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Quantitative rigidity of differential inclusions in two dimensions

Xavier Lamy*

Andrew Lorent†

Guanying Peng‡

Abstract

For any compact connected one-dimensional submanifold $K \subset \mathbb{R}^{2 \times 2}$ without boundary which has no rank-one connection and is elliptic, we prove the quantitative rigidity estimate

$$\inf_{M \in K} \int_{B_{1/2}} |Du - M|^2 dx \leq C \int_{B_1} \text{dist}^2(Du, K) dx, \quad \forall u \in H^1(B_1; \mathbb{R}^2).$$

This is an optimal generalization, for compact connected submanifolds of $\mathbb{R}^{2 \times 2}$ without boundary, of the celebrated quantitative rigidity estimate of Friesecke, James and Müller for the approximate differential inclusion into $SO(n)$. The proof relies on the special properties of elliptic subsets $K \subset \mathbb{R}^{2 \times 2}$ with respect to conformal-anticonformal decomposition, which provide a quasilinear elliptic PDE satisfied by solutions of the exact differential inclusion $Du \in K$. We also give an example showing that no analogous result can hold true in $\mathbb{R}^{n \times n}$ for $n \geq 3$.

1 Introduction

In 1850, Liouville [23] proved that, given a domain $\Omega \subset \mathbb{R}^3$, any smooth map $u: \Omega \rightarrow \mathbb{R}^3$ satisfying the differential inclusion $Du(x) \in \mathbb{R}_+O(n)$ for all $x \in \Omega$ must be either affine or a Möbius transform. A corollary to Liouville's Theorem is that a C^3 function whose gradient belongs everywhere to $SO(n)$ is an affine mapping. This phenomenon of being able to globally control a map satisfying a certain differential inclusion $Du \in K$ is known as “rigidity”.

In [14] Friesecke, James and Müller solved a long standing open problem by proving an optimal quantitative rigidity estimate for $K = SO(n)$. Specifically, they showed that for every bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, there exists a constant $C(\Omega)$ such that, for $K = SO(n)$,

$$\inf_{R \in K} \|Dv - R\|_{L^2(\Omega)} \leq C(\Omega) \|\text{dist}(Dv, K)\|_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega; \mathbb{R}^n). \quad (1)$$

Here and below, $\text{dist}(M, K)$ denotes the distance from a matrix $M \in \mathbb{R}^{n \times n}$ to a subset $K \subset \mathbb{R}^{n \times n}$ measured in the Euclidean norm. This result strengthened earlier work of a series

*Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Xavier.Lamy@math.univ-toulouse.fr

†Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA. Email: lorentaw@uc.edu

‡Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA. Email: gpeng@wpi.edu

of authors, including John [18], Rešetnjak [27], and Kohn [20], and it has had a number of important applications, in particular to thin film limits of elastic structures [14, 15]. One of the remarkable things about this result is that it is a striking fact about classical mathematical objects that could be understood by the mathematicians of hundreds of years ago.

A number of works have extended the above result (1) to cover various larger classes of matrices than $K = SO(n)$. Chaudhuri and Müller [8] and later De Lellis and Székelyhidi [10] considered a set of the form $K = SO(n)A \cup SO(n)B$ where A and B are strongly incompatible in the sense of Matos [25]. Faraco and Zhong [13] proved an analogous quantitative rigidity result with $K = \mathfrak{m} \cdot SO(n)$ where $\mathfrak{m} \subset (0, +\infty)$ is compact. There the infimum in the left-hand side of (1) also needs to include the gradients of Möbius transforms, and the integral is over a smaller subset $\Omega' \subset \subset \Omega$. Similar results for maps defined on the sphere have been obtained recently by Luckhaus and Zemas [24]. Best constants for (1) are investigated by Lewicka and Müller in [22].

Our main result is an optimal generalization of the quantitative rigidity estimate of [14] in the context of compact connected submanifolds $K \subset \mathbb{R}^{2 \times 2}$ without boundary.

Theorem 1.1. *Let $K \subset \mathbb{R}^{2 \times 2}$ be a smooth, compact and connected 1-manifold without boundary. Assume that K has no rank-one connections, and is elliptic in the sense that there exists $C_* > 0$ such that*

$$|M - M'|^2 \leq C_* \det(M - M') \quad \forall M, M' \in K. \quad (2)$$

Then for any $u \in H^1(B_1; \mathbb{R}^2)$ we have

$$\inf_{M \in K} \int_{B_{1/2}} |Du - M|^2 dx \leq C \int_{B_1} \text{dist}^2(Du, K) dx, \quad (3)$$

for some constant $C = C(K) > 0$.

Remark 1.2. A covering argument as in [13] shows that the estimate (3) in the balls $B_{1/2} \subset B_1$ automatically improves to

$$\inf_{M \in K} \int_{\Omega'} |Du - M|^2 dx \leq C \int_{\Omega} \text{dist}^2(Du, K) dx, \quad \forall u \in H^1(\Omega; \mathbb{R}^2),$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ and open subset $\Omega' \subset \subset \Omega \subset \mathbb{R}^2$, and a constant $C = C(K, \Omega', \Omega) > 0$.

This result is optimal among compact connected submanifolds $K \subset \mathbb{R}^{2 \times 2}$ without boundary for the following reasons:

- First, it is classical that the no-rank-one-connections assumption is necessary for the rigidity of the exact differential inclusion (see e.g. [26, 19]).
- Second, ellipticity is necessary for the validity of the linearized version of (3) (see Remark 2.3), because non-ellipticity would imply that the tangent space $T_M K$ has a rank-one connection for some $M \in K$.
- Third, the two previous conditions (no rank-one connections and ellipticity) imply that the connected submanifold K must be of dimension 1 [32, Corollary 3.5 & 3.6].

Moreover, we provide in Section 5 an example showing that the two-dimensional setting is also optimal: there exists an elliptic 1-submanifold $K \subset \mathbb{R}^{3 \times 3}$ without rank-one connection but which contains a so-called \mathcal{T}_4 configuration, a well-known obstruction to compactness of sequences $\{u_k\} \subset H^1$ satisfying $\text{dist}(Du_k, K) \rightarrow 0$ in L^2 [7], and therefore to any type of quantitative rigidity estimate.

One of our motivations for studying differential inclusion into general submanifolds $K \subset \mathbb{R}^{2 \times 2}$ is our previous work [21] where we obtained a rigidity result for a non-elliptic differential inclusion related to the so-called Aviles-Giga functional, and pointed out the nice consequences that a corresponding quantitative rigidity estimate would have. Theorem 1.1 is not valid for non-elliptic differential inclusions, but the ideas in the present work should be relevant to attain that goal.

While the statements of the quantitative rigidity results of [14, 8, 13] are elementary, their proofs are not. Their starting point, in addition to rigidity of the exact differential inclusion, is a linearized version of (1) for the differential inclusion $Du \in T_{M_0}K$ into a tangent space $T_{M_0}K$. For $K = SO(n)$ and $M_0 = I$, this is Korn's inequality. A natural linearization procedure then provides a quantitative rigidity estimate, but in terms of the L^∞ norm of $\text{dist}(Du, K)$, rather than L^2 . Strengthening the L^2 bound on $\text{dist}(Du, K)$ into an L^∞ bound constitutes therefore the main difficulty. A key idea, introduced in [14], is to use the regularity of an elliptic PDE satisfied by solutions of the exact differential inclusion: for $K = SO(n)$ the exact differential inclusion $Du \in SO(n)$ implies that the coordinate functions u_k are harmonic. For $K \subset \mathbb{R}_+SO(n)$ the coordinate functions satisfy the $(n-2)$ -Laplace equation $\text{div}(|\nabla u_k|^{n-2} \nabla u_k) = 0$. Such PDE follows from the universal identity $\text{div} \text{cof}(Du) = 0$ (where cof denotes the cofactor matrix), together with identities satisfied by matrices in the specific set K . It is satisfied by solutions of the exact differential inclusion, and for a general map u the error from solving that PDE can be controlled in terms of the right-hand side of (1). This allows to reduce the proof of (1) to maps solving that PDE. Elliptic regularity then provides, via a compactness argument, a uniform bound on $\text{dist}(Du, K)$ and the linearization can be performed.

