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► **To cite this version:**

| Thai Ha-Huy. A tale of two Rawlsian criteria. 2022. ⟨hal-04126416⟩

**HAL Id: hal-04126416**

**<https://hal.science/hal-04126416v1>**

Preprint submitted on 13 Jun 2023

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# A tale of two Rawlsian criteria\*

Thai Ha-Huy<sup>†</sup>

14 January 2022

## ABSTRACT

This article considers optimization problems under two Rawls criteria. The first one is the classical Rawls criterion in literature and the second one as a result of the maximin criteria with multiple discount factors presented in Chambers and Echenique [8]. Though these criteria are different, they have the same optimal value and solution.

**Keywords:** Rawls criterion, maximin principle, multiple discount rates.

**JEL** classification numbers: C61, D11, D90.

## 1 INTRODUCTION

Consider the following classical question: given a stock of renewable resources, what would be the best inter-temporal exploitation of it, considering the welfare of both current and future generations?

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\*The author would like to thank Stefano Bosi, for many helpful comments. He is also grateful to the LABEX MME-DII (ANR-11-LBX-0023-01) for supports during the writing of this article. This article is finished during the period the author worked as *délegué* in the Theoretical Economics Team UMR 8545 at CNRS, Paris School of Economics (campus Jourdan). He acknowledges the members of PSE and CNRS for the warm welcome and excellent working conditions.

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The famous Ramsey criterion, which uses a constant discount rate and is used largely in research into economic dynamics, is criticized for its weak weighting parameters for generations in the distant future. The evaluation of each utility stream is quasi-determined by a finite number of generations. This raises the concerns that following the Ramsey criterion, the economy does not leave enough resource for the future generation. We can use as an illustration the comment of Ramsey, "*discounting the interests of future people is ethically indefensible and arises merely from the weakness of imagination.*"

In the classical work "Theory of justice", Rawls [21] assumes that if one is hidden behind a *veil of ignorance*, with total lack of information about the condition into which she/he will be born, the economic agent should choose the maximization of the least favoured generation. Specifically, given an inter-temporal consumption streams, her/his evaluation criterion of inter-temporal utilities streams should be

$$U(c_0, c_1, c_2, \dots) = \inf_{s \geq 0} u(c_s),$$

where  $u(c_t)$  is the utility of the  $t^{th}$  generation, given  $c_t$  as the consumed resource.

We can consider the Rawls's question in another way: the economic agent may be ambiguous about what is the "good" discount factor to choose in evaluating utilities streams. Her/his set of possible discount factors is  $(0, 1)$ . Having total lack of information, for a given consumption stream  $\{c_s\}_{s=0}^{\infty}$ , she/he should evaluate it as<sup>1</sup>

$$U(c_0, c_1, c_2, \dots) = \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(c_s) \right].$$

This criterion can also be considered as an application of Rawls's spirit in the configuration where disagreements exist between people in the economy about how to discount the future. The social planner chooses a criterion that maximizes the least favoured person.

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<sup>1</sup>For the axiomatic foundation and discussion about the importance of the normalizing term  $1 - \delta$ , see Chambers and Echenique [8] and Drugeon *et al*[12]. Observe that for any  $0 < \delta < 1$ , we have  $(1 - \delta) \sum_{s=0}^{\infty} \delta^s = 1$ .

This is not the only reason which urges us to consider this study. In recent decades, a large body of literature has risen in decision theory, enlarging the world of Savage [22], where the famous *sure-thing principle* is not satisfied. The seminar contribution of Gilboa and Schmeidler [14] considers the behaviour under which the economic agent, facing ambiguities, maximizes the worst scenario. This allows us to make a link to the Rawlsian criteria. Assume that the economic agent must choose a time discounting system to evaluate the inter-temporal consumption streams. The set of possible time discounting systems is  $\Delta = (\pi_0, \pi_1, \pi_2, \dots)$  such that  $\pi_s > 0$  for any  $s$  and  $\sum_{s=0}^{\infty} \pi_s = 1$ . Behind the *veil of ignorance*, every time discounting system is possible. Hence, the criterion under *ambiguity aversion* is

$$\begin{aligned} U(c_0, c_1, c_2, \dots) &= \inf_{\pi \in \Delta} \left[ \sum_{s=0}^{\infty} \pi_s u(c_s) \right] \\ &= \inf_{s \geq 0} u(c_s), \end{aligned}$$

which is the first Rawls criterion.

