an a-proper interval. Therefore, $\widehat{g}^{-1}(K_a) \subset I \setminus V_\eta(0)$ for some positive η. We conclude from the 0-proximality of BG_0 that there exists k_0 such that $\widehat{k_0}(I \setminus V_\eta(0)) \subset J'$, hence that $\widehat{k_0} \circ \widehat{g}^{-1}(K_a) \subset J'$ and finally that $\widehat{g} \circ \widehat{k_0} \circ \widehat{g}^{-1}(K_a) \subset \widehat{g}(J') \subset J$ with $g \circ k_0 \circ g^{-1} \in BG_a$.

We turn now to the proof of Theorem 4. We recall

Theorem 4. *Let $a \in I$ and G be an a-LBS subgroup of $\mathcal{PC}(I)$ such that*

(1) the regular G-orbit of a is infinite and
(2) the subgroup BG_a is perfect and a-proximal.

- *If G does not contain any involution then G is 12-uniformly simple.*
- *If G has the NCI property then G is 24-uniformly simple.*

W.l.o.g. we can suppose that G satisfies the hypothesis of Theorem 4 with $a = 0$ and let $f, \phi \in G \setminus \{\mathrm{Id}\}$.

By Hypothesis (1), the regular G-orbit of 0 is infinite, therefore it contains some point $a \notin \{0, \widehat{\phi}^{-1}(0)\} \cup \mathrm{Disc}(\widehat{\phi})$. Since G is 0-LBS, Lemma 2.3 implies that there exist $\phi_0 \in BG_0$ and $\phi_a \in BG_a$ such that $\phi = \phi_0\phi_a$.

We claim that $\phi = g_0 b_a$ with $g_0 \in BG_0$ and b_a a commutator in BG_a.

Indeed, by the definition of BG_a, there exist K_a an a-proper interval and $\delta > 0$ such that $\mathrm{Supp}(\widehat{\phi_a}) \subset K_a \subset I \setminus V_\delta(a)$.

According to Lemma 4.3, the group BG_a is a-proximal. Then given any $\frac{1}{2} > \eta > 0$, there exists $k_a \in BG_a$ such that $\widehat{k_a}(K_a) \subset I \setminus V_\eta(0)$. Therefore $\mathrm{Supp}(C_{\widehat{k_a}}(\widehat{\phi_a})) = \widehat{k_a}(\mathrm{Supp}(\widehat{\phi_a})) \subset I \setminus V_\eta(0)$ and then $C_{k_a}(\phi_a) \in BG_0$. Finally

$$\phi = \phi_0\phi_a = \phi_0 C_{k_a}(\phi_a) C_{k_a}(\phi_a^{-1})\phi_a = g_0 b_a \text{ with } \begin{cases} g_0 = \phi_0 C_{k_a}(\phi_a) \in BG_0 \text{ and} \\ b_a = C_{k_a}(\phi_a^{-1})\phi_a \text{ a commutator in } BG_a. \end{cases}$$

As BG_0 is 0-proximal it is non-trivial. As BG_0 is perfect as well it follows from the second point of Remark 1.14 that it contains some f' that is not an involution.

In addition, as BG_0 is 0-proximal then Lemma 4.1 and Proposition 2.2 (changing J for $\widehat{f'}(J)$) imply that g_0 is a product of 2 commutators in BG_0. Therefore ϕ is a product of 2 commutators in BG_0 and one commutator in BG_a for some $a \in I$.

If f is not an involution, applying Lemma 4.2 to BG_0 and BG_a and Proposition 3.3 to G, we obtain that each commutator is a product of two f-commutators. As every f-commutator is a product of 2 conjugates of f or f^{-1}, we finally get that ϕ is a product of 12 conjugates of f or f^{-1}.

If f is an involution, by the NCI-property, there exists $h \in G$ such that $F = f \circ (hfh^{-1})$ is not an involution. Applying the previous case to F, we get that ϕ is the product of 12 conjugates of F or F^{-1} that is a product of 24 conjugates of f. □

5. Proof of Corollaries.

In this section, we check that many groups satisfy the hypothesis of Theorem 4.

According to Arnoux and Lacourte $\mathcal{PC}^+(I)$, $\mathcal{PC}(I)$ and $\mathcal{A}(I)$ are simple (see [Arn81b] III Proposition 1.7 and [Lac22] Theorem 1.4). In Section 6 we will prove that $\mathcal{A}^+(I)$ is simple.

Epstein ([Eps70]) established that $\mathrm{PL}^+(\mathbb{S}^1)$ is simple and Thompson showed that T_2 is simple (see e.g. [CFP96]) and in Section 7 we will prove that the Stein-Thompson groups $T_{(n_1,n_2,\ldots,n_p)}$ with $n_2 = n_1^{2k} - 1$ are simple.

All the groups previously mentioned are infinite, hence they are NCI, by Remark 1.14. It is easy to check that they also are 0-LBS, the regular G-orbit of 0 is infinite and have an associated BG_0 which is 0-proximal.

It remains to prove that the corresponding BG_0 are perfect.
If $G = \mathrm{PL}^+(\mathbb{S}^1)$ then $B\mathrm{PL}^+(\mathbb{S}^1)_0 = [\mathrm{PL}^+(I), \mathrm{PL}^+(I)]$ is perfect, by Epstein ([Eps70]).
If $G = T_2$ then by Theorem 4.1 of [CFP96] and related comments, $(BT_2)_0 = [F_2, F_2]$ is perfect. If $G = T_{(n_1,n_2,\ldots,n_p)}$ with $n_2 = n_1^{2k} - 1$, this is provided by Lemma 7.2.

