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KÄHLER DIFFERENTIALS AND \mathbf{Z}_p -EXTENSIONS

by

Laurent Berger

Abstract. — Let K be a p -adic field, and let K_∞/K be a Galois extension that is almost totally ramified, and whose Galois group is a p -adic Lie group of dimension 1. We prove that K_∞ is not dense in $(\mathbf{B}_{\mathrm{dR}}^+/\mathrm{Fil}^2 \mathbf{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_\infty)}$. Moreover, the restriction of θ to the closure of K_∞ is injective, and its image via θ is the set of vectors of \widehat{K}_∞ that are C^1 with zero derivative for the action of $\mathrm{Gal}(K_\infty/K)$. The main ingredient for proving these results is the construction of an explicit lattice of \mathcal{O}_{K_∞} that is commensurable with $\mathcal{O}_{K_\infty}^{d=0}$, where $d : \mathcal{O}_{K_\infty} \rightarrow \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ is the differential.

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Introduction

Let K be a p -adic field, namely a finite extension of $W(k)[1/p]$ where k is a perfect field of characteristic p . Let \mathbf{C} be the p -adic completion of an algebraic closure \overline{K} of K . Let K_∞/K be a Galois extension that is almost totally ramified, and whose Galois group is a p -adic Lie group of dimension 1. Let $\mathbf{B}_{\mathrm{dR}}(\widehat{K}_\infty) = \mathbf{B}_{\mathrm{dR}}(\mathbf{C})^{\mathrm{Gal}(\overline{K}/K_\infty)}$ be Fontaine's field of periods attached to K_∞/K , and for $n \geq 1$, let $\mathbf{B}_n(\widehat{K}_\infty) = \mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/\mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)$.

This note is motivated by Ponsinet's paper [Pon20], in which he relates the study of universal norms for the extension K_∞/K to the question of whether K_∞ is dense in

$\mathbf{B}_n(\widehat{K}_\infty)$ for $n \geq 1$. The density result holds for $n = 1$ by the Ax-Sen-Tate theorem. Our main result is the following.

Theorem A. — *The field K_∞ is not dense in $\mathbf{B}_2(\widehat{K}_\infty)$.*

By the constructions of Fontaine and Colmez (see [Fon94] and [Col12]), $\mathbf{B}_2(\mathbf{C})$ is the completion of \overline{K} for a topology defined using the Kähler differentials $\Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. Some partial results towards theorem A have been proved by Iovita-Zaharescu in [IZ99], by studying these Kähler differentials. Let $\Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ be the Kähler differentials of $\mathcal{O}_{K_\infty}/\mathcal{O}_K$ and let $d : \mathcal{O}_{K_\infty} \rightarrow \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ be the differential. Our main technical result is the construction of a lattice of \mathcal{O}_{K_∞} that is commensurable with $\mathcal{O}_{K_\infty}^{d=0}$. Since the inertia subgroup of $\text{Gal}(K_\infty/K)$ is a p -adic Lie group of dimension 1, there exists a finite subextension K_0/K of K_∞ such that K_∞/K_0 is a totally ramified \mathbf{Z}_p -extension. Let K_n be the n -th layer of this \mathbf{Z}_p -extension.

Theorem B. — *The lattices $\sum_{n \geq 0} p^n \mathcal{O}_{K_n}$ and $\mathcal{O}_{K_\infty}^{d=0}$ are commensurable.*

In order to prove this, we use Tate's results on ramification in \mathbf{Z}_p -extensions. As a corollary of theorem B, we can say more about the completion of K_∞ in $\mathbf{B}_2(\widehat{K}_\infty)$. The field \widehat{K}_∞ is a Banach representation of the p -adic Lie group $\text{Gal}(K_\infty/K)$. Let $\theta : \mathbf{B}_2(\mathbf{C}) \rightarrow \mathbf{C}$ be the usual map from p -adic Hodge theory.

