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► **To cite this version:**

Marc Bonnet, Stéphanie Chaillat, Alice Nassor. Solvability results for the transient acoustic scattering by an elastic obstacle. *Journal of Mathematical Analysis and Applications*, 2024, 536 (128198), 10.1016/j.jmaa.2024.128198 . hal-04128900

**HAL Id: hal-04128900**

**<https://hal.science/hal-04128900>**

Submitted on 14 Jun 2023

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# SOLVABILITY RESULTS FOR THE TRANSIENT ACOUSTIC SCATTERING BY AN ELASTIC OBSTACLE

MARC BONNET, STÉPHANIE CHAILLAT, ALICE NASSOR

ABSTRACT. The well-posedness of the linear evolution problem governing the transient scattering of acoustic waves by an elastic obstacle is investigated. After using linear superposition in the acoustic domain, the analysis focuses on an equivalent causal transmission problem. The proposed analysis provides existence and uniqueness results, as well as continuous data-to-solution maps. Solvability results are established for three cases, which differ by the assumed regularity in space on the transmission data on the acoustic-elastic interface  $\Gamma$ . The first two results consider data with “standard”  $H^{-1/2}(\Gamma)$  and improved  $H^{1/2}(\Gamma)$  regularity in space, respectively, and are established using the Hille-Yosida theorem and energy identities. The third result assumes data with  $L^2(\Gamma)$  regularity in space and follows by Sobolev interpolation. Obtaining the latter result was motivated by the key role it plays (in a separate study) in the justification of an iterative numerical solution method based on domain decomposition. A numerical example is presented to emphasize the latter point.

KEYWORDS: Transient acoustic-elastic scattering, Fluid-solid interaction, well-posedness

## 1 Introduction.

Mathematical models for the scattering of waves by penetrable obstacles play a key role in many applications arising in geophysics, health sciences, aerospace and naval industry, among others. This study addresses the solvability of the transient acoustic-elastic scattering problem, where an elastic solid immersed in an unbounded acoustic fluid scatters a given incident acoustic wave. It is motivated by applications to fluid-structure interaction (FSI), specifically by our ongoing research activities in the context of naval engineering [22, 24] where accurately assessing the potential effects of remote underwater explosions on submarines rests on computationally solving the acoustic-elastic problem. Those studies have highlighted the importance of efficient numerical algorithms and that of rigorous mathematical studies to derive and justify them.

Following a previous stage [22] which focused on transient acoustic scattering by a rigid hull, our ongoing efforts towards solving acoustic-elastic scattering arising in fluid-structure interaction problems (FSIPs) involve defining coupling algorithms, where the unbounded fluid medium is modelled using boundary integral equations (BIEs) with the time variable dealt with using a convolution quadrature based approach while finite elements and classical time-stepping algorithms are used for the elastic solid. It turns out that the knowledge of data-to-solution mappings for the continuous transient FSIP, and also for initial-boundary value problems (IBVPs) involved in a domain decomposition approach, is essential in designing coupling algorithms. For example, available results on the solvability of transient Neumann IBVPs [17, 18, 27, 28] show that the Sobolev interior regularity of the solution is about  $1/2$  above that of the Neumann data, with same regularity in time for both (whereas the corresponding regularity gain is  $3/2$  for time-harmonic Neumann BVPs), which precludes their use in transient coupling schemes as solution iterates cannot remain in the same function space. In our ongoing computational work [5], we formulate a coupling scheme for the transient FSIP based on Robin-Robin iterations, inspired by a related work [10] on frequency-domain acoustic-acoustic coupling, and prove its convergence to the FSIP solution assuming the latter exists and belongs to the requisite function space.

The goal of this work is thus to provide solvability results for the continuous transient acoustic-elastic scattering problem, in such a way that those results in particular complete the convergence

proof of our coupling iterative approach given in [5]. Generally speaking, a mathematical problem (such as the FSIP model studied here) is well-posed if its solution (a) exists, (b) is unique and (c) depends continuously on the problem data. In our context (where in particular the FSIP is linear), requirement (c) typically leads to the data  $g$  and the solution  $\phi$  verifying  $\|\phi\|_{\mathcal{X}} \leq C\|g\|_{\mathcal{D}}$  for specified pairs  $\mathcal{D}, \mathcal{X}$  of data and solution spaces and some constant  $C$ , and multiple data-to-solution mappings may be established. Knowing such data-to-solution mappings is very important for proving that computational coupling algorithms are well-behaved.

The well-posedness of the time-harmonic acoustic-elastic scattering is established in [3] in a standard variational setting (where in particular, Lipschitz regularity for the fluid-solid interface is found to be sufficient). While mathematical methods and results for the analysis of transient IBVPs are available since over a half-century [21], investigations on the solvability of the transient acoustic-elastic coupled problem are scarcer and recent. Solvability results for the 2D transient FSIP in variational form are given in [2], the 2D setting there resulting mainly from the chosen viewpoint of solving for the total field and using a 2D Dirichlet-to-Neumann map for domain truncation purposes. A solvability result for the 3D problem is given in [19] by extending the framework of [12, Chap. 7] based on the convergence of Galerkin approximations for IBVPs.

This paper aims at contributing solvability results for the transient acoustic-elastic scattering problem that fit in the framework of our coupling methodology [5]. The analysis of the latter in continuous form results in the convergence in  $L^2([0, T], L^2(\Gamma))$  norm of iteration residuals written in terms of velocities and conormal derivative of fluid and solid quantities on the shared interface  $\Gamma$ ,  $T$  being the chosen finite analysis duration. Our present goals thus include establishing solvability results such that the relevant parts of the solution have the above regularity on  $\Gamma$ , while keeping regularity requirements on the data as low as possible to allow severe loadings potentially undergone in practice by submerged structures.

We reformulate the acoustic-elastic scattering problem using the total elastic field but the scattered acoustic field, which are governed by a PDE system with transmission data on  $\Gamma$  and initial-rest conditions; as a result, the fluid-structure response for a finite analysis time is compactly supported. Our main contribution then consists of three solvability results for the latter problem. The first one assumes “standard”  $H^{-1/2}(\Gamma)$  space regularity of the transmission data, whereas improved  $H^{1/2}(\Gamma)$  space regularity for that data is assumed for the second one. Those two results in turn yield a third solvability result by Sobolev interpolation, where the transmission data has  $L^2(\Gamma)$  regularity in space. The latter result is the one most useful for the justification of our coupling method under current development. For the first two results, we follow a proof methodology whose basic steps are well established, namely:

- First consider data with sufficient regularity in time to allow classical time derivatives whenever needed.
- Recast the FSIP as a first-order in time system with values in an appropriate time-independent function space, then check that the Hille-Yosida theorem applies (see e.g. [7, Chap. 7], [13], [15, Sec. 3]), yielding a strong (in time) solution by semigroup theory arguments.
- Derive an energy identity where a weaker norm of the above solution is controlled by a weaker norm of the data.
- Using density arguments, extend the foregoing solvability results to a mild (in time) solution.

The space regularity setting for the first result corresponds to the mainstream variational treatment of second-order linear PDEs. By contrast, the second result, which emphasizes extra regularity in space for the transmission data and the solution, is of a kind for which the available literature on transient IBVPs appears much scarcer. Elliptic regularity arguments cannot be readily invoked due to insufficient regularity of the accelerations, so that carrying out the above proof steps entails modifying the operator domain and image spaces, showing that the former is a dense subspace of the latter, and revisiting the subsequent energy identity. Then, the third result relies on a Rellich integral identity in addition to invoking available space interpolation results. In addition, since we consider transmission problems with interfacial data, whereas semigroup methods for evolution PDEs usually assume homogeneous BCs and body sources and nonzero

initial data or body sources, proofs rely on a prior recasting of the transmission problem to the canonical format (homogeneous BCs, inhomogeneous body data) by lifting the interfacial data and formulating a problem for the complementary fields involving body sources and homogeneous BCs (as done in e.g. [14, Sec. 8.2] for the Neumann acoustic IBVP).

This article is organized as follows. In Section 2, the main FSIP and its weak formulation are described. The main solvability results are then given (as Theorems 1 to 3) in Section 3. Section 4 then presents a numerical illustration on a 2D test case which highlights the effect of data regularity on the solution. The ensuing Sections 5 to 7 give the proofs of Theorems 1 to 3, respectively, with the help of auxiliary results whose proofs are deferred to Section 8.

## 2 FSI problem formulation.

We first define the setting for the considered fluid-structure interaction (FSI) problems, introduce notation and outline goals. Boldface letters, e.g.  $\mathbf{x}, \mathbf{u}, \boldsymbol{\sigma}$ , denote points, vectors or tensors in the physical space  $\mathbb{R}^d$  ( $d = 2, 3$ ).

**2.1 Setting and governing equations.** A bounded elastic solid (mass density  $\rho_s$ , elasticity tensor  $\mathcal{C}$ ) occupying the domain  $\Omega_s \subset \mathbb{R}^d$  is submerged in an acoustic fluid (mass density  $\rho$ , acoustic wave velocity  $c$ ) occupying the unbounded fluid region  $\Omega := \mathbb{R}^d \setminus \bar{\Omega}_s$  (where  $d = 2$  or  $d = 3$  is the spatial dimension), see Figure 1. We denote by  $\Gamma := \partial\Omega = \partial\Omega_s$  the Lipschitz continuous boundary separating the solid and fluid domains, and by  $\mathbf{n}$  the unit *outward* normal to  $\Gamma$  with respect to the solid domain. The elasticity tensor  $\mathcal{C}$  is elliptic, the other coefficients  $\rho, c, \rho_s$  being positive and bounded away from 0. Our goal is to compute over a finite time interval  $[0, T]$  the scattering of a given acoustic wave by the elastic solid.

We choose the velocity potential  $\phi$ , assumed to satisfy the wave equation

$$-\Delta\phi + \frac{1}{c^2}\phi'' = 0 \quad \text{in } \Omega \times [0, T],$$

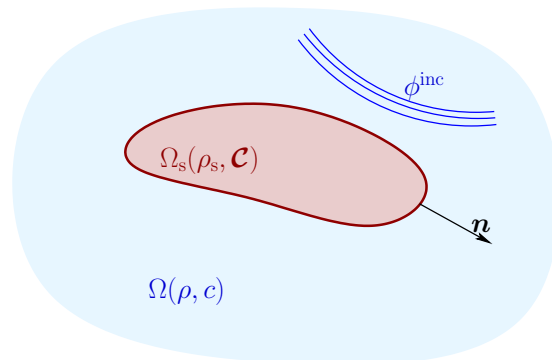
as the main variable describing the fluid motion, the velocity and pressure being then respectively given by  $\mathbf{v} = \nabla\phi$  (by definition of  $\phi$ ) and  $p = -\rho\phi'$ , with the prime symbol denoting time derivatives throughout. The primary excitation consists in an incident wave which would propagate undisturbed in the absence of the submerged solid (i.e. in an infinite fluid domain  $\mathbb{R}^d$ ), defined by the given velocity potential  $\phi_{\text{inc}}$  solving at all times the homogeneous wave equation in  $\mathbb{R}^d$ :

$$-\Delta\phi_{\text{inc}} + \frac{1}{c^2}\phi_{\text{inc}}'' = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}. \quad (1)$$

Moreover,  $\phi_{\text{inc}}$  is assumed to verify  $\phi_{\text{inc}}(\cdot, t) = 0$  in a neighborhood  $D$  of  $\Omega_s$  for any  $t \leq 0$ , so that the immersed elastic solid causes no disturbance before  $t = 0$  (i.e.  $\phi = \phi_{\text{inc}}$  in  $\Omega \times (-\infty, 0]$ ).

The main variable describing the solid motion is taken as the displacement  $\mathbf{u}$ , verifying the elastic wave equation

$$-\Delta_s \mathbf{u} + \rho_s \mathbf{u}'' = \mathbf{0} \quad \text{in } \Omega_s \times [0, T],$$



**Figure 1.** FSI configuration: geometry and notation

where  $\Delta_s$  is the Navier differential operator such that  $\Delta_s \mathbf{u} = \operatorname{div}(\mathbf{C} : \nabla^s \mathbf{u})$ . The fluid-structure interaction (FSI) then occurs due to the kinematical and dynamical transmission conditions

$$(a) \partial_n \phi = \mathbf{u}' \cdot \mathbf{n}, \quad (b) \rho \phi' \mathbf{n} = \mathbf{t}[\mathbf{u}] \quad \text{in } \Gamma \times [0, T], \quad (2)$$

which express the continuity across  $\Gamma$  of (a) the normal velocity and (b) the traction vector, given for the solid by

$$\mathbf{t}[\mathbf{u}] := (\mathbf{C} : \nabla \mathbf{u}) \cdot \mathbf{n}.$$

Since the excitation triggering the FSI originates in the fluid and  $\phi_{\text{inc}}(\cdot, t)$  needs not be compactly supported, we set

$$\phi = \phi_{\text{inc}} + \phi_{\text{sc}},$$

where the scattered field  $\phi_{\text{sc}}$  is causal and compactly supported at any finite time by virtue of the foregoing assumptions. The total fields are retained in  $\Omega_s$ . The transmission conditions (2) expressed in terms of  $\phi_{\text{sc}}$  then read

$$(a) \partial_n \phi_{\text{sc}} = \mathbf{u}' \cdot \mathbf{n} - \partial_n \phi_{\text{inc}}, \quad (b) \mathbf{t}[\mathbf{u}] = \rho(\phi'_{\text{sc}} + \phi'_{\text{inc}}) \mathbf{n} \quad \text{on } \Gamma \times [0, T]. \quad (3)$$

The FSI problem is thereafter formulated in terms of  $\phi_{\text{sc}}$  as the main unknown for the fluid response; we omit from now on the superscript “sc”, all fluid variables (e.g.  $\phi$ ) being understood to pertain to that contribution (the complete fluid motion being recovered by adding back the appropriate “inc” quantities). With this convention, the complete FSI problem for  $(\phi, \mathbf{u})$  consists of the initial value transmission problem (IVTP) defined by (a) the acoustic wave equation and initial rest conditions in  $\Omega$ , (b) the elastic wave equation and initial rest conditions in  $\Omega_s$ , and (c) the kinematic and dynamic transmission conditions (3).

We will in fact focus on an equivalent non-dimensional version of the FSI problem, obtained by expressing the coordinates  $\mathbf{x}, t$ , field variables  $\phi, \mathbf{u}, \mathbf{t}$  and material parameters  $\mathbf{C}, \rho_s$  as

$$\hat{\mathbf{x}} = b\hat{\mathbf{x}}, \quad t = b\hat{t}/c, \quad \phi = bc\hat{\phi}, \quad \mathbf{u} = b\hat{\mathbf{u}}, \quad \mathbf{t} = \rho c^2 \hat{\mathbf{t}}, \quad \mathbf{C} = \rho c^2 \hat{\mathbf{C}}, \quad \rho_s = \rho \hat{\rho}_s \quad (4)$$

(where  $b$  is a characteristic length, e.g.  $2b = \operatorname{diam}(\Gamma)$ ). Our main objective will hence be to investigate the solvability of the dimensionless IVTP defined by

$$\begin{aligned} -\Delta \phi + \phi'' &= 0 & \text{in } \Omega \times ]0, T[, & \quad \phi(0) = \phi'(0) = 0 & \text{in } \Omega, \\ -\Delta_s \mathbf{u} + \rho_s \mathbf{u}'' &= \mathbf{0} & \text{in } \Omega_s \times ]0, T[, & \quad \mathbf{u}(0) = \mathbf{u}'(0) = \mathbf{0} & \text{in } \Omega_s, \\ \partial_n \phi &= \mathbf{u}' \cdot \mathbf{n} + \mathbf{v}, & & \quad \mathbf{t}[\mathbf{u}] = \phi' \mathbf{n} + \mathbf{h} & \text{in } \Gamma \times ]0, T[ \end{aligned} \quad (5)$$

(having dropped all hat symbols), with (4) allowing recovery of the dimensional variables. Generic transmission data in the form of arbitrary prescribed values of the discontinuities  $\mathbf{v}$  of normal velocity and  $\mathbf{h}$  of traction across  $\Gamma$  is considered in problem (5) for greater generality, the target FSI problem corresponding to

$$\mathbf{v} = -\partial_n \phi_{\text{inc}}, \quad \mathbf{h} = \phi'_{\text{inc}} \mathbf{n}. \quad (6)$$

**Incident plane wave.** The case of an incident plane wave field  $\phi_{\text{inc}}$  of the form

$$\phi_{\text{inc}}(\mathbf{x}, t) = f\left(t - \frac{1}{c} \hat{\mathbf{p}} \cdot \mathbf{x}\right), \quad (7)$$

where the unit vector  $\hat{\mathbf{p}}$  defines the direction of propagation and the univariate function  $t \mapsto f(t)$  specifying the time modulation of the propagating pulse is assumed to have a compact support, is of frequent practical interest. Any plane wave of the form (7) solves (1) in  $\mathbb{R}^d \times \mathbb{R}$ . The transmission data (6) is then given by

$$\mathbf{v} = \frac{1}{c} f'(t - \frac{1}{c} \hat{\mathbf{p}} \cdot \mathbf{x}) \hat{\mathbf{p}} \cdot \mathbf{n}, \quad \mathbf{h} = f'(t - \frac{1}{c} \hat{\mathbf{p}} \cdot \mathbf{x}) \mathbf{n}. \quad (8)$$

in particular, both quantities involve the derivative of the incident pulse. See also Remark 6.

**2.2 Variational formulation of the FSIP.** Before stating and proving well-posedness results for problem (5), we need to specify the underlying mathematical framework. Our goal being to obtain solvability results under the least restrictive assumptions on the data, we adopt the common mathematical setting for transient PDEs whereby time-dependent fields are treated as functions of time with values in a Sobolev space (see e.g. [12], Chap. 7 or [21], Chaps. 3-5). The solvability analysis will as a result largely consist in proving the well-posedness of a variational form of the FSIP. In this section, we hence specify the framework (function spaces, bilinear forms, relevant Green identities, variational FSI problem) that underlies the subsequent analysis.