Following this scheme, the main ingredient to prove Theorem 1.1 is to embed K into the graph of a uniformly monotone vector field: this will be enough to turn the identity $\text{div} \text{cof}(Du) = 0$ into a quasilinear elliptic equation for the exact differential inclusion $Du \in K$.

Proposition 1.3. *Let K be as in Theorem 1.1. There exist $G_1, G_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ smooth, globally Lipschitz vector fields such that*

$$K \subset \left\{ \begin{pmatrix} A \\ iG_1(A) \end{pmatrix} : A \in \mathbb{R}^2 \right\} \cap \left\{ \begin{pmatrix} -iG_2(B) \\ B \end{pmatrix} : B \in \mathbb{R}^2 \right\},$$

and G_1, G_2 are uniformly monotone, that is

$$(G_j(X) - G_j(X')) \cdot (X - X') \geq \lambda |X - X'|^2 \quad \forall X, X' \in \mathbb{R}^2,$$

for some constant $\lambda > 0$ depending only on K .

Proposition 1.3 relies on remarkable properties of elliptic subsets of $\mathbb{R}^{2 \times 2}$ with respect to the decomposition into conformal and anticonformal parts, discovered in [32] and exploited in a striking manner in [5, 11, 12] (see also [19]). It is also related to the classical link between two-dimensional elliptic PDEs of second order and complex Beltrami equations, see e.g. the

introduction of [6], and [4, 2, 1, 3] for recent developments on nonlinear Beltrami equations. The proof of Proposition 1.3 is in fact extremely close to [3, Theorem 5]. Then the proof of Theorem 1.1 follows the scheme outlined above.

The article is organized as follows. In Section 2 we establish the two basic prerequisites to Theorem 1, rigidity for the exact differential inclusion and the linearized estimate. In Section 3 we give the proof of Proposition 1.3. In Section 4 we gather these ingredients to prove Theorem 1.1. In Section 5 we describe the counterexample in $\mathbb{R}^{3 \times 3}$.

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2 Basic ingredients: rigidity of the exact inclusion and linearized estimate

In this section, let $K \subset \mathbb{R}^{2 \times 2}$ be as in Theorem 1.1. We prove the following two Lemmas.

Lemma 2.1. *If $u \in H^1(B_1; \mathbb{R}^2)$ is such that $Du \in K$ a.e., then $Du \equiv M$ for some $M \in K$.*

Lemma 2.2. *For all $M \in K$ and $u \in H^1(B_1; \mathbb{R}^2)$ we have*

$$\inf_{X \in T_M K} \int_{B_1} |Du - X|^2 dx \leq C \int_{B_1} \text{dist}^2(Du, T_M K) dx \quad (4)$$

for some constant $C = C(K) > 0$, where $T_M K$ denotes the linear tangent space to K at M .

Remark 2.3. The linearized estimate (4), or rather its weaker interior version

$$\inf_{X \in T_M K} \int_{B_{1/2}} |Du - X|^2 dx \leq C \int_{B_1} \text{dist}^2(Du, T_M K) dx, \quad (5)$$

is a necessary condition for (3) to be valid. Assume indeed that (3) is verified, fix $u \in C^1(\overline{B}_1; \mathbb{R}^2)$, and apply (3) to $v_\epsilon(x) = Mx + \epsilon u(x)$ for $\epsilon \ll 1$. There exists $M_\epsilon \in K$ such that

$$\begin{aligned} \int_{B_{1/2}} |M - M_\epsilon - \epsilon Du|^2 dx &\leq C \int_{B_1} \text{dist}^2(M + \epsilon Du, K) dx \\ &\leq C\epsilon^2 \int_{B_1} \text{dist}^2(Du, T_M K) dx + o(\epsilon^2). \end{aligned}$$

Hence, letting $X_\epsilon = \epsilon^{-1}(M - M_\epsilon)$, we have

$$\int_{B_{1/2}} |X_\epsilon - Du|^2 dx \leq C \int_{B_1} \text{dist}^2(Du, T_M K) dx + o(1).$$

In particular X_ϵ is bounded, and extracting a converging subsequence we obtain $X \in T_M K$ showing the validity of (5) for $u \in C^1(\overline{B}_1; \mathbb{R}^2)$, and then by density for $u \in H^1(\overline{B}_1; \mathbb{R}^2)$.

Proof of Lemma 2.1. Let $\ell = |K|$ and $\gamma: \mathbb{R}/\ell\mathbb{Z} \rightarrow K$ be an arc-length parametrization of K . The ellipticity assumption (2) ensures that $\gamma'(t)$ is invertible for all $t \in \mathbb{R}$. Let $u \in H^1(B_1; \mathbb{R}^2)$ such that $Du \in K$ a.e., then u is smooth by [31], and since B_1 is simply connected there exists a smooth lifting $\theta: B_1 \rightarrow \mathbb{R}$ such that $Du = \gamma(\theta)$. Using that $\operatorname{div} \operatorname{cof}(Du) = 0$, where cof denotes the cofactor matrix, we find $\operatorname{cof}(\gamma'(\theta))\nabla\theta = 0$, hence $\nabla\theta = 0$ since $\operatorname{cof}(\gamma'(\theta))$ is invertible. Therefore Du is constant. \square

Proof of Lemma 2.2. For $M \in K$ we denote by $P_M: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ the orthogonal projection onto $(T_M K)^\perp$, so that

$$\operatorname{dist}^2(X, T_M K) = |P_M X|^2.$$

We denote by $P_M^{\alpha\beta jk} \in \mathbb{R}$ the coefficients of P_M , that is,

$$(P_M X)_{\alpha\beta} = \sum_{jk} P_M^{\alpha\beta jk} X_{jk} \quad \forall X = (X_{jk}) \in \mathbb{R}^{2 \times 2}.$$

Define $\mathbb{P}_M(i\xi) \in \mathcal{L}(\mathbb{C}^2; \mathbb{C}^{2 \times 2})$ by

$$(\mathbb{P}_M(i\xi)v)_{\alpha\beta} = \sum_{jk} P_M^{\alpha\beta jk} v_j \xi_k \quad \forall \xi, v \in \mathbb{C}^2,$$

so the differential operator $u \mapsto P_M Du$ has symbol $\mathbb{P}_M(i\xi)$, i.e.,

$$(P_M(Df))_{\alpha\beta} = \frac{1}{2\pi} \int \left(\mathbb{P}_M(i\xi) \hat{f}(\xi) \right)_{\alpha\beta} e^{ix \cdot \xi} d\xi.$$

We claim that $\mathbb{P}_M(\xi)$ has trivial kernel for all non-zero $\xi \in \mathbb{C}^2$. Let indeed $v \in \mathbb{C}^2$ such that $\mathbb{P}_M(\xi)v = 0$. This implies that $P_M \operatorname{Re}(v \otimes i\xi) = P_M \operatorname{Im}(v \otimes i\xi) = 0$, because the coefficients $P_M^{\alpha\beta jk}$ are real-valued. In other words, the real and imaginary parts of $v \otimes i\xi$ both belong to $\ker P_M = T_M K$. Since $T_M K$ is a one-dimensional subspace of $\mathbb{R}^{2 \times 2}$ which doesn't contain any rank-one matrix, we have $T_M K = \mathbb{R}X_0$ for some invertible matrix X_0 . Hence we deduce that $v \otimes i\xi = \lambda X_0$ for some $\lambda \in \mathbb{C}$ and an invertible matrix $X_0 \in T_M K$. But $v \otimes i\xi$ has zero determinant, so $\lambda = 0$ and we must have $v = 0$. This proves that $\mathbb{P}_M(\xi)$ has trivial kernel. Therefore we have the representation formula [29, Theorem 4.1] and the coercive inequality [29, Theorem 8.15] that follows from it,

$$\int_{B_1} |Du|^2 dx \leq C \int_{B_1} |P_M Du|^2 dx + C \int_{B_1} |u|^2 dx, \quad (6)$$

for all $u \in H^1(B_1; \mathbb{R}^2)$. (In the notation of [29], $N = 4$, $M = 2$, and the index set $\{1, 2, 3, 4\}$ for j is in our case given by $\{1, 2\}^2$, and we can take $m_j = 1$ for $j \in \{1, 2\}^2$ and $l_i = 0$ for $i \in \{1, 2\}$.) The constant $C > 0$ in (6) depends a priori on the fixed matrix $M \in K$. Denote by $C(M)$ the best possible constant in (6). Then for any $M, M' \in K$ we have

$$\begin{aligned} \int_{B_1} |Du|^2 dx &\leq 2C(M) \int_{B_1} |P_{M'} Du|^2 dx + C(M) \int_{B_1} |u|^2 dx \\ &\quad + 2C(M) \|P_M - P_{M'}\|^2 \int_{B_1} |Du|^2 dx. \end{aligned}$$

