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# On the application of the Schoenberg quasi-interpolant for complexity reduction in trajectory generation

[Poster]

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## Highlights

- Employ the Schoenberg operator to approximate nonlinear functions (which stem from cost and/or constraints representation in an optimisation-based trajectory generation problem);
- Present a motion planning algorithm which efficiently provides feasible trajectories via Schoenberg-based approximations.

## B-spline curves

We consider the parametrization of a trajectory  $z(t)$  with B-spline curves as a linear combination of control points and B-spline basis functions:

$$z(t) = \sum_{k=0}^{n-1} P_k B_{k,p,\xi}(t) = \mathbf{P} \mathbf{B}_{p,\xi}(t), \forall t \in [0, T], \quad (1)$$

with  $\mathbf{P} = [P_0 \dots P_{n-1}]$  the matrix which gathers the control points,  $\mathbf{B}_{p,\xi}(t) = [B_{0,p,\xi}(t) \dots B_{n-1,p,\xi}(t)]^T$  the basis vector,  $\xi = \{\tau_0 \leq \tau_1 \leq \dots \leq \tau_m\}$  a knot sequence starting at 0 and ending at  $T$ . If  $m \geq p + 2$ , we can define B-splines of order up to  $p$  over the knot sequence.

## Schoenberg operator

The Schoenberg operator [Lyche et al. 2018] approximates a function, denoted as  $f(\cdot)$ , by interpolating it as weighted sum of B-splines basis functions and the function's values in the Greville points of the B-spline basis:

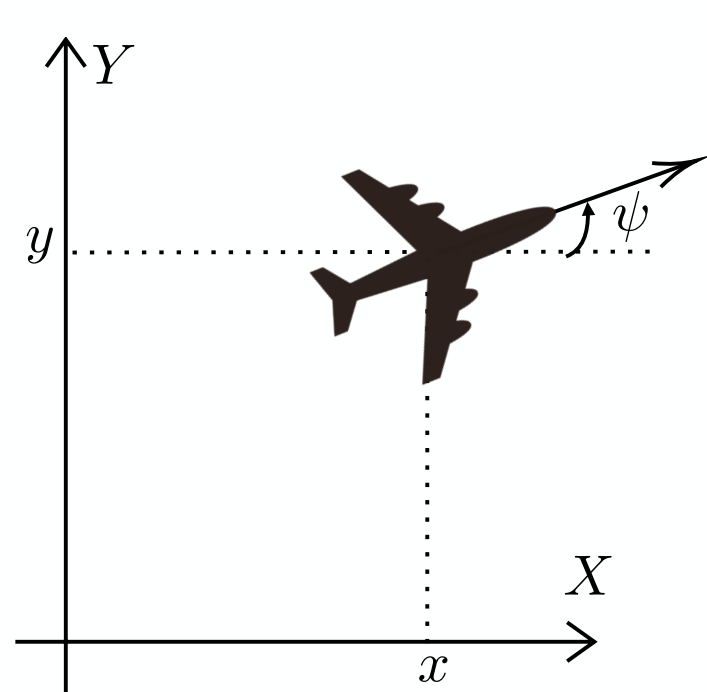
$$\tilde{f}(t) = \sum_{k=0}^{\tilde{n}-1} f(\tilde{t}_k) \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t), \quad (2)$$

where the Greville points are

$$\tilde{t}_k = \frac{\tilde{\tau}_{k+1} + \dots + \tilde{\tau}_{k+\tilde{p}-1}}{\tilde{p} - 1}. \quad (3)$$

## Trajectory generation for a fixed-wing UAV

Let us consider a 2D 3-DOF model of a fixed wing UAV, [Beard and McLain 2012], where the position  $(x(t), y(t))$  and the heading (yaw) angle  $\psi(t) \in [0; 2\pi]$  rad are the state variables. Their dynamics are described by the following equations, where  $V_a(t)$  and  $\phi(t)$  are the inputs, the airspeed velocity and the roll angle, respectively:



$$\dot{x}(t) = V_a(t) \cos \psi(t), \quad (4a)$$

$$\dot{y}(t) = V_a(t) \sin \psi(t), \quad (4b)$$

$$\dot{\psi}(t) = \frac{g \tan \phi(t)}{V_a(t)}. \quad (4c)$$

We consider as flat output (for more details on flatness and the UAV's flat representation see [Fliess et al. 1995], [Prodan et al. 2013]) the position of the aircraft  $z(t) = [z_1(t) \ z_2(t)]^T = [x(t) \ y(t)]^T$ .