Finally, let $G \in \{\mathcal{PC}(I), \mathcal{PC}^+(I), \mathcal{A}(I), \mathcal{A}^+(I)\}$ and f_0 in BG_0. There exist c, d with $0 < c < d < 1$ such that $\mathrm{Supp}(f_0) \subset [c, d)$. The group $G([c, d))$ of elements of G with support in $[c, d)$ is isomorphic to G which is a simple group. In particular, $G([c, d))$ is perfect and we

get that f_0 is a product of commutators of elements in G having support in $[c,d)$, hence of elements in BG_0.

6. $\mathcal{A}^+(I)$ IS SIMPLE

6.1. **Preliminaries.** The aim of this section is to fix notation and terminology, to collect a few results and to prove some basic results to be used for establishing the simplicity of $\mathcal{A}^+(I)$. In particular, we describe the conjugacy classes of involutions in $\mathcal{A}^+(I)$.

Definition 1.9 can be extended to every half open real interval J (see the appendix by P. Arnoux) and the corresponding groups are denoted by $\mathrm{PL}^+(J) < \mathrm{PL}^+(\mathbb{S}_J) < \mathcal{A}^+(J)$, where $\mathbb{S}_J$ is the circle obtained by identifying the endpoints of J and $\mathrm{PL}^+(J)$ is identified with the stabilizer of the left endpoint of J in $\mathrm{PL}^+(\mathbb{S}_J)$. It is plain that $\mathrm{PL}^+(\mathbb{S}_J)$ is isomorphic to $\mathrm{PL}^+(\mathbb{S}^1)$.

Definition 6.1.

- An IET that has at most one interior discontinuity point is called a **rotation** and it is denoted by R_a, where a is the image of 0.
- An IET g whose support is a half-open interval $J = [a,b) \subset [0,1)$ is a **restricted rotation** if the orientation preserving affine map that sends J to $[0,1)$ conjugates $g_{|J}$ to a rotation. We denote it by $R_{\alpha,J}$ where α is defined by $R_{\alpha,J}(x) = x+\alpha \pmod{|b-a|}$ for $x \in J$.

Lemma 6.2.

Every non-trivial involution $i \in \mathcal{A}^+(I)$ is conjugated in $\mathcal{A}^+(I)$ to either $R_{\frac{1}{2}}$ or to $RR_{\frac{1}{2}}$ the order 2 restricted rotation of support $[\frac{1}{2},1)$ that exchanges $[\frac{1}{2},\frac{3}{4})$ and $[\frac{3}{4},1)$.

Proof. As i is a non-trivial involution, the interval I can be decomposed into a finite union of pairwise disjoint half-open intervals: $I_1,\cdots,I_p$ and $J_1,\cdots,J_q$ satisfying the following:

(1) The map i is continuous on these intervals.
(2) The integers p and q are such that $p = 2k \geq 2$, $q \geq 0$ and in the case that $q = 0$ there is no J_j.
(3) $J_j \subset \mathrm{Fix}(i)$ and if $j \leq k$ then $i(I_j) = I_{j+k}$.

Let H be the AIET defined by:

- Whenever $q \neq 0$, the map H sends affinely J_j to $[\frac{j-1}{2q},\frac{j}{2q})$ for $j = 1,\cdots,q$.
- H sends affinely I_j to $\begin{cases} [\frac{j-1}{p},\frac{j}{p}) & \text{for } j=1,\cdots,p \text{ if } q=0, \\ [\frac{1}{2}+\frac{j-1}{2p},\frac{1}{2}+\frac{j}{2p}) & \text{for } j=1,\cdots,p \text{ if } q\neq 0. \end{cases}$

We can check that H conjugates i to a map with support $[0,1)$ if $q=0$ or $[\frac{1}{2},1)$ if $q\neq 0$ which also is an IET (this can be verified by computing the slope of $H\circ i\circ H^{-1}$ on each $H(I_j)$). Moreover by definition, $H\circ i\circ H^{-1}$ sends any two cyclic-consecutive intervals among the $H(I_j)$, $j=1,\cdots,k$ to cyclic-consecutive ones so it is continuous except at $\frac{1}{2}$ if $q=0$ and at $\frac{1}{2}$ and $\frac{3}{4}$ if $q\neq 0$.

In conclusion, $H\circ i\circ H^{-1} = R_{\frac{1}{2}}$ if $q=0$ or $H = RR_{\frac{1}{2}}$ if $q\neq 0$. □

6.2. **The group $\mathcal{A}^+(I)$ is perfect and generated by its involutions.**

We first exhibit generators of $\mathcal{A}^+(I)$.

Proposition 6.3.

Every $f \in \mathcal{A}^+(I)$ can be written as $f = g\circ h$ with $h \in \mathrm{PL}^+(I)$ and g an IET.

Proof. Let $f \in \mathcal{A}^+(I)$, we denote by $I_1, \cdots, I_p$ the maximal continuity intervals of f and we denote by $J_{\pi(i)}$ the interval $f(I_i)$. We consider the IET E defined by the partition $\{J_i\}$ and the permutation π^{-1} that tells us how the J_i are rearranged. By construction, the AIET $h = E \circ f$ is continuous on I and $f = E^{-1} \circ h$ has the required form. □

According to [Arn81b], [Nov09] or [Vor17] (see the appendix for a proof), any interval exchange transformation g is a product of restricted rotations. Therefore, we will see below that Proposition 6.3 insures that every $f \in \mathcal{A}^+(I)$ is a product of commutators [resp. involutions] if this property holds for any $h \in \mathrm{PL}^+(\mathbb{S}^1)$.