We denote by  $(v, w)_\Gamma$  or  $(\mathbf{v}, \mathbf{w})_\Gamma$  the  $L^2(\Gamma)$  scalar product. The variational formulation of problem (5) involves the bilinear forms  $a, b$  (for the fluid domain) and  $A, B$  (for the solid domain) defined by

$$\begin{aligned} a(\phi, \tilde{\phi}) &:= \int_{\Omega} \nabla \phi \cdot \nabla \tilde{\phi} \, dV, & b(\phi, \tilde{\phi}) &:= \int_{\Omega} \phi \tilde{\phi} \, dV, \\ A(\mathbf{u}, \tilde{\mathbf{u}}) &:= \int_{\Omega_s} \nabla^s \mathbf{u} : \mathcal{C} : \nabla^s \tilde{\mathbf{u}} \, dV, & B(\mathbf{u}, \tilde{\mathbf{u}}) &:= \int_{\Omega_s} \rho_s \mathbf{u} \cdot \tilde{\mathbf{u}} \, dV. \end{aligned} \quad (9)$$

where  $\nabla^s \mathbf{u} := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$  is the linearized strain tensor associated with a displacement  $\mathbf{u}$  in  $\Omega_s$ , and  $\mathcal{C}$  and  $\rho_s$  are the dimensionless material parameters of the solid introduced in (4). In addition to the standard Sobolev spaces  $L^2(\Omega)$ ,  $\mathbf{L}^2(\Omega_s) := L^2(\Omega_s; \mathbb{R}^d)$ ,  $H^1(\Omega)$  and  $\mathbf{H}^1(\Omega_s) := H^1(\Omega_s; \mathbb{R}^d)$ , we will use the spaces  $H_\Delta^1(\Omega)$  and  $\mathbf{H}_\Delta^1(\Omega_s)$  defined by

$$H_\Delta^1(\Omega) = \{ \phi \in H^1(\Omega), \Delta \phi \in L^2(\Omega) \}, \quad \mathbf{H}_\Delta^1(\Omega_s) = \{ \mathbf{u} \in \mathbf{H}^1(\Omega_s), \Delta_s \mathbf{u} \in L^2(\Omega_s) \}.$$

The Green identities for both the fluid and the solid media (e.g. [25, Thm. 4.4]) for Lipschitz domains  $\Omega_s$  and  $\Omega$  are given by: for any  $(\phi, \tilde{\phi}) \in H_\Delta^1(\Omega) \times H^1(\Omega)$ , it holds

$$a(\phi, \tilde{\phi}) = -b(\Delta \phi, \tilde{\phi}) - (\partial_n \phi, \tilde{\phi})_\Gamma \quad (10a)$$

and for any  $(\mathbf{u}, \tilde{\mathbf{u}}) \in \mathbf{H}_\Delta^1(\Omega_s) \times \mathbf{H}^1(\Omega_s)$ , we have

$$A(\mathbf{u}, \tilde{\mathbf{u}}) = -B(\rho_s^{-1} \Delta_s \mathbf{u}, \tilde{\mathbf{u}}) + (\mathbf{t}[\mathbf{u}], \tilde{\mathbf{u}})_\Gamma. \quad (10b)$$

The differing signs in front of the last term of (10a,b) result from the chosen orientation convention for  $\Gamma$ . The (co)normal derivatives  $\partial_n u$  and  $\mathbf{t}[\mathbf{u}]$  are elements of  $H^{-1/2}(\Gamma)$  and  $\mathbf{H}^{-1/2}(\Gamma)$ , respectively, with the notation  $(\cdot, \cdot)_\Gamma$  understood as the  $H^{-1/2}(\Gamma), H^{1/2}(\Gamma)$  duality pairing extensions of the  $L^2(\Gamma)$  scalar product. All functions involved in this work are real-valued.

By the classical process of applying the Green identities (10a,b) to weighted-residual forms of the PDEs and using the transmission conditions in the resulting interfacial integrals, the FSI problem (5) can be recast in variational form: find  $(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^1) \cap C_T^1(\mathbf{L}^2)$  such that

$$\left. \begin{aligned} \text{(a)} \quad & a(\phi(t), \tilde{\phi}) + b(\phi''(t), \tilde{\phi}) + A(\mathbf{u}(t), \tilde{\mathbf{u}}) + B(\mathbf{u}''(t), \tilde{\mathbf{u}}) + I(\mathbf{u}'(t), \tilde{\phi}) + I_s(\phi'(t), \tilde{\mathbf{u}}) \\ & = (\mathbf{h}(t), \tilde{\mathbf{u}})_\Gamma - (\mathbf{v}(t), \tilde{\phi})_\Gamma \quad \text{in } \mathcal{D}'([0, T]) \quad \text{for all } (\tilde{\phi}, \tilde{\mathbf{u}}) \in \mathbf{H}^1, \\ \text{(b)} \quad & (\phi, \mathbf{u})(0) = 0 \quad \text{in } \mathbf{H}^1, \quad (\phi', \mathbf{u}') = 0 \quad \text{in } \mathbf{L}^2, \end{aligned} \right\} \quad (11)$$

where we have abbreviated as  $C_T^m(X)$  the space  $C^m([0, T]; X)$  of  $m$  times continuously differentiable functions  $f : [0, T] \rightarrow X$  with values in a Hilbert space  $X$ , equipped with the norm

$$\|\phi\|_{C_T^m(H)} = \sum_{k=0}^m \left( \sup_{t \in [0, T]} \|\partial_t^k \phi(t)\|_H \right).$$

The bilinear forms  $a, b, A, B$  are as defined in (9), the coupling occurs through the additional bilinear forms

$$I(\mathbf{v}, \tilde{\phi}) := (\mathbf{n} \cdot \mathbf{v}, \tilde{\phi})_\Gamma, \quad I_s(\psi, \tilde{\mathbf{u}}) := -(\psi, \mathbf{n} \cdot \tilde{\mathbf{u}})_\Gamma, \quad (12)$$

and the shorthand notations  $\mathbf{L}^2, \mathbf{H}^s$  ( $s \in \mathbb{R}$ ) refer for convenience to the product spaces

$$\mathbf{L}^2 := L^2(\Omega) \times L^2(\Omega_s), \quad \mathbf{H}^s := H^s(\Omega) \times \mathbf{H}^s(\Omega_s).$$

We also use similar notation for product spaces of pairs of interfacial traces or transmission data:

$$\mathbf{H}^s(\Gamma) := H^s(\Gamma) \times \mathbf{H}^s(\Gamma) \quad (\mathbf{L}^2(\Gamma) := \mathbf{H}^0(\Gamma)).$$

The equality (11a) is to be satisfied in the sense of distributions in the time variable with support in  $[0, T]$  (i.e. in  $\mathcal{D}'([0, T])$ , as indicated there).

The FSI problem (5) and the variational problem (11) are equivalent in the following sense (see proof in Sec. 8.1):

**Proposition 1**  $(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^1) \cap C_T^1(\mathbf{L}^2)$  solves the IVTP (5) if and only if  $(\phi, \mathbf{u})$  solves the variational problem (11).

### 3 Solvability results.

We now state the solvability results which constitute the main focus of this work. For the reasons discussed in the Introduction, they consist in two primary solvability results (Theorems 1 and 2) and a subsequent result obtained by invoking interpolation theorems for the previous solvability mappings (Theorem 3). Those results are formulated in terms of spaces of functions  $f : [0, T] \rightarrow X$  with values in a Hilbert space  $X$ , which are taken as either  $C_T^m(X)$  (see Section 2.2) or  $L^2([0, T]; X)$ ,  $H^m([0, T]; X)$  (often abbreviated as  $L_T^2(X)$ ,  $H_T^m(X)$ , respectively hereafter) equipped with the respective norms

$$\|\phi\|_{L^2([0, T]; H)}^2 = \int_0^T \|\phi(t)\|_H^2 dt, \quad \|\phi\|_{H^m([0, T]; H)}^2 = \sum_{k=0}^m \int_0^T \|\partial_t^k \phi(t)\|_H^2 dt.$$

**3.1 Primary solvability result, data with  $H^{-1/2}(\Gamma)$  space regularity.** For this first result, proved in Section 5, we consider transmission data with the “standard” space regularity for variational problems.

**Theorem 1** *Let  $\Gamma$  be a Lipschitz closed surface.*

(i) *Let  $(\mathbf{v}, \mathbf{h}) \in H_T^1(\mathbf{H}^{-1/2}(\Gamma))$  with  $(\mathbf{v}, \mathbf{h})(0) = (0, \mathbf{0})$ . Then, the transmission problem (5) admits a unique solution  $(\phi, \mathbf{u})$ , with*

$$(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^1), \quad (\phi', \mathbf{u}') \in C_T^0(\mathbf{L}^2).$$

(ii) *If  $(\mathbf{v}, \mathbf{h}) \in H_T^2(\mathbf{H}^{-1/2}(\Gamma))$  with  $(\mathbf{v}, \mathbf{h})(0) = (\mathbf{v}', \mathbf{h}')(0) = (0, \mathbf{0})$ , we have*

$$(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}_\Delta^1), \quad (\phi', \mathbf{u}') \in C_T^0(\mathbf{H}^1), \quad (\phi'', \mathbf{u}'') \in C_T^0(\mathbf{L}^2).$$

*with  $\mathbf{H}_\Delta^1 := H_\Delta^1(\Omega) \times \mathbf{H}_\Delta^1(\Omega_s)$ .*

**3.2 Primary solvability result, data with  $H^{1/2}(\Gamma)$  space regularity.** Motivated in particular by the later objective of applying Sobolev interpolation, we now consider cases where the data has identical regularity in time but improved regularity in space (relative to Theorem 1). Taking advantage of this to obtain corresponding improvement on the solution space regularity is not straightforward. For instance, elliptic regularity arguments cannot be readily invoked due to insufficient regularity of the accelerations, while energy estimates obtained for proving Theorem 1 also do not directly adapt to data with extra spatial smoothness (see Remark 4). This called for substantial modifications of the main steps of the proof method used for Theorem 1, allowing to obtain by the same general approach the following additional set of primary solvability results, proved in Section 6:

**Theorem 2** *Let  $\Gamma$  be a  $C^{1,1}$  closed surface.*

(i) *If  $(\mathbf{v}, \mathbf{h}) \in L_T^2(\mathbf{H}^{1/2}(\Gamma))$  with  $(\mathbf{v}, \mathbf{h})(0) = (0, \mathbf{0})$ . Then, the transmission problem (5) admits a unique solution  $(\phi, \mathbf{u})$ , with*

$$(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^1), \quad (\phi', \mathbf{u}') \in C_T^0(\mathbf{L}^2).$$

- (ii) If  $(\mathbf{v}, \mathbf{h}) \in H_T^1(\mathbf{H}^{1/2}(\Gamma))$  with  $(\mathbf{v}, \mathbf{h})(0) = (\mathbf{v}', \mathbf{h}')(0) = (0, \mathbf{0})$ , we have
- $$(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^2), \quad (\phi', \mathbf{u}') \in C_T^0(\mathbf{H}^1), \quad (\phi'', \mathbf{u}'') \in C_T^0(\mathbf{L}^2).$$

**3.3 Interpolation solvability result.** Finally, the design and convergence study of domain decomposition algorithms applied to the IVTP (5) make it convenient to consider transmission data whose space regularity is  $L^2(\Gamma)$  rather than  $H^{1/2}(\Gamma)$  or  $H^{-1/2}(\Gamma)$ . Motivated by this, we apply interpolation space arguments (see e.g. [21, Chaps. 1, 4]) to mappings given in Theorems 1 and 2 (or other similar pairs of results), which yields additional solvability mappings. In the following theorem, proved in Section 7, we give mappings for which the data has  $L^2(\Gamma)$  space regularity while the solution has Neumann traces  $\partial_n \phi, \mathbf{t}[\mathbf{u}]$  and boundary traces of velocities  $\phi', \mathbf{u}'$  in  $L_T^2(\Gamma)$ . Such mappings are very useful as they provide supporting arguments in the analysis and justification of coupling algorithms based on iteratively solving IBVPs in each domain [5].

**Theorem 3** *Let  $\Gamma$  be a  $C^{1,1}$  closed surface. Let  $(\mathbf{v}, \mathbf{h}) \in H_T^1(\mathbf{L}^2(\Gamma))$  with  $(\mathbf{v}, \mathbf{h})(0) = (0, \mathbf{0})$ .*

- (i) *Then, the transmission problem (5) admits a unique weak solution  $(\phi, \mathbf{u})$ , with*

$$(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^{3/2}), \quad (\phi', \mathbf{u}') \in C_T^0(\mathbf{H}^{1/2}).$$

- (ii) *In addition, the velocities have boundary traces, with  $\phi'|_\Gamma \in L_T^2(\Gamma)$  and  $\mathbf{u}'|_\Gamma \in \mathbf{L}_T^2(\Gamma)$ , and we also have  $\partial_n \phi \in L_T^2(\Gamma)$  and  $\mathbf{t}[\mathbf{u}] \in \mathbf{L}_T^2(\Gamma)$ .*

### 3.4 Discussion.

**Remark 1 (Loss of regularity relative to the time-harmonic FSIP)** *The solvability result of [3] for the time-harmonic FSIP states that  $(\phi, \mathbf{u}) \in \mathbf{H}^1$  if the incident wave is such that  $(\partial_n \phi_{inc}, \phi_{inc} \mathbf{n}) \in \mathbf{H}^{-1/2}(\Gamma)$ . By contrast, in the present transient case, a  $L_T^2(\mathbf{H}^{-1/2}\Gamma)$  data regularity is not sufficient for obtaining a  $L_T^2(\mathbf{H}^1)$  solution regularity, which is achieved at the cost of one extra unit of data regularity either in space (Theorem 1, item (i)) or in time (Theorem 2, item (ii)).*

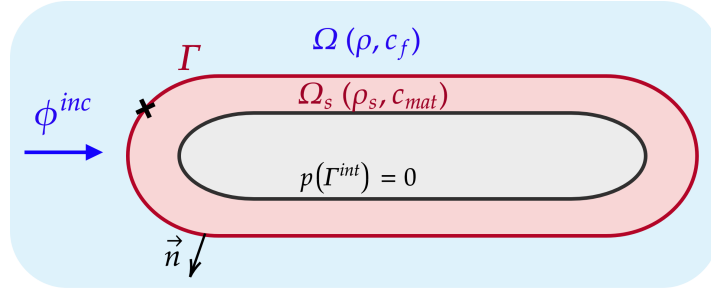
*This behavior is essentially the same as that affecting the solvability mappings of Neumann IBVPs, thoroughly studied in [17, 18, 27, 28] where the achievable solution regularity is also shown to be fractionally higher (or the required data regularity fractionally lower) compared to solvability mappings obtained using the more-classical approach based on strong solutions and energy identities used in this work. Such considerations are very important for iterative domain decomposition methods that proceed by generating convergent sequences of IBVP solutions in each domain: iterates must remain in a fixed space, a prerequisite that sequences of Robin IBVP solutions [11, 5] satisfy while sequences of Neumann IBVP solutions do not due to the aforementioned reasons. Regarding the present context, Theorem 3 implies that both the acoustic and elastodynamic components of the FSIP are Robin solutions whose respective Robin data is in  $L_T^2(\Gamma)$ .*

**Remark 2 (Interfacial data for incident plane waves)** *Consider incident plane waves of the form (7). If  $f \in C^2(\mathbb{R})$ , the associated interfacial data (8) has enough regularity for Theorem 3, and a fortiori for Theorem 1(i). If  $f \in C^1(\mathbb{R})$  only, even Theorem 1(i) does not in general apply.*

*If  $f \in H^m(\mathbb{R})$ , we have  $\phi_{inc}|_{D \times [0, T]} \in H_T^p(H^q(D))$  if  $p + q \leq m$ , where  $D \subset \mathbb{R}^d$  is a bounded neighborhood of  $\Omega_s$ . For  $m = 2$ , the data (8) therefore has adequate regularity for Theorem 2(i), while  $m = 3$  is suitable for Theorem 2(ii).*

**Remark 3 (Hollow solids)** *The problem setting of Section 2 assumes for definiteness that the solid domain  $\Omega_s$  fills the whole region interior to  $\Gamma$ . However, the results of this work apply without modification to hollow elastic solids  $\Omega_s$  such that  $\partial\Omega_s = \Gamma \cup \Gamma_{int}$ , where the additional boundary component  $\Gamma_{int}$  is interior to (and disconnected from)  $\Gamma$  and supports homogeneous boundary conditions. The test case configuration of Section 4 uses such a configuration. Similarly, additional given excitations applied to  $\Omega_s$  away from  $\Gamma$  may be accounted for with straightforward modifications.*