For all $M \in K$, there exists $\delta(M) > 0$ sufficiently small such that for all $M' \in K \cap B_{\delta(M)}(M)$, we have $2C(M)\|P_M - P_{M'}\|^2 < 1/2$. It follows that

$$C(M') \leq \frac{2C(M)}{1 - 2C(M)\|P_M - P_{M'}\|^2} < 4C(M) \quad \forall M' \in K \cap B_{\delta(M)}(M).$$

By compactness this implies that $M \mapsto C(M)$ is bounded on K , so we can take the constant C in (6) to depend only on K .

Moreover, if $u \in H^1(B_1; \mathbb{R}^2)$ satisfies $P_M Du = 0$ a.e., then $Du = \lambda X_0$ for some $\lambda \in L^2(B_1; \mathbb{R})$, and the distributional identity $0 = \operatorname{div} \operatorname{cof}(Du) = \operatorname{cof}(X_0) \nabla \lambda$ implies that λ is constant, hence $Du \equiv X$ for some $X \in T_M K$.

Therefore (4) follows from (6) via a compactness argument: assume by contradiction the existence of sequences $M^k \in K$, and $u^k \in H^1(B_1; \mathbb{R}^2)$ such that

$$\inf_{X \in T_{M^k} K} \int_{B_1} |Du^k - X|^2 dx = 1, \quad \int_{B_1} |P_{M^k} Du^k|^2 dx \longrightarrow 0.$$

Subtracting from u^k its average and $X^k x$ for the matrix X^k at which the infimum in the left-hand side is attained, we may in fact assume

$$\int_{B_1} u^k dx = \int_{B_1} Du^k \cdot X dx = 0 \quad \forall X \in T_{M^k} K,$$

and

$$\int_{B_1} |Du^k|^2 dx = 1, \quad \int_{B_1} |P_{M^k} Du^k|^2 dx \longrightarrow 0.$$

Thus we may extract subsequences (not relabeled) $u^k \rightharpoonup u$ weakly in $H^1(B_1; \mathbb{R}^2)$ and strongly in $L^2(B_1; \mathbb{R}^2)$, and $M^k \rightarrow M \in K$. It follows that $P_{M^k} Du^k \rightharpoonup P_M Du$ in $L^2(B_1; \mathbb{R}^{2 \times 2})$, and thus by lower semicontinuity of the L^2 norm under weak convergence, we have $P_M Du = 0$ a.e., which implies $Du \equiv X$ for some $X \in T_M K$. Approximating X by a sequence $X^k \in T_{M^k} K$ and using $Du^k \rightharpoonup Du$ in $L^2(B_1; \mathbb{R}^{2 \times 2})$, we deduce that $0 = \int_{B_1} Du \cdot X dx = |B_1| |X|^2$, and thus $Du \equiv X = 0$. Further u satisfies $\int_{B_1} u dx = 0$, which implies $u \equiv 0$. Plugging u^k into (6) gives

$$1 = \int_{B_1} |Du^k|^2 dx \leq C \int_{B_1} |P_{M^k} Du^k|^2 dx + C \int_{B_1} |u^k|^2 dx.$$

Passing to the limit as $k \rightarrow \infty$ and using the strong L^2 convergence, we have $1 \leq C \int_{B_1} |u|^2 dx = 0$, which gives a contradiction. \square

3 Proof of Proposition 1.3

We only prove the existence of G_1 , the existence of G_2 is obtained by the same arguments. The proof relies on the properties of the conformal and anticonformal projections of K uncovered in [32, 12]. For any $z_+, z_- \in \mathbb{C}$, we denote by

$$[z_+, z_-] = \begin{pmatrix} \operatorname{Re} z_+ & -\operatorname{Im} z_+ \\ \operatorname{Im} z_+ & \operatorname{Re} z_+ \end{pmatrix} + \begin{pmatrix} \operatorname{Re} z_- & \operatorname{Im} z_- \\ \operatorname{Im} z_- & -\operatorname{Re} z_- \end{pmatrix} \in \mathbb{R}^{2 \times 2},$$

the 2×2 matrix whose conformal, respectively anticonformal, part is represented by z_+ , respectively z_- . For any $A \in \mathbb{R}^{2 \times 2}$, the decomposition $A = [z_+, z_-]$ is unique, and we have the identities

$$\det A = |z_+|^2 - |z_-|^2, \quad |A|^2 = 2|z_+|^2 + 2|z_-|^2, \quad \|A\| = |z_+| + |z_-|,$$

where $|A|$ and $\|A\|$ denote the Hilbert-Schmidt and the operator norms of A , respectively. We denote by $p_+ : [z_+, z_-] \mapsto z_+$ the projection onto the conformal part.

Using these notations, the ellipticity assumption (2) is equivalent to

$$2|z_+ - z'_+|^2 + 2|z_- - z'_-|^2 \leq C_* (|z_+ - z'_+|^2 + |z_- - z'_-|^2),$$

for all $[z_+, z_-], [z'_+, z'_-] \in K$. This implies that the curve K is C_* -elliptic in the sense of [12, Def. 1], and it follows from the analysis in [32] (see also the proofs of Lemmas 1 and 2 in [12]) that $p_+(K)$ is a Jordan curve and

$$K = \{[z, H(z)]: z \in p_+(K)\},$$

for some k -Lipschitz function $H: p_+(K) \rightarrow \mathbb{C}$ with $0 \leq k = (C_* - 1)/(C_* + 1) < 1$. Further the explicit formula $p_+(A) = (a_{11} + a_{22})/2 - i(a_{12} - a_{21})/2$ for $A = (a_{ij})$ and the smoothness of K imply that $p_+(K) \subset \mathbb{C}$ is smooth.

Lemma 3.1. *The function H admits a smooth extension $H: \mathbb{C} \rightarrow \mathbb{C}$ which is k -Lipschitz for some (possibly larger) $0 \leq k < 1$.*

Proof of Lemma 3.1. We first fix, thanks to Kirszbraun's theorem, a k -Lipschitz extension $\widehat{H}: \mathbb{C} \rightarrow \mathbb{C}$. In the rest of the proof we modify \widehat{H} to make it smooth while still agreeing with H on $p_+(K)$, at the cost of slightly increasing its Lipschitz constant.

Let $[z_+(t), z_-(t)]$, $t \in \mathbb{R}/\ell\mathbb{Z}$, denote a smooth arc-length parametrization of K , so that $2|\dot{z}_+|^2 + 2|\dot{z}_-|^2 = 1$, where $\dot{z}_\pm = \frac{d}{dt}z_\pm$. As $z_- = H(z_+)$ and H is k -Lipschitz, it follows that $|\dot{z}_-| \leq k|\dot{z}_+|$, so $|\dot{z}_+|^2 \geq \frac{1}{2(1+k^2)} > 0$. Therefore we may reparametrize and consider

$$K = \{[z(s), H(z(s))]: s \in \mathbb{R}/\ell_+\mathbb{Z}\},$$

with $z(s)$ an arc-length parametrization of $p_+(K)$ and ℓ_+ its length, and the map $s \mapsto H(z(s))$ is smooth by smoothness of K .