Now assume that the economic agent is just ambiguous about the set of time discounting systems satisfying the usual properties such as *impatience*, and *stability*. Let  $\mathcal{D}$  be that set. In Chambers and Echenique [8], this set is described as:

$$\mathcal{D} = \{ \pi \in \Delta \text{ such that } \exists \delta \in (0, 1) : \pi_s = (1 - \delta)\delta^s \text{ for all } s \geq 0 \}.$$

The *maximin* criterion then becomes the second Rawlsian one.

Naturally, this raises the question of the behaviour of the economy under the Rawls criteria. The first Rawls criterion is well studied in the seminar contributions of Arrow [4], Solow [19] and Calvo [7]. The result is clear: the behaviour of the economy depends strongly on the initial stock. If the stock of a renewable resource is below the *golden rule* (the stock allowing a maximal level of constant consumption), the optimal exploitation strategy is to ensure that the stock remains constant over time. In the case of abundant stock of resources, which is higher than the *golden rule*, there is an infinite number of solutions and every optimal

path converges decreasingly to this level.

The purpose of this work is to study the same question under the second Rawls criterion. First, we prove that there is a lower bound for the speed of convergence to steady state of Ramsey models, even in cases where the discount factor is very near to 1. Basing on this result, we prove that the two Rawlsian criteria have the same value function. The solutions coincide if the stock of resource is low (under the *golden rule*). In this case, the optimal choice is to remain constant over time. For the case where the resource is abundant (higher than the *golden rule*), the solution under the first criterion is also a solution under the second one.

This article is organized as follows. Section 2 introduces the two Rawlsian problems, the main properties of the first one and solves the second one. Section 3 discusses different criteria studied in the literature. Proofs of Proposition 2.3 and Corollary 2.1 are given in the Appendix.

## 2 THE TWO RAWLSIAN CRITERIA

### 2.1 FUNDAMENTALS

Denote by  $u$  the instantaneous utility function and  $f$  the regeneration function of the renewable resource. These two functions are supposed to be strictly increasing and concave. Assume further that the concavity of utility function is strict, and  $f'(0) = \infty$ ,  $f'(\infty) < 1$ .

Let  $x_s$  be the stock of resource at time  $s$ , the agent divides  $f(x_s)$  into two parts: the consumptions  $c_s$  and the investment for tomorrow,  $x_{s+1}$ . For any date  $s$ , we have  $c_s + x_{s+1} \leq f(x_s)$ . The agent must make a trade-off between consumption today and investment in tomorrow.

For any given capital stock  $x_0 \geq 0$ , denote by  $\Pi(x_0)$  the set of feasible paths of stock  $\{x_s\}_{s=0}^{\infty}$ : for any  $s$ ,  $0 \leq x_{s+1} \leq f(x_s)$ .

Denote by  $\bar{x}$  the golden rule, the capital accumulation corresponding to the max-

imum level of constant consumption<sup>2</sup>:  $\bar{x} = \operatorname{argmax}(f(x) - x)$ . By the concavity of  $f$ , this value  $\bar{x}$  is solution to the equation  $f'(x) = 1$ .

## 2.2 THE RAMSEY PROBLEM

The classical dynamic programming literature, culminating in Stokey and Lucas (with Prescott) [20], considers the following problem. For given  $x_0 \geq 0$  and discount factor  $0 < \delta < 1$ , the agent solves the optimization program, well-known as *the Ramsey problem*<sup>3</sup>:

$$v(x_0) = \max_{\Pi(x_0)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(c_s) \right],$$

s.t  $c_s + x_{s+1} \leq f(x_s)$ , for all  $s \geq 0$ ,

$c_s, x_s \geq 0$  for all  $s \geq 0$ .