Indeed, on one side, adding a discontinuity, $\mathrm{PL}^+(\mathbb{S}^1)$ can be seen as a subgroup of $\mathcal{A}^+(I)$ and $\mathrm{PL}^+(I)$ is a subgroup of $\mathrm{PL}^+(\mathbb{S}^1)$. Hence a map $h \in \mathrm{PL}^+(I)$ that is a product of commutators [resp. involutions] in $\mathrm{PL}^+(\mathbb{S}^1)$ is a product of commutators [resp. involutions] in $\mathcal{A}^+(I)$.

On the other side, the map $f \mapsto f|_J$ sends the restricted rotations of support J into $\mathrm{PL}^+(\mathbb{S}_J)$ and it is an isomorphism onto its image, the subgroup of $\mathrm{PL}^+(\mathbb{S}_J)$ consisting of its rotations. In addition, if any $h \in \mathrm{PL}^+(\mathbb{S}^1)$ is a product of commutators [resp. involutions] in $\mathrm{PL}^+(\mathbb{S}^1)$, then any $h \in \mathrm{PL}^+(\mathbb{S}_J)$ is a product of commutators in $\mathrm{PL}^+(\mathbb{S}_J)$ and then in $\mathcal{A}^+(J)$. Finally, extending maps by Id on the complement of J, we get that writing a restricted rotation of support J as product of commutators [resp. involutions] reduces to doing that for a rotation in $\mathrm{PL}^+(\mathbb{S}^1)$.

As Theorem 3.2 of [Eps70] states that $\mathrm{PL}^+(\mathbb{S}^1)$ is simple, $\mathrm{PL}^+(\mathbb{S}^1)$ is generated by either its commutators or its involutions, so

$$\mathcal{A}^+(I) = \langle \text{commutators} \rangle = \langle \text{involutions} \rangle.$$

6.3. The group $\mathcal{A}^+(I)$ is simple.

Let N be a non-trivial normal subgroup of $\mathcal{A}^+(I)$. The problem reduces to proving that N contains a non-trivial involution τ_1 having fix points and a fix point free involution τ_2 since $\mathcal{A}^+(I) = \langle \text{involutions} \rangle$ will be the normal closure of $\langle \tau_1, \tau_2 \rangle$, by Lemma 6.2.

Let f be a non-trivial element of N, then there exists a non empty half-open interval J such that $f(J) \cap J = \emptyset$ and J and $f(J)$ have length less than $\frac{1}{2}$.

Let $i \in \mathcal{A}^+(I)$ be an involution with support $\mathrm{Supp}(i) = J$. Therefore $\mathrm{Supp}(f \circ i \circ f^{-1}) = f(\mathrm{Supp}(i)) = f(J)$ is disjoint from $\mathrm{Supp}(i)$. Consequently $f \circ i \circ f^{-1}$ and i commute, hence $\tau_1 = [f, i] = f \circ i \circ f^{-1} \circ i$ is an involution of support $J \cup f(J)$ and it belongs to N. Then we have proved that N contains a non-trivial involution τ_1 having fixed points.

For constructing a fix point free involution in N, we consider h_1, h_2 in $\mathcal{A}^+(I)$ such that

$$(\star) \quad h_1(J) = [0, \frac{1}{4}), \quad h_1(f(J)) = [\frac{1}{2}, \frac{3}{4}), \quad h_2(J) = [\frac{1}{4}, \frac{1}{2}) \text{ and } \quad h_2(f(J)) = [\frac{3}{4}, 1).$$

The map $i_1 = h_1 \circ \tau_1 \circ h_1^{-1}$ [resp. $i_2 = h_2 \circ \tau_1 \circ h_2^{-1}$] is an involution, it belongs to N and its support is $h_1(J \cup f(J)) = [0, \frac{1}{4}) \cup [\frac{1}{2}, \frac{3}{4})$ [resp. $h_2(J \cup f(J)) = [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1)$].
Therefore i_1 and i_2 have disjoint supports and $\tau_2 = i_1 \circ i_2 \in N$ is an involution of full support. Then we have also proved that N contains a fix point free involution τ_2.

7. Simplicity of certain Stein-Thompson groups

In this section, we prove Theorem 2, using results of Stein ([Ste92]) and Bieri-Strebel's Lemma C12.8 and Theorem C12.14 of [BS16] that, in our context, can be stated as

Theorem C12.14 of [BS16].
The group $(BT_{\{n_1,n_2,\dots,n_p\}})_0 = \{f \in T_{\{n_1,n_2,\dots,n_p\}} : f(0) = 0, f(1) = 1,\ Df(0) = Df(1) = 1\}$ *is perfect provided that the following properties* (i) *and* (ii) *hold.*

- ***(i)*** $(1-\Lambda)A = A$, *where* $(1-\Lambda)A = \{\ \sum (1-\lambda_i)a_i\ ,\ \lambda_i \in \Lambda = \langle n_i\rangle,\ a_i \in A = \mathbb{Z}[\Lambda]\ \}$,
- ***(ii)*** Λ *contains a rational number* $\frac{n}{q} > 1$ *so that* $n^2 - q^2 \in \Lambda$.

Lemma 7.1. *Let* $\Lambda = \langle n_1, \cdots, n_p\rangle$, $A = \mathbb{Z}[\Lambda]$ *and* $d = \gcd(n_i - 1)$. *Then* $(1-\Lambda)A = dA$.