For small enough $\delta > 0$, the map

$$\begin{aligned} \varphi: \mathbb{R}/\ell_+\mathbb{Z} \times (-2\delta, 2\delta) &\rightarrow \mathcal{U}_{2\delta} = \{z \in \mathbb{C}: \text{dist}(z, p_+(K)) < 2\delta\} \\ (s, r) &\mapsto z(s) + ri\dot{z}(s), \end{aligned}$$

is a smooth diffeomorphism. We first modify \widehat{H} by setting

$$\widetilde{H} = \widehat{H} \circ \Phi, \quad \Phi(Z) = \begin{cases} z(s) + \lambda(r)i\dot{z}(s) & \text{if } Z = \varphi(s, r) \in \mathcal{U}_{2\delta}, \\ Z & \text{otherwise,} \end{cases}$$

where λ is the odd $(1 - \delta)^{-1}$ -Lipschitz function given for $r > 0$ by

$$\lambda(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq \delta^2, \\ -\frac{\delta^2}{1-\delta} + \frac{r}{1-\delta} & \text{for } \delta^2 < r \leq \delta, \\ r & \text{for } r > \delta. \end{cases}$$

In particular we have

$$\tilde{H}(Z) = H(z(s)) \quad \forall Z = \varphi(s, r) \in \mathcal{U}_{\delta^2},$$

so \tilde{H} is smooth in \mathcal{U}_{δ^2} (by smoothness of $s \mapsto H(z(s))$ and φ^{-1}) and agrees with H on $p_+(K)$. Note that by definition of φ and λ we have $\Phi(Z) = Z$ in $\mathbb{C} \setminus U_\delta$ and therefore Φ is Lipschitz in \mathbb{C} . Since $D\varphi(s, 0) \in SO(2)$ for all $s \in \mathbb{R}/\ell_+\mathbb{Z}$, we have $\|D\varphi\| \leq 1 + C\delta$ on $\mathbb{R}/\ell_+\mathbb{Z} \times (-2\delta, 2\delta)$. Further, we have $\|D(\varphi^{-1})\| = 1$ on $p_+(K)$, and $\|D(\varphi^{-1})\| \leq 1 + C\delta$ on $\mathcal{U}_{2\delta}$. Denoting by ψ the $(1 - \delta)^{-1}$ -Lipschitz map $(s, r) \mapsto (s, \lambda(r))$, we write $\Phi(Z) = \varphi(\psi(\varphi^{-1}(Z)))$ and deduce that $\|D\Phi\| \leq 1 + C\delta$ a.e. in $\mathcal{U}_{2\delta}$. This inequality is also true in the rest of \mathbb{C} by definition of Φ , so we conclude that \tilde{H} is \tilde{k} -Lipschitz in \mathbb{C} , with $\tilde{k} = (1 + C\delta)k < 1$ for small enough $\delta > 0$. Now δ is fixed and we define, for $\epsilon \in (0, \delta^2/4)$,

$$H_\epsilon(z) = \int_{\mathbb{C}} \tilde{H}(z + \epsilon\chi(z)y)\rho(y) dy,$$

for a smooth kernel $\rho \geq 0$ with support in B_1 and $\int \rho(y) dy = 1$, and some smooth cut-off function χ with $\mathbf{1}_{\mathcal{U}_{\delta^2/4}} \leq 1 - \chi \leq \mathbf{1}_{\mathcal{U}_{\delta^2/2}}$. In $\mathcal{U}_{3\delta^2/4}$, the map H_ϵ is smooth thanks to the smoothness of \tilde{H} in \mathcal{U}_{δ^2} . In $\mathbb{C} \setminus \bar{\mathcal{U}}_{\delta^2/2}$, we have $H_\epsilon(z) = \int_{\mathbb{C}} \tilde{H}(z + \epsilon y)\rho(y) dy$ is also smooth. Therefore the map H_ϵ is smooth in \mathbb{C} . Further, $H_\epsilon(z) = \tilde{H}(z)$ for $z \in \mathcal{U}_{\delta^2/4}$, and thus agrees with H on $p_+(K)$. Finally, denoting by L the Lipschitz constant of χ , we have

$$\begin{aligned} |H_\epsilon(z) - H_\epsilon(z')| &\leq \int_{B_1} |\tilde{H}(z + \epsilon\chi(z)y) - \tilde{H}(z' + \epsilon\chi(z')y)|\rho(y) dy \\ &\leq \int_{B_1} \tilde{k}(1 + \epsilon L|y|)|z - z'|\rho(y) dy \leq \tilde{k}(1 + \epsilon L)|z - z'|. \end{aligned}$$

So H_ϵ is k_ϵ -Lipschitz with $k_\epsilon \leq \tilde{k}(1 + \epsilon L) < 1$ for small enough ϵ . \square

Lemma 3.2. *The map $F: z \mapsto \bar{z} + H(z)$ is a smooth diffeomorphism from \mathbb{C} onto \mathbb{C} . Moreover F is $(1 + k)$ -Lipschitz and F^{-1} is $(1 - k)^{-1}$ -Lipschitz.*

Proof of Lemma 3.2. For any $w \in \mathbb{C}$ the equation

$$w = \bar{z} + H(z) \quad \Leftrightarrow \quad z = \bar{w} - \bar{H}(z),$$

admits a unique solution $z \in \mathbb{C}$ thanks to the fixed point theorem, since $z \mapsto \bar{w} - \bar{H}(z)$ is k -Lipschitz and $0 \leq k < 1$. This shows that F is bijective. The inequalities

$$(1 - k)|z - z'| \leq |F(z) - F(z')| \leq (1 + k)|z - z'|,$$

follow directly from the fact that H is k -Lipschitz and imply the announced Lipschitz constants of F and F^{-1} . The inverse F^{-1} is smooth thanks to the Inverse Function Theorem, since $DF(z): h \mapsto \bar{h} + DH(z)h$ is invertible for all $z \in \mathbb{C}$ because $\|DH\| < 1$. \square

Proof of Proposition 1.3 completed. Then, identifying \mathbb{C} with \mathbb{R}^2 , we define

$$G_1(A) = \overline{F^{-1}(A)} - H(F^{-1}(A)),$$

so that a short calculation shows that

$$[z, H(z)] = \begin{pmatrix} A \\ iG_1(A) \end{pmatrix} \quad \text{for } A = F(z).$$

The map G_1 is smooth and globally Lipschitz with Lipschitz constant $\Lambda = (1+k)/(1-k)$. Moreover, for all $A = F(z)$, $A' = F(z')$, we have

$$\begin{aligned} (G_1(A) - G_1(A')) \cdot (A - A') &= \det \begin{pmatrix} A - A' \\ i(G_1(A) - G_1(A')) \end{pmatrix} \\ &= \det([z - z', H(z) - H(z')]) \\ &= |z - z'|^2 - |H(z) - H(z')|^2 \\ &\geq (1 - k^2)|z - z'|^2. \end{aligned}$$

Since F is $(1+k)$ -Lipschitz, this implies

$$(G_1(A) - G_1(A')) \cdot (A - A') \geq \lambda |A - A'|^2,$$

for $\lambda = (1-k^2)/(1+k)^2 = (1-k)/(1+k) > 0$. This concludes the proof of Proposition 1.3. \square

4 Proof of Theorem 1.1

Step 1. We may assume that u is Lipschitz, thanks to the truncation argument of [14, Proposition A.1]. Let $K \subset B_R \subset \mathbb{R}^{2 \times 2}$ for some $R > 0$. Then for all $X \in \mathbb{R}^{2 \times 2}$ with $|X| > 2R$, we have $|X| \leq 2 \operatorname{dist}(X, K)$. An application of [14, Proposition A.1] with $\lambda = 2R$ gives $v : B_1 \rightarrow \mathbb{R}^2$ satisfying

$$\begin{aligned} \|Dv\|_{L^\infty(B_1)} &\leq 2C_0R, \\ \int_{B_1} |Du - Dv|^2 dx &\leq C_0 \int_{\{x \in B_1 : |Du| > 2R\}} |Du|^2 dx \\ &\leq 4C_0 \int_{B_1} \operatorname{dist}^2(Du, K) dx, \end{aligned}$$

for some constant C_0 (depending only on B_1). If there exists $M \in K$ such that

$$\int_{B_{1/2}} |Dv - M|^2 dx \leq \tilde{C}(K) \int_{B_1} \operatorname{dist}^2(Dv, K) dx,$$

then repeated applications of the triangle inequality give

$$\int_{B_{1/2}} |Du - M|^2 dx \leq \left(16\tilde{C}(K)C_0 + 4\tilde{C}(K) + 8C_0\right) \int_{B_1} \operatorname{dist}^2(Du, K) dx.$$

Thus if Theorem 1.1 holds for all Lipschitz mappings v for some constant $\tilde{C}(K)$, then it is also valid for all H^1 mappings with $C(K) = 16\tilde{C}(K)C_0 + 4\tilde{C}(K) + 8C_0$.