Under suitable conditions, the value function  $v$  is solution to the following functional equation<sup>4</sup>:

$$v(x) = \max_{0 \leq y \leq f(x)} [(1 - \delta)u(f(x) - y) + \delta v(y)],$$

for every  $x \geq 0$ .

The strict concavity of the utility function and the regeneration function implies that the optimal program has unique optimal path  $\{x_s\}_{s=0}^{\infty} \in \Pi(x_0)$ , satisfying

$$v(x_s) = (1 - \delta)u(f(x_s) - x_{s+1}) + \delta v(x_{s+1}),$$

for every  $s \geq 0$ .

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<sup>2</sup>The constant sequence  $(\bar{x}, \bar{x}, \dots)$  generates a constant sequence of consumption  $(\bar{c}, \bar{c}, \dots)$ , where  $\bar{c} = f(\bar{x}) - \bar{x}$ .

<sup>3</sup>In general, when we work with only one discount factor  $\delta$ , for the sake of simplicity, the term  $1 - \delta$  can be relaxed. In contrast, this term plays an important rôle in multiple discount rates configuration. For details, see the axiomatization base in Chamber and Echenique [8], and comments about time discounting parameters satisfying *temporal stability* property in Drugeon *et al* [12].

<sup>4</sup>See Stokey and Lucas (with Prescott) [20], chapter 4.

For each discount factor  $\delta$ , the optimal path of the Ramsey problem corresponding to  $\delta$  converges monotonically<sup>5</sup> to  $x^\delta$ , the solution to

$$f'(x) = \frac{1}{\delta}.$$

Moreover, we verify easily that:

$$\begin{aligned} \lim_{\delta \rightarrow 0} x^\delta &= 0, \\ \lim_{\delta \rightarrow 1} x^\delta &= \bar{x}. \end{aligned}$$

We will use these properties in Section 2.4.

### 2.3 THE CLASSICAL RAWLS CRITERION

The famous Rawls criterion, embedded in an optimal growth context, can be considered as the following program. For given  $x_0 > 0$ , the economic agent solves:

$$\begin{aligned} &\max \left[ \inf_{s \geq 0} u(c_s) \right], \\ &\text{s.t } c_s + x_{s+1} \leq f(x_s), \text{ for all } s \geq 0, \\ &c_s, x_s \geq 0 \text{ for all } s \geq 0. \end{aligned}$$

For each feasible stock path  $\mathbf{x} = \{x_s\}_{s=0}^\infty$ , let

$$\nu(\mathbf{x}) = \inf_{s \geq 0} u(c_s).$$

In Stockey, Lucas (with Prescott) [20], the compactness of feasible set  $\Pi(x_0)$ , and the upper semi-continuity of the value function  $\nu$  with respect to the product topology are ensured. This implies the existence of an optimal path  $x^* \in \Pi(x_0)$  such that  $\nu(x^*) = \max_{\mathbf{x} \in \Pi(x_0)} \nu(\mathbf{x})$ .

The properties of optimal paths are well studied in Arrow [4], Solow [19] and

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<sup>5</sup>The monotonicity of the optimal path can be established for a larger class of utility function satisfying the super-modularity property. For details, see Amir [2] and [3].

Calvo [7] whose some results can be resumed in Proposition 2.1. The behaviour of the economy depends strongly on the initial condition, with the golden rule  $\bar{x}$  as critical threshold.