Proof.
First, we prove the inclusion $(1-\Lambda)A \subset dA$.

By the definition of $(1-\Lambda)A$, it suffices to show that $(1-\lambda)a \in dA$, for any $a \in A$ and $\lambda = n_1^{s_1}...n_p^{s_p} \in \Lambda$. By converting the fractions to have the same denominator, there exist $q_i, t_i \in \mathbb{N}$ and $a' \in A$ such that

$$(1-\lambda)a = (1 - n_1^{s_1}...n_p^{s_p})a = (n_1^{q_1}...n_p^{q_p} - n_1^{t_1}...n_p^{t_p})a'.$$

By replacing the n_i's by $k_i d + 1$ and calculating, we obtain $(1-\lambda)a = (dN)a'$ with $N \in \mathbb{N}$.

Next, we show that $dA \subset (1-\Lambda)A$.

From Bezout's identity, we obtain $d = u_1(n_1 - 1) + \cdots + u_p(n_p - 1)$ with $u_i \in \mathbb{Z}$.

Thus, for any $a \in A$, we have $da = \sum(n_i - 1)(u_i a) \in (1-\Lambda)A$. □

Lemma 7.2. $(BT_{\{n_1,n_2\cdots,n_p\}})_0$ *is perfect provided that* $n_2 = n_1^{2k} - 1$.

Proof. We check that $(BT_{\{n_1,n_2,\dots,n_p\}})_0$ satisfies the properties (i) and (ii) of Bieri and Strebel's Theorem. Indeed, by Lemma 7.1, the property (i) is equivalent to that $d = gcd(n_i - 1) = 1$.

As n_1 is congruent to 1 modulo $n_1 - 1$, we know that n_1^{2k} is congruent to 1 modulo $n_1 - 1$ so $n_1^{2k} - 2$ is congruent to -1 modulo $n_1 - 1$ and then $gcd(n_1 - 1, n_1^{2k} - 2) = 1$. This implies that Property (i) of Lemma 7.1 is satisfied.

Moreover, considering $\dfrac{n}{q} = \dfrac{n_1^k}{1}$ yields $n^2 - q^2 = n_1^{2k} - 1 \in \Lambda$, so (ii) holds for $(BT_{\{n_1,n_2,\dots,n_p\}})_0$.
□

Lemma 7.3. *If* $(BT_{\{n_1,n_2,\dots,n_p\}})_0$ *is perfect then* $T_{\{n_1,n_2,\dots,n_p\}}$ *is perfect.*

Proof. Let $f \in T_{\{n_1,n_2,\dots,n_p\}}$. As $T_{\{n_1,n_2,\dots,n_p\}}$ is a 0-LBS group, Lemma 2.3 with $a \in A \setminus \{0, f^{-1}(0)\}$ implies that $f = f_0 f_a$ with $f_0 \in (BT_{\{n_1,n_2,\dots,n_p\}})_0$ and $f_a \in (BT_{\{n_1,n_2,\dots,n_p\}})_a$.

It is a simple matter to prove that $(BT_{\{n_1,n_2,\dots,n_p\}})_a = R_a(BT_{\{n_1,n_2,\dots,n_p\}})_0 R_a^{-1}$ (see the proof of Lemma 4.3) and Lemma 7.2 now implies that both $(BT_{\{n_1,n_2,\dots,n_p\}})_0$ and $(BT_{\{n_1,n_2,\dots,n_p\}})_a$ are perfect. We conclude that $f = f_0 f_a$ is a product of commutators in $T_{\{n_1,n_2,\dots,n_p\}}$ and finally that $T_{\{n_1,n_2,\dots,n_p\}}$ is perfect. □

We turn now on to the proof of the simplicity of $T_{\{n_1,n_2,\dots,n_p\}}$. According to [Ste92], the group $T''_{\{n_1,n_2,\dots,n_p\}}$ is simple and $T''_{\{n_1,n_2,\dots,n_p\}} = T_{\{n_1,n_2,\dots,n_p\}}$ by the previous lemma, so we have that $T_{\{n_1,n_2,\dots,n_p\}}$ is simple.

Appendix A. Simplicity of groups of interval exchange transformations

by Pierre Arnoux

In this appendix, we prove the simplicity of some groups of piecewise continuous maps. Recall the definitions:

Definition 1. An interval exchange transformation on an interval $J = [a, b)$ is a bijection of J which is everywhere right continuous, and, except on a finite number of points, continuous and derivable with derivative 1; alternatively, it can be defined as a permutation by translations on a finite collection of semi-open subintervals of J.

More generally, an affine (resp. generalised) interval exchange transformation is a bijection defined by a finite partition of half open intervals, such that the restriction of the map to each interval is an orientation preserving affine map (resp. an orientation preserving homeomorphism).

An interval exchange transformation with flips is a bijection on J, except maybe for a finite set, which is derivable except for this finite set, with derivative $+1$ or -1. As noted in the introduction, it is defined up to a finite set.

From now on, we fix an interval J. As before, we denote by $\mathcal{G}^+(J)$ the group of interval exchange transformations on the interval J, by $\mathcal{A}^+(J)$ [resp. $\mathcal{PC}^+(J)$] the group of affine [resp. generalised] interval exchanges transformations and by $\mathcal{G}(J)$ the group of classes of interval exchange transformations with flips.

The simplicity of $[\mathcal{G}^+(J), \mathcal{G}^+(J)]$, $\mathcal{PC}^+(J)$ and $\mathcal{G}(J)$ was obtained in [Arn81b], but are not easily available.