**Figure 2.** Configuration for the numerical test case.

#### 4 Numerical example.

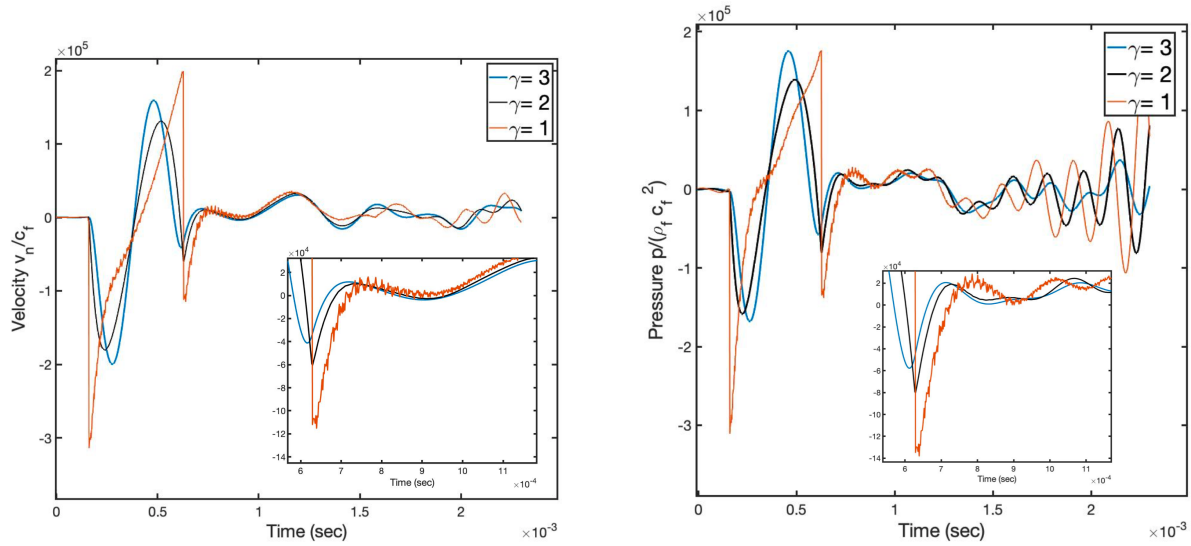
To illustrate the effect of data space-time regularity on the solution of the coupled problem, we consider a representative 2D test configuration, where  $\Omega_s$  is a hollow solid made of two straight parallel walls closed by two semicircular parts and immersed in a fluid occupying the unbounded surrounding region  $\Omega$  (Figure 2), so that the coupling interface  $\Gamma$  is the outer boundary of  $\Omega_s$ . We assume translational invariance along  $e_3$  and plane strain deformations for the solid, so that  $\phi = \phi(x_1, x_2)$  and  $\mathbf{u} = u_1(x_1, x_2)\mathbf{e}_1 + u_2(x_1, x_2)\mathbf{e}_2$ . The acoustic-elastic IVTP to be solved thus consists of the two-dimensional form of (5) supplemented with a traction-free (i.e. homogeneous Neumann) BC on the interior solid boundary. The transmission data  $\mathbf{v}, \mathbf{h}$  results from the incident acoustic wave defined by

$$\phi_{\text{inc}}(\mathbf{x}, t) = f\left(t - \frac{1}{c}x_1\right), \quad f(t) = \begin{cases} a\left(\frac{2t}{T_p}\right)^\gamma \left(2 - \frac{2t}{T_p}\right)^\gamma & \text{if } t \in [0, T_p], \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

The exponent  $\gamma$  allows to tune the space-time regularity of the modulating pulse  $f$  of the incident plane wave (see Remark 2), since  $f \in C^{\gamma-1}(\mathbb{R})$  by construction. In the numerical results, the pulse duration and amplitude are set to  $T_p = 4.675 \cdot 10^{-4}$  s and  $a = 10^8$ , the fluid acoustic velocity and mass density to  $c_f = 1500$  m/s and  $\rho_f = 1000$  kg/m<sup>3</sup>, and the elastic mass density and Young's modulus to  $\rho_{\text{mat}} = 7800$  kg.m<sup>-3</sup> and  $E = 210$  GPa. The simulation time interval is  $[0, T]$  with  $T = 2.3 \cdot 10^{-3}$  s; it is discretised into  $M = 1118$  uniform time steps ( $\Delta t = 1.7 \cdot 10^{-6}$  s).

We solve the fluid-structure interaction problem with the convergent iterative global-in-time method proposed in the companion investigation [5], based on solving IBVPs with Robin boundary conditions in each domain. The latter is implemented, for this proof-of-concept stage, using an in-house finite element method (FEM) solver based on the programming material provided in [6] for the elastic body and an in-house boundary element method (BEM) solver, based as described in [23] on the convolution quadrature method and the Z-transform, for the acoustic medium. The latter relies on numerically solving a finite set of boundary integral equations in the complex frequency domain, a task that is accelerated by means of a Hierarchical matrix compression [4, 8] and a high-frequency approximation (HFA) [22]. For each transient BEM solution,  $2M_{\text{HFA}} = 70$  BEM problems corresponding to complex frequencies with modulus below a chosen cut-off value are solved using GMRES with a relative tolerance of  $10^{-6}$ . Solutions at the remaining  $2(M - M_{\text{HFA}})$  frequencies are approximated by a heuristic HFA. The transient BEM solutions yielded with or without recourse to a HFA by the inverse Z transform, whose accuracy parameter is set to  $10^{-5}$  [23], agree within less than 1% in relative  $L_T^2(\Gamma)$  norm. The fluid-solid interface is discretized into 1778 boundary elements (with piecewise-linear and continuous interpolation), while the finite element model has 15 400 nodal displacement unknowns.

We solve numerically the above-described problem for three cases defined by setting  $\gamma = 1, 2, 3$  in the incident wave (14). The interfacial data being given in terms of  $\phi_{\text{inc}}$  by (6), the case  $\gamma = 3$  meets the data regularity requirement for Theorem 2, while  $\gamma = 1$  fails to meet that of Theorem 1. In Figure 3, we plot the computed time histories at the evaluation point indicated in Fig. 14 of



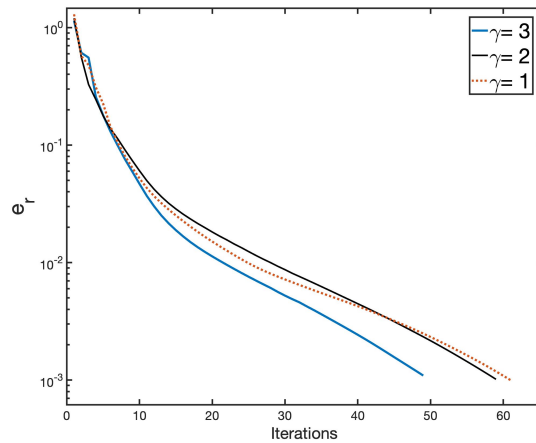
**Figure 3.** Scattering by a 2D cylinder: normalized fluid normal velocities (left) and normalized fluid pressures (right) obtained for three different load regularities.

(a) the normalized elastic normal velocity and (b) the normalized total fluid pressure, obtained upon convergence of the coupling iterations. As expected, the discontinuous incident load (6) having discontinuities for  $\gamma = 1$ , the solution is discontinuous and visibly affected by oscillations. The loads  $\gamma = 2$  (with a discontinuous derivative) and  $\gamma = 3$  produce increasingly better behaved numerical time histories.

The space-time regularity of the data also influences the convergence rate of the iterative algorithm. To observe this, we evaluate for each solution iterate  $(\phi^i, \mathbf{u}^i)$  the dimensionless transmission residual on  $\Gamma$  given by

$$e_r(\phi^i, \mathbf{u}^i) := \frac{\|\partial_n \phi - \partial_t \mathbf{u}^i \cdot \mathbf{n} - \mathbf{v}\|_{L_T^2(\Gamma)}^2 + \|\mathbf{t}^i - \partial_t \phi^i \mathbf{n} - \mathbf{h}\|_{L_T^2(\Gamma)}^2}{\|\mathbf{v}\|_{L_T^2(\Gamma)}^2 + \|\mathbf{h}\|_{L_T^2(\Gamma)}^2}$$

Plots of  $e_r(\phi^i, \mathbf{u}^i)$  against the iteration number  $i$  for  $\gamma = 1, 2, 3$  on Figure 4 show that the convergence speed improves with the regularity of the incident wave.



**Figure 4.** Scattering by a 2D cylinder: convergence of the normalized transmission residual indicator  $e_r$  with the coupling iterations, for three different load regularities.

## 5 Proof of Theorem 1.

This section is now devoted to the proof of Theorem 1 by means of the Hille-Yosida theorem. Its main steps are as follows: (i) recast the IVTP in a first-order form; (ii) define a lifting of the interfacial data; (iii) apply the Hille-Yosida theorem for the complementary part of the FSI solution, which obeys a non-homogeneous first-order system with homogeneous interface conditions; (iv) show that a weaker norm of the strong solution is controlled by a weaker norm of the data; and finally (v) define by density a weak solution satisfying the variational formulation (11). Many of the ingredients and notations introduced thereafter will be also used for the proofs of Theorem 2 (Section 6) and Theorem 3 (Section 7).

**5.1 First-order form of the FSIP.** In preparation to applying the Hille-Yosida theorem, we begin by recasting the IVTP in first-order form. Setting  $\psi := \phi'$  and  $\mathbf{v} := \mathbf{u}'$  to treat the velocities as separate unknowns, the system (5) yields the following first-order system for  $\mathbb{U}(t) := (\phi, \mathbf{u}, \psi, \mathbf{v})^T(t)$ :

$$\mathbb{U}' + \mathbf{A}\mathbb{U} = 0, \quad \mathbb{U}(0) = 0, \quad \mathbf{B}\mathbb{U} = \mathbb{H} \quad (15)$$

having set  $\mathbb{H} := (\mathbf{v}, \mathbf{h})^T$  and with the (unbounded) differential operator  $\mathbf{A}$  in  $\Omega \times \Omega_s \times \Omega \times \Omega_s$  and boundary operator  $\mathbf{B}$  in  $\Gamma \times \Gamma$  defined by

$$\mathbf{A}\mathbb{U} = \begin{Bmatrix} -\psi \\ -\mathbf{v} \\ -\Delta\phi \\ -\rho_s^{-1}\Delta_s\mathbf{u} \end{Bmatrix}, \quad \mathbf{B}\mathbb{U} = \begin{Bmatrix} \partial_n\phi - \mathbf{n}\cdot\mathbf{v} \\ \mathbf{t}[\mathbf{u}] - \mathbf{n}\psi \end{Bmatrix}. \quad (16)$$

Let the Hilbert space  $\mathcal{H}$  be defined by  $\mathcal{H} := \mathbf{H}^1 \times \mathbf{L}^2 = H^1(\Omega) \times \mathbf{H}^1(\Omega_s) \times L^2(\Omega) \times \mathbf{L}^2(\Omega_s)$  and equipped with the scalar product given, for any  $\mathbb{U}, \tilde{\mathbb{U}}$  in  $\mathcal{H}$ , by

$$(\mathbb{U}, \tilde{\mathbb{U}})_{\mathcal{H}} := a(\phi, \tilde{\phi}) + b(\phi, \tilde{\phi}) + A(\mathbf{u}, \tilde{\mathbf{u}}) + B(\mathbf{u}, \tilde{\mathbf{u}}) + b(\psi, \tilde{\psi}) + B(\mathbf{v}, \tilde{\mathbf{v}}), \quad (17)$$

in terms of the bilinear forms (9). Then, let the space  $\mathcal{H}_A$  and its scalar product  $(\cdot, \cdot)_{\mathcal{H}_A}$  be defined by

$$\mathcal{H}_A := \{\mathbb{U} \in \mathcal{H}, \mathbf{A}\mathbb{U} \in \mathcal{H}\} = \mathbf{H}_\Delta^1 \times \mathbf{H}^1, \quad (\mathbb{U}, \tilde{\mathbb{U}})_{\mathcal{H}_A} = (\mathbb{U}, \tilde{\mathbb{U}})_{\mathcal{H}} + (\mathbf{A}\mathbb{U}, \mathbf{A}\tilde{\mathbb{U}})_{\mathcal{H}},$$

with the product space  $\mathbf{H}_\Delta^1$  as defined in Theorem 1. We set the domain  $\mathcal{D}(\mathbf{A})$  of the operator  $\mathbf{A}$  defined by (16) as  $\mathcal{D}(\mathbf{A}) = \{\mathbb{U} \in \mathcal{H}_A, \mathbf{B}\mathbb{U} = 0\}$ , in particular embedding as essential conditions the interfacial constraints  $\mathbf{B}\mathbb{U} = 0$ , i.e.

$$(a) \quad \partial_n\phi - \mathbf{n}\cdot\mathbf{v} = 0, \quad (b) \quad \mathbf{t}[\mathbf{u}] - \mathbf{n}\psi = \mathbf{0}. \quad (18)$$

The following density property, proved in Section 8.2, is crucial for applying the Hille-Yosida theorem:

**Lemma 1** *The space  $\mathcal{D}(\mathbf{A})$  is a dense subspace of  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ .*

We note that the norm  $\|\cdot\|_{\mathcal{H}}$  arising from (17) is equivalent to the standard Sobolev product norm of  $H^1(\Omega) \times \mathbf{H}^1(\Omega_s) \times L^2(\Omega) \times \mathbf{L}^2(\Omega_s) = \mathcal{H}$ . This follows from the fact that relevant Sobolev norms in  $\Omega$  or  $\Omega_s$  can be expressed in terms of the bilinear forms (9), the  $L^2(\Omega)$  and  $H^1(\Omega)$  norms being then given by

$$\|\phi\|_{\Omega}^2 = b(\phi, \phi), \quad \|\phi\|_{1,\Omega}^2 = a(\phi, \phi) + b(\phi, \phi), \quad (19a)$$

and the  $\mathbf{L}^2(\Omega_s)$  and  $\mathbf{H}^1(\Omega_s)$  norms by

$$\|\mathbf{u}\|_{\Omega_s}^2 = B(\mathbf{u}, \mathbf{u}), \quad \|\mathbf{u}\|_{1,\Omega_s}^2 = A(\mathbf{u}, \mathbf{u}) + B(\mathbf{u}, \mathbf{u}). \quad (19b)$$

The norms given by (19b,b) are equivalent to the usual  $L^2$  or  $H^1$  norms, for the solid thanks to Korn's inequality (e.g. [9, Thm. 6.15-1]) and the assumed properties of the material parameters.

**5.2 Interface data lifting.** To put the system (15) in a form allowing invocation of the Hille-Yosida theorem, we need to define a data lifting and obtain from (15) a first-order system featuring homogeneous transmission conditions. To this end, let  $\mathbb{U}_L$  satisfy

$$(\mu I + \mathbf{A})\mathbb{U}_L = 0, \quad \mathbf{B}\mathbb{U}_L = \mathbb{H}, \quad (20)$$

for some  $\mu \in \mathbb{R}$ ,  $\mu \neq 0$ . The problem (20) is uniquely solvable for  $\mathbb{U}_L$ :

**Lemma 2** *Let  $\mathbb{H} \in X([0, T]; \mathbf{H}^{-1/2}(\Gamma))$  for some Banach space  $X$ . Then, the system (20) has a unique solution  $\mathbb{U}_L \in X([0, T]; \mathcal{H})$ , verifying  $\|\mathbb{U}_L\|_{X([0, T]; \mathcal{H})} \leq C\|\mathbb{H}\|_{X([0, T]; \mathbf{H}^{-1/2}(\Gamma))}$ .*

*Proof.* Writing  $\mathbb{U}_L = (\phi_L, \mathbf{u}_L, \psi_L, \mathbf{v}_L)^\top$  and eliminating  $\psi_L$  and  $\mathbf{v}_L$ , the remaining equations of system (20) are

$$\begin{aligned} -\Delta\phi_L + \mu^2\phi_L &= 0 \quad \text{in } Q, & -\Delta_s \mathbf{u} + \rho_s \mu^2 \mathbf{u}_L &= \mathbf{0} \quad \text{in } Q_s, \\ \partial_n \phi_L - \mu \mathbf{u}_L \cdot \mathbf{n} &= \mathbf{v}, & \mathbf{t}[\mathbf{u}_L] - \mu \phi_L \mathbf{n} &= \mathbf{h} \quad \text{in } \Sigma. \end{aligned}$$

They define for each  $t \in [0, T]$  a transmission problem, which is set in weak form as

$$\text{Find } (\phi, \mathbf{u})(t) \in \mathbf{H}^1, \quad \mathcal{A}(\phi(t), \mathbf{u}(t); \tilde{\phi}, \tilde{\mathbf{u}}) = \mathcal{F}_t(\tilde{\phi}, \tilde{\mathbf{u}}) \quad \text{for all } (\tilde{\phi}, \tilde{\mathbf{u}}) \in \mathbf{H}^1 \quad (21)$$

with

$$\begin{aligned} \mathcal{A}(\phi, \mathbf{u}; \tilde{\phi}, \tilde{\mathbf{u}}) &= a(\phi, \tilde{\phi}) + \mu^2 b(\phi, \tilde{\phi}) + A(\mathbf{u}, \tilde{\mathbf{u}}) + \mu^2 B(\mathbf{u}, \tilde{\mathbf{u}}) + \mu(\mathbf{u} \cdot \mathbf{n}, \tilde{\phi})_\Gamma - \mu(\phi \mathbf{n}, \tilde{\mathbf{u}})_\Gamma \\ \mathcal{F}_t(\tilde{\phi}, \tilde{\mathbf{u}}) &= (\mathbf{h}(t), \tilde{\mathbf{u}})_\Gamma - (\mathbf{v}(t), \tilde{\phi})_\Gamma \end{aligned}$$

We note that  $(\mathbf{u} \cdot \mathbf{n}, \tilde{\phi})_\Gamma - (\phi \mathbf{n}, \tilde{\mathbf{u}})_\Gamma = 0$  if  $(\tilde{\phi}, \tilde{\mathbf{u}}) = (\phi, \mathbf{u})$ . Together with definitions (9), this implies  $\mathcal{A}(\phi, \mathbf{u}; \phi, \mathbf{u}) \geq C\|(\phi, \mathbf{u})\|_{\mathbf{H}^1}^2$ , i.e., that the bilinear form  $\mathcal{A}$  is coercive on  $\mathbf{H}^1$ . Moreover, the linear functional  $\mathcal{F}_t$  is continuous on  $\mathbf{H}^1$  by assumption on  $(\mathbf{v}, \mathbf{h})$ . Problem (21) is therefore uniquely solvable in  $\mathbf{H}^1$  by Lax-Milgram's theorem. The remaining equations of (20) then give  $(\psi_L, \mathbf{v}_L) = \mu(\phi_L, \mathbf{u}_L) \in \mathbf{H}^1$ . Finally, we have  $\|\mathbb{U}_L(t)\|_{\mathcal{H}} \leq C\|\mathbb{H}(t)\|_{\mathbf{H}^{-1/2}(\Gamma)}$  for each  $t \in [0, T]$ , hence the claimed space-time estimate.  $\square$

We can now reformulate the FSI system (15) using the new unknown  $\mathbb{U}_c := \mathbb{U} - \mathbb{U}_L$ . Since  $\mathbb{U}$  and  $\mathbb{U}_L$  respectively verify (15) and (20) and introducing the generic system

$$\mathbb{Z}' + \mathbf{A}\mathbb{Z} = \mathbb{F}, \quad \mathbf{B}\mathbb{Z} = 0, \quad \mathbb{Z}(0) = 0 \quad t \in [0, T] \quad (22)$$

with unknown  $\mathbb{Z}$  and datum  $\mathbb{F}$ , it follows that  $\mathbb{U}_c$  solves system (22) with  $\mathbb{F} := \mu\mathbb{U}_L - \mathbb{U}'_L$ . This reformulation of the initial IVTP as a non-homogeneous first-order system with homogeneous interface conditions allows now to apply the Hille-Yosida theorem.