PROPOSITION 2.1. i) Consider the case  $0 \leq x_0 \leq \bar{x}$ . The problem has unique solution  $\mathbf{x}^* = (x_0, x_0, \dots)$ , and

$$\begin{aligned} \max_{\mathbf{x} \in \Pi(x_0)} \nu(\mathbf{x}) &= \nu(\mathbf{x}^*) \\ &= u(f(x_0) - x_0). \end{aligned}$$

ii) Consider the case  $x_0 > \bar{x}$ . The problem has an infinite number of solutions which all converge to  $\bar{x}$ , and

$$\max_{\mathbf{x} \in \Pi(x_0)} \nu(\mathbf{x}) = u(f(\bar{x}) - \bar{x}).$$

For initial capital stock  $x_0$  smaller than  $\bar{x}$ , the optimal choice is to remain in the *status quo*. The unique solution  $\mathbf{x}^*$  satisfies  $x_s^* = x_0$  for any  $s \geq 0$ . The optimal value is  $u(f(x_0) - x_0)$ . For  $x_0$  bigger than  $\bar{x}$ , there exists an infinite number of solution, every optimal stock path converges to  $\bar{x}$  and the optimal value is  $u(f(\bar{x}) - \bar{x})$ .

## 2.4 THE SECOND RAWLSIAN CRITERION

This section is devoted to the study of the second Rawls criterion. For each feasible stock path  $\mathbf{x} = \{x_s\}_{s=0}^{\infty}$ , let

$$\hat{\nu}(\mathbf{x}) = \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(c_s) \right].$$

In [12], Drugeon *et al* consider the optimization problem with multiple discount factors under the *maximin* criteria<sup>6</sup>. Let  $\mathcal{D} = [\underline{\delta}, \bar{\delta}]$  represents the set of possible

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<sup>6</sup>An axiomatic foundation for this criterion can be found in the work of Chambers and Echenique [8].

discount factors, for  $x_0 > 0$ , the economic agent solves:

$$\begin{aligned} & \max_{\delta \in \mathcal{D}} \min \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(c_s) \right] \\ & \text{s.c } c_s + x_{s+1} \leq f(x_s), \\ & c_s, x_s \geq 0 \text{ for any } s. \end{aligned}$$

By technical difficulties relying with the fixed point arguments, Druegeon *et al* [12] assume that  $\mathcal{D}$  is a closed set belonging to  $(0, 1)$ :  $0 < \underline{\delta} \leq \bar{\delta} < 1$ . Proposition 2.2, proven in Druegeon *et al* [12], gives a detailed description of the optimal path under the maximin criteria with multiple discount factors.

PROPOSITION 2.2. *Assume that  $0 < \underline{\delta} \leq \bar{\delta} < 1$ . Let  $\chi^*$  denote the unique optimal path for the maximin problem.*

- i) *For  $x_0 \leq x^{\underline{\delta}}$ ,  $\chi^*$  coincides with the optimal path of the Ramsey problem with discount factor  $\underline{\delta}$ , is increasing and converges to  $x^{\underline{\delta}}$ .*
- ii) *For  $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$ , for any  $s$ ,  $x_s^* = x_0$ . The optimal path  $\chi^*$  coincides with the optimal solution of Ramsey problem with discount factor  $\delta$  satisfying  $x^{\delta} = x_0$ .*
- iii) *For  $x_0 \geq x^{\bar{\delta}}$ ,  $\chi^*$  coincides with the optimal path of the Ramsey problem with discount factor  $\bar{\delta}$ , is decreasing and converges to  $x^{\bar{\delta}}$ .*

Proposition 2.2, and Figure 1, taken in Druegeon *et al*, provide us an illustration how optimal paths depend in initial condition under the the maximin criterion.

Naturally, with the results in Proposition 2.2, we may conjecture that if the under bound  $\underline{\delta}$  converges to zero, and the upper bound  $\bar{\delta}$  converges to 1, the two Ralwsian problems have the same value function:

$$\max_{\mathbf{x} \in \Pi(x_0)} \nu(\mathbf{x}) = \max_{\mathbf{x} \in \Pi(x_0)} \hat{\nu}(\mathbf{x}).$$

Propositions 2.3 confirms this conjecture, and represents the main result of this article.