Theorem C. — *The completion of K_∞ in $\mathbf{B}_2(\widehat{K}_\infty)$ is isomorphic via θ to the set of vectors of \widehat{K}_∞ that are C^1 with zero derivative for the action of $\text{Gal}(K_\infty/K)$.*

This is a field, and it is also the set of $y \in \widehat{K}_\infty$ that can be written as $y = \sum_{n \geq 0} p^n y_n$ with $y_n \in K_n$ and $y_n \rightarrow 0$.

We also prove that $d(\mathcal{O}_{K_\infty})$ contains no nontrivial p -divisible element (coro 3.5), and that $d : \mathcal{O}_{K_\infty} \rightarrow \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ is not surjective (coro 3.6). These two statements are equivalent to theorem A by the results of [IZ99]; using our computations, we give a short independent proof.

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1. Kähler differentials

Let K be a p -adic field. If L/K is a finite extension, let $\mathfrak{d}_{L/K} \subset \mathcal{O}_L$ denote its different.

Proposition 1.1. — *Let K be a p -adic field, and let L/K be an algebraic extension.*

1. If L/K is a finite extension, then $\Omega_{\mathcal{O}_L/\mathcal{O}_K} = \mathcal{O}_L/\mathfrak{d}_{L/K}$ as \mathcal{O}_L -modules.
2. If $M/L/K$ are finite extensions, the map $\Omega_{\mathcal{O}_L/\mathcal{O}_K} \rightarrow \Omega_{\mathcal{O}_M/\mathcal{O}_K}$ is injective.
3. If L/K is an algebraic extension, and $\omega_1, \omega_2 \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, then there exists $x \in \mathcal{O}_L$ such that $\omega_2 = x\omega_1$ if and only if $\text{Ann}(\omega_1) \subset \text{Ann}(\omega_2)$.

Proof. — See for instance §2 of [Fon82]. □

Recall (see §2 of [CG96]) that an algebraic extension L/K is deeply ramified if the set $\{\text{val}_p(\mathfrak{d}_{F/K})\}_F$ is unbounded, as F runs through the set of finite extensions of K contained in L . Alternatively (remark 3.3 of [Sch12]), L/K is deeply ramified if and only if \widehat{L} is a perfectoid field. An extension K_∞/K as in the introduction is deeply ramified.

Corollary 1.2. — *If L/K is deeply ramified, then $\Omega_{\mathcal{O}_L/\mathcal{O}_K} = L/\mathcal{O}_L$ as \mathcal{O}_L -modules.*

Proposition 1.3. — *If L/K is deeply ramified, then $d : \mathcal{O}_L \rightarrow \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective if and only if $d(\mathcal{O}_L)$ is p -divisible.*

Proof. — Since L/K is deeply ramified, $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is p -divisible. This proves one implication. Assume now that $d(\mathcal{O}_L)$ is p -divisible, so that there exists a sequence $\{\alpha_i\}_{i \geq 1}$ of \mathcal{O}_L such that $d\alpha_1 \neq 0$ and $d\alpha_i = p \cdot d\alpha_{i+1}$ for all $i \geq 1$. If $\omega \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, prop 1.1 implies that there exists $i \geq 1$ and $x_i \in \mathcal{O}_L$ such that $\omega = x_i \cdot d\alpha_i$. Take $k \geq 0$ such that $d(p^k x_i) = 0$. We then have $\omega = x_i \cdot d\alpha_i = p^k x_i \cdot d\alpha_{i+k} = d(p^k x_i \alpha_{i+k})$. Hence d is surjective. □

Proposition 1.4. — *Let L/K be a deeply ramified extension, and let $K' \subset L$ be a finite extension of K .*

1. $d : \mathcal{O}_L \rightarrow \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective if and only if $d' : \mathcal{O}_L \rightarrow \Omega_{\mathcal{O}_L/\mathcal{O}_{K'}}$ is surjective.
2. $\mathcal{O}_L^{d=0}$ and $\mathcal{O}_L^{d'=0}$ are commensurable.