**5.3 Application of the Hille-Yosida theorem.** The format of (22), together with the density property of Lemma 1, allow its solvability to be decided for any right-hand side  $\mathbb{F}$  having appropriate regularity, if it satisfies the conditions of the Hille-Yosida theorem [13]. In the present context, we need to verify that there exists  $\lambda \in \mathbb{R}$  such that  $\mathbf{A}_\lambda := \mathbf{A} + \lambda I : \mathcal{D}(\mathbf{A}) \rightarrow \mathcal{H}$  is maximal monotone, i.e., satisfies

$$(\mathbf{A}_\lambda \mathbb{U}, \mathbb{U})_{\mathcal{H}} \geq 0 \quad \text{for any } \mathbb{U} \in \mathcal{D}(\mathbf{A}) \quad (\mathbf{A}_\lambda \text{ monotone}),$$

$$\text{for any } \mathbb{F} \in \mathcal{H}, \text{ there exists } \mathbb{U} \in \mathcal{D}(\mathbf{A}) \text{ such that } (\mathbf{A}_\lambda + I)\mathbb{U} = \mathbb{F} \quad (\mathbf{A}_\lambda + I \text{ surjective}).$$

Proving that the above conditions are indeed met will be facilitated by the following lemma:

**Lemma 3** *For any  $\mathbb{U} \in \mathcal{D}(\mathbf{A})$ , we have*

$$(a) \quad (\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}} = -b(\psi, \phi) - B(\mathbf{v}, \mathbf{u}), \quad (b) \quad 2|(\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}}| \leq \|\mathbb{U}\|_{\mathcal{H}}^2.$$

*Proof of Lemma 3.* Using the Green identities (10a,b) in  $(\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}}$  and the essential conditions (18) yields (a). Then, (b) results from applying Young's inequality to the right-hand side of (a) and recalling the definition (17) of  $\|\mathbb{U}\|_{\mathcal{H}}^2$ .  $\square$

The proof of the monotonicity and surjectivity then follows.

1. *Monotonicity.* Using definitions (16) of  $\mathbf{A}$  and (17) of the scalar product in  $\mathcal{H}$  and applying Lemma 3a, we obtain after rearrangement

$$\begin{aligned} (\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}} + \lambda(\mathbb{U}, \mathbb{U})_{\mathcal{H}} &= \lambda a(\phi, \phi) + \lambda A(\mathbf{u}, \mathbf{u}) + \frac{1}{2}b(\phi - \psi, \phi - \psi) + \frac{1}{2}B(\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) \\ &\quad + (\lambda - \frac{1}{2})[b(\phi, \phi) + b(\psi, \psi)] + (\lambda - \frac{1}{2})[B(\mathbf{u}, \mathbf{u}) + B(\mathbf{v}, \mathbf{v})], \end{aligned} \quad (23)$$

implying that  $(\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}} + \lambda(\mathbb{U}, \mathbb{U})_{\mathcal{H}} \geq 0$ , i.e. monotonicity holds, for any  $\lambda \geq \frac{1}{2}$ .

2. *Surjectivity.* Now, for  $\mu = \lambda + 1$ , we investigate whether the equation  $(\mathbf{A} + \mu\mathbf{l})\mathbb{U} = \mathbb{F}$  is solvable for  $\mathbb{U} = (\phi, \mathbf{u}, \psi, \mathbf{v})^{\top} \in \mathcal{D}(\mathbf{A})$  given  $\mathbb{F} = (f, \mathbf{f}, g, \mathbf{g})^{\top} \in \mathcal{H}$ . Eliminating  $\psi$  and  $\mathbf{v}$  via  $\psi = \mu\phi - f$  and  $\mathbf{v} = \mu\mathbf{u} - \mathbf{f}$  with the first two equations, the problem on  $\phi, \mathbf{u}$  reads

$$\begin{aligned} \text{(b)} \quad \mu^2\phi - \Delta\phi &= g + \mu f & \text{in } \Omega, & \quad \text{(e)} \quad \partial_n\phi - \mu\mathbf{u} \cdot \mathbf{n} = -\mathbf{f} \cdot \mathbf{n} & \text{on } \Gamma, \\ \text{(d)} \quad \mu^2\mathbf{u} - \rho_s^{-1}\Delta_s\mathbf{u} &= \mathbf{g} + \mu\mathbf{f} & \text{in } \Omega_s, & \quad \text{(f)} \quad \mathbf{t}[\mathbf{u}] - \mu\phi\mathbf{n} = -f\mathbf{n} & \text{on } \Gamma, \end{aligned}$$

and is set in variational form as

$$\text{Find } (\phi, \mathbf{u}) \in \mathbf{H}_{\Delta}^1, \quad \mathcal{A}(\phi, \mathbf{u}; \tilde{\phi}, \tilde{\mathbf{u}}) = \mathcal{F}(\tilde{\phi}, \tilde{\mathbf{u}}) \quad \text{for all } (\tilde{\phi}, \tilde{\mathbf{u}}) \in \mathbf{H}^1 \quad (24)$$

where  $\mathcal{A}$  is the bilinear form in problem (21) and

$$\mathcal{F}(\tilde{\phi}, \tilde{\mathbf{u}}) = (g + \mu f, \tilde{\phi}) + (\mathbf{f} \cdot \mathbf{n}, \tilde{\phi})_{\Gamma} + (\mathbf{g} + \mu\mathbf{f}, \tilde{\mathbf{u}}) - (f\mathbf{n}, \tilde{\mathbf{u}})_{\Gamma}.$$

The bilinear form  $\mathcal{A}$  is already known to be coercive on  $\mathbf{H}^1$ , and the linear functional  $\mathcal{F}$  is continuous on  $\mathbf{H}^1$  for any  $\mathbb{F} \in \mathcal{H}$ . Problem (24) is hence uniquely solvable by the Lax-Milgram theorem. Elliptic regularity then shows that  $(\phi, \mathbf{u}) \in \mathbf{H}_{\Delta}^1$ , whereupon reconstructing  $(\psi, \mathbf{v})$  with the remaining equations yields  $(\psi, \mathbf{v}) \in \mathbf{H}^1$ . The system  $(\mathbf{A} + \mu\mathbf{l})\mathbb{U} = \mathbb{F}$  is therefore uniquely solvable in  $\mathcal{D}(\mathbf{A})$  for any  $\mathbb{F} \in \mathcal{H}$ .

3. *Conclusion.* Choosing  $\mu = \lambda + 1$ , items 1, 2 above show that  $\mathbf{A} + \lambda\mathbf{l} : \mathcal{D}(\mathbf{A}) \rightarrow \mathcal{H}$  is maximal monotone for any  $\lambda \geq \frac{1}{2}$ . The Hille-Yosida theorem hence applies to the generic system (22):

**Proposition 2** [13, Chap. II, Theorem 1.3] *Assume that either  $\mathbb{F} \in C_T^1(\mathcal{H})$  or  $\mathbb{F} \in C_T^0(\mathcal{D}(\mathbf{A}))$ . Then, the system (22) has a unique solution  $\mathbb{Z} \in C_T^1(\mathcal{H}) \cap C_T^0(\mathcal{D}(\mathbf{A}))$ .*

At this point, we have shown that the system (22) has a unique solution for any  $\mathbb{F}$  with sufficient regularity. We now aim at finding a strong solution  $\mathbb{U}$  of the first-order form IVTP (15) under weaker regularity assumption on  $\mathbb{H}$  (which corresponds to our data). According to Lemma 2, we know that if  $\mathbb{H} \in C_T^2(\mathbf{H}^{-1/2})$  with  $\mathbb{H}(0) = 0$ , we have  $\mathbb{U}_L \in C_T^2(\mathcal{H})$  with  $\mathbb{U}_L(0) = 0$ . Hence, system (22) with  $\mathbb{F} = \mu\mathbb{U}_L - \mathbb{U}'_L$  has a unique strong solution  $\mathbb{U}_c$ . Moreover, by Lemma 4, the FSI strong solution is given by  $\mathbb{U} = \mathbb{U}_L + \mathbb{U}_c$  and  $\mathbb{U} \in C_T^1(\mathcal{H}) \cap C_T^0(\mathcal{D}(\mathbf{A}))$ .

**5.4 Energy estimates.** To show the energy estimates and then define the weak solution for problem (11), we need in fact to show that  $\mathbb{U}$  admits another representation. We will need the following Lemma:

**Lemma 4** *Let  $\mathbb{Z}$  solve (22) for given  $\mathbb{F}$ . If  $\mathbb{F}(0) = 0$  and either  $\mathbb{F} \in C_T^2(\mathcal{H})$  or  $\mathbb{F} \in C_T^1(\mathcal{D}(\mathbf{A}))$ , we have  $\mathbb{Z} \in C_T^1(\mathcal{D}(\mathbf{A}))$ , and its time derivative  $\mathbb{Z}' \in C_T^1(\mathcal{H}) \cap C_T^0(\mathcal{D}(\mathbf{A}))$  solves (i)  $(\mathbb{Z}')' + \mathbf{A}\mathbb{Z}' = \mathbb{F}'$  and  $\mathbb{B}\mathbb{Z}' = 0$  in  $[0, T]$ , (ii)  $\mathbb{Z}'(0) = 0$ .*

*Proof of Lemma 4.* The system (22) may be differentiated in time (since all quantities are  $C^1$  in time with values in the requisite spaces); moreover,  $\mathbb{Z}'(0) = \mathbb{F}(0) - \mathbf{A}\mathbb{Z}(0) = 0$ .  $\square$

Using the Lemma 4 and introducing  $\mathbb{Z}$  that solves (22) with  $\mathbb{F} = \mathbb{U}_L$ , we have that

$$\mathbb{U} = \mathbb{U}_L + \mu\mathbb{Z} - \mathbb{Z}'. \quad (25)$$

We now establish estimates for the strong solution given by (25) to show that a weaker norm of the strong solution is controlled by a weaker norm of the data.

**Lemma 5** For any  $\mathbb{H} \in C_T^2(\mathbf{H}^{-1/2}(\Gamma))$  with  $\mathbb{H}(0) = 0$ , the strong solution  $\mathbb{U}$  of the FSI system (15) verifies the estimate

$$\sup_{t \in [0, T]} \|\mathbb{U}(t)\|_{\mathcal{H}}^2 \leq C \|\mathbb{H}\|_{H_T^1(\mathbf{H}^{-1/2}(\Gamma))}^2. \quad (26)$$

If  $\mathbb{H} \in C_T^3(\mathbf{H}^{-1/2}(\Gamma))$  with  $\mathbb{H}(0) = \mathbb{H}'(0) = 0$ , we have the higher-regularity estimate

$$\sup_{t \in [0, T]} \|\mathbb{U}(t)\|_{\mathcal{H}_A}^2 \leq C \|\mathbb{H}\|_{H_T^2(\mathbf{H}^{-1/2}(\Gamma))}^2. \quad (27)$$

The constant  $C > 0$  in each estimate depends on  $T$  and  $\Gamma$  but not on  $\mathbb{H}$ .

*Proof of Lemma 5.* We recall the representation (25) of  $\mathbb{U}$ . Testing the system  $\mathbb{Z}' + \mathbf{A}\mathbb{Z} = \mathbb{U}_L$  against  $\mathbb{Z}$ , we have  $(\mathbb{Z}', \mathbb{Z})_{\mathcal{H}} = (\mathbb{U}_L, \mathbb{Z})_{\mathcal{H}} - (\mathbf{A}\mathbb{Z}, \mathbb{Z})_{\mathcal{H}}$  which, using Lemma 3 for  $\mathbb{U} = \mathbb{Z} \in C_T^1(\mathcal{D}(\mathbf{A}))$ , gives

$$2(\mathbb{Z}'(\tau), \mathbb{Z}(\tau))_{\mathcal{H}} \leq 2(\mathbb{U}_L(\tau), \mathbb{Z}(\tau))_{\mathcal{H}} + \|\mathbb{Z}(\tau)\|_{\mathcal{H}}^2$$

Integrating over  $\tau \in [0, t]$ , we thus obtain

$$\|\mathbb{Z}(t)\|_{\mathcal{H}}^2 \leq 2 \int_0^t \|\mathbb{Z}(\tau)\|_{\mathcal{H}}^2 d\tau + \|\mathbb{U}_L\|_{L_T^2(\mathcal{H})}^2 \quad t \in [0, T]. \quad (a)$$

The same derivation applies to  $\mathbb{Z}'$  (by testing against  $\mathbb{Z}'$  the system  $(\mathbb{Z}')' + \mathbf{A}\mathbb{Z}' = \mathbb{U}'_L$  obeyed by  $\mathbb{Z}'$ , see Lemma 4), to obtain

$$\|\mathbb{Z}'(t)\|_{\mathcal{H}}^2 \leq 2 \int_0^t \|\mathbb{Z}'(\tau)\|_{\mathcal{H}}^2 d\tau + \|\mathbb{U}'_L\|_{L_T^2(\mathcal{H})}^2, \quad t \in [0, T]. \quad (b)$$

As usual, Gronwall's lemma plays a key role in the derivation of such estimates. The following version [11, Chap. 18] is used:

**Lemma 6 (Gronwall)** Let the univariate function  $\Phi \in L^\infty([0, T])$  verify  $\Phi(t) \geq 0$  a.e. in  $[0, T]$ . Assume in addition that the inequality

$$\Phi(t) \leq C_1 \int_0^t \Phi(s) ds + C_2$$

holds a.e. in  $[0, T]$  for some constants  $C_1, C_2 \geq 0$ . Then:

$$\Phi(t) \leq C_2 \exp(C_1 t).$$

Lemma 6 is applicable to inequalities (a) and (b) with, respectively,  $\Phi(t) = \|\mathbb{Z}(t)\|_{\mathcal{H}}^2$  and  $\Phi(t) = \|\mathbb{Z}'(t)\|_{\mathcal{H}}^2$ . Consequently there exists  $C > 0$  such that

$$\|\mathbb{Z}(t)\|_{\mathcal{H}}^2 \leq C \|\mathbb{U}_L\|_{L_T^2(\mathcal{H})}^2, \quad \|\mathbb{Z}'(t)\|_{\mathcal{H}}^2 \leq C \|\mathbb{U}'_L\|_{L_T^2(\mathcal{H})}^2 \quad t \in [0, T]. \quad (c)$$

Since  $\mathbb{U}_L \in C_T^2(\mathcal{H})$ , estimate (26) follows by using (c) in the solution representation (25) and Lemma 2.

If  $\mathbb{H}$  in fact verifies the given higher-regularity assumptions and initial conditions, the estimates (26) apply to both  $\mathbb{U}$  (with datum  $\mathbb{H}$ ) and  $\mathbb{U}'$  (with datum  $\mathbb{H}'$ ). Since  $\mathbb{U}'(t) = \mathbf{A}\mathbb{U}(t)$  holds in  $\mathcal{H}$ , we similarly obtain estimate (27) from

$$\|\mathbb{U}(t)\|_{\mathcal{H}_A}^2 = \|\mathbb{U}(t)\|_{\mathcal{H}}^2 + \|\mathbf{A}\mathbb{U}(t)\|_{\mathcal{H}}^2 = \|\mathbb{U}(t)\|_{\mathcal{H}}^2 + \|\mathbb{U}'(t)\|_{\mathcal{H}}^2 \leq C \|\mathbb{H}\|_{H_T^2(\mathbf{H}^{-1/2}(\Gamma))}^2 \quad t \in [0, T]. \quad \square$$

**5.5 Existence and uniqueness of a weak solution.** Estimate (26) shows that the strong solution  $\mathbb{U}$  with data  $\mathbb{H} \in C_T^2(\mathbf{H}^{-1/2})$  found in Section 5.3 in fact has its weaker  $C_T^0(\mathcal{H})$  norm controlled by the weaker  $H_T^1(\mathbf{H}^{-1/2}(\Gamma))$  norm of  $\mathbb{H}$ . This allows to obtain a solvability result for the FSIP in variational form, under weaker regularity assumptions on  $\mathbb{H}$  (which is the main goal of this contribution).