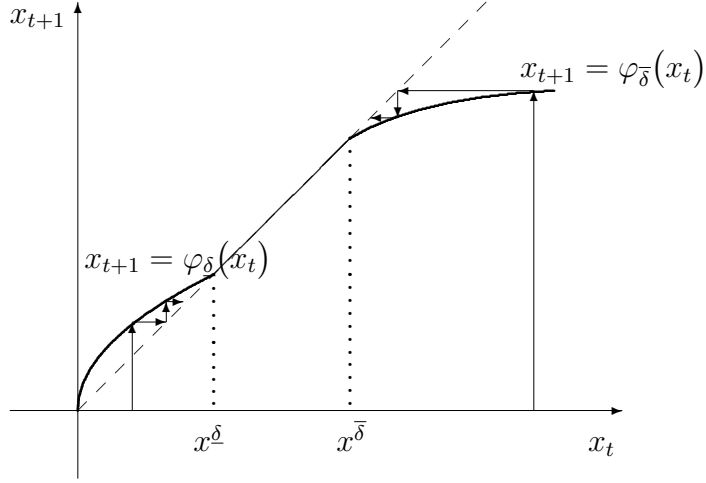


Figure 1: The optimal policy function in multiple discount rates configuration

PROPOSITION 2.3. *For any  $x_0 \geq 0$ , we have*

$$\max_{\chi \in \Pi(x_0)} \inf_{s \geq 0} u(f(x_s) - x_{s+1}) = \max_{\chi \in \Pi(x_0)} \left[ \inf_{\delta \in (0,1)} (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right].$$

Using Proposition 2.3, we can provide a description of the solutions under two criteria. For the case  $x_0$  is smaller than the *the golden rule*, the solutions of two problems coincide. The stock remains constant over time. For the other case, we are ensured that every solution to the problem under the first Rawls criterion is a solution under the second one.

COROLLARY 2.1. *For any  $x_0 \geq 0$ ,*

- i) *For  $0 \leq x_0 \leq \bar{x}$ , the two Rawlsian problems have the same solution  $\mathbf{x}^* = (x_0, x_0, x_0, \dots)$ .*
- ii) *For  $x_0 > \bar{x}$ , every solution under the first Rawlsian criterion is a solution under the second one.*

### 3 COMMENTS

### 3.1 CONVEX COMBINATION BETWEEN CRITERIA

The Ramsey criterion is criticized about putting privileges for the generations in the present and close future. In another way, other criteria, for example, the lim inf take into account only the distant future. As a way to reconcile these to extremes, Chichilnisky in [9], [10] proposes a criterion satisfying her *No-dictatorship of the present* and *No-dictatorship of the future* properties. Her criterion is a convex combination of a Ramsey part and a lim inf part<sup>7</sup>.

However, because the path optimizing the Ramsey part converges to the inverse of discount factor, otherwise the path optimizing the lim inf part converges to the *golden rule*, the solution under the combination of these criteria may not exist. This was established by Heal [16] in an economy with renewable resources and by Ayong le Kama and al [6] in a one sector economy context. It is always difficult taking into account at the same time the efficiency and the equality.

As a response for this challenge, Alvarez-Cuadrado and Van Long [1] consider the convex combination between a Ramsey part and a Rawlsian part, in the continuous time configuration. They give a detailed description of the behaviour of the economy<sup>8</sup>, which keeps consumption to be constant in the early periods of time, and after that, behaves as optimal solution of a Ramsey economy. Another approach is due to Asheim and Ekeland [5], who consider the linear *markovian* solutions of the problem under Chichilnisky's criterion, and conclude that the lim inf part has no effect on the optimal choice.

The *overtaking* criterion of Gale [13] satisfies the two *non-dictatorship* properties of Chichilnisky, but this criterion is not complete. If we focus only on the *good programs*, the set of feasible paths which differ not too much from the *golden rule*<sup>9</sup>, the optimal path exists and converges to the *golden rule*. As an attempt to avoid the non-completeness problem, Le Van and Morhaim [18] consider the

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<sup>7</sup>For a detailed discussion about Chichilnisky's criterion, see Alvarez-Cuadrado and Van Long [1].