Proof. — We have an exact sequence of \mathcal{O}_L -modules, compatible with d and d'

$$\mathcal{O}_L \otimes \Omega_{\mathcal{O}_{K'}/\mathcal{O}_K} \xrightarrow{f} \Omega_{\mathcal{O}_L/\mathcal{O}_K} \xrightarrow{g} \Omega_{\mathcal{O}_L/\mathcal{O}_{K'}} \rightarrow 0.$$

Let us prove (1). If $d : \mathcal{O}_L \rightarrow \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective, then clearly $d' : \mathcal{O}_L \rightarrow \Omega_{\mathcal{O}_L/\mathcal{O}_{K'}}$ is surjective. Conversely, there exists $r \geq 0$ such that $p^r \cdot \Omega_{\mathcal{O}_{K'}/\mathcal{O}_K} = \{0\}$. If $\omega \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, write it as $\omega = p^r \omega_r$. By hypothesis, there exists $\alpha_r \in \mathcal{O}_L$ such that $\omega_r = d' \alpha_r$ in $\Omega_{\mathcal{O}_L/\mathcal{O}_{K'}}$. Hence $p^r(\omega_r - d\alpha_r) = 0$ in $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ so that $\omega = d(p^r \alpha_r)$. We now prove (2). The exact sequence above implies that $\mathcal{O}_L^{d=0} \subset \mathcal{O}_L^{d'=0}$. Conversely, if $x \in \mathcal{O}_L^{d'=0}$, then $dx \in \ker g = \text{im } f$, so that $p^r \cdot dx = 0$. Hence $p^r \cdot \mathcal{O}_L^{d'=0} \subset \mathcal{O}_L^{d=0}$. □

Corollary 1.5. — *In order to prove theorem B, we can replace K by any finite subextension K' of K . In particular, we can assume that K_∞/K is a totally ramified \mathbf{Z}_p -extension.*

2. Ramification in \mathbf{Z}_p -extensions

Let K_∞/K be a totally ramified \mathbf{Z}_p -extension. We recall some of the results of §3.1 of [Tat67] concerning the ramification of K_∞/K and the action of $\text{Gal}(K_\infty/K)$ on K_∞ . Let K_n be the n -th layer of K_∞/K , so that $[K_n : K] = p^n$.

Proposition 2.1. — *There are constants a, b such that $|\text{val}_p(\mathfrak{D}_{K_n/K}) - n - b| \leq p^{-n}a$ for $n \geq 0$.*

Proof. — See §3.1 of [Tat67]. □

The notation $\sum_{n \geq 0} p^n \mathcal{O}_{K_n}$ denotes the set of elements of K_∞ that are finite sums of elements of $p^n \mathcal{O}_{K_n}$.

Corollary 2.2. — *There exists $n_0 \geq 0$ such that $\sum_{n \geq 0} p^{n+n_0} \mathcal{O}_{K_n} \subset \mathcal{O}_{K_\infty}^{d=0}$.*

Proposition 2.3. — *There exists $c(K_\infty/K) > 0$ such that for all $n, k \geq 0$ and $x \in \mathcal{O}_{K_{n+k}}$, we have $\text{val}_p(\text{N}_{K_{n+k}/K_n}(x)/x^{[K_{n+k}:K_n]} - 1) \geq c(K_\infty/K)$.*

Proof. — The result follows from the fact (see 1.2.2 of [Win83]) that the extension K_∞/K is strictly APF. One can then apply 1.2.1, 4.2.2 and 1.2.3 of [Win83]. □

If $n \geq 0$ and $x \in K_\infty$, then $R_n(x) = p^{-k} \cdot \text{Tr}_{K_{n+k}/K_n}(x)$ is independent of $k \gg 0$ such that $x \in K_{n+k}$, and is the normalized trace of x .