To this aim, let  $\mathbb{H} \in H_T^1(\mathbf{H}^{-1/2}(\Gamma))$  with  $\mathbb{H}(0) = 0$  be some transmission data. By a density argument<sup>1</sup> there exists a sequence  $\mathbb{H}_n \in C_T^2(\mathbf{H}^{-1/2}(\Gamma))$  with  $\mathbb{H}_n(0) = 0$  such that  $\|\mathbb{H}_n - \mathbb{H}\|_{H_T^1(\mathbf{H}^{-1/2}(\Gamma))} \rightarrow 0$ , and a lifting  $\mathbb{U}_L^n$  can be associated using (20) to each  $\mathbb{H}_n$ . Using Proposition 2, the FSI system (15) has for each datum  $\mathbb{H}_n$  a unique solution  $\mathbb{U}_n \in C_T^1(\mathcal{H}) \cap C_T^0(\mathcal{D}(\mathbf{A}))$ , given by (25) with  $\mathbb{Z}$  solving (22) for  $\mathbb{F} = \mathbb{U}_L^n$ . Applying (by linear superposition) the estimate (26) to the data  $\mathbb{H}_n - \mathbb{H}_m$  and corresponding solution  $\mathbb{U}_n - \mathbb{U}_m$ , we readily find that  $(\mathbb{U}_n)$  is a Cauchy sequence in  $C_T^0(\mathcal{H})$ . Upon taking the limit  $n \rightarrow \infty$  in that estimate, the limit  $\mathbb{U} \in C_T^0(\mathcal{H})$  of  $\mathbb{U}^n$  satisfies

$$\sup_{t \in [0, T]} \|\mathbb{U}(t)\|_{\mathcal{H}}^2 \leq C \|\mathbb{H}\|_{H_T^1(\mathbf{H}^{-1/2}(\Gamma))}^2,$$

and defines the expected solution of the variational problem (11). We now need to prove that this is actually the case, and that  $\mathbb{U}$  is the only such solution.

1.  $\mathbb{U}$  defines a solution of the variational problem (11). Let  $(\tilde{\phi}, \tilde{\mathbf{u}}) \in \mathbf{H}^1$  be a pair of time-independent functions, and let  $\varphi \in C_0^\infty([0, T])$ . Testing against  $\tilde{\phi}\varphi$  and  $\tilde{\mathbf{u}}\varphi$ , respectively, the wave equations verified (by virtue of (15)) in  $L^2(\Omega \times [0, T])$  by each  $\phi_n$  and in  $L^2(\Omega_s \times [0, T])$  by each  $\mathbf{u}_n$ , we have

$$\int_0^T (-\Delta\phi_n + \phi_n'', \tilde{\phi})_{\Omega} \varphi(t) dt + \int_0^T (-\Delta_s \mathbf{u}_n + \rho_s \mathbf{u}_n'', \tilde{\mathbf{u}})_{\Omega_s} \varphi(t) dt = 0.$$

We first apply the Green identities (10a,b) (which is valid since each  $(\phi_n, \mathbf{u}_n)$  belongs to  $C_T^0(\mathbf{H}_\Delta^1)$ ) and express  $\partial_n \phi_n$  and  $\mathbf{t}[\mathbf{u}_n]$  by means of the transmission conditions (which they verify in the  $L_T^2(H^{-1/2}(\Gamma))$  sense), to obtain

$$\begin{aligned} \int_0^T [a(\phi_n, \tilde{\phi}) + b(\phi_n'', \tilde{\phi}) + A(\mathbf{u}_n, \tilde{\mathbf{u}}) + B(\mathbf{u}_n'', \tilde{\mathbf{u}}) + I(\mathbf{u}', \tilde{\phi}) + I_s(\phi', \tilde{\mathbf{u}})] \varphi(t) dt \\ = \int_0^T [(\mathbf{h}_n(t), \tilde{\mathbf{u}})_{\Gamma} - (\mathbf{v}_n(t), \tilde{\phi})_{\Gamma}] \varphi(t) dt, \end{aligned}$$

in terms of the bilinear forms  $a, b, A, B$  defined by (9) and  $I, I_s$  by (12). Then, all time derivatives are transferred to  $\varphi$  via integrations by parts, yielding (since  $\varphi \in C_0^\infty([0, T])$ )

$$\begin{aligned} \int_0^T [a(\phi_n, \tilde{\phi}) + A(\mathbf{u}_n, \tilde{\mathbf{u}})] \varphi(t) dt + \int_0^T [b(\phi_n, \tilde{\phi}) + B(\mathbf{u}_n, \tilde{\mathbf{u}})] \varphi''(t) dt \\ - \int_0^T [I(\mathbf{u}_n, \tilde{\phi}) + I_s(\phi_n, \tilde{\mathbf{u}})] \varphi'(t) dt = \int_0^T [(\mathbf{h}_n(t), \tilde{\mathbf{u}})_{\Gamma} - (\mathbf{v}_n(t), \tilde{\phi})_{\Gamma}] \varphi(t) dt. \end{aligned}$$

Since  $\mathbb{H}_n \rightarrow \mathbb{H}$  in  $L_T^2(\mathbf{H}^{-1/2}(\Gamma))$  (by assumption), which implies  $\mathbb{U}_n \rightarrow \mathbb{U}$  in  $L_T^2(\mathcal{H})$ , and using the continuity of the bilinear forms  $a, b, A, B, I, I_s$ , taking the limit  $n \rightarrow \infty$  in the above identity gives

$$\begin{aligned} \int_0^T [a(\phi, \tilde{\phi}) + A(\mathbf{u}, \tilde{\mathbf{u}})] \varphi(t) dt + \int_0^T [b(\phi, \tilde{\phi}) + B(\mathbf{u}, \tilde{\mathbf{u}})] \varphi''(t) dt \\ - \int_0^T [I(\mathbf{u}, \tilde{\phi}) + I_s(\phi, \tilde{\mathbf{u}})] \varphi'(t) dt = \int_0^T [(\mathbf{h}(t), \tilde{\mathbf{u}})_{\Gamma} - (\mathbf{v}(t), \tilde{\phi})_{\Gamma}] \varphi(t) dt \end{aligned}$$

for any  $\varphi \in C_0^\infty([0, T])$ . The components of  $\mathbb{U}$  therefore satisfy the variational formulation (11) as an equality in  $\mathcal{D}'([0, T])$ .

2. *Uniqueness.* Assume that the variational formulation (11) has, for the same datum  $\mathbb{H}$ , two distinct solutions  $\mathbb{U}_1$  and  $\mathbb{U}_2$  satisfying initial-rest conditions. By linearity, the components of  $\mathbb{W} := \mathbb{U}_1 - \mathbb{U}_2$  must then solve the variational problem (11) with  $\mathbb{H} = 0$ . The function

<sup>1</sup>Take any  $g \in H_T^1(H^{-1/2}(\Gamma))$  with  $g(0) = 0$ . Letting  $\hat{g}(t) = g(T)t/T$ , we have  $g - \hat{g} \in H_0^1([0, T]; H^{-1/2}(\Gamma))$  and  $\hat{g} \in C_T^2(H^{-1/2}(\Gamma)) \subset H_T^1(H^{-1/2}(\Gamma))$ . Since  $C_c^2([0, T]; H^{-1/2}(\Gamma))$  is dense in  $H_0^1([0, T]; H^{-1/2}(\Gamma))$ , see e.g. [7, Sec. 8.3], approximating sequences  $g_n$  of  $g$  exist as claimed.

$\mathbb{Z}(t) := \int_0^t \mathbb{W}(s) ds$  is also at initial-rest and (by integration over the time interval  $[0, t]$ ) solves the same homogeneous variational problem. Moreover, due to the integration in time,  $\mathbb{Z} \in C_T^0(\mathcal{D}(\mathbf{A})) \cap C_T^1(\mathcal{H})$ , i.e., is a strong solution of the homogeneous evolution problem. By Proposition 2, we must hence have  $\mathbb{Z} = 0$ , implying  $\mathbb{W} = 0$ . This proves the uniqueness of the weak solution. The proof of the first part of Theorem 1 is complete.

3. *Data with higher time regularity.* If in fact  $\mathbb{H} \in H_T^2(\mathbf{H}^{-1/2}(\Gamma))$  with  $\mathbb{H}(0) = \mathbb{H}'(0) = 0$ , the previous analysis applies to both  $\mathbb{H}$  and  $\mathbb{H}'$ , so that  $(\phi, \mathbf{u}) \in C_T^1(\mathbf{H}^1)$  and  $(\phi', \mathbf{u}') \in C_T^1(\mathbf{L}^2)$ . In particular,  $(\phi'', \mathbf{u}'') \in C_T^0(\mathbf{L}^2)$ . Since, in addition, each  $(\phi_n, \mathbf{u}_n)$  in the limiting process verifies the homogeneous wave equation, the limit  $(\phi, \mathbf{u})$  satisfies  $(\Delta\phi, \mathbf{\Delta}_s \mathbf{u}) \in C_T^0(\mathbf{L}^2)$ . Hence  $(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}_\Delta^1)$ , and the claimed regularity for the second part of Theorem 1 follows.

## 6 Proof of Theorem 2.

The proof method for Theorem 2 relies on the general steps previously used for Theorem 1. The first-order form (15) of the IVTP, which reads

$$\mathbb{U}' + \mathbf{A}\mathbb{U} = 0, \quad \mathbb{U}(0) = 0, \quad \mathbb{B}\mathbb{U} = \mathbb{H} \quad (28)$$

(repeated for convenience), again serves as the point of departure. To account for the assumed additional regularity in space of the data  $\mathbb{H}$ ,  $\mathbf{A}$  is now considered as an operator on  $\mathcal{D}(\mathbf{A})$  with domain  $\mathcal{D}(\mathbf{A}^2)$ , where

$$\mathcal{D}(\mathbf{A}^2) := \{ \mathbb{U} \in \mathcal{D}(\mathbf{A}) \text{ such that } \mathbf{A}\mathbb{U} \in \mathcal{D}(\mathbf{A}) \subset \mathcal{H}_\mathbf{A} \}, \quad \|\mathbb{U}\|_{\mathcal{D}(\mathbf{A}^2)}^2 := \|\mathbb{U}\|_{\mathcal{H}_\mathbf{A}}^2 + \|\mathbf{A}\mathbb{U}\|_{\mathcal{H}_\mathbf{A}}^2. \quad (29)$$

The main modifications in the proof steps, relative to Section 5, then result from replacing the spaces  $\mathcal{H}$  and  $\mathcal{D}(\mathbf{A})$  with  $\mathcal{D}(\mathbf{A})$  and  $\mathcal{D}(\mathbf{A}^2)$ , respectively, in the application of the Hille-Yosida theorem to the system (28) (i.e., in Proposition 2). We note that the space  $\mathcal{D}(\mathbf{A}^2)$  embeds not only the interfacial constraints  $\mathbb{B}\mathbb{U} = 0$  given by (18), but also the additional higher-order constraints  $\mathbb{B}(\mathbf{A}\mathbb{U}) = 0$ , i.e.

$$(a) \quad \partial_n \psi - \rho_s^{-1} \mathbf{n} \cdot \mathbf{\Delta}_s \mathbf{u} = 0, \quad (b) \quad \mathbf{t}[\mathbf{v}] - \mathbf{n} \Delta \phi = \mathbf{0}, \quad (30)$$

as essential conditions. The following counterpart of the density property of Lemma 1, proved in 8.3, is verified by  $\mathcal{D}(\mathbf{A}^2)$ :

**Lemma 7** *The space  $\mathcal{D}(\mathbf{A}^2)$  is a dense subspace of both  $(\mathcal{H}_\mathbf{A}, \|\cdot\|_{\mathcal{H}_\mathbf{A}})$  and  $(\mathcal{D}(\mathbf{A}), \|\cdot\|_{\mathcal{H}_\mathbf{A}})$ .*

**6.1 Interface data lifting.** Towards applying the Hille-Yosida theorem, problem (28) needs as before to be recast as a first-order system with homogeneous transmission conditions. We define again the lifting  $\mathbb{U}_L$  of  $\mathbb{H}$  by the system (20). The extra regularity in space of  $\mathbb{H}$  directly translates into corresponding extra regularity for  $\mathbb{U}_L$ : adapting Lemma 2 and using elliptic regularity at each  $t$ , if  $\mathbb{H} \in X([0, T]; \mathbf{H}^{1/2}(\Gamma))$  for some Banach space  $X$ , the system (20) has a unique solution  $\mathbb{U}_L \in X([0, T]; \mathcal{H}_\mathbf{A})$ , verifying  $\|\mathbb{U}_L\|_{X([0, T]; \mathcal{H}_\mathbf{A})} \leq C \|\mathbb{H}\|_{X([0, T]; \mathbf{H}^{1/2}(\Gamma))}$ .

**6.2 Application of the Hille-Yosida theorem.** We apply again the Hille-Yosida theorem to the generic problem (22) using the unknown  $\mathbb{U}_c := \mathbb{U} - \mathbb{U}_L$  and the data  $\mathbb{F} := \mu \mathbb{U}_L - \mathbb{U}'_L$ . The definition (29) of  $\mathcal{D}(\mathbf{A})$  is such that the Hille-Yosida theorem still applies to  $\mathbf{A} : \mathcal{D}(\mathbf{A}^2) \rightarrow \mathcal{D}(\mathbf{A})$ , via a direct transposition of the arguments of Section 5 and with the help of the following adaptation of Lemma 3:

**Lemma 8** *For any  $\mathbb{U} \in \mathcal{D}(\mathbf{A}^2)$ , we have*

$$(a) \quad (\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}_\mathbf{A}} = -b(\psi, \phi) - B(\mathbf{v}, \mathbf{u}) - b(\Delta\phi, \psi) - B(\mathbf{\Delta}_s \mathbf{u}, \mathbf{v}), \quad (b) \quad 2|(\mathbf{A}\mathbb{U}, \mathbb{U})_{\mathcal{H}_\mathbf{A}}| \leq \|\mathbb{U}\|_{\mathcal{H}_\mathbf{A}}^2.$$

*Proof.* Lemma (8) is obtained by using identity (a) of Lemma 3 with  $\mathbb{U}$  replaced by  $\mathbf{A}\mathbb{U}$ .  $\square$



Then we can verify that all conditions of the Hille-Yosida theorem are satisfied:

1. *Monotonicity.* Thanks to Lemma 8, identity (23) also holds for  $\mathbf{A}\mathbb{U}$  instead of  $\mathbb{U}$ , replacing all components of  $\mathbb{U}$  by those of  $\mathbf{A}\mathbb{U}$  in the right-hand side. The monotonicity of  $\mathbf{A}_\lambda := \mathbf{A} + \lambda\mathbf{l} : \mathcal{D}(\mathbf{A}^2) \rightarrow \mathcal{D}(\mathbf{A})$  follows in the same way, still subject to  $\lambda \geq \frac{1}{2}$ .

2. *Surjectivity.* The solvability in  $\mathcal{D}(\mathbf{A}^2)$  of  $(\mathbf{A} + \mu\mathbf{l})\mathbb{U} = \mathbb{F}$  for given  $\mathbb{F} \in \mathcal{D}(\mathbf{A})$  is still decided by the variational problem (24). The regularity assumption  $\mathbb{F} \in \mathcal{D}(\mathbf{A})$  implies the desired smoothness of its unique solution  $\mathbb{U}$ , namely  $\mathbb{U}, \mathbf{A}\mathbb{U}, \mathbf{A}^2\mathbb{U}$  being in  $\mathcal{H}$ , by elliptic regularity. This regularity in turn allows each component of the equality  $\mathbf{A}\mathbb{U} + \eta\mathbb{U} = \mathbb{F}$  to hold in  $H^{1/2}(\Gamma)$ , and we use those relations to compute

$$\mathbf{t}[v_L] - (\Delta\phi_L)\mathbf{n} = \mathbf{g}\mathbf{n} - \mathbf{t}[\mathbf{f}] = \mathbf{0}, \quad \partial_n\psi_L - (\rho_s^{-1}\Delta_s\mathbf{u}_L) \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} - \partial_n f = 0 \quad \text{on } \Gamma,$$

since the constraints (18) are satisfied by both  $\mathbb{U}$  (by the definition of problem (24)) and  $\mathbb{F} \in \mathcal{D}(\mathbf{A})$ . Concluding,  $(\mathbf{A} + \mu\mathbf{l})\mathbb{U} = \mathbb{F}$  has a unique solution  $\mathbb{U} \in \mathcal{D}(\mathbf{A}^2)$  for any  $\mathbb{F} \in \mathcal{D}(\mathbf{A})$ .

3. *Conclusion.* Choosing  $\mu = \lambda + 1$ ,  $\mathbf{A} + \lambda\mathbf{l} : \mathcal{D}(\mathbf{A}^2) \rightarrow \mathcal{D}(\mathbf{A})$  is maximal monotone for any  $\lambda \geq \frac{1}{2}$ . The Hille-Yosida theorem hence applies to the generic system (22). Proposition 2 on its strong solvability now holds with  $\mathcal{H}, \mathcal{D}(\mathbf{A})$  replaced by  $\mathcal{D}(\mathbf{A}), \mathcal{D}(\mathbf{A}^2)$ .

Following the same arguments as in Section 5, the FSI solution  $\mathbb{U} = \mathbb{U}_L + \mathbb{U}_c$  is unique and the FSI system (15) then admits a unique solution  $\mathbb{U} \in C_T^1(\mathcal{D}(\mathbf{A})) \cap C_T^0(\mathcal{D}(\mathbf{A}^2))$ .