The simplicity of $\mathcal{A}^+(I)$ in not established in [Arn81b], however its tools provide a different proof and this is detailed here.

Namely, in this appendix, we prove the following:

Proposition 2. The groups $\mathcal{A}^+(J), \mathcal{PC}^+(J)$ and $\mathcal{G}(J)$ are simple. The group $\mathcal{G}^+(J)$ is not simple, but its commutator subgroup is simple.

The proof of the proposition consists, using a lemma due to Epstein, in proving first that the commutator subgroup of all these groups is the smallest normal subgroup, and then, for the first three, in proving that they are perfect.

A.1. **A condition implying that every normal subgroup contains the commutator subgroup.** Recall that two transformations with disjoint support commute.

Remark that, if H is a normal subgroup of a group G, and $h \in H$, then for all $a \in G$, $[a, h] = aha^{-1}h^{-1}$ is in H, as product of two elements of H: aha^{-1} which is a conjugate of h, hence in H by normality, and the inverse of h. Remark also that, if a commutes with c, then $[a, bc] = abca^{-1}c^{-1}b^{-1} = [a, b]$. We will use these properties to prove the following lemma, due to Epstein [Eps68]

Lemma 3. Let G be a group of transformations of a manifold endowed with a measure μ. Suppose that G satisfies the two conditions:

(1) For all $\epsilon > 0$, any element of G is the product of a finite number of elements whose support has measure less than ϵ.
(2) For all $h \in G \setminus \{\mathrm{Id}\}$, there exist $E \subset \mathrm{Supp}(h)$ such that $h(E) \cap E = \emptyset$, and $\epsilon > 0$ such that, if g_1 and g_2 are two elements of G whose support has measure less than ϵ, we can find $f \in G$ such that $f(\mathrm{Supp}(g_i)) \subset E$ for $i = 1, 2$.

Then $[G, G]$ is the smallest normal subgroup of G.

Proof. Let H be a normal subgroup of G; we want to prove that any commutator belongs to H. Let h be a non-trivial element of H, and let E and ϵ be as in Condition (2).
We claim that Condition (1) implies that for all $\epsilon > 0$, any element of $[G, G]$ is the product of a finite number of conjugates of commutators of elements with support of measure less than ϵ.

Indeed, let $\varepsilon > 0$ and $g, k \in G$.
We first prove that $[g, k]$ is a product of conjugates of $[a, k]$ with $\mu(\mathrm{Supp}(a)) \leq \varepsilon$.

By the condition (1), g can be written as $g = g_1 \cdots g_p$ where $\mu(\mathrm{Supp}(g_i)) \leq \varepsilon$.
We argue by induction on p, supposing that for any $f \in G$ that is a product of at most $p-1$ elements whose support has measure less than ϵ, the commutator $[f, k]$ is a product of conjugates of $[a, k]$ with $\mu(\mathrm{Supp}(a)) \leq \varepsilon$.

A straightforward calculus leads to $[g_1 \cdots g_p, k] = g_1[g_2 \cdots g_p, k]g_1^{-1}[g_1, k]$ and by induction hypothesis, it holds that $[g_2 \cdots g_p, k] = \prod h_i[a_i, k]h_i^{-1}$.

It remains to prove that any $[a, k]$ decomposes in commutators of elements with support of measure less than ϵ. This follows from the fact that $[a, g] = [g, a]^{-1}$ and the previous argument shows that $[g, a]$ has the required decomposition.

By condition (1) and the previous claim, it is enough to prove that the commutator of two elements g_1, g_2 with support of measure less than ϵ belongs to H.

Let f be as in condition (2), and $h' = f^{-1}hf$. We have $h' \in H$ by normality. If S_i, for $i = 1, 2$, is the support of g_i, one checks that $h'(S_i) \subset f^{-1}(h(E))$ is disjoint from $S_1 \cup S_2 \subset f^{-1}(E)$. This implies that g_1 and the conjugate $h'g_2^{-1}h'^{-1}$ of the inverse of g_2 have disjoint support, hence commute. This fact, and the remarks above, imply that

$$[g_1, g_2] = [g_1, g_2h'g_2^{-1}h'^{-1}] = [g_1, [g_2, h']] \in H$$

We have proved that the commutator of any element with small support belongs to H; since any element of $[G, G]$ is the product of a finite number of conjugates of commutators with small support and H is normal we have that $[G, G] \subset H$. □

A.2. **Interval exchange transformations are product of transformations with small support.** We will prove that, in all the considered groups of interval exchange transformations, any element can be written as a product of a finite number of elements with a support of arbitrarily small size.

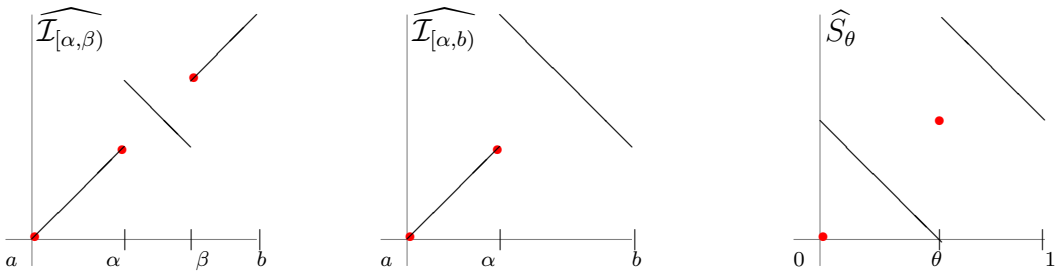
Now, we list some definitions and properties that are easily available in [GL22] and we add proofs for sake of completeness.