Proposition 2.4. — *There exists $c_2 \in \mathbf{Z}_{\geq 0}$ such that $\text{val}_p(R_n(x)) \geq \text{val}_p(x) - c_2$ for all $n \geq 0$ and $x \in K_\infty$.*

Proof. — See §3.1 of [Tat67] (including the remark at the bottom of page 172). □

In particular, $R_n(\mathcal{O}_{K_\infty}) \subset p^{-c_2} \mathcal{O}_{K_n}$ for all $n \geq 0$. Let $K_0^\perp = K_0$ and for $n \geq 1$, let K_n^\perp be the kernel of $R_{n-1} : K_n \rightarrow K_{n-1}$, let $R_n^\perp = R_n - R_{n-1}$, and $R_0^\perp = R_0$. If $x \in K_\infty$ and $i \geq 0$, then $R_n^\perp(x) = 0$ for $n \gg 0$, and $x = (\sum_{n \geq i+1} R_n^\perp(x)) + R_i(x)$. Prop 2.4 implies that $R_n^\perp(\mathcal{O}_{K_\infty}) \subset p^{-c_2} \mathcal{O}_{K_n}$ for all $n \geq 0$. Let $\mathcal{O}_{K_n}^\perp = \mathcal{O}_{K_n} \cap K_n^\perp$.

Corollary 2.5. — *For all $i \geq 0$, we have $\mathcal{O}_{K_\infty} \subset (\oplus_{m \geq i+1} p^{-c_2} \mathcal{O}_{K_m}^\perp) \oplus p^{-c_2} \mathcal{O}_{K_i}$.*

Proof. — If $x \in \mathcal{O}_{K_\infty}$, write $x = \sum_{m \geq i+1} R_m^\perp(x) + R_i(x)$. □

For $n \geq 0$, let g_n denote a topological generator of $\text{Gal}(K_\infty/K_n)$.

Lemma 2.6. — *There exists a constant c_3 such that for all $n \geq 1$ and $x \in K_{n+1}^\perp$, we have $\text{val}_p(x) \geq \text{val}_p((1 - g_n)(x)) - c_3$.*

Proof. — See §3.1 of [Tat67] (including the remark at the bottom of page 172). □

3. The lattice $\mathcal{O}_{K_\infty}^{d=0}$

We now prove theorem B. Thanks to corollary 1.5, we assume that K_∞/K is a totally ramified \mathbf{Z}_p -extension. Let $\{\rho_n\}_{n \geq 0}$ be a norm compatible sequence of uniformizers of the K_n . Let $m_c \geq 0$ be the smallest integer such that $p^{m_c} \cdot c(K_\infty/K) \geq 1/(p-1)$.

Proposition 3.1. — *We have $\text{val}_p(\rho_{n+1}^{p^k} - \rho_n^k) \geq \text{val}_p(k) - m_c$.*

Proof. — Note that if $x, y \in \mathbf{C}$ with $\text{val}_p(x - y) \geq v$, then $\text{val}_p(x^p - y^p) \geq \min(v + 1, pv)$. Let $c = c(K_\infty/K)$ and $m = m_c$. We have $\text{val}_p(\rho_{n+1}^p - \rho_n) \geq c$ by prop 2.3, so that $\text{val}_p(\rho_{n+1}^{p^{j+1}} - \rho_n^{p^j}) \geq p^j c$ for all j such that $p^{j-1}c \leq 1/(p-1)$.

In particular, $\text{val}_p(\rho_{n+1}^{p^{m+1}} - \rho_n^{p^m}) \geq p^m c \geq 1/(p-1)$, so that $\text{val}_p(\rho_{n+1}^{p^{m+j+1}} - \rho_n^{p^{m+j}}) \geq j + 1/(p-1)$ if $j \geq 0$. This implies the result. \square

Theorem 3.2. — *There exists $n_1 \in \mathbf{Z}_{\geq 0}$ such that $\mathcal{O}_{K_\infty}^{d=0} \subset \sum_{m \geq n_1} p^{m-n_1} \mathcal{O}_{K_m}$.*