<sup>8</sup>An analysis for the discrete time configuration is presented in Ha-Huy and Nguyen [15].

<sup>9</sup>See Dana and Le Van [11].

Ramsey problem and study the properties of the solution when the discount rate converges to 1. They prove that the sequence of solutions converges to the solution of the problem under Gale's criterion.

### 3.2 TECHNICAL CONCERNS

The result for the first Rawlsian criterion is based only on the concavity of the function  $f$ , and we can obtain Proposition 2.1 without imposing concavity on the utility function  $u$ . However, in order to apply results in dynamic programming literature, for solving the problem under the second Rawlsian criterion, we must assume the concavity property for the utility function.

And, consider the case where  $f'(\infty) \geq 1$ . Under this assumption,  $\bar{x} = \infty$ . For the two Rawlsian criteria, the only solution is to remain constant. The only remark is that, since the feasible paths could be unbounded, we must assume conditions ensuring the determination of value function and its continuity. For the details, curious readers can refer to the article of Le Van and Morhaim [17], with the most important condition being *tail insensitivity* property.

If  $f'(0) \leq 1$ , every feasible path converges to zero, the two problems become trivial.

## 4 APPENDIX

### 4.1 PROOF OF PROPOSITION 2.3

To facilitate the exposition, for each  $0 < \delta < 1$ , denote by  $\{x_s(\delta)\}_{s=0}^{\infty}$  the optimal path of Ramsey problem corresponding to the discount factor  $\delta$ .

Observe that for any feasible path of stock  $\{x_s\}_{s=0}^{\infty}$  belonging to  $\Pi(x_0)$ :

$$\inf_{s \geq 0} u(f(x_s) - x_{s+1}) \leq \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right].$$

This implies

$$\max_{\chi \in \Pi(x_0)} \min_{s \geq 0} u(f(x_s) - x_{s+1}) \leq \max_{\chi \in \Pi(x_0)} \left[ \inf_{\delta \in (0,1)} (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right].$$

Now we will prove the converse inequality.

Consider first the case  $0 < x_0 < \bar{x}$ . Fix  $0 < \underline{\delta} < \bar{\delta} < 1$  such that  $x^{\underline{\delta}} < x_0 < x^{\bar{\delta}}$ .

Define  $\chi^* = (x_0, x_0, \dots)$ , which is the unique optimal path for the *maximin* criterion with the set of discount rates  $\mathcal{D} = [\underline{\delta}, \bar{\delta}]$ . For any feasible path  $\chi \neq \chi^*$ , following Drugeon *et al* [12], we have

$$\begin{aligned} \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right] &\leq \inf_{\underline{\delta} \leq \delta \leq \bar{\delta}} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right] \\ &< \inf_{\underline{\delta} \leq \delta \leq \bar{\delta}} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s^*) - x_{s+1}^*) \right] \\ &= u(f(x_0) - x_0) \\ &= \max_{\chi \in \Pi(x_0)} \left[ \inf_{s \geq 0} u(f(x_s) - x_{s+1}) \right]. \end{aligned}$$

This implies that the two Rawlsian problems have the same maximum value and unique solution  $\chi^*$ .

Now consider the case  $x_0 > \bar{x}$ . The idea of the proof is that for any  $\delta$ , the sequence  $\{x_s(\delta)\}_{s=0}^{\infty}$  converges to  $x^{\delta}$  with a speed that is sufficiently high and independent with the choice of  $\delta$ .

We prove that for any  $\epsilon > 0$ , there exists  $T(\epsilon)$  such that for any  $T \geq T(\epsilon)$ , any  $0 < \delta < 1$ , we have

$$x^{\delta} < x_T(\delta) < \bar{x} + \epsilon.$$

For each  $0 < \delta < 1$ , consider a time  $s$  satisfying  $x_0 \geq x_1(\delta) \geq \dots \geq x_{s+1}(\delta) \geq \bar{x} + \epsilon$ . Observe that  $f'(\bar{x} + \epsilon) < 1$ . Let  $f'(\bar{x} + \epsilon) = 1 - \epsilon_1$ , with  $\epsilon_1 > 0$ .