Definition 4. Let $\alpha, \beta \in \overline{J} = [a, b]$ and $0 \leq \theta < \beta - \alpha$.

The **symmetry** of $[\alpha, \beta)$, denoted by $\mathcal{I}_{[\alpha,\beta)}$, is the element of $\mathcal{G}(J)$ represented by the FIET $i = \widehat{\mathcal{I}_{[\alpha,\beta)}}$ given by $i(x) = x$ if $x \notin (\alpha, \beta)$ and $i(x) = \alpha + \beta - x$ if $x \in (\alpha, \beta)$.

A **distinguished involution** is a product of finitely many symmetries having disjoint supports.

Remark 5. Let $\theta \in [0, 1)$, set $R_\theta = R_{\theta,[0,1)}$ and $S_\theta = \mathcal{I}_{[0,\theta)} \circ \mathcal{I}_{(\theta,1)}$, it is easy to check that $S_\theta \circ S_{\theta'} = R_{\theta-\theta'}$ and $R_\alpha \circ S_\theta \circ R_\alpha^{-1} = S_{\theta+2\alpha}$.



Lemma 6. Every element of $\mathcal{G}^+(J)$ is the product of a finite number of restricted rotations. Every element of $\mathcal{G}(J)$ is the product of a distinguished involution and an element of $\mathcal{G}^+(J)$.

Proof. For clarity, given $K = [c, d)$ and $L = [d, e)$ two consecutive half-open intervals, we denote by $R_{K,L}$ the restricted rotation of support $K \sqcup L$ whose interior discontinuity point is d. Let $g \in \mathcal{G}^+(J)$ with continuity intervals $I_1, \cdots I_m$ and let $g(I_i) = J_{\pi(i)}$. We consider $R_1 = R_{K,L}$, where $K = J_1 \cup \cdots \cup J_{\pi(1)-1}$ and $L = J_{\pi(1)}$. One directly has that $R_1 \circ g|_{I_1} = \mathrm{Id}$ and $g_1 := R_1 \circ g|_{I_2 \cup \cdots \cup I_m}$ has at most $m-1$ continuity intervals.

Starting with g_1, we define similarly R_2 and we get that $R_2 \circ g_1|_{I_2} = \mathrm{Id}$ and $g_2 := R_2 \circ g_1|_{I_3 \cup \cdots \cup I_m}$ has at most $m-2$ continuity intervals.

Repeating the previous argument $m-1$ times leads to a g_{m-1} having at most 1 continuity interval, so $g_{m-1} = \mathrm{Id}$.

Extending the restricted rotations R_i to J by the identity map, we conclude that $R_{m-1} \circ \cdots \circ R_1 \circ g = \mathrm{Id}$ and then g is a product of a finitely many restricted rotations.

Let $f \in \mathcal{G}(J)$, we denote by $I_1, \cdots I_m$ the continuity intervals of f and by

$$F = \{i \in \{1, ...m\} : f \text{ is orientation reversing on } I_i\}.$$

It is easy to check that $f \circ \prod_{i \in F} \mathcal{I}_{I_i}$ belongs to $\mathcal{G}^+(J)$ and that the $\mathcal{I}_{I_i}$'s have disjoint supports, so $\prod_{i \in F} \mathcal{I}_{I_i}$ is a distinguished involution and the second item of Lemma 6 directly follows. □

Lemma 7. (Proposition 6.3) Every element of $\mathcal{A}^+(J)$ (resp. $\mathcal{PC}^+(J)$) is the product of an element of $\mathcal{G}^+(J)$ and an orientation preserving PL-homeomorphism of J (resp. homeomorphism of J)

Proposition 6.3 is only stated for $\mathcal{A}^+(J)$, but the proof is exactly the same for $\mathcal{PC}^+(J)$.

Lemma 8. For any $\epsilon > 0$, any restricted rotation can be written in $\mathcal{G}^+(J)$ as the product of elements of $\mathcal{G}^+(J)$ with support of measure less than ϵ.

Proof. It suffices to prove it for a rotation on $[0, 1)$. Let R_α be a rotation on $[0, 1)$. We can construct an element $f \in \mathcal{G}^+(J)$ with support on $[0, \frac{1}{4}) \cup R_\alpha[0, \frac{1}{4}]$, which coincides with R_α on $[0, \frac{1}{4})$. The measure of $\mathrm{Supp}(f)$ is less than half the measure of $\mathrm{Supp}(R_\alpha)$. Let $g = f^{-1}R_\alpha$; it is by construction the identity on $[0, \frac{1}{4})$, hence the measure of its support is at most $\frac{3}{4}$ that of the support of R_α. Hence we have written $R_\alpha = fg$, where f and g have a support whose measure is at most $\frac{3}{4}$ that of the support of R_α. Since they are elements of $\mathcal{G}^+(J)$, we can again decompose them in restricted rotations, which can be similarly decomposed. By iteration, we can write a rotation as a finite product of elements with arbitrarily small support. □

Lemma 9. For any $\epsilon > 0$, any distinguished involution can be written as the product of elements of $\mathcal{G}(J)$ with support of measure less than ϵ.