Step 2. We may assume in addition that $u \in C^2(B_1; \mathbb{R}^2)$ solves

$$\operatorname{div} G_1(Du_1) = \operatorname{div} G_2(Du_2) = 0 \quad \text{in } B_1. \tag{7}$$

Consider indeed $w \in C^2(B_1)$ such that $w = u$ on ∂B_1 and

$$\operatorname{div} G_1(Dw_1) = \operatorname{div} G_2(Dw_2) = 0 \quad \text{in } B_1.$$

The existence of such w is guaranteed by the ellipticity of the equation $0 = \operatorname{div} G_j(Dw_j) = \operatorname{tr}(DG_j(Dw_j)D^2w_j) = 0$, invoking e.g. [16, Theorem 12.5]: the inequality $\lambda|\xi|^2 \leq DG_j(A)\xi \cdot \xi \leq \Lambda|\xi|^2$, valid for all $A, \xi \in \mathbb{R}^2$ thanks to Proposition 1.3, ensures that the eigenvalues of the symmetric part $[DG_j(A)]_s = (DG_j(A) + DG_j(A)^T)/2$ of $DG_j(A)$ are bounded above and below (since $DG_j(A)\xi \cdot \xi = [DG_j(A)]_s\xi \cdot \xi$ for all $\xi \in \mathbb{R}^2$) and in particular condition (ii) in [16, Theorem 12.5] is satisfied. Letting $v = u - w$ and using the uniform monotonicity of G_1 we find

$$\lambda \int_{B_1} |Dv_1|^2 dx \leq \int_{B_1} (G_1(Du_1) - G_1(Dw_1)) \cdot Dv_1 dx.$$

Since $\operatorname{div} G_1(Dw_1) = 0$ and $\operatorname{div}(iDu_2) = 0$ we rewrite this as

$$\begin{aligned} \lambda \int_{B_1} |Dv_1|^2 dx &\leq \int_{B_1} (G_1(Du_1) + iDu_2) \cdot Dv_1 dx \\ &\leq \frac{1}{2\lambda} \int_{B_1} |G_1(Du_1) + iDu_2|^2 dx + \frac{\lambda}{2} \int_{B_1} |Dv_1|^2 dx, \end{aligned}$$

and infer

$$\int_{B_1} |Dv_1|^2 dx \leq \frac{1}{\lambda^2} \int_{B_1} |G_1(Du_1) + iDu_2|^2 dx.$$

According to Proposition 1.3 the function $M \mapsto G_1(A) + iB$, where A, B denote the first and second row of the matrix M , vanishes on K . Since that function is Lipschitz we deduce that $|G_1(Du_1) + iDu_2| \leq C \operatorname{dist}(Du, K)$, and therefore

$$\int_{B_1} |Dv_1|^2 dx \leq C \int_{B_1} \operatorname{dist}^2(Du, K) dx.$$

Applying a similar argument to v_2 we obtain

$$\int_{B_1} |Dv|^2 dx \leq C \int_{B_1} \operatorname{dist}^2(Du, K) dx.$$

Recalling that $v = u - w$ and using the triangle inequality we deduce

$$\begin{aligned} \int_{B_1} \operatorname{dist}^2(Dw, K) dx &\leq C \int_{B_1} \operatorname{dist}^2(Du, K) dx, \\ \int_{B_{1/2}} |Du - M|^2 dx &\leq 2 \int_{B_{1/2}} |Dw - M|^2 dx + C \int_{B_1} \operatorname{dist}^2(Du, K) dx. \end{aligned}$$

As a consequence, if Theorem 1.1 is valid for w then we obtain it for u . This proves Step 2.

Step 3. As $u_j \in C^2(B_1)$ satisfies (7), it is a weak solution of

$$\operatorname{div}(\partial_i(G_j(Du_j))) = \operatorname{div}(DG_j(Du_j)D(\partial_i u_j)) = 0 \quad \text{for } i = 1, 2.$$

Invoking the De Giorgi-Nash estimates [9] (see e.g. [17, Theorem 4.11] for the precise statement we use here) for $\partial_i u_j$, $i, j \in \{1, 2\}$, we obtain

$$\|Du\|_{C^\alpha(\bar{B}_{1/2})} \leq C\|Du\|_{L^2(B_1)},$$

for some $\alpha > 0$ and some constant $C = C(K)$. Thanks to this estimate and the exact rigidity obtained in Lemma 2.1, we may argue exactly as in [13, Lemma 4.5] to deduce that

$$\inf_{M \in K} \|Du - M\|_{L^\infty(B_{1/2})} \leq \rho \left(\int_{B_1} \text{dist}^2(Du, K) dx \right), \quad (8)$$

for some function ρ depending only on K and satisfying $\rho(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Step 4. We finally combine Step 3 with the linearized estimate of Lemma 2.2, to obtain our main estimate (3). The basic idea, as in [14, 13], is to linearize $\text{dist}^2(\cdot, K)$ around $M_0 \in K$ such that $|Du - M_0|$ is uniformly small. When doing so, (3) formally turns into the linearized estimate (4) of Lemma 2.2, and it remains to control the error terms. Due to the modification of u arising from the translation $X \in T_M K$ in the left-hand side of the linearized estimate (4), it is not directly obvious that the error terms are negligible. In [14] this problem is absent because their equivalent of (8) comes with an explicit $\rho(\epsilon) = C\epsilon^{\frac{1}{4}}$. In [13] it is taken care of via a topological degree argument [13, Proposition 4.7] (see also [28]) which allows to avoid the translation. It is however not obvious to us how to apply that topological degree argument in the absence of a group structure on K , and we present here an alternative method relying on elementary estimates.

We assume without loss of generality that

$$\int_{B_1} \text{dist}^2(Du, K) dx = \epsilon \leq \epsilon_0, \quad (9)$$

where $\epsilon_0 = \epsilon_0(K)$ is to be chosen in the course of the proof. If (9) is not valid then (3) is automatically satisfied for a large enough constant C because the left-hand side of (3) is bounded thanks to Step 1.

We fix $\delta_0 > 0$ depending only on K , such that the nearest-point projection Π_K onto K is uniquely defined and smooth in the neighborhood $\mathcal{N}_{2\delta_0}(K)$. We first choose ϵ_0 small enough that $\rho(\epsilon_0) \leq \delta_0$, so thanks to (8) the projection $\Pi_K(Du)$ is well-defined.

We claim that, for every $M \in K$, there exists $Y_M \in K$ such that

$$\begin{aligned} \int_{B_{1/2}} |Du - Y_M|^2 dx &\leq C \int_{B_{1/2}} \text{dist}^2(Du, K) dx + C \int_{B_{1/2}} |Du - M|^4 dx, \\ \text{and } |M - Y_M|^2 &\leq C \int_{B_{1/2}} |Du - M|^2 dx. \end{aligned} \quad (10)$$

Here and in the rest of this proof we denote by $C > 0$ a generic constant depending only on K .