**6.3 Energy estimates.** To derive energy estimates, we start by observing that for any  $\mathbb{F} \in C_T^1(\mathcal{D}(\mathbf{A}))$ , the solution  $\mathbb{Z} \in C_T^0(\mathcal{D}(\mathbf{A}^2))$  of the generic system (22) verifies

$$(a) \quad \sup_{t \in [0, T]} \|\mathbb{Z}(t)\|_{\mathcal{H}_A}^2 \leq C \|\mathbb{F}\|_{L_T^2(\mathcal{H}_A)}^2, \quad (b) \quad \sup_{t \in [0, T]} \|\mathbb{Z}'(t)\|_{\mathcal{H}_A}^2 \leq C \|\mathbb{F}'\|_{L_T^2(\mathcal{H}_A)}^2. \quad (31)$$

Both inequalities stem from repeating the proof of Lemma 5 leading to inequalities (c) there, with  $\mathcal{H}, \mathcal{D}(\mathbf{A})$  replaced by  $\mathcal{H}_A, \mathcal{D}(\mathbf{A}^2)$  and invoking the improved Green identity of Lemma 8.

Let now  $\mathbb{H} \in C_T^2(\mathbf{H}^{1/2}(\Gamma))$  with  $\mathbb{H}(0) = 0$ , so that  $\mathbb{U}_L \in C_T^2(\mathcal{H}_A)$  and  $\mathbb{U}_L(0) = 0$ . Invoking Lemma 7, there exists a sequence  $\mathbb{U}_L^m \in C_T^2(\mathcal{D}(\mathbf{A}^2))$  with  $\mathbb{U}_L^m(0) = 0$  such that  $\|\mathbb{U}_L - \mathbb{U}_L^m\|_{C_T^2(\mathcal{H}_A)} \rightarrow 0$ . Let then  $\mathbb{Z}_m \in C_T^1(\mathcal{D}(\mathbf{A})) \cap C_T^0(\mathcal{D}(\mathbf{A}^2))$  solve (22) with  $\mathbb{F} = \mathbb{U}_L^m$ . The representation (25) of  $\mathbb{U}$  solving the FSI system (15) suggests to define the approximating sequence  $\mathbb{U}^m$  given by either

$$(a) \quad \mathbb{U}^m = \eta\mathbb{Z}_m - \mathbb{Z}'_m + \mathbb{U}_L, \quad (b) \quad \mathbb{U}^m = (\eta\mathbf{l} + \mathbf{A})\mathbb{Z}_m + (\mathbb{U}_L - \mathbb{U}_L^m) \quad (32)$$

(with case (b) obtained by using in (a) the equality  $\mathbb{Z}'_m = \mathbb{U}_L^m - \mathbf{A}\mathbb{Z}_m$  in  $\mathcal{H}_A$ ).

Applying estimate (31a) with  $\mathbb{F} = \mathbb{U}_L^m$  to (32b), we have

$$\sup_{t \in [0, T]} \|\mathbb{U}^m(t)\|_{\mathcal{H}}^2 \leq \sup_{t \in [0, T]} C \|\mathbb{Z}_m(t)\|_{\mathcal{H}_A}^2 \leq C \|\mathbb{U}_L^m\|_{L_T^2(\mathcal{H}_A)}^2. \quad (33)$$

Alternatively, estimates (31a,b) with  $\mathbb{F} = \mathbb{U}_L^m$  applied to (32a) give

$$\sup_{t \in [0, T]} \|\mathbb{U}^m(t)\|_{\mathcal{H}_A}^2 \leq C \|\mathbb{U}_L^m\|_{H_T^1(\mathcal{H}_A)}^2. \quad (34)$$

**Remark 4** *By contrast with the proof of Theorem 1, estimates (33) and (34) cannot be directly obtained for the FSI solution  $\mathbb{U}$  from its representation (25), as  $\mathbb{Z}$  in the latter does not verify  $\mathbb{B}(\mathbf{A}\mathbb{Z}) = 0$  and thus is not in  $\mathcal{D}(\mathbf{A}^2)$ , preventing the use of Lemma 8b. Identity (a) in Lemma 8 could of course be augmented with interfacial terms (so as to remove the requirement  $\mathbb{B}(\mathbf{A}\mathbb{U}) = 0$  there) but the resulting appearance of higher-order derivatives in interfacial terms would prove equally problematic. Hence our recourse, permitted by the density result of Lemma 7, to approximations (32) of  $\mathbb{U}$  that have the correct interfacial traces while allowing to invoke Lemma 8.*

**6.4 Variational problem.** We note that  $\mathbb{U}^m \in C_T^1(\mathcal{D}(\mathbf{A})) \cap C_T^0(\mathcal{D}(\mathbf{A}^2))$  defined by (32) solves the system (i)  $\mathbb{U}'_m + \mathbf{A}\mathbb{U}_m = \mathbb{F}_m$ , (ii)  $\mathbb{U}_m(0) = 0$ , (iii)  $\mathbf{B}\mathbb{U} = \mathbb{H}$ , with the components  $(f_m, \mathbf{f}_m, g_m, \mathbf{g}_m)$  of  $\mathbb{F}_m$  given by

$$\begin{aligned} f_m &= \mu(\phi_L^m - \phi_L) - (\phi_L^{m'} - \phi_L'), & \mathbf{f}_m &= \mu(\mathbf{u}_L^m - \mathbf{u}_L) - (\mathbf{u}_L^{m'} - \mathbf{u}_L'), \\ g_m &= \mu(\psi_L^m - \psi_L) - (\psi_L^{m'} - \psi_L'), & \mathbf{g}_m &= \mu(\mathbf{v}_L^m - \mathbf{v}_L) - (\mathbf{v}_L^{m'} - \mathbf{v}_L'). \end{aligned}$$

Consequently, after eliminating  $\psi_m, \mathbf{v}_m$ , the remaining unknowns  $\phi_m, \mathbf{u}_m$  of  $\mathbb{U}_m$  are found to satisfy the inhomogeneous wave equations

$$-\Delta\phi_m + \phi_m'' = f_m' + g_m \quad \text{in } \Omega, \quad -\Delta_s \mathbf{u}_m + \rho_s \mathbf{u}_m'' = \rho_s(\mathbf{f}_m' + \mathbf{g}_m) \quad \text{in } \Omega_s.$$

Proceeding as in Section 5.5, we take weighted residuals of the above equations using the same test functions  $\tilde{\phi}\varphi$  and  $\tilde{\mathbf{u}}\varphi$ , apply the Green identities (10a,b), use the transmission conditions and transfer all time derivatives to  $\varphi$  via integration by parts. This results in the identity

$$\begin{aligned} & \int_0^T \left\{ [a(\phi_m, \tilde{\phi}) + A(\mathbf{u}_m, \tilde{\mathbf{u}})]\varphi(t) + [b(\phi_m, \tilde{\phi}) + B(\mathbf{u}_m, \tilde{\mathbf{u}})]\varphi''(t) - [I(\mathbf{u}_m, \tilde{\phi}) + I_s(\phi_m, \tilde{\mathbf{u}})]\varphi'(t) \right\} dt \\ &= \int_0^T [(\mathbf{h}(t), \tilde{\mathbf{u}})_\Gamma - (\mathbf{v}(t), \tilde{\phi})_\Gamma]\varphi(t) dt + \int_0^T [b(f_m' + g_m, \tilde{\phi}) + B(\mathbf{f}_m' + \mathbf{g}_m, \tilde{\mathbf{u}})]\varphi(t) dt. \end{aligned} \quad (35)$$

Either estimate (33) or (34) implies that  $\mathbb{U}^m$  is a Cauchy sequence in  $C_T^0(\mathcal{H})$ , while the last integral in (35) vanishes as  $m \rightarrow \infty$  by the definition of  $\mathbb{U}_L^m$ . Taking the limit  $m \rightarrow \infty$  in (35), the limit  $(\phi, \mathbf{u}) \in \mathbf{H}^1$  of  $(\phi_m, \mathbf{u}_m)_m$  is as a result found to verify the variational formulation (11) for any datum  $\mathbb{H} \in C_T^2(\mathbf{H}^{1/2}(\Gamma))$ . Moreover, taking the limit  $m \rightarrow \infty$  in estimates (33) and (34), the limit  $\mathbb{U}$  verifies the estimates

$$(a) \sup_{t \in [0, T]} \|\mathbb{U}(t)\|_{\mathcal{H}}^2 \leq C \|\mathbb{U}_L\|_{L_T^2(\mathcal{H}_A)}^2, \quad (b) \sup_{t \in [0, T]} \|\mathbb{U}(t)\|_{\mathcal{H}_A}^2 \leq C \|\mathbb{U}_L\|_{H_T^1(\mathcal{H}_A)}^2. \quad (36)$$

Then, considering some transmission data  $\mathbb{H} \in L_T^2(\mathbf{H}^{1/2}(\Gamma))$ , there exists a sequence  $\mathbb{H}_n \in C_T^2(\mathbf{H}^{1/2}(\Gamma))$  with  $\mathbb{H}_n(0) = 0$  such that  $\|\mathbb{H}_n - \mathbb{H}\|_{L_T^2(\mathbf{H}^{1/2}(\Gamma))} \rightarrow 0$ . A lifting  $\mathbb{U}_L^n \in C_T^2(\mathcal{H}_A)$  can be associated to each  $\mathbb{H}_n$ , leading to  $\mathbb{U}_n$  solving the variational problem (11) for that data. The estimate (36a) shows that  $\mathbb{U}_n$  defines a Cauchy sequence in  $C_T^0(\mathcal{H})$ , whose limit  $\mathbb{U} \in C_T^0(\mathcal{H})$  also satisfies the variational formulation (11) and depends continuously on  $\|\mathbb{H}\|_{L_T^2(\mathbf{H}^{1/2}(\Gamma))}$ .

Similarly, approximating  $\mathbb{H} \in H_T^1(\mathbf{H}^{1/2}(\Gamma))$  with  $\mathbb{H}(0) = 0$  by a sequence  $\mathbb{H}_n \in C_T^2(\mathbf{H}^{1/2}(\Gamma))$  with  $\mathbb{H}_n(0) = 0$ ,  $\mathbb{U}_n$  is, by the estimate (36b) applied to  $(\mathbb{U}_n, \mathbb{U}_L^n)$ , a Cauchy sequence in  $C_T^0(\mathcal{H}_A)$ . Its limit  $\mathbb{U}$  satisfies the variational formulation (11) and, by taking the limit  $n \rightarrow \infty$  in (36b), depends continuously on  $\|\mathbb{H}\|_{H_T^1(\mathbf{H}^{1/2}(\Gamma))}$ . Moreover, we have  $(\phi'', \mathbf{u}'') = (\Delta\phi, \rho_s^{-1}\Delta_s \mathbf{u}) \in C_T^0(\mathbf{L}^2)$ . Thanks to the latter and the assumed regularity of  $\Gamma$ , elliptic regularity provides  $(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^2)$ .

**6.5 Conclusion.** The IVTP (5) and the variational problem (11) being equivalent (by Prop. 1), the proof of Theorem 2 is complete for the two considered cases.

## 7 Proof of Theorem 3.

(i) *Proof of regularity in  $\Omega$  from interpolation.* Interpolating the data and solution spaces given by Theorem 1 (first case) and Theorem 2 (second case) and invoking the interpolation property for spaces of continuous functions with Hilbert range (see e.g. [21, Chap. 1, Sec. 14.2]), we deduce the continuity of the following data-to-solution mappings:

$$\begin{aligned} (\mathbf{v}, \mathbf{h}) \in [H_T^1(\mathbf{H}^{1/2}(\Gamma)), H_T^1(\mathbf{H}^{-1/2}(\Gamma))]_{1/2} &\rightarrow (\phi, \mathbf{u}) \in [C_T^0(\mathbf{H}^2), C_T^0(\mathbf{H}^1)]_{1/2} \\ &\rightarrow (\phi', \mathbf{u}') \in [C_T^0(\mathbf{H}^1), C_T^0(\mathbf{L}^2)]_{1/2} \end{aligned}$$

where  $[X, Y]_\theta$  denotes the interpolation space with weight  $\theta \in [0, 1]$  (in particular  $[X, Y]_0 = X$  and  $[X, Y]_1 = Y$ ). Moreover, we have

$$\begin{aligned} H_T^1(\mathbf{H}^{1/2}(\Gamma)), H_T^1(\mathbf{H}^{-1/2}(\Gamma)) \Big|_{1/2} &= H_T^1([\mathbf{H}^{1/2}(\Gamma), \mathbf{H}^{-1/2}(\Gamma)]_{1/2}) = H_T^1(\mathbf{L}^2(\Gamma)), \\ [C_T^0(\mathbf{H}^2), C_T^0(\mathbf{H}^1)]_{1/2} &= C_T^0([\mathbf{H}^2, \mathbf{H}^1]_{1/2}) = C_T^0(\mathbf{H}^{3/2}), \\ C_T^0(\mathbf{H}^1), H_T^1(\mathbf{L}^2) \Big|_{1/2} &= C_T^0([\mathbf{H}^1, \mathbf{L}^2]_{1/2}) = C_T^0(\mathbf{H}^{1/2}). \end{aligned}$$

Using these equalities in the above mappings yields the claimed continuous mappings:

$$(\mathbf{v}, \mathbf{h}) \in H_T^1(\mathbf{L}^2(\Gamma)) \rightarrow (\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^{3/2}), \quad (\mathbf{v}, \mathbf{h}) \in H_T^1(\mathbf{L}^2(\Gamma)) \rightarrow (\phi', \mathbf{u}') \in C_T^0(\mathbf{H}^{1/2}).$$

(ii) *Proof of boundary trace of velocity.* The above data-to-velocity mapping falls just short of the applicability of the trace theorem. Useful estimates for boundary traces may however be obtained using the following lemma, proved in Sec. 8.4:

**Lemma 9 (Integral identities on boundary traces)** *Let the interface  $\Gamma$  be a  $C^{1,1}$  closed surface. Let  $\theta \in C_c^1(\bar{D}; \mathbb{R}^d)$  be an extension in a neighborhood  $D$  of  $\Omega_s$  of the unit normal on  $\Gamma$  (such an extension exists, see [20, Chap. I, Lemma 3.1]). Any pair  $(\phi, \mathbf{u})$  solving  $-\Delta\phi + \phi'' = 0$  in  $Q$  and  $-\rho_s^{-1}\Delta_s \mathbf{u} + \mathbf{u}'' = \mathbf{f}$  in  $Q_s$  verifies the integral identity*

$$\|\phi'\|_{\Gamma, T}^2 + \|\mathbf{u}'\|_{\Gamma, T}^2 + \|\partial_n \phi\|_{\Gamma, T}^2 + \|\partial_n \mathbf{u}\|_{\Gamma, T, \mathbf{Q}^{-1}}^2 = \|\nabla_S \phi\|_{\Gamma, T}^2 + \|\nabla_S \mathbf{u}\|_{\Gamma, T, \mathbf{C}}^2 + \mathcal{C}(\phi, \mathbf{u}; \theta), \quad (37)$$

where  $\nabla_S \phi := \nabla \phi - \partial_n \phi \mathbf{n}$  is the tangential gradient of  $\phi$  ( $\nabla_S \phi$  is entirely determined by the boundary trace  $\phi|_\Gamma$  of  $\phi$ , see e.g. [26, Sec. 2.5.6]),  $\mathbf{Q}$  is the symmetric positive definite (spd) Christoffel matrix (defined by  $Q_{ik} = C_{ijkl} n_k n_l$ ),  $\|\mathbf{f}\|_{\Gamma, T, \mathbf{A}}^2 := (\mathbf{A} \cdot \mathbf{f}, \mathbf{f})_{\Gamma, T}$  for  $\mathbf{A}$  spd, and with

$$\begin{aligned} \mathcal{C}(\phi, \mathbf{u}; \theta) &= (|\phi'|^2 - |\nabla \phi|^2, \operatorname{div} \theta)_{\Omega, T} + 2((\nabla \phi \otimes \nabla \phi), \nabla \theta)_{\Omega, T} + 2(\phi', \nabla \phi \cdot \theta)_\Omega \Big|_0^T \\ &+ (|\mathbf{u}'|^2 - \nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u}, \operatorname{div} \theta)_{\Omega_s, T} + 2(\nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u}, \nabla \theta)_{\Omega_s, T} + 2(\mathbf{u}', \nabla \mathbf{u} \cdot \theta)_{\Omega_s} \Big|_0^T. \end{aligned} \quad (38)$$

We then temporarily assume that the data is in  $H_T^1(\mathbf{H}^{1/2}(\Gamma))$ , i.e., sufficiently smooth to lead to a solution smooth enough to justify all integrations by parts producing (37). We first note that

$$|\mathcal{C}(\phi, \mathbf{u}; \theta)| \leq C \|(\mathbf{v}, \mathbf{h})\|_{H_T^1(\mathbf{L}^2(\Gamma))}^2. \quad (a)$$

This follows directly from the interior regularity result of (i) for the space-time norms, and from Theorem 1 for the space norms at  $t = T$ . We also have

$$\|\nabla_S \phi\|_{\Gamma, T}^2 + \|\nabla_S \mathbf{u}\|_{\Gamma, T, \mathbf{C}}^2 \leq C \|(\mathbf{v}, \mathbf{h})\|_{H_T^1(\mathbf{L}^2(\Gamma))}^2 \quad (b)$$

by similar arguments together with:

- available estimates for surface gradients, see e.g. [25, Lemma 4.23]
- the (uniform on  $\Gamma$ ) inequality  $|\partial_n \mathbf{u}|^2 \leq C |\partial_n \mathbf{u} \cdot \mathbf{Q}^{-1} \cdot \partial_n \mathbf{u}| \leq C |\partial_n \mathbf{u}|^2$  exploiting classical ellipticity properties of the elasticity tensor  $\mathbf{C}$ .