Proof. It is enough to prove it for the involution $\mathcal{I} : x \mapsto 1 - x$ on $[0, 1)$. Let n be such that $\frac{1}{n} < \epsilon$, and let f_i be such that $f_i(x) = 1 - x$ if $x \in [\frac{i}{2n}, \frac{i+1}{2n}) \cup (1 - \frac{i+1}{2n}, 1 - \frac{i}{2n}]$, and $f_i(x) = x$ otherwise. It is clear that all the f_i have support of measure $\frac{1}{n} < \epsilon$, and by construction $\mathcal{I} = f_0 f_1 \dots f_{n-1}$. □

Lemma 10. For any $\epsilon > 0$, any homeomorphism (resp. PL-homeomorphism) of $[0, 1)$ can be written as the product of homeomorphisms (resp. PL-homeomorphisms) whose support are contained in intervals of measure less than ϵ.

Proof. We do the proof for a homeomorphism, it works, *mutatis mutandis*, for a PL- homeomorphism.

Let h be such a homeomorphism; without loss of generality, we can suppose that $h(\frac{1}{2}) < \frac{1}{2}$. One can then construct a homeomorphism g with support in $[0, \frac{3}{4})$ such that $g(h(\frac{1}{2})) = \frac{1}{2}$.

The homeomorphism gh fixes the point $\frac{1}{2}$; hence it can be naturally decomposed in a product $f_1 f_2$, where $\mathrm{Supp}(f_1) \subset [0, \frac{1}{2}]$ and $\mathrm{Supp}(f_2) \subset [\frac{1}{2}, 1]$. We have written $h = g^{-1} f_1 f_2$ as the product of 3 elements whose supports have measure at most $\frac{3}{4}$ that of h. By iterating this construction, we can make the support of the maps contained in intervals as small as we want. □

If h is a transformation in any of the groups $\mathcal{G}^+(J)$, $\mathcal{G}(J)$ and $\mathcal{A}^+(J)$ which is not the identity, we can find an interval E which is disjoint from $h(E)$. Let ϵ be less than half the length of this interval. Since the support of an element of $\mathcal{G}^+(J)$, $\mathcal{G}(J)$ and $\mathcal{A}^+(J)$ is a finite union of intervals, if one has two elements g_1, g_2 with support of measure less than ϵ, it is clear that we can find an element of $\mathcal{G}^+(J)$ which sends the supports of g_1, g_2 into E.

All these lemmas imply the following :

Proposition 11. The groups $\mathcal{G}^+(J)$, $\mathcal{G}(J)$ and $\mathcal{A}^+(J)$ satisfy the conditions of Lemma 3.

Things are slightly more complicated for the group $\mathcal{PC}^+(J)$, since the support of a homeomorphism does not need to be a finite union of intervals. However, the reader will check that the proof of Lemma 3 is still valid if we reformulate condition (1), by asking the support to be contained in a finite union of intervals with total measure less than ϵ, and change accordingly the condition (2). This is precisely the condition proved in Lemma 10. Hence the group $\mathcal{PC}^+(J)$ also satisfies the conclusion of Lemma 3.

A.3. **Commutators in groups of interval exchange transformations.** We now want to prove that specific elements are commutators.

Lemma 12. Distinguished involutions and restricted rotations are commutators in $\mathcal{G}(J)$.

Lemma 13. Let $\alpha, \beta \in J$.

The maps $\mathcal{I}_{[\alpha,\beta)}$ and $R_{\theta,[\alpha,\beta)}$ are commutators in $\mathcal{G}\big([\alpha, \beta)\big)$ and then in $\mathcal{G}(J)$.

Proof. Conjugating by a homothety, it is sufficient to prove that $\mathcal{I}_{[0,1)}$ and $R_{\theta,[0,1)}$ are commutators in $\mathcal{G}([0, 1))$ and it is easy to see that $\mathcal{I}_{[0,1)}$ is the product of the involutions f_1 and f_2 whose best representatives are described as below:

As f_2 is conjugated to $f_1 = f_1^{-1}$ by $r = R_{\frac{1}{2}}$, one has $\mathcal{I}_{[0,1)} = f_1 r f_1^{-1} r^{-1}$ is a commutator.

In addition, according to Remark 5, any rotation is the product of 2 involutions that are conjugated by a rotation; thus $R_{\theta,[0,1)}$ is a commutator. □

Since any element of $\mathcal{G}(J)$ is a product of a distinguished involution and restricted rotations, this implies that any element of $\mathcal{G}(J)$ is a product of commutators; and since we have proved that the commutator subgroup is the smallest normal subgroup of $\mathcal{G}(J)$, we have proved

Proposition 14. The group $\mathcal{G}(J)$ is simple.

Let us now consider the group $\mathcal{A}^+(J)$. Conjugating by a homothety, it is sufficient to consider the group $\mathcal{A}^+([0, 1))$.

Lemma 15. Every element of $[\mathrm{PL}^+([0,1]), \mathrm{PL}^+([0,1])]$ is a product of commutators.

Proof. This result is proved in [Eps70]; for completeness, we give the main point of the argument. Any element of $[\mathrm{PL}^+([0,1]), \mathrm{PL}^+([0,1])]$ can be written as a product of maps which are the identity out of an interval $[a,b]$, and which are affine on 2 intervals $[a,c]$ and $[c,b]$. It suffices to write such a map as a commutator.