To prove (10), we first invoke Lemma 2.2, according to which we have

$$\inf_{X \in T_M K} \int_{B_{1/2}} |Du - M - X|^2 dx \leq C \int_{B_{1/2}} \text{dist}^2(Du - M, T_M K) dx.$$

Choosing $X = X_M \in T_M K$ attaining the infimum in the left-hand side, we obtain

$$\int_{B_{1/2}} |Du - M - X_M|^2 dx \leq C \int_{B_{1/2}} \text{dist}^2(Du - M, T_M K) dx. \quad (11)$$

Moreover the minimizing property of X_M implies that $\int_{B_{1/2}} (Du - M - X_M) dx$ is orthogonal to X_M . We deduce

$$|X_M|^2 = \int_{B_{1/2}} X_M \cdot (Du - M) dx \leq \frac{1}{2} |X_M|^2 + C \int_{B_{1/2}} |Du - M|^2 dx,$$

and therefore

$$|X_M|^2 \leq C \int_{B_{1/2}} |Du - M|^2 dx. \quad (12)$$

Recalling from the proof of Lemma 2.2 that P_M denotes the orthogonal projection onto $(T_M K)^\perp$, we estimate the integrand in the right-hand side of (11) using

$$\begin{aligned} \text{dist}(Du - M, T_M K) &= |P_M(Du - M)| \\ &\leq |Du - \Pi_K(Du)| + |\Pi_K(Du) - M - (I - P_M)(Du - M)| \\ &\leq |Du - \Pi_K(Du)| + C|Du - M|^2. \end{aligned}$$

The last inequality follows from the fact that $I - P_M = D\Pi_K(M)$. Since $|Du - \Pi_K(Du)| = \text{dist}(Du, K)$, plugging this into (11) we find

$$\int_{B_{1/2}} |Du - M - X_M|^2 dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) dx + C \int_{B_{1/2}} |Du - M|^4 dx \quad (13)$$

Now we may choose $Y_M \in K$ such that

$$|M + X_M - Y_M| \leq C|X_M|^2. \quad (14)$$

Indeed, if $|X_M| \leq \delta_0$ then one can simply take $Y_M = \Pi_K(M + X_M)$ and use the fact that $D\Pi_K(M)X_M = (I - P_M)X_M = X_M$, and if $|X_M| \geq \delta_0$ one may take $Y_M = M$. From (13) and (14) we infer

$$\int_{B_{1/2}} |Du - Y_M|^2 dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) dx + C \int_{B_{1/2}} |Du - M|^4 dx + C|X_M|^4.$$

Using (12) and Cauchy-Schwarz to estimate the last term, we deduce the first inequality in (10), and the estimate on $|M - Y_M|$ in (10) follows from (14) and (12) (taking into account that $|X_M| \leq C$ thanks to (12) and Step 1).

Our goal is to find $M \in K$ for which we can discard the last term in (10). To that end, we apply (10) to define by induction a sequence $(M_j) \subset K$ satisfying

$$M_{j+1} = Y_{M_j} \quad \forall j \geq 0,$$

and M_0 to be chosen later. According to (10) we have

$$\int_{B_{1/2}} |Du - M_{j+1}|^2 dx \leq C \int_{B_{1/2}} \text{dist}^2(Du, K) dx + C \int_{B_{1/2}} |Du - M_j|^4 dx,$$

$$\text{and } |M_j - M_{j+1}|^2 \leq C \int_{B_{1/2}} |Du - M_j|^2 dx,$$

Therefore, setting

$$q_j = \int_{B_{1/2}} |Du - M_j|^2 \quad \text{and} \quad s_j = \sup_{B_{1/2}} |Du - M_j|, \quad (15)$$

and recalling the definition of $\epsilon = \int_{B_1} \text{dist}^2(Du, K) dx$ (9), we find

$$q_{j+1} \leq C\epsilon + Cs_j^2 q_j \quad \text{and} \quad s_{j+1} \leq s_j + Cq_j^{\frac{1}{2}}. \quad (16)$$

We wish to find $j_0 \geq 0$ such that

$$q_{j_0} \leq \Lambda\epsilon, \quad (17)$$

for some constant $\Lambda > 0$ depending only on K . Recalling the definitions of q_j (15) and ϵ (9), this concludes the proof of (3). We prove by induction on j_0 that, if $s_0 \leq 1/(2\Lambda)$ and

$$q_j > \Lambda\epsilon \quad \text{for } j = 0, \dots, j_0, \quad (18)$$

then

$$q_j \leq (\Lambda s_0)^{2j} s_0^2 \quad \text{for } j = 0, \dots, j_0. \quad (19)$$

The constant $\Lambda > 0$ depends only on K and will be adjusted in the course of the proof. Initialization at $j_0 = 0$ is obvious since $|B_{1/2}| = \pi/4 \leq 1$. Next we assume (18)-(19) and prove that either (17) or (19) is satisfied at $j_0 + 1$. Combining (19) with the second inequality in (16), we find

$$s_{j_0} \leq s_0 + Cs_0 \sum_{j=0}^{j_0-1} (\Lambda s_0)^j \leq (1+C)s_0,$$

provided $2\Lambda s_0 \leq 1$. Therefore the first inequality in (16) gives

$$q_{j_0+1} \leq C\epsilon + C(1+C)^2 s_0^2 q_{j_0},$$

and using (19) we obtain

$$\begin{aligned} q_{j_0+1} &\leq C\epsilon + C(1+C)^2 s_0^4 (\Lambda s_0)^{2j_0} \\ &= C\epsilon + \frac{C(1+C)^2}{\Lambda^2} (\Lambda s_0)^{2j_0+2} s_0^2. \end{aligned}$$

Therefore we have

$$\text{either } q_{j_0+1} \leq 2C\epsilon, \quad \text{or} \quad q_{j_0+1} \leq 2 \frac{C(1+C)^2}{\Lambda^2} (\Lambda s_0)^{2j_0+2} s_0^2.$$

In the first case (17) is satisfied for $j = j_0 + 1$ provided $\Lambda \geq 2C$, and in the second case (19) is satisfied at $j_0 + 1$, provided $\Lambda \geq (2C)^{1/2}(1 + C)$. This concludes the proof of (19) by induction.

As a consequence, assuming by contradiction that there is no $j_0 \geq 0$ satisfying (17), we would have

$$q_j > \Lambda\epsilon \quad \text{and} \quad q_j \leq (\Lambda s_0)^{2j} s_0^2 \quad \forall j \geq 0.$$

As $s_0 \leq 1/(2\Lambda)$, this implies $q_j \rightarrow 0$ and, passing to the limit in $q_j > \Lambda\epsilon$ we deduce $\epsilon = 0$. But in that case by Lemma 2.1 we have $Du \equiv M_0$ and this choice of M_0 gives $q_0 = 0$.

We conclude that there exists $j_0 = j_0(\epsilon)$ satisfying (17), if we can choose M_0 in such a way that $s_0 \leq 1/(2\Lambda)$. But thanks to (8) we do have $s_0 \leq \rho(\epsilon) \leq 1/(2\Lambda)$ provided ϵ_0 is small enough. \square

5 A 3×3 counter-example

In this section we prove that the two-dimensional setting of Theorem 1.1 is optimal in the following sense: a connected 1-submanifold of $\mathbb{R}^{3 \times 3}$ which has no rank-one connection and is elliptic may not satisfy Šverák's compactness result [31], and even less a quantitative rigidity estimate.

We recall (see e.g. [30, § 2]) that an ordered set of $N \geq 4$ matrices $\{T_i\}_{i=1}^N \subset \mathbb{R}^{m \times n}$ without rank-one connections is said to form a \mathcal{T}_N configuration if there exist matrices $P_i, C_i \in \mathbb{R}^{m \times n}$ and numbers $\kappa_i > 1$ such that

$$\begin{aligned} T_1 &= P + \kappa_1 C_1, \\ T_2 &= P + C_1 + \kappa_2 C_2, \\ &\dots \\ T_N &= P + C_1 + C_2 + \dots + C_{N-1} + \kappa_N C_N, \end{aligned}$$

where C_i is rank-one for all i and $\sum_{i=1}^N C_i = 0$.