Since in addition  $\mathbf{t}[\mathbf{u}] = \mathbf{n} \cdot \mathbf{C} : \nabla_S \mathbf{u} + \mathbf{Q} \cdot \partial_n \mathbf{u}$ , we have

$$\|\mathbf{t}[\mathbf{u}]\|_{\Gamma, T}^2 \leq C (\|\partial_n \mathbf{u}\|_{\Gamma, T}^2 + \|\nabla_S \mathbf{u}\|_{\Gamma, T}^2) \leq C (\|\partial_n \mathbf{u}\|_{\Gamma, T}^2 + \|(\mathbf{v}, \mathbf{h})\|_{H_T^1(\mathbf{L}^2(\Gamma))}^2) \quad (c)$$

We finally use (a), (b) and (c) in (38) and obtain

$$\|\phi'\|_{\Gamma, T}^2 + \|\partial_n \phi\|_{\Gamma, T}^2 + \|\mathbf{u}'\|_{\Gamma, T}^2 + \|\mathbf{t}[\mathbf{u}]\|_{\Gamma, T}^2 \leq C \|(\mathbf{v}, \mathbf{h})\|_{H_T^1(\mathbf{L}^2(\Gamma))}^2.$$

All claims of (ii) finally follow by the density of  $H_T^1(\mathbf{H}^{1/2}(\Gamma))$  in  $H_T^1(\mathbf{L}^2(\Gamma))$ .

**Remark 5** *Equalities such as (37), sometimes referred to as Rellich-Nečas identities, are the basis of hidden regularity results (for the scalar wave case, see [20], Theorem 4.1 and Lemma 3.7 in Chap. I and Lemma 1.3 in Chap. III), as they allow in some cases to infer the regularity boundary traces of weak solutions when their right-hand sides remain well-defined. An elastodynamic identity of this type is given in [16].*

## 8 Additional proofs.

**8.1 Proof of Proposition 1.** 1. *Space-time Green identities* [11, Chap. XVIII]. We recall space-time Green identities for both the fluid and the solid media, for Lipschitz domains  $\Omega_s$  and  $\Omega$  and with the geometrical notations and conventions otherwise as specified in Section 2. Let  $\phi, \psi \in H^1(Q)$ , with  $\psi(0) = \psi(T) = 0$  in  $\Omega$ . If  $-\Delta\phi + \phi'' \in L^2(Q)$ , we have

$$\int_0^T [a(\phi, \psi) - b(\phi', \psi')] dt = (-\Delta\phi + \phi'', \psi)_Q - (\partial_n \phi, \psi)_\Sigma \quad (39a)$$

Likewise, let  $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(Q_s)$ , with  $\mathbf{v}(0) = \mathbf{v}(T) = \mathbf{0}$  in  $\Omega_s$ . If in addition  $-\Delta_s \mathbf{u} + \mathbf{u}'' \in \mathbf{L}^2(Q_s)$ , we have

$$\int_0^T [A(\mathbf{u}, \mathbf{v}) - B(\mathbf{u}', \mathbf{v}')] dt = (-\Delta_s \mathbf{u} + \rho_s \mathbf{u}'', \mathbf{v})_{Q_s} + (\mathbf{t}[\mathbf{u}],_s \mathbf{v})_\Sigma. \quad (39b)$$

2. Let  $(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^1) \cap C_T^1(\mathbf{L}^2)$  solve the IPVB (5). Let  $(\tilde{\phi}, \tilde{\mathbf{u}}) \in \mathbf{H}^1$  be a pair of time-independent functions, and let  $\varphi \in C_0^\infty([0, T])$ . The space-time Green identities (39a) and (39b) applied to  $\phi, \mathbf{u}$  (which in particular satisfy their respective homogeneous wave equation) and the space-time test functions  $\psi(\mathbf{x}, t) := \tilde{\phi}(\mathbf{x})\varphi(t) \in H^1(Q)$  and  $\mathbf{v}(\mathbf{x}, t) := \tilde{\mathbf{u}}(\mathbf{x})\varphi(t) \in \mathbf{H}^1(Q_s)$  yield

$$\begin{aligned} \int_0^T [a(\phi, \tilde{\phi})\varphi - b(\phi', \tilde{\phi})\varphi'] dt &= -(\partial_n \phi, \psi)_\Sigma = \int_0^T [(\mathbf{u} \cdot \mathbf{n}, \tilde{\phi})_\Gamma \varphi' - (\mathbf{v}, \tilde{\phi})_\Gamma \varphi] dt, \\ \int_0^T [A(\mathbf{u}, \tilde{\mathbf{u}})\varphi - B(\mathbf{u}', \tilde{\mathbf{u}})\varphi'] dt &= (\mathbf{t}[\mathbf{u}], \mathbf{v})_\Sigma = \int_0^T [-(\phi \mathbf{n}, \tilde{\mathbf{u}})_\Gamma \varphi' + (\mathbf{h}, \tilde{\mathbf{u}})_\Gamma \varphi] dt \end{aligned}$$

each second equality resulting from the transmission conditions of (5). The above equalities hold for all  $\varphi \in C_0^\infty([0, T])$ , implying (upon introducing the bilinear forms  $I$  and  $I_s$  defined by (12)) that the identities

$$\begin{aligned} a(\phi(t), \tilde{\phi}) + b(\phi''(t), \tilde{\phi}) + I(\mathbf{u}'(t), \tilde{\phi}) &= -(\mathbf{v}(t), \psi)_\Gamma & \text{for all } \tilde{\phi} \in H^1(\Omega), \\ A(\mathbf{u}(t), \tilde{\mathbf{u}}) + B(\mathbf{u}''(t), \tilde{\mathbf{u}}) + I_s(\phi'(t), \tilde{\mathbf{u}}) &= (\mathbf{h}(t), \tilde{\mathbf{u}})_\Gamma & \text{for all } \tilde{\mathbf{u}} \in H^1(\Omega_s), \end{aligned} \quad t \in [0, T]$$

with time-independent test functions  $\tilde{\phi}, \tilde{\mathbf{u}}$  hold in  $\mathcal{D}'([0, T])$ . The variational formulation (11) finally results from summing the above equalities.

3. Conversely, let  $(\phi, \mathbf{u}) \in C_T^0(\mathbf{H}^1) \cap C_T^1(\mathbf{L}^2)$  solve the variational problem (11). Taking  $(\tilde{\phi}, \tilde{\mathbf{u}}) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega_s; \mathbb{R}^d) \subset \mathbf{H}^1$  in (11) (which cancels all interfacial integrals) and testing the resulting (distributional in time) equality by  $\varphi \in C_0^\infty([0, T])$ , the distributions  $(-\Delta\phi + \phi'') \in \mathcal{D}'(Q)$  and  $(-\Delta_s \mathbf{u} + \rho_s \mathbf{u}'') \in \mathcal{D}'(Q_s)$  verify

$$\begin{aligned} \langle -\Delta\phi + \phi'', \tilde{\phi}(\mathbf{x})\varphi(t) \rangle &= 0 & \text{for all } \tilde{\phi} \in C_0^\infty(\Omega), \varphi \in C_0^\infty([0, T]), \\ \langle -\Delta_s \mathbf{u} + \rho_s \mathbf{u}'', \tilde{\mathbf{u}}(\mathbf{x})\varphi(t) \rangle &= 0 & \text{for all } \tilde{\mathbf{u}} \in C_0^\infty(\Omega_s; \mathbb{R}^d), \varphi \in C_0^\infty([0, T]). \end{aligned}$$

Since the set of all linear combinations of tensor-product test functions  $\tilde{\phi}(\mathbf{x})\varphi(t)$  (resp.  $\tilde{\mathbf{u}}(\mathbf{x})\varphi(t)$ ) is dense in  $C_0^\infty(Q)$  (resp.  $C_0^\infty(\Omega_s; \mathbb{R}^d)$ ), the above equalities imply that  $-\Delta\phi + \phi'' = 0$  (in  $\mathcal{D}'(Q)$ ) and  $-\Delta_s \mathbf{u} + \rho_s \mathbf{u}'' = \mathbf{0}$  (in  $\mathcal{D}'(Q_s)$ ), and hence in  $L^2(Q)$  and  $\mathbf{L}^2(Q_s)$  ( $C_0^\infty(Q)$  being dense in  $L^2(Q)$ , and  $C_0^\infty(\Omega_s; \mathbb{R}^d)$  in  $\mathbf{L}^2(Q_s)$ ); the respective homogeneous wave equations are therefore satisfied in the  $L^2$  sense:

$$-\Delta\phi + \phi'' = 0 \quad \text{a.e. in } Q, \quad -\Delta_s \mathbf{u} + \rho_s \mathbf{u}'' = \mathbf{0} \quad \text{a.e. in } Q_s. \quad (40)$$

We now turn to the behavior of  $(\phi, \mathbf{u})$  on  $\Gamma$ , by taking  $(\tilde{\phi}, \tilde{\mathbf{u}}) \in C_0^\infty(\bar{\Omega}) \times C_0^\infty(\bar{\Omega}_s; \mathbb{R}^d) \subset \mathbf{H}^1$  in (11) and testing the resulting (distributional in time) equality by  $\varphi \in C_0^\infty([0, T])$ ; this produces

$$\begin{aligned} \int_0^T [a(\phi(t), \tilde{\phi}) + A(\mathbf{u}(t), \tilde{\mathbf{u}})] \varphi(t) dt - \int_0^T [b(\phi'(t), \tilde{\phi}) + B(\mathbf{u}'(t), \tilde{\mathbf{u}})] \varphi'(t) dt \\ = \int_0^T [(\mathbf{h}, \tilde{\mathbf{u}})_\Gamma - (\mathbf{v}, \tilde{\phi})_\Gamma] \varphi(t) dt - \int_0^T [(\phi \mathbf{n}, \tilde{\mathbf{u}})_\Gamma - (\mathbf{u} \cdot \mathbf{n}, \psi)_\Gamma] \varphi'(t) dt. \end{aligned}$$

In view of (40), the above left-hand side may be evaluated using the space-time Green identities (39a,b), yielding

$$\begin{aligned} \int_0^T \left\{ (\partial_n \phi + \mathbf{v}, \tilde{\phi})_\Gamma \varphi(t) + (\mathbf{u} \cdot \mathbf{n}, \tilde{\phi})_\Gamma \varphi'(t) \right\} dt &= 0 \quad \text{for all } \tilde{\phi} \in C_0^\infty(\Omega), \varphi \in C_0^\infty([0, T]), \\ \int_0^T \left\{ (\mathbf{t}[\mathbf{u}] + \mathbf{h}, \tilde{\mathbf{u}})_\Gamma \varphi(t) + (\phi \mathbf{n}, \tilde{\mathbf{u}})_\Gamma \varphi'(t) \right\} dt &= 0 \quad \text{for all } \tilde{\mathbf{u}} \in C_0^\infty(\Omega_s; \mathbb{R}^d), \varphi \in C_0^\infty([0, T]), \end{aligned}$$

from which we conclude that the transmission conditions in (5) are verified in  $\mathcal{D}'([0, T]; \mathbf{H}^{-1/2}(\Gamma))$ .

**8.2 Proof of Lemma 1.** We prove the lemma by showing that, given any  $\mathbb{U} \in \mathcal{H}$  and  $\varepsilon > 0$ , there exists  $\mathbb{U}_\varepsilon \in \mathcal{D}(\mathbf{A})$  such that  $\|\mathbb{U} - \mathbb{U}_\varepsilon\|_{\mathcal{H}} \leq C\varepsilon$  for some  $C > 0$ . The requirement  $\mathbb{U}_\varepsilon \in \mathcal{D}(\mathbf{A})$  entails that  $\mathbb{U}_\varepsilon$  satisfies the interfacial constraints (18). We may thus settle the issue by separately seeking a pair  $(\phi_\varepsilon, \mathbf{v}_\varepsilon) \in H_\Delta^1(\Omega) \times \mathbf{H}^1(\Omega_s)$  that approximates  $(\phi, \mathbf{v}) \in H^1(\Omega) \times \mathbf{L}^2(\Omega_s)$  and satisfies (18a), and another pair  $(\psi_\varepsilon, \mathbf{u}_\varepsilon) \in H^1(\Omega) \times \mathbf{H}_\Delta^1(\Omega_s)$  that approximates  $(\psi, \mathbf{u}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega_s)$  and satisfies (18b).

*Construction of  $(\phi_\varepsilon, \mathbf{v}_\varepsilon)$ .* We begin by choosing some  $\mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega_s)$  such that  $\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\Omega_s} \leq \varepsilon$  (using the density of  $\mathbf{H}^1(\Omega_s)$  in  $\mathbf{L}^2(\Omega_s)$ ), and then some  $\mathbf{q}_\varepsilon \in \mathbf{H}^{\text{div}}(\Omega)$  such that  $\|\mathbf{q}_\varepsilon - \nabla \phi\|_\Omega \leq \varepsilon$  and  $\mathbf{q}_\varepsilon \cdot \mathbf{n} = \mathbf{v}_\varepsilon \cdot \mathbf{n}$  in  $H^{-1/2}(\Gamma)$  (e.g. by defining an elliptic lifting  $\hat{\phi}$  of the boundary condition  $\partial_n \hat{\phi} = \mathbf{v}_\varepsilon \cdot \mathbf{n}$  on  $\Gamma$ , which is in  $H_\Delta^1(\Omega)$  since  $\mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega_s)$ , then choosing in  $\mathbf{H}_0^{\text{div}}(\Omega)$  an approximation of  $\nabla(\phi - \hat{\phi})$ ). We recall that  $\mathbf{H}_0^{\text{div}}(\Omega)$ , the closure of  $C_0^\infty(\Omega)$  in  $\mathbf{H}^{\text{div}}(\Omega) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \text{div } \mathbf{w} \in L^2(\Omega)\}$ , is dense in  $\mathbf{L}^2(\Omega)$  (see e.g. [1, Thm. 13.1.5]); moreover, the following Green identities hold for any  $(\mathbf{w}, \tilde{\phi}) \in \mathbf{H}^{\text{div}}(\Omega) \times H^1(\Omega)$  and  $(\mathbf{s}, \tilde{\mathbf{u}}) \in \mathbf{H}^{\text{div}}(\Omega_s) \times \mathbf{H}^1(\Omega_s)$ :

$$b(\mathbf{w}, \nabla \tilde{\phi}) + b(\text{div } \mathbf{w}, \tilde{\phi}) = -(\mathbf{w} \cdot \mathbf{n}, \tilde{\phi})_\Gamma, \quad B(\mathbf{s}, \nabla \tilde{\mathbf{u}}) + B(\text{div } \mathbf{s}, \tilde{\mathbf{u}}) = (\rho_s \mathbf{s} \cdot \mathbf{n}, \tilde{\mathbf{u}})_\Gamma \quad (41)$$

We then define  $\phi_\varepsilon$  as the solution of the well-posed variational problem

$$a(\phi_\varepsilon, \tilde{\phi}) + \mu b(\phi_\varepsilon, \tilde{\phi}) = \mu b(\phi, \tilde{\phi}) + b(\mathbf{q}_\varepsilon, \nabla \tilde{\phi}) \quad \text{for all } \tilde{\phi} \in H^1(\Omega), \quad (42)$$

(where  $\mu > 0$  may be chosen arbitrarily) whose data is such that  $\phi_\varepsilon \in H_\Delta^1(\Omega)$ . Now, we use  $b(\mathbf{q}_\varepsilon, \nabla \tilde{\phi}) = b(\mathbf{q}_\varepsilon - \nabla \phi, \nabla \tilde{\phi}) + a(\phi, \tilde{\phi})$  in (42) and rearrange terms, to obtain

$$a(\phi_\varepsilon - \phi, \tilde{\phi}) + \mu b(\phi_\varepsilon - \phi, \tilde{\phi}) = b(\mathbf{q}_\varepsilon - \nabla \phi, \nabla \tilde{\phi}) \quad \text{for all } \tilde{\phi} \in H^1(\Omega).$$

Setting  $\tilde{\phi} = \phi - \phi_\varepsilon$  in the above equality, recalling the definition of  $\mathbf{q}_\varepsilon$ , and invoking the coercivity of  $a + \mu b$  and the continuity of  $b$ , we obtain

$$\|\phi_\varepsilon - \phi\|_{1,\Omega}^2 \leq C \|\mathbf{q}_\varepsilon - \nabla \phi\|_\Omega \|\phi_\varepsilon - \phi\|_\Omega \leq C\varepsilon \|\phi_\varepsilon - \phi\|_{1,\Omega}$$

and hence

$$\|\phi_\varepsilon - \phi\|_{1,\Omega} \leq C\varepsilon$$

for some constant  $C > 0$ . Finally, on applying Green identities (10a) to  $a(\phi_\varepsilon, \tilde{\phi})$  and (41) to  $b(\mathbf{q}_\varepsilon, \nabla \tilde{\phi})$ , the variational problem (42) becomes

$$-b(\Delta \phi_\varepsilon, \tilde{\phi}) + \mu b(\phi_\varepsilon, \tilde{\phi}) - (\partial_n \phi_\varepsilon, \tilde{\phi})_\Gamma = \mu b(\phi, \tilde{\phi}) - b(\text{div } \mathbf{q}_\varepsilon, \tilde{\phi}) - (\mathbf{v}_\varepsilon \cdot \mathbf{n}, \tilde{\phi})_\Gamma \quad \text{for all } \tilde{\phi} \in H^1(\Omega),$$

from which  $\phi_\varepsilon$  is found to satisfy the desired interfacial condition (18a). Concluding, we have constructed a suitable approximation  $(\phi_\varepsilon, \mathbf{v}_\varepsilon) \in H_\Delta^1(\Omega) \times \mathbf{H}^1(\Omega_s)$  of  $(\phi, \mathbf{v}) \in H^1(\Omega) \times \mathbf{L}^2(\Omega_s)$ .

*Construction of  $(\psi_\varepsilon, \mathbf{u}_\varepsilon)$ .* The proof duplicates the previous steps, up to straightforward transpositions due to the roles of the solid and fluid media being reversed; we omit the details.