By Euler equations, we have

$$\begin{aligned}
u'(f(x_s(\delta)) - x_{s+1}(\delta)) &= \delta u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) f'(x_{s+1}(\delta)) \\
&\leq u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) f'(x_{s+1}(\delta)) \\
&\leq u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) f'(\bar{x} + \epsilon) \\
&\leq u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) - \epsilon_1 u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) \\
&\leq u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) - \epsilon_2,
\end{aligned}$$

for  $\epsilon_2 = \epsilon_1 u'(f(x_0))$ , since  $f(x_0) \geq f(x_{s+1}(\delta)) - x_{s+2}(\delta)$ . Observe that  $\epsilon_2$  does not depend on  $\delta$ .

We then deduce

$$\begin{aligned}
\epsilon_2 &\leq u'(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) - u'(f(x_s(\delta)) - x_{s+1}(\delta)) \\
&= u''(\xi) [(f(x_{s+1}(\delta)) - x_{s+2}(\delta)) - (f(x_s(\delta)) - x_{s+1}(\delta))] \\
&= (-u''(\xi)) [(f(x_s(\delta)) - x_{s+1}(\delta)) - (f(x_{s+1}(\delta)) - x_{s+2}(\delta))],
\end{aligned}$$

with some  $f(x_{s+1}(\delta)) - x_{s+2}(\delta) \leq \xi \leq f(x_s(\delta)) - x_{s+1}(\delta)$ . This implies

$$x_{s+1}(\delta) - x_{s+2}(\delta) \leq f(x_s(\delta)) - f(x_{s+1}(\delta)) - \frac{\epsilon_2}{-u''(\xi)}.$$

As  $x_{s+1}(\delta) \geq \bar{x} + \epsilon$ , it is easy to verify that

$$\begin{aligned}
f(\bar{x}) - \bar{x} &\leq f(x_{s+1}(\delta)) - x_{s+2}(\delta) \\
&\leq \xi \\
&\leq f(x_s(\delta)) - x_{s+1}(\delta) \\
&\leq f(x_0).
\end{aligned}$$

Let

$$a = \sup_{f(\bar{x}) - \bar{x} \leq \xi \leq f(x_0)} (-u''(\xi)),$$

and

$$\epsilon_3 = \frac{\epsilon_2}{a}.$$

The value  $\epsilon_3$  is strictly positive and is independent with respect to  $\delta$ . Moreover,

$$\begin{aligned} x_{s+1}(\delta) - x_{s+2}(\delta) &\leq f(x_s(\delta)) - f(x_{s+1}(\delta)) - \epsilon_3 \\ &\leq f'(x_{s+1}(\delta))(x_s(\delta) - x_{s+1}(\delta)) - \epsilon_3 \\ &\leq x_s(\delta) - x_{s+1}(\delta) - \epsilon_3. \end{aligned}$$

Hence for  $T(\epsilon)$  big enough such that  $x_0 - T(\epsilon)\epsilon_3 < 0$ , we have  $x_T(\delta) < \bar{x} + \epsilon$  for any  $T \geq T(\epsilon)$  and for any  $0 < \delta < 1$ . Otherwise we will have  $x_T(\delta) - x_{T+1}(\delta) \leq 0$  for some  $T \geq T(\epsilon)$ : a contradiction<sup>10</sup>.