Proof. — Let $n_1 = \lceil a - b + m_c + 2 \rceil$. Take $x \in \mathcal{O}_{K_n}^{d=0}$ and write $x = \sum_{i=0}^{p^n-1} x_i \rho_n^i$ with $x_i \in \mathcal{O}_K$, so that $dx = \sum_{i=0}^{p^n-1} i x_i \rho_n^{i-1} \cdot d\rho_n$. If $dx = 0$, then $\sum_{i=0}^{p^n-1} i x_i \rho_n^{i-1} \in \mathfrak{d}_{K_n/K}$ so that by prop 2.1 (and since $\text{val}_p(\rho_n^{p^n}) \leq 1$), for all i we have

$$\text{val}_p(x_i) \geq n - a + b - \text{val}_p(i) - 1.$$

For $k \geq 1$, let

$$y_k = \sum_{p \nmid j} x_{p^{k-1}j} \rho_{n-(k-1)}^j + \sum_{\ell} x_{p^{k\ell}} (\rho_{n-(k-1)}^{p^\ell} - \rho_{n-k}^\ell).$$

Note that $y_k \in \mathcal{O}_{K_{n-k+1}}$. Let us bound $\text{val}_p(y_k)$. We have $\text{val}_p(x_{p^{k-1}j} \rho_{n-(k-1)}^j) \geq n - a + b - k$. We also have $\text{val}_p(x_{p^{k\ell}}) \geq n - a + b - k - \text{val}_p(\ell) - 1$, and $\text{val}_p(\rho_{n-(k-1)}^{p^\ell} - \rho_{n-k}^\ell) \geq \text{val}_p(\ell) - m_c$ by prop 3.1. Hence $\text{val}_p(y_k) \geq n - a + b - k - 1 - m_c$ and therefore $y_k \in p^{n-k+1-n_1} \mathcal{O}_{K_{n-k+1}}$. Finally, we have $x = y_1 + \dots + y_{n-1} + \sum_{\ell} x_{p^{n-n_1}\ell} \rho_{n_1}^\ell$, and $\sum_{\ell} x_{p^{n-n_1}\ell} \rho_{n_1}^\ell \in \mathcal{O}_{K_{n_1}}$, which implies the result. \square

Remark 3.3. — Compare with lemma 4.3.2 of [Fou05].

Corollary 3.4. — *We have $\mathcal{O}_{K_\infty}^{d=0} \subset (\oplus_{m \geq n_1+1} p^{m-n_1-c_2} \mathcal{O}_{K_m}^\perp) \oplus p^{-c_2} \mathcal{O}_{K_{n_1}}$.*

Proof. — By theorem 3.2, it is enough to prove that $p^n \mathcal{O}_{K_n} \subset (\oplus_{m \geq n_1+1} p^{m-c_2} \mathcal{O}_{K_m}^\perp) \oplus p^{n_1-c_2} \mathcal{O}_{K_{n_1}}$ for all $n \geq n_1$. If $x \in p^n \mathcal{O}_{K_n}$, write $x = R_n^\perp(x) + R_{n-1}^\perp(x) + \dots + R_{n_1+1}^\perp(x) + R_{n_1}(x)$. We have $R_{n-k}^\perp(x) \in p^{n-c_2} \mathcal{O}_{K_{n-k}}^\perp \subset p^{(n-k)-c_2} \mathcal{O}_{K_{n-k}}^\perp$ and likewise $R_{n_1}(x) \in p^{n-c_2} \mathcal{O}_{K_{n_1}} \subset p^{n_1-c_2} \mathcal{O}_{K_{n_1}}$. \square

Corollary 3.5. — *There are no nontrivial p -divisible elements in $d(\mathcal{O}_{K_\infty})$.*