Proposition 5.1. *There exists a smooth, compact and connected 1-submanifold $K \subset \mathbb{R}^{3 \times 3}$ without boundary which is elliptic and has no rank-one connection, but contains a \mathcal{T}_4 configuration.*

By a known construction, see e.g. [7, Theorem 3.1], Proposition 5.1 implies the existence of a sequence of maps $u_k: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\int_{B_1} \text{dist}^2(Du_k, K) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

but (Du_k) is not precompact in $L^2(B_{1/2})$. In particular, one certainly cannot hope for a quantitative estimate

$$\inf_{M \in K} \int_{B_{1/2}} |Du_k - M|^2 dx \leq \rho \left(\int_{B_1} \text{dist}^2(Du_k, K) dx \right),$$

for any function $\rho(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proposition 5.1 is a consequence of the construction below. Let $a > 0$ and define matrices T_1, T_2, T_3, T_4 by

$$T_1 = -T_3 = \begin{pmatrix} 1+a & 0 & 0 \\ 0 & 1 & 0 \\ 1+a & 0 & 0 \end{pmatrix}, \quad T_2 = -T_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1+a & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and C_1, C_2, C_3, C_4 by

$$C_1 = -C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_2 = -C_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We have that $T_k - C_k$ is rank-one for $k = 1, 2, 3, 4$, and (with the convention that $C_5 = C_1$)

$$T_k - \frac{2+a}{a} (T_k - C_k) = C_{k+1} \quad \text{for } k = 1, 2, 3, 4,$$

so $\{T_1, T_2, T_3, T_4\}$ forms a \mathcal{T}_4 configuration. Next we construct a curve $K = K_a$ as in Proposition 5.1, which contains this \mathcal{T}_4 configuration. Let $\theta_a = \arctan(1/(1+a))$, so

$$\cos \theta_a = \frac{1+a}{r_a}, \quad \sin \theta_a = \frac{1}{r_a}, \quad r_a = \sqrt{1 + (1+a)^2}.$$

Let $\rho: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth monotonically increasing function to be determined later that satisfies

- $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$ for $\theta \in \mathbb{R}$,
- $\rho(\hat{\theta}_k) = \hat{\theta}_k$ for $\hat{\theta}_k = \theta_a + (k-1)\frac{\pi}{2}$, $k = 1, 2, 3, 4$.

Define $K_a = \Gamma_a(\mathbb{R}/2\pi\mathbb{Z})$, with

$$\Gamma_a(\theta) := \begin{pmatrix} r_a \cos(\theta) & -\sin(8\theta - 8\theta_a) & \sin(6\rho(\theta) - 6\theta_a) \\ \sin(6\theta - 6\theta_a) & r_a \sin(\theta) & \sin(8\rho(\theta) - 8\theta_a) \\ r_a \cos(\theta) & \sin(8\theta - 8\theta_a) & \sin(6\rho(\theta) - 6\theta_a) \end{pmatrix} \quad (20)$$

Then we have

$$T_k = \Gamma_a(\hat{\theta}_k) \text{ for } k = 1, 2, 3, 4,$$

so K_a contains the \mathcal{T}_4 configuration $\{T_1, T_2, T_3, T_4\}$. Next we adjust the parameter $a > 0$ and the function ρ in order to ensure that K_a has no rank-one connection and is elliptic.

Notation. With the $M_{i_1 i_2, j_1 j_2}$ minor we mean the determinant of the 2×2 submatrix corresponding to the rows i_1, i_2 and columns j_1, j_2 .

Lemma 5.2. *If $a > 0$ is such that $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$, the curve K_a is elliptic, i.e. $\text{Rank } \Gamma'_a(\theta) > 1$ for all $\theta \in \mathbb{R}$.*

Proof. The derivative Γ'_a is given by

$$\Gamma'_a(\theta) := \begin{pmatrix} -r_a \sin(\theta) & -8 \cos(8\theta - 8\theta_a) & 6\rho'(\theta) \cos(6\rho(\theta) - 6\theta_a) \\ 6 \cos(6\theta - 6\theta_a) & r_a \cos(\theta) & 8\rho'(\theta) \cos(8\rho(\theta) - 8\theta_a) \\ -r_a \sin(\theta) & 8 \cos(8\theta - 8\theta_a) & 6\rho'(\theta) \cos(6\rho(\theta) - 6\theta_a) \end{pmatrix}$$

Assume $\text{Rank } \Gamma'_a(\theta) \leq 1$ for some $\theta \in \mathbb{R}$. Then calculating the $M_{12,12}$ minor we have

$$-r_a^2 \sin(\theta) \cos(\theta) + 48 \cos(8\theta - 8\theta_a) \cos(6\theta - 6\theta_a) = 0$$

and calculating the $M_{23,12}$ minor we have

$$48 \cos(8\theta - 8\theta_a) \cos(6\theta - 6\theta_a) + r_a^2 \sin(\theta) \cos(\theta) = 0.$$

Adding and subtracting these two equations we obtain that

$$\sin(\theta) \cos(\theta) = 0 \text{ and } \cos(8\theta - 8\theta_a) \cos(6\theta - 6\theta_a) = 0.$$

The first equality implies $\theta \in \frac{\pi}{2}\mathbb{Z}$, and then the second equality becomes

$$\cos(8\theta_a) \cos(6\theta_a) = 0,$$

which is impossible for $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$. □

Lemma 5.3. *For any $a > 0$ such that $\theta_a \notin \frac{\pi}{24}\mathbb{Z}$, for any $\epsilon > 0$, there exists a smooth monotonic function $\rho: \mathbb{R} \rightarrow \mathbb{R}$ with the following properties*

- $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$ for $\theta \in \mathbb{R}$,
- $\rho(\hat{\theta}_k) = \hat{\theta}_k$ for $\theta_k = \hat{\theta}_a + (k-1)\frac{\pi}{2}$, $k = 1, 2, 3, 4$.
- For any $\theta, \theta' \in [0, 2\pi) \cap \frac{\pi}{24}\mathbb{Z}$, we have $\rho(\theta) - \rho(\theta') \notin \frac{\pi}{12}\mathbb{Z}$
- $\sup_{x \in \mathbb{R}} |\rho(x) - x| < \epsilon$

Proof. Let $\delta = \frac{1}{2} \text{dist}(\theta_a, \frac{\pi}{24}\mathbb{Z}) = \frac{1}{2} \text{dist}(\{\hat{\theta}_k\}, \frac{\pi}{24}\mathbb{Z}) > 0$ and fix a smooth function φ such that $\text{supp } \varphi \subset (-\delta, \delta)$, $0 \leq \varphi \leq \varphi(0) = 1$ and $|\varphi'| \leq 2/\delta$. Define ρ on $[0, 2\pi)$ by setting

$$\rho(\theta) = \theta + \sum_{j=1}^{47} t_j \varphi\left(\theta - j\frac{\pi}{24}\right) \quad \text{for } \theta \in [0, 2\pi),$$

where $t_1, \dots, t_{47} \in (-\eta, \eta)$ are to be fixed later and $\eta = \min(\epsilon, \delta/2)$. The choice of $\delta > 0$ ensures that $\rho(\hat{\theta}_k) = \hat{\theta}_k$, also since $|t_j| < \epsilon$ we have $|\rho - id| < \epsilon$, and finally since $|t_j| < \delta/2$ we have $\rho' > 0$ on $[0, 2\pi)$. Moreover the function $\rho - id$ is identically zero near 0 and 2π , so it can be extended to a smooth 2π periodic function, thus yielding a smooth monotonic extension $\rho: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\rho(\theta + 2\pi) = \rho(\theta) + 2\pi$ for $\theta \in \mathbb{R}$. It remains to argue that we can pick $t_1, \dots, t_{47} \in [0, \eta)$ to ensure that the third condition in Lemma 5.3 is satisfied.

