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Context : boiling flows

Nuclear thermal hydraulics

- Multicomponent flows : liquid, vapor, solid obstacles
- Stiff source terms (heat source, boiling, singular pressure loss)
- Flow in subsonic or supersonic regime
 - Note : define the Mach number for two phase flows

In this context, staggered schemes seem more robust and precise than Riemann solvers.

Objectives :

- Study of stability and TVD properties of staggered schemes
- Extension to multiphase flows with complex terms (drift models)
- Extension to porous model (reactor core and steam generator)
- Extension to more general meshes

Numerical diffusion of staggered schemes

Numerical diffusion : (1D case for simplicity)

$$\partial_t U + A(U)\partial_x U = 0, \quad \text{non conservative form.} \quad (1)$$

For a given two point semi-discretisation *disc*

$$\partial_t U_i + A_{disc}^+ \frac{U_i - U_{i-1}}{\Delta x} + A_{disc}^- \frac{U_{i+1} - U_i}{\Delta x} = 0. \quad (2)$$

we have when $\Delta x \rightarrow 0$:

$$\partial_t U + A\partial_x U = \mathcal{D}(U, \Delta x) + o(\Delta x) \quad (3)$$

$$= -\Delta x D_{disc} \partial_{xx} U + o(\Delta x), \quad (4)$$

where

- $A = A_{disc}^+ + A_{disc}^-$ (consistency of (2) with (1))
- $D = A_{disc}^+ - A_{disc}^-$ represent the **numerical diffusion operator**

Examples with the wave system :

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + c^2 \partial_x \rho = 0 \end{cases} \quad (5)$$

Case 1 : centered scheme

$$\begin{cases} \partial_t \rho_i + \frac{q_{i+1} - q_{i-1}}{2\Delta x} = 0 \\ \partial_t q_i + c^2 \frac{\rho_{i+1} - \rho_{i-1}}{2\Delta x} = 0 \end{cases}$$

- $D_{centre} = 0$ (order 2 scheme), D is diagonalisable on \mathbb{R}
- Odd/even cells decoupling ✗

Case 2 : upwind scheme

$$\begin{cases} \partial_t \rho_i + \frac{q_{i+1} - q_{i-1}}{2\Delta x} = c \frac{\rho_{i+1} - 2\rho_i + \rho_{i-1}}{\Delta x} \\ \partial_t q_i + c^2 \frac{\rho_{i+1} - \rho_{i-1}}{2\Delta x} = c \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x} \end{cases}$$

$$D_{upwind} = |A_{wave}| = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad A_{wave} = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}.$$

- Numerical diffusion is too strong if $c \gg \|\mathbf{u}\|$ ✗
- The numerical diffusion matrix D_{upwind} is diagonalisable on \mathbb{R}

Case 3 : « simple » staggered scheme

$$\begin{cases} \partial_t \rho_i + \frac{q_{i+1/2} - q_{i-1/2}}{\Delta x} = 0 \\ \partial_t q_{i+1/2} + c^2 \frac{\rho_{i+1} - \rho_i}{\Delta x} = 0 \end{cases}, \quad U_i = \begin{pmatrix} \rho_i \\ q_{i+1/2} \end{pmatrix}. \quad (6)$$

$$D_{stag} = \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}, \quad A_{wave} = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}. \quad (7)$$

- The numerical diffusion is mostly on the second equation
- D_{stag} is not diagonalisable on \mathbb{R}

Most staggered discretisations yield non linearisable schemes → difficult linear stability analysis

A new class of staggered schemes

Numerical diffusion of conservative schemes

$$\partial_t U + \partial_x F(U) = 0 \quad \text{conservative form (1D for simplicity).}$$

A **conservative scheme** takes the form

$$\partial_t U_i + \frac{F_{i,i+1} - F_{i-1,i}}{\Delta x} = 0 \quad (8)$$

The **numerical fluxes** F_{ij} can be cast into the sum of two terms

$$F_{i,j} = \frac{F(U_i) + F(U_j)}{2} + D \frac{U_i - U_j}{2}, \quad (9)$$

where D is the **upwind operator** of the scheme.