Proof. — By props 1.3 and 1.4, we can assume that K_∞/K is a totally ramified \mathbf{Z}_p -extension. Let $\{\alpha_i\}_{i \geq 1}$ be a sequence of \mathcal{O}_{K_∞} such that $d\alpha_i = p \cdot d\alpha_{i+1}$ for all $i \geq 1$. Write $\alpha_i = \sum \alpha_{i,m}$ with $\alpha_{i,m} \in p^{-c_2} \mathcal{O}_{K_m}^\perp$ for $m \geq n_1 + 1$ and $\alpha_{i,n_1} \in p^{-c_2} \mathcal{O}_{K_{n_1}}$. Since $p^k \alpha_{k+i} - \alpha_i \in \mathcal{O}_{K_\infty}^{d=0}$, corollary 3.4 implies that $p^k \alpha_{k+i,m} - \alpha_{i,m} \in p^{m-n_1-c_2} \mathcal{O}_{K_m}$ for all $m \geq n_1$. Taking $k \gg 0$ now implies that $\alpha_{i,m} \in p^{m-n_1-c_2} \mathcal{O}_{K_m}$ for all $m \geq n_1$. Corollary 2.2 gives $p^{n_0+n_1+c_2} \alpha_i \in \mathcal{O}_{K_\infty}^{d=0}$. Taking $i = n_0 + n_1 + c_2 + 1$ gives $d\alpha_1 = 0$. \square

Corollary 3.6. — *The differential $d : \mathcal{O}_{K_\infty} \rightarrow \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ is not surjective.*

Proof. — This follows from corollary 3.5 and proposition 1.3. \square

4. The completion of K_∞ in \mathbf{B}_2

We now prove theorems A and C. Since we are concerned with the completion of K_∞ , we can once again replace K with a finite subextension of K_∞ and assume that K_∞/K is a totally ramified \mathbf{Z}_p -extension. Let \widehat{K}_∞^2 denote the completion of K_∞ in \mathbf{B}_2 , so that $R = \theta(\widehat{K}_\infty^2)$ is a subring of \widehat{K}_∞ . Let $\Gamma = \text{Gal}(K_\infty/K)$, and let $c : \Gamma \rightarrow \mathbf{Z}_p$ be an isomorphism of p -adic Lie groups. Let w_2 be the valuation on K_∞ defined by $w_2(x) = \min\{n \in \mathbf{Z} \text{ such that } p^n x \in \mathcal{O}_{K_\infty}^{d=0}\}$. The restriction of the natural valuation of \mathbf{B}_2 to K_∞ is w_2 (see §1.4 and §1.5 of [Fon94], or theorem 3.1 of [Col12]). The map $\theta : \mathbf{B}_2 \rightarrow \mathbf{C}$ has the following property (see §1.4 of [Fon94])

Lemma 4.1. — *If $\{x_k\}_{k \geq 1}$ is a sequence of K_∞ that converges to $x \in \mathbf{B}_2$ for w_2 , then $\{x_k\}_{k \geq 1}$ is Cauchy for val_p , and $\theta(x) = \lim_{k \rightarrow +\infty} x_k$ for the p -adic topology.*

Let $M = \bigoplus_{n \geq 0} p^n \mathcal{O}_{K_n}^\perp$. Corollary 2.2 and theorem 3.2 imply that M and $\mathcal{O}_{K_\infty}^{d=0}$ are commensurable. Hence \widehat{K}_∞^2 is the M -adic completion of K_∞ . Let w'_2 be the M -adic valuation on K_∞ , so that w'_2 and w_2 are equivalent.

Lemma 4.2. — *If $x \in K_\infty$, then $\text{val}_p(R_n^\perp(x)) \geq w'_2(x) + n$.*

Proof. — Write $x = \sum_{n \geq 0} R_n^\perp(x)$. If $x \in p^w M$, then $R_n^\perp(x) \in p^{n+w} \mathcal{O}_{K_n}$. \square

Proposition 4.3. — *Every element $x \in \widehat{K}_\infty^2$ can be written in one and only one way as $\sum_{n \geq 0} x_n^\perp$ where $x_n^\perp \in K_n^\perp$ and $p^{-n} x_n^\perp \rightarrow 0$ for val_p .*

Proof. — Note that such a series converges for w_2 . The map $R_n^\perp : K_\infty \rightarrow K_n^\perp$ sends $p^w M \subset K_\infty$ to $p^{w+n} \mathcal{O}_{K_n}^\perp$, so that it is uniformly continuous for the w_2 -adic topology, and therefore extends to a continuous map $R_n^\perp : \widehat{K}_\infty^2 \rightarrow K_n^\perp$.