**8.3 Proof of Lemma 7.** Similarly to Section 8.2, our goal is to show that, given any  $\mathbb{U} \in \mathcal{H}_A$  and  $\varepsilon > 0$ , there exists  $\mathbb{U}_\varepsilon \in \mathcal{D}(\mathbf{A}^2)$  such that  $\|\mathbb{U} - \mathbb{U}_\varepsilon\|_{\mathcal{H}_A} \leq C\varepsilon$  for some  $C > 0$ . The requirement  $\mathbb{U}_\varepsilon \in \mathcal{D}(\mathbf{A})$  includes the interfacial constraints (18) and (30), to be respectively verified in  $\mathbf{H}^{1/2}(\Gamma)$  and  $\mathbf{H}^{-1/2}(\Gamma)$ . As before, we may separately seek an approximation  $(\phi_\varepsilon, \mathbf{v}_\varepsilon) \in H^1(\Omega, \Delta) \times \mathbf{H}_\Delta^1(\Omega_s)$  of  $(\phi, \mathbf{v}) \in H_\Delta^1(\Omega) \times \mathbf{H}^1(\Omega_s)$  satisfying (18a), (30b) and an approximation  $(\psi_\varepsilon, \mathbf{u}_\varepsilon) \in H_\Delta^1(\Omega) \times \mathbf{H}^1(\Omega_s, \Delta_s)$  of  $(\psi, \mathbf{u}) \in H^1(\Omega) \times \mathbf{H}_\Delta^1(\Omega_s)$  satisfying (18b), (30a), where  $H^1(\Omega, \Delta) = \{\phi \in H^1(\Omega), \Delta\phi \in H^1(\Omega)\}$  and similarly for  $\mathbf{H}^1(\Omega_s, \Delta_s)$ .

*Construction of  $(\phi_\varepsilon, \mathbf{v}_\varepsilon)$ .* We start by choosing some  $r_\varepsilon \in H^1(\Omega)$  such that  $\|\Delta\phi - r_\varepsilon\|_\Omega \leq \varepsilon$  (by the density of  $\mathbf{H}^1(\Omega)$  in  $\mathbf{L}^2(\Omega)$ ), then some stress tensor field  $\mathbf{s}_\varepsilon \in \mathbf{H}^{\text{div}}(\Omega_s)$  such that  $\|\mathbf{s}_\varepsilon - \mathbf{C} : \nabla \mathbf{v}\|_{\Omega_s} \leq \varepsilon$  and  $\rho_s \mathbf{s}_\varepsilon \cdot \mathbf{n} = -r_\varepsilon \mathbf{n}$  in  $H^{-1/2}(\Gamma)$ . Then, let  $(\phi_\varepsilon, \mathbf{v}_\varepsilon)$  be defined as the solution of the following coupled variational problem: find  $(\phi_\varepsilon, \mathbf{v}_\varepsilon) \in H^1(\Omega) \times \mathbf{H}^1(\Omega_s)$  such that

$$\begin{aligned} A(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}}) + \mu B(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}}) - \mu(\phi_\varepsilon \mathbf{n}, \tilde{\mathbf{u}})_\Gamma &= B(\mathbf{s}_\varepsilon, \nabla \tilde{\mathbf{u}}) + \mu B(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}}) - \mu(\phi \mathbf{n}, \tilde{\mathbf{u}})_\Gamma \quad (\text{a}), \\ a(\phi_\varepsilon, \tilde{\phi}) + \mu b(\phi_\varepsilon, \tilde{\phi}) + (\mathbf{v}_\varepsilon \cdot \mathbf{n}, \tilde{\phi})_\Gamma &= \mu b(\phi, \tilde{\phi}) - b(r_\varepsilon, \tilde{\phi}) \quad (\text{b}) \end{aligned} \quad (43)$$

(where  $\mu > 0$  may again be chosen arbitrarily) for all  $(\tilde{\phi}, \tilde{\mathbf{u}}) \in H^1(\Omega) \times \mathbf{H}^1(\Omega_s)$ . As setting  $(\tilde{\phi}, \tilde{\mathbf{u}}) = (\mu\phi_\varepsilon, \mathbf{v}_\varepsilon)$  and summing (a) and (b) yields in the left-hand side a quadratic form that is coercive on  $H^1(\Omega) \times \mathbf{H}^1(\Omega_s)$ , the above problem is readily found to be well-posed by Lax-Milgram's theorem.

To complete the proof, we need to verify that  $(\phi_\varepsilon, \mathbf{v}_\varepsilon)$  achieves the sought approximation of  $(\phi, \mathbf{v})$ . We start by recasting problem (43a) in the form

$$A(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}}) + \mu B(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}}) = ([\mu(\phi_\varepsilon - \phi) - r_\varepsilon] \mathbf{n}, \tilde{\mathbf{u}})_\Gamma - B(\text{div } \mathbf{s}_\varepsilon, \tilde{\mathbf{u}}) + \mu B(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}}) \quad \text{for all } \tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega_s),$$

which results from applying Green identity (41) to  $B(\mathbf{s}_\varepsilon, \nabla \tilde{\mathbf{u}})$ . As  $\text{div } \mathbf{s}_\varepsilon \in \mathbf{L}^2(\Omega_s)$ ,  $\mathbf{v}_\varepsilon \in \mathbf{H}^1(\Omega_s)$  and  $(\phi_\varepsilon - \phi + r_\varepsilon)_\Gamma \in H^{1/2}(\Gamma)$ , the above variational equation implies by elliptic regularity that  $\mathbf{v}_\varepsilon \in \mathbf{H}^2(\Omega_s) \subset \mathbf{H}_\Delta^1(\Omega_s)$ . Then, Green identity (10b) applied to  $A(\mathbf{v}_\varepsilon, \tilde{\mathbf{u}})$  provides

$$-\mathbf{t}[\mathbf{v}_\varepsilon] = [\mu(\phi - \phi_\varepsilon) + r_\varepsilon] \mathbf{n} \quad \text{in } H^{1/2}(\Gamma) \quad (44)$$

Next, turning to the variational equation (43b), we find that  $\phi_\varepsilon \in H^2(\Omega)$  (since  $\mu\phi - r_\varepsilon \in H^1(\Omega)$  and  $\mathbf{v}_\varepsilon \cdot \mathbf{n} \in \mathbf{H}^{3/2}(\Gamma)$ ), from which we infer that  $\nabla \phi_\varepsilon \in H_\Delta^1(\Omega)$  as the solution of a well-posed variational problem (found by taking test functions of the form  $\tilde{\phi} = \text{div } \boldsymbol{\psi}$  with  $\boldsymbol{\psi} \in H^1(\Omega)$  in (43b) and using Green identities). Hence  $\Delta\phi_\varepsilon \in H^1(\Omega)$  and we obtain

$$\Delta\phi_\varepsilon = r_\varepsilon + \mu(\phi_\varepsilon - \phi) \quad \text{in } H^1(\Omega), \quad \partial_n \phi_\varepsilon = \mathbf{v}_\varepsilon \cdot \mathbf{n} \quad \text{in } H^{1/2}(\Gamma). \quad (45)$$

by applying Green identity (10a) to  $a(\phi_\varepsilon, \tilde{\phi})$ . The above boundary condition is the interfacial condition (18a), while condition (30b) results from using the trace of the above differential equation in (44). Now, we may choose some  $\mathbf{q}_\varepsilon \in \mathbf{H}^{\text{div}}(\Omega)$  such that  $\|\mathbf{q}_\varepsilon - \nabla \phi\|_\Omega \leq \varepsilon$  and  $\mathbf{q}_\varepsilon \cdot \mathbf{n} = \mathbf{v}_\varepsilon \cdot \mathbf{n}$  in  $H^{-1/2}(\Gamma)$ . Setting  $r_\varepsilon = (r_\varepsilon - \Delta\phi) + \text{div}(\nabla\phi - \mathbf{q}_\varepsilon) + \text{div } \mathbf{q}_\varepsilon$  in (43b), applying the Green identity (41) to  $b(\text{div } \mathbf{q}_\varepsilon, \tilde{\phi})$  and rearranging terms, we find

$$-b(r_\varepsilon, \tilde{\phi}) = b(\Delta\phi - r_\varepsilon, \tilde{\phi}) + b(\text{div } \mathbf{q}_\varepsilon - \Delta\phi, \tilde{\phi}) + (\mathbf{v}_\varepsilon \cdot \mathbf{n}, \tilde{\phi})_\Gamma + b(\mathbf{q}_\varepsilon - \nabla\phi, \nabla\tilde{\phi}) + a(\phi, \tilde{\phi})$$

so that (43b) yields, for any  $\tilde{\phi} \in H^1(\Omega)$ :

$$a(\phi_\varepsilon - \phi, \tilde{\phi}) + \mu b(\phi_\varepsilon - \phi, \tilde{\phi}) = b(\Delta\phi - r_\varepsilon, \tilde{\phi}) + b(\text{div}(\mathbf{q}_\varepsilon - \nabla\phi), \tilde{\phi}) - b(\nabla\phi - \mathbf{q}_\varepsilon, \nabla\tilde{\phi}). \quad (46a)$$

In a similar manner, (43a) becomes

$$A(\mathbf{v}_\varepsilon - \mathbf{v}, \tilde{\mathbf{u}}) + \mu B(\mathbf{v}_\varepsilon - \mathbf{v}, \tilde{\mathbf{u}}) = \mu((\phi_\varepsilon - \phi) \mathbf{n}, \tilde{\mathbf{u}})_\Gamma + B(\mathbf{s}_\varepsilon - \mathbf{C} : \nabla \mathbf{v}, \nabla \tilde{\mathbf{u}}) \quad (46b)$$

for all  $\tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega_s)$ . Proceeding as in Section 8.2, we select  $\tilde{\phi} = \phi_\varepsilon - \phi$  in (46a) and  $\tilde{\mathbf{u}} = \mathbf{v}_\varepsilon - \mathbf{v}$  in (46b). After using the trace theorem for both factors in the interfacial integral, we successively obtain

$$\begin{aligned} \|\phi_\varepsilon - \phi\|_{1,\Omega} &\leq C(\|\Delta\phi - r_\varepsilon\|_\Omega + \|\text{div}(\mathbf{q}_\varepsilon - \nabla\phi)\|_\Omega + \|\nabla\phi - \mathbf{q}_\varepsilon\|_\Omega) \leq C\varepsilon, \\ \|\mathbf{v}_\varepsilon - \mathbf{v}\|_{1,\Omega_s} &\leq C(\|\phi_\varepsilon - \phi\|_{1,\Omega} + \|\mathbf{s}_\varepsilon - \mathbf{C} : \nabla \mathbf{v}\|_{\Omega_s}) \leq C\varepsilon \end{aligned}$$

Finally, the second estimate above used in (45) provides

$$\|\Delta(\phi_\varepsilon - \phi)\|_\Omega \leq \|r_\varepsilon - \Delta\phi\|_\Omega + \mu\|\phi_\varepsilon - \phi\|_\Omega \leq C\varepsilon,$$

and the proof is complete.

*Construction of  $(\psi_\varepsilon, \mathbf{u}_\varepsilon)$ .* The proof, here again, follows the same method up to appropriate modifications due to the roles of the solid and fluid media being reversed; we omit the details.

**8.4 Proof of Lemma 9.** Let  $D$  be a bounded neighborhood of  $\Omega_s$  in  $\mathbb{R}^d$ , and let  $\boldsymbol{\theta} \in C^1(\bar{D}; \mathbb{R}^d)$ . We start from the weighted residual identities

$$\left(-\Delta\phi + \phi'', \nabla\phi \cdot \boldsymbol{\theta}\right)_{\Omega, T} = 0, \quad \left(-\Delta_s \mathbf{u} + \mathbf{u}'', \nabla \mathbf{u} \cdot \boldsymbol{\theta}\right)_{\Omega_s, T} = 0, \quad (\text{WR})$$

where  $\phi$  and  $\mathbf{u}$  respectively solve the homogeneous acoustic wave equation in  $\Omega$  and the homogeneous elastic wave equation in  $\Omega_s$  and are smooth enough to allow all integrations by parts to follow. Using the equality

$$\begin{aligned} (2\phi'' - 2\Delta\phi) \nabla\phi \cdot \boldsymbol{\theta} &= 2\partial_t [\phi'(\nabla\phi \cdot \boldsymbol{\theta})] + \operatorname{div} [ (|\nabla\phi|^2 - |\phi'|^2)\boldsymbol{\theta} - 2(\nabla\phi \cdot \boldsymbol{\theta})\nabla\phi ] \\ &\quad + (|\phi'|^2 - |\nabla\phi|^2)\operatorname{div}\boldsymbol{\theta} + 2(\nabla\phi \otimes \nabla\phi) : \nabla\boldsymbol{\theta}, \end{aligned}$$

which can be checked by inspection (e.g. using component notation), we write

$$\begin{aligned} 2(\phi'' - \Delta\phi, \nabla\phi \cdot \boldsymbol{\theta})_{\Omega, T} &= 2(\phi', \nabla\phi \cdot \boldsymbol{\theta})_{\Omega} \Big|_0^T + (|\nabla\phi|^2 - |\phi'|^2, \boldsymbol{\theta} \cdot \mathbf{n})_{\Gamma, T} - 2(\partial_n \phi, \nabla\phi \cdot \boldsymbol{\theta})_{\Gamma, T} \\ &\quad + (|\phi'|^2 - |\nabla\phi|^2, \operatorname{div}\boldsymbol{\theta})_{\Omega, T} + 2((\nabla\phi \otimes \nabla\phi), \nabla\boldsymbol{\theta})_{\Omega, T}. \end{aligned} \quad (\text{a})$$

For the contribution of  $\Omega_s$  in (WR), we similarly find

$$\begin{aligned} 2(\mathbf{u}'' - \Delta_s \mathbf{u}, \nabla \mathbf{u} \cdot \boldsymbol{\theta})_{\Omega_s, T} &= 2(\mathbf{u}', \nabla \mathbf{u} \cdot \boldsymbol{\theta})_{\Omega_s} \Big|_0^T + (\nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u} - |\mathbf{u}'|^2, \boldsymbol{\theta} \cdot \mathbf{n})_{\Gamma, T} - 2(\mathbf{t}[\mathbf{u}], \nabla \mathbf{u} \cdot \boldsymbol{\theta})_{\Gamma, T} \\ &\quad + (|\mathbf{u}'|^2 - \nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u}, \operatorname{div}\boldsymbol{\theta})_{\Omega_s, T} + 2(\nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u}, \nabla \boldsymbol{\theta})_{\Omega_s, T}, \end{aligned} \quad (\text{b})$$

We then substitute (a) and (b) into (WR). In addition, we set  $|\nabla\phi|^2 = (\partial_n \phi)^2 + |\nabla_S \phi|^2$  in the interfacial integral of (a), and likewise introduce the decomposition  $\nabla \mathbf{u} = \nabla_S \mathbf{u} + \partial_n \mathbf{u} \otimes \mathbf{n}$  and express the traction vector as  $\mathbf{t}[\mathbf{u}] = \mathbf{n} \cdot \mathbf{C} : (\nabla_S \mathbf{u} + \partial_n \mathbf{u} \otimes \mathbf{n}) = \mathbf{n} \cdot \mathbf{C} : \nabla_S \mathbf{u} + \mathbf{Q} \cdot \partial_n \mathbf{u}$ , yielding

$$\nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u} - 2\mathbf{t}[\mathbf{u}] \cdot \partial_n \mathbf{u} = \nabla_S \mathbf{u} : \mathbf{C} : \nabla_S \mathbf{u} - \partial_n \mathbf{u} \cdot \mathbf{Q} \cdot \partial_n \mathbf{u},$$

in the interfacial integrals of (b). These manipulations, and subsequent rearrangement, produce the identities

$$\begin{aligned} &((\partial_n \phi)^2 + |\nabla_S \phi|^2 - |\phi'|^2, \boldsymbol{\theta} \cdot \mathbf{n})_{\Gamma, T} - 2(\partial_n \phi, \nabla\phi \cdot \boldsymbol{\theta})_{\Gamma, T} \\ &= (|\nabla\phi|^2 - |\phi'|^2, \operatorname{div}\boldsymbol{\theta})_{\Omega, T} - 2((\nabla\phi \otimes \nabla\phi), \nabla\boldsymbol{\theta})_{\Omega, T} - 2(\phi', \nabla\phi \cdot \boldsymbol{\theta})_{\Omega} \Big|_0^T, \end{aligned} \quad (\text{c})$$

$$\begin{aligned} &(\nabla_S \mathbf{u} : \mathbf{C} : \nabla_S \mathbf{u} - \partial_n \mathbf{u} \cdot \mathbf{Q} \cdot \partial_n \mathbf{u} - |\mathbf{u}'|^2, \boldsymbol{\theta} \cdot \mathbf{n})_{\Gamma, T} - 2(\mathbf{t}[\mathbf{u}], \nabla \mathbf{u} \cdot \boldsymbol{\theta})_{\Gamma, T} \\ &= (\nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u} - |\mathbf{u}'|^2, \operatorname{div}\boldsymbol{\theta})_{\Omega_s, T} - 2(\nabla \mathbf{u} : \mathbf{C} : \nabla \mathbf{u}, \nabla \boldsymbol{\theta})_{\Omega_s, T} - 2(\mathbf{u}', \nabla \mathbf{u} \cdot \boldsymbol{\theta})_{\Omega_s} \Big|_0^T, \end{aligned} \quad (\text{d})$$

from which the claimed identity (38) follows from summing (c), (d) and setting  $\boldsymbol{\theta}$  equal to a two-sided extension in  $D$  of the unit normal on  $\Gamma$  in the resulting equality.

**Acknowledgement.** The doctoral thesis of A. Nassor is funded by AID (Agence de l'Innovation de Défense) and Naval Group. We thank G. Barras, O. Grosset (DGA) and B. Leblé (Naval Group) for their input and discussions regarding the engineering applications of transient acoustic scattering solution methods.

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