Denote by σ_a the piecewise affine homeomorphism of $[0,1]$ which fixes 0 and 1, sends $\frac{1}{2}$ to a, and is affine on $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. If we choose a,b with $0<a<b<\frac{1}{2}$, it is easily checked that the commutator $\sigma_a^{-1}\sigma_b^{-1}\sigma_a\sigma_b$ is a PL map which is the identity out of the interval $\frac{1}{2}, \frac{3-4a}{4-4a}$, and which is linear on 2 intervals. A simple study shows that, up to conjugacy, one obtains in this way any piecewise affine map on 2 intervals. □

Lemma 16. Any rotation is a product of commutator in $\mathcal{A}^+([0,1))$

Proof. The group $\mathrm{PL}^+(\mathbb{S}^1)$ of piecewise affine homeomorphisms of the circle can be embedded in $\mathcal{A}^+([0,1))$ as piecewise affine transformations of the interval $[0,1)$. But (see [Eps70]) this group is simple; hence any rotation of $[0,1)$ can be written as a product of commutators. □

Hence any element of $\mathcal{A}^+(J)$ is a product of commutators, and we prove as above:

Proposition 17. The group $\mathcal{A}^+(J)$ is simple.

Remark 18. The proof that rotations are product of commutators is fundamentally different in $\mathcal{A}^+(J)$ and $\mathcal{G}(J)$; and indeed, the property is false in their intersection $\mathcal{G}^+(J)$.

We end with the proof of the simplicity of $\mathcal{PC}^+(J)$ reduced to that of $\mathcal{PC}^+([0,1))$.

Lemma 19. Any rotation and any orientation preserving homeomorphism of $[0,1]$ is a product of commutators in $\mathcal{PC}^+([0,1))$.

Proof. The proof given for rotations in $\mathcal{A}^+([0,1))$ is still valid in $\mathcal{PC}^+([0,1))$, since this group contains $\mathcal{A}^+([0,1))$.

We proved above that any homeomorphism of $[0,1]$ is a product of homeomorphisms whose supports are contained in small intervals; conjugating by a rotation, we can consider a homeomorphism h of $[0,1]$ whose support is included in $[\frac{1}{4},\frac{1}{2}]$.

Define ϕ on $[0,1]$ by $\phi(x)=\frac{x}{2}$ if $x<\frac{1}{2}$, $\phi(x)=\frac{3x}{2}-\frac{1}{2}$ if $x>\frac{1}{2}$. We define a sequence of functions f_i by $f_1=\phi h\phi^{-1}$, $f_{i+1}=\phi f_i\phi^{-1}$ for $i>1$. The sequence f_i converges uniformly to the identity, and they have disjoint support; hence the sequence $g_n=f_1f_2\dots f_n$ converges to a function g which is a homeomorphism of $[0,1]$ and verifies $\phi^{-1}g\phi=hg$; hence h is a commutator. □

As above, this proves that $\mathcal{PC}^+(J)$ is simple.

It is well-known that the group $\mathcal{G}^+(J)$ is not simple, and not equal to its commutator subgroup, this is provided by the following

Theorem 5. *(Arnoux-Fathi-Sah 1981* [Arn81a]*) Let λ be the Lebesgue measure on $[0,1)$,*

$$\text{the application } \mathrm{SAF}: \begin{cases} \mathcal{G}^+(J) \to \mathbb{R}\otimes_{\mathbb{Q}}\mathbb{R} \\ f \quad \mapsto \mathrm{SAF}(f)=\displaystyle\sum_{\alpha\in\mathbb{R}} \alpha\otimes\lambda((f-\mathrm{Id})^{-1}(\{\alpha\})) \end{cases}$$

is a morphism and its kernel is the commutator subgroup of $[\mathcal{G}^+(J),\mathcal{G}^+(J)]$.

But we have:

Proposition 20. The group $[\mathcal{G}^+(J),\mathcal{G}^+(J)]$ is simple.

Proof.

The group $\mathcal{G}^+(J)$ satisfies the conditions of Lemma 3, which implies that its group of commutator is the smallest normal subgroup. Since the commutator subgroup of $[\mathcal{G}^+(J), \mathcal{G}^+(J)]$ is normal in $\mathcal{G}^+(J)$, the group $[\mathcal{G}^+(J), \mathcal{G}^+(J)]$ is perfect.

It remains to prove that the commutator subgroup satisfies also the conditions of Lemma 3. Let $f \in [\mathcal{G}^+(J), \mathcal{G}^+(J)]$. The SAF-invariant of an involution, being equal to its own opposite, is zero so any involution is a product of commutators, and eventually composing with an involution which exchanges a small interval and its image by f, we can assume that $\mathrm{Supp}(f) \neq [0, 1)$. By section A.2, f can be decomposed as $f = g_1 \dots g_n$, a product of interval exchange transformations with small support included in $\mathrm{Supp}(f)$. There is no reason for g_i to be a product of commutators; but in that case, its invariant $\mathrm{SAF}(g_i)$ is not 0, and we can find maps h_i with small support disjoint from the supports of the g_i such that $\mathrm{SAF}(h_i) = \mathrm{SAF}(g_i)$; hence h_i commute with all the g_k, and we can write:

$$f = g_1 h_1^{-1} \dots g_n h_n^{-1} h_n \dots h_1.$$

Since f is a product of commutators, $\mathrm{SAF}(f) = 0$. By construction, $\mathrm{SAF}(g_i h_i^{-1}) = 0$, hence it is a product of commutator; if we define $k = h_n \dots h_1$, we have, taking the invariant of both sides, $\mathrm{SAF}(k) = 0$, hence k is a product of commutator, and $f = (g_1 h_1^{-1}) \dots (g_n h_n^{-1}) k$ is a decomposition in product of commutators with arbitrarily small size.

This proves the first condition; to prove the second condition we can find involutions sending a finite union of intervals inside an interval of larger measure. □

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