By the independence of  $T(\epsilon)$  in respect to  $\delta$ , combining with result that for  $s \geq T(\epsilon)$ , we have  $x^\delta \leq x_s(\delta) \leq \bar{x} + \epsilon$ , we get the following inequality:

$$\begin{aligned} &\lim_{\delta \rightarrow 1} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s(\delta)) - x_{s+1}(\delta)) \right] \\ &= \lim_{\delta \rightarrow 1} \left[ (1 - \delta) \sum_{s=0}^{T(\epsilon)} \delta^s u(f(x_s(\delta)) - x_{s+1}(\delta)) \right] \\ &+ \lim_{\delta \rightarrow 1} \left[ \delta^{T(\epsilon)+1} (1 - \delta) \sum_{s=T(\epsilon)+1}^{\infty} \delta^{s-T(\epsilon)-1} u(f(x_s(\delta)) - x_{s+1}(\delta)) \right] \\ &= \lim_{\delta \rightarrow 1} \left[ \delta^{T(\epsilon)+1} (1 - \delta) \sum_{s=T(\epsilon)+1}^{\infty} \delta^{s-T(\epsilon)-1} u(f(x_s(\delta)) - x_{s+1}(\delta)) \right] \\ &\leq \lim_{\delta \rightarrow 1} u(f(\bar{x} + \epsilon) - x^\delta) \\ &= u(f(\bar{x} + \epsilon) - \bar{x}). \end{aligned}$$

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<sup>10</sup>It is well known that the solution of Ramsey problem converges monotonically to the steady state.

For any feasible path  $\chi \in \Pi(x_0)$ ,

$$\begin{aligned} \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right] &\leq \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s(\delta)) - x_{s+1}(\delta)) \right] \\ &\leq \lim_{\delta \rightarrow 1} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s(\delta)) - x_{s+1}(\delta)) \right] \\ &\leq u(f(\bar{x} + \epsilon) - \bar{x}). \end{aligned}$$

Since  $\epsilon > 0$  is chosen arbitrarily, this implies

$$\inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right] \leq u(f(\bar{x}) - \bar{x}).$$

We then have

$$\max_{\mathbf{x} \in \Pi(x_0)} \nu(\mathbf{x}) = \max_{\mathbf{x} \in \Pi(x_0)} \hat{\nu}(\mathbf{x}) = u(f(\bar{x}) - \bar{x}).$$

For a solution of the problem with the second Rawlsian criterion, take for example the sequence  $\hat{\chi} \in \Pi(x_0)$  such that  $\hat{x}_s = \bar{x}$  for any  $s \geq 1$ . For each  $\delta$ ,

$$(1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(\hat{x}_s) - \hat{x}_{s+1}) = (1 - \delta)u(f(x_0) - \bar{x}) + \delta u(f(\bar{x}) - \bar{x}).$$

Since  $x_0 > \bar{x}$ , the function  $(1 - \delta)u(f(x_0) - \bar{x}) + \delta u(f(\bar{x}) - \bar{x})$  is strictly decreasing in respect to  $\delta$ . This implies

$$\begin{aligned} \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(\hat{x}_s) - \hat{x}_{s+1}) \right] &= \lim_{\delta \rightarrow 1} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(\hat{x}_s) - \hat{x}_{s+1}) \right] \\ &= u(f(\bar{x}) - \bar{x}). \end{aligned}$$

The proof is completed.

## 4.2 PROOF OF COROLLARY 2.1

(i) This property is proven using the same the arguments as in the proof of Proposition 2.3.

(ii) Consider some feasible path  $\mathbf{x}^*$  which is a solution of the problem under first Rawls criterion. Since  $u(f(x_s^*) - x_{s+1}^*) \geq u(f(\bar{x}) - \bar{x})$  for any  $s \geq 0$ , for any  $0 < \delta < 1$ ,

$$(1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s^*) - x_{s+1}^*) \geq u(f(\bar{x}) - \bar{x}).$$

This implies

$$\begin{aligned} \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s^*) - x_{s+1}^*) \right] &\geq u(f(\bar{x}) - \bar{x}) \\ &= \max_{\mathbf{x} \in \Pi(x_0)} \inf_{\delta \in (0,1)} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s u(f(x_s) - x_{s+1}) \right]. \end{aligned}$$

Hence  $\mathbf{x}^*$  is a solution of the problem under second Rawls criterion.

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