Using symmetrising variables ($U \rightarrow \bar{U}$ such that ∇F is symmetric), D becomes \bar{D} and the L^2 energy evolution inequality is

$$\partial_t \frac{1}{2} \int_{\mathbb{R}} \|\bar{U}\|^2 = - \int_{\mathbb{R}} {}^t \partial_h \bar{U} (\bar{D} + {}^t \bar{D}) \partial_h \bar{U} \quad (10)$$

Therefore we seek matrices D satisfying

- $\bar{D} + {}^t \bar{D} \geq 0$ (stability)
- $\bar{D} + {}^t \bar{D}$ does not involve the sound speed c (precision)

Wave system (5) : with « simple » staggered discretisation (6):

$$D_{stag} = \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}, \quad \bar{D}_{stag} = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad \bar{D}_{stag} + {}^t \bar{D}_{stag} = 0.$$

- Diffusion numérique nulle ✓
- Fort couplage entre cellules ✓

Euler equations : We consider the following Euler equations :

$$\partial_t \rho + \nabla \cdot (\rho \bar{u}) = 0 \quad (11)$$

$$\partial_t (\rho \bar{u}) + \nabla \cdot (\rho \bar{u} \otimes \bar{u}) + \bar{\nabla} p = 0 \quad (12)$$

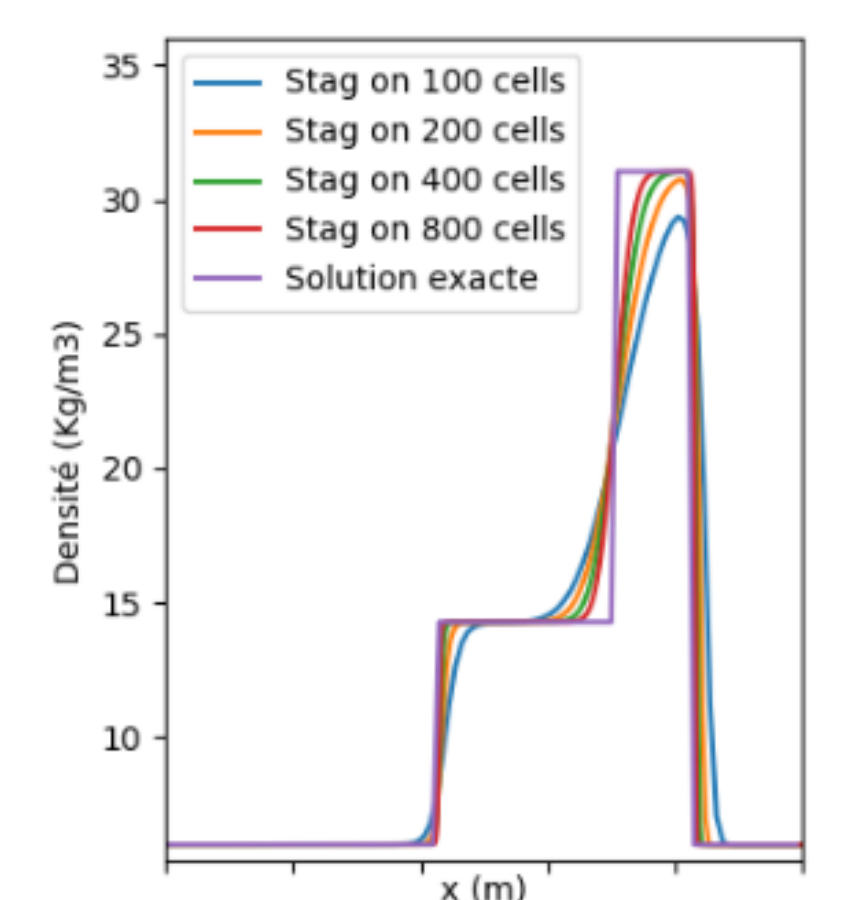
$$\partial_t (\rho (e + \frac{1}{2} \|\bar{u}\|^2)) + \nabla \cdot ((\rho e + \frac{1}{2} \rho \|\bar{u}\|^2 + p) \bar{u}) = 0 \quad (13)$$

Symmetrisation with $\frac{\partial p}{\partial \rho}|_e, \frac{\partial p}{\partial e}|_\rho$ not appearing on the diagonal :

$$\bar{D}_{Stag} = \begin{pmatrix} |u| & \sqrt{\frac{\partial p}{\partial \rho}|_e} & 0 \\ -\sqrt{\frac{\partial p}{\partial \rho}|_e} & |u| & -\sqrt{\frac{\partial p}{\partial e}|_\rho} \frac{p}{\rho^2} \\ 0 & \sqrt{\frac{\partial p}{\partial e}|_\rho} \frac{p}{\rho^2} & |u| \end{pmatrix}$$

Shock tube (high Mach) : Toro benchmark

- Convergence of 6 of the 7 tests
- Failure of test number 2 (near vacuum → negative densities)
- No checkerboard modes



Lid driven cavity (low Mach)

- Good low Mach asymptotic
- No checkerboard modes
- extension to two phase flow models