Denote $t_0 = 0$. By induction, we may for each $j = 1, \dots, 47$ choose $t_j \in (-\eta, \eta)$ to ensure that

$$\rho(j\pi/24) - \rho(\ell\pi/24) = t_j - t_\ell + (j - \ell)\pi/24 \notin \frac{\pi}{12}\mathbb{Z}$$

for all $\ell \in \{0, \dots, j-1\}$. This is possible because at each step there is only a discrete set of values of t_j to avoid. □

Lemma 5.4. *If $a > 0$ is such that $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$, $\epsilon > 0$ is small enough and ρ is as in Lemma 5.3, then the curve $K_a \subset \mathbb{R}^{3 \times 3}$ does not contain Rank-1 connections.*

Proof. Note as in Lemma 5.2 that the assumption $\theta_a \notin \frac{\pi}{48}\mathbb{Z}$ implies

$$\cos(8\theta_a) \neq 0 \quad \text{and} \quad \cos(6\theta_a) \neq 0.$$

We assume that there exist $\theta \neq \theta' \in \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\text{Rank}(\Gamma_a(\theta) - \Gamma_a(\theta')) = 1, \tag{21}$$

and we obtain a contradiction. We do this in several steps.

Step 1. We have

$$\theta + \theta' \in \pi\mathbb{Z} \quad \text{and} \quad \theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cup \frac{\pi}{4}\mathbb{Z}. \tag{22}$$

Proof of Step 1. From (20) calculating the $M_{12,12}$ minor we have

$$\begin{aligned} 0 &= r_a^2 (\cos(\theta) - \cos(\theta')) (\sin(\theta) - \sin(\theta')) \\ &\quad + (\sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a)) (\sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a)) \\ &= -4r_a^2 \sin\left(\frac{\theta + \theta'}{2}\right) \cos\left(\frac{\theta + \theta'}{2}\right) \sin^2\left(\frac{\theta - \theta'}{2}\right) \\ &\quad + 4 \sin(3(\theta - \theta')) \sin(4(\theta - \theta')) \cos(3(\theta + \theta') - 6\theta_a) \cos(4(\theta + \theta') - 8\theta_a). \end{aligned} \tag{23}$$

And calculating the $M_{23,12}$ minor we have

$$\begin{aligned} 0 &= -r_a^2 (\cos(\theta) - \cos(\theta')) (\sin(\theta) - \sin(\theta')) \\ &\quad + (\sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a)) (\sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a)) \\ &= 4r_a^2 \sin\left(\frac{\theta + \theta'}{2}\right) \cos\left(\frac{\theta + \theta'}{2}\right) \sin^2\left(\frac{\theta - \theta'}{2}\right) \\ &\quad + 4 \sin(3(\theta - \theta')) \sin(4(\theta - \theta')) \cos(3(\theta + \theta') - 6\theta_a) \cos(4(\theta + \theta') - 8\theta_a). \end{aligned} \tag{24}$$

Adding and subtracting (23) and (24) we obtain the equations

$$0 = \sin(3(\theta - \theta')) \sin(4(\theta - \theta')) \cos(3(\theta + \theta') - 6\theta_a) \cos(4(\theta + \theta') - 8\theta_a) \tag{25}$$

and

$$0 = \sin\left(\frac{\theta + \theta'}{2}\right) \cos\left(\frac{\theta + \theta'}{2}\right) \sin^2\left(\frac{\theta - \theta'}{2}\right). \tag{26}$$

Since $\theta \neq \theta'$ in $\mathbb{R}/2\pi\mathbb{Z}$, the last factor of (26) is nonzero, so either the first or the second must be zero. This implies $\theta + \theta' \in \pi\mathbb{Z}$. As a consequence, the last two factors in (25) are equal to $\pm \cos(6\theta_a)$ and $\cos(8\theta_a)$ and are nonzero by our choice of a . So one of the first two factors of (25) must vanish, that is, $\theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cup \frac{\pi}{4}\mathbb{Z}$.

Step 2. We have $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$.

Proof of Step 2. Considering the $M_{13,23}$ minor of $\Gamma_a(\theta) - \Gamma_a(\theta')$ we obtain the equation

$$\begin{aligned} 0 &= (\sin(8\theta - 8\theta_a) - \sin(8\theta' - 8\theta_a)) \\ &\quad \times (\sin(6\rho(\theta) - 6\theta_a) - \sin(6\rho(\theta') - 6\theta_a)) \\ &= 4 \sin(4(\theta - \theta')) \cos(4(\theta + \theta') - 8\theta_a) \\ &\quad \times \sin(3(\rho(\theta) - \rho(\theta'))) \cos(3(\rho(\theta) + \rho(\theta')) - 6\theta_a). \end{aligned}$$

From Step 1 we have $\theta + \theta' \in \pi\mathbb{Z}$ so the second factor is $\cos(8\theta_a) \neq 0$. The last factor is arbitrarily close to $\pm \cos(6\theta_a) \neq 0$ since $|\rho - id| \leq \epsilon$. The third factor is nonzero by construction of ρ (recall Lemma 5.3) because $\theta, \theta' \in \frac{\pi}{24}\mathbb{Z}$ by (22). So we must have $\sin(4(\theta - \theta')) = 0$ hence $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$.

Step 3. We have $\theta - \theta' \in \pi\mathbb{Z}$.

Proof of Step 3. Considering the $M_{12,13}$ minor of $\Gamma_a(\theta) - \Gamma_a(\theta')$ we obtain the equation

$$\begin{aligned} 0 &= r_a (\cos(\theta) - \cos(\theta')) (\sin(8\rho(\theta) - 8\theta_a) - \sin(8\rho(\theta') - 8\theta_a)) \\ &\quad - (\sin(6\theta - 6\theta_a) - \sin(6\theta' - 6\theta_a)) \\ &\quad \times (\sin(6\rho(\theta) - 6\theta_a) - \sin(6\rho(\theta') - 6\theta_a)), \end{aligned}$$

which we rewrite as

$$\begin{aligned} &- 4r_a \sin\left(\frac{\theta + \theta'}{2}\right) \sin\left(\frac{\theta - \theta'}{2}\right) \\ &\quad \times \sin(4(\rho(\theta) - \rho(\theta'))) \cos(4(\rho(\theta) + \rho(\theta')) - 8\theta_a) \\ &= 4 \sin(3(\theta - \theta')) \cos(3(\theta + \theta') - 6\theta_a) \\ &\quad \times \sin(3(\rho(\theta) - \rho(\theta'))) \cos(3(\rho(\theta) + \rho(\theta')) - 6\theta_a). \end{aligned} \tag{27}$$

Since $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$ and $|\rho - id| \leq \epsilon$, the third factor in the left-hand side of (27) has absolute value $\leq 4\epsilon$. Since $\theta + \theta' \in \pi\mathbb{Z}$, the second factor in the right-hand side of (27) has absolute value equal to $|\cos(6\theta_a)| > 0$, and for small enough ϵ the absolute value of the last factor is $\geq |\cos(6\theta_a)|/2 > 0$. Taking also into account that the first and third factor in the right-hand side of (27) differ from each other by an error $\leq 3\epsilon$, we must have

$$\sin^2(3(\theta - \theta')) \leq c_a \epsilon,$$

for some $c_a > 0$ depending only on a . Because $\theta - \theta' \in \frac{\pi}{4}\mathbb{Z}$ by Step 2, provided ϵ is chosen small enough, this implies $\theta - \theta' \in \frac{\pi}{3}\mathbb{Z} \cap \frac{\pi}{4}\mathbb{Z} = \pi\mathbb{Z}$.

Step 4: Conclusion. From Step 1 and Step 3 we have $\theta + \theta', \theta - \theta' \in \pi\mathbb{Z}$, so $\theta, \theta' \in \pi\mathbb{Z}$. Since we may without loss of generality exchange the roles of θ and θ' and choose arbitrary representants in $\mathbb{R}/2\pi\mathbb{Z}$, this amounts to $\theta = 0$ and $\theta' = \pi$. Considering the $M_{12,13}$ minor of $\Gamma_a(0) - \Gamma_a(\pi)$ as in (27) we deduce

$$\sin(4(\rho(\pi) - \rho(0))) \cos(4(\rho(\pi) + \rho(0)) + 8\theta_a) = 0.$$

Using again that $|\rho - id| \leq \epsilon$, the second factor has absolute value $\geq |\cos(8\theta_a)|/2$ provided ϵ is small enough, and the first factor is nonzero by construction of ρ , so we conclude that (21) is not possible for $\theta \neq \theta' \in \mathbb{R}/2\pi\mathbb{Z}$. \square

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