Let $x \in \widehat{K}_\infty^2$ be the w_2 -adic limit of $\{x_k\}_{k \geq 1}$ with $x_k \in K_\infty$. For a given k , the sequence $\{p^{-n}R_n^\perp(x_k)\}_{n \geq 0} \in \prod_{n \geq 0} K_n^\perp$ has finite support. As $k \rightarrow +\infty$, these sequences converge uniformly in $\prod_{n \geq 0} K_n^\perp$ to $\{p^{-n}R_n^\perp(x)\}_{n \geq 0}$, so that $p^{-n}R_n^\perp(x) \rightarrow 0$ as $n \rightarrow +\infty$. Hence $\sum_{n \geq 0} R_n^\perp(x)$ converges for w_2 . Since $x_k = \sum_{n \geq 0} R_n^\perp(x_k)$ for all k , we have $x = \sum_{n \geq 0} R_n^\perp(x)$. Finally, if $x = \sum_{n \geq 0} x_n^\perp$ with $x_n^\perp \in K_n^\perp$ and $p^{-n}x_n^\perp \rightarrow 0$ for val_p , then $x_n^\perp = R_n^\perp(x)$ which proves unicity. \square

Corollary 4.4. — *The map $\theta : \widehat{K}_\infty^2 \rightarrow \widehat{K}_\infty$ is injective.*

Proof. — If $x_n^\perp \in K_n^\perp$ and $x_n^\perp \rightarrow 0$ and $\sum_{n \geq 0} x_n^\perp = 0$ in \widehat{K}_∞ , then $x_n^\perp = 0$ for all n . \square

Corollary 4.5. — *The ring R is the set of $y \in \widehat{K}_\infty$ that can be written as $y = \sum_{n \geq 0} p^n y_n$ with $y_n \in K_n$ and $y_n \rightarrow 0$.*

Proposition 4.6. — *The ring R is a field, and $R = \{x \in \widehat{K}_\infty \text{ such that } g(x) - x = o(c(g)) \text{ as } g \rightarrow 1 \text{ in } \Gamma\}$.*

Proof. — The fact that R is a field results from the second statement, since $g(1/x) - 1/x = (x - g(x))/(xg(x))$. Take $y = \sum_{n \geq 0} p^n y_n$ with $y_n \in K_n$ and $y_n \rightarrow 0$. If $m \geq 1$, then for all $k \geq 0$, we have $y_n \in p^{m+n} \mathcal{O}_{K_n}$. We can write $y = x_k + \sum_{n > k} p^n y_n$ and then $(g - 1)(y) \in p^{k+m} \mathcal{O}_{K_\infty}$ if $g \in \text{Gal}(K_\infty/K_k)$. This proves one implication.

Conversely, take $x \in \widehat{K}_\infty$ such that $g(x) - x = o(c(g))$. Write $x = \sum_{k \geq 0} x_k$ with $x_0 = R_0(x) \in K_0$ and $x_k = R_k^\perp(x) \in K_k^\perp$ for all $k \geq 1$. For $n \geq 0$, let g_n denote a topological generator of $\text{Gal}(K_\infty/K_n)$. Take $m \geq 0$, and $n \gg 0$ such that we have $\text{val}_p((g_n - 1)(x)) \geq m + n$. We have $(1 - g_n)(x) = \sum_{k \geq n+1} (1 - g_n)x_k$, so that by lemma 2.6 and prop 2.4: $\text{val}_p(x_{n+1}) \geq \text{val}_p((1 - g_n)(x_{n+1})) - c_3 \geq \text{val}_p((1 - g_n)(x)) - c_2 - c_3 \geq n + m - c_2 - c_3$. This implies the result. \square

Remark 4.7. — Prop 4.6 says that R is the set of vectors of \widehat{K}_∞ that are C^1 with zero derivative (flat to order 1) for the action of Γ .

Theorem A follows from cor 4.4 since $\theta : \mathbf{B}_2(\widehat{K}_\infty) \rightarrow \widehat{K}_\infty$ is not injective. Finally, cor 4.4, cor 4.5, and prop 4.6 imply theorem C.

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