Thus, we are able to express all the state variables and inputs via the flat output and its first and second derivatives:

$$x(t) = z_1(t), \quad (5a)$$

$$y(t) = z_2(t), \quad (5b)$$

$$\psi(t) = \arctan \left( \frac{\dot{z}_2(t)}{\dot{z}_1(t)} \right), \quad (5c)$$

$$V_a(t) = \sqrt{\dot{z}_1^2(t) + \dot{z}_2^2(t)}, \quad (5d)$$

$$\phi(t) = \arctan \left( \frac{1}{g} \frac{\ddot{z}_2(t)\dot{z}_1(t) - \dot{z}_2(t)\ddot{z}_1(t)}{\sqrt{\dot{z}_1^2(t) + \dot{z}_2^2(t)}} \right). \quad (5e)$$

This will allow to reformulate the various costs/constraints in terms of the flat output  $z(t)$  which will be further parameterized as in (1).

## Schoenberg-based approximation of the optimization problem

Motion planning often reduces to solving a constrained optimization problem of the form

$$\min_{z(t)} \int_0^T \mathcal{C}(z(t)) dt \quad (6a)$$

$$\text{s.t. } g_i(z(t)) \leq 0, \quad 0 \leq t \leq T, \quad (6b)$$

$$h_j(z(t)) = 0, \quad 0 \leq t \leq T. \quad (6c)$$

The goal is to find a trajectory  $z(t)$  which minimizes the cost  $\mathcal{C}(\cdot)$  from (6a) while simultaneously respecting the inequality (6b) and equality constraints (6c) over the time interval  $[0, T]$ .

To better illustrate the subsequent ideas let us particularize (6) by taking  $z(t)$  as in (5). Then, we have that:

- the cost  $\mathcal{C}(\cdot)$  minimizes the trajectory length along the time horizon  $[0, T]$ ;
- the bounds on the velocity's magnitude are denoted by  $\underline{v}$  and  $\bar{v}$ , respectively;
- the bounds on the roll angle are denoted by  $\underline{\phi}$  and  $\bar{\phi}$ , respectively;
- the way-points passing (through  $p_j$  at  $t_j$ ) conditions.

Thus, (6) is instantiated to the particular form:

$$\min_{z(t)} \int_0^T \|\dot{z}(t)\| dt \quad (7a)$$

$$\text{s.t. } \underline{v} \leq \|\dot{z}(t)\| \leq \bar{v}, \quad \forall t \in [0, T] \quad (7b)$$

$$\underline{\phi} \leq \phi(t) \leq \bar{\phi}, \quad \forall t \in [0, T] \quad (7c)$$

$$z(t_j) = p_j, \quad \forall j. \quad (7d)$$

Parameterizing  $z(t)$  as in (1) allows to reformulate (7) into

$$\min_{P_k, B_k(t)} \int_0^T f_c(t) dt, \quad (8a)$$

$$\text{s.t. } \underline{v}^2 \leq f_v(t) \leq \bar{v}^2, \quad (8b)$$

$$\tan(\underline{\phi}) \leq f_\phi(t) \leq \tan(\bar{\phi}), \quad (8c)$$

$$\sum_{k=0}^{n-1} P_k B_{k,p}(t_j) = p_j, \quad \forall j, \quad (8d)$$

with the notations

$$f_c(t) = \sqrt{\sum_{k_1, k_2=0}^{n-2} (P'_{k_1})^T P'_{k_2} B_{k_1, p-1}(t) B_{k_2, p-1}(t)}, \quad (9a)$$

$$f_v(t) = \sum_{k_1, k_2=0}^{n-2} (P'_{k_1})^T P'_{k_2} B_{k_1, p-1}(t) B_{k_2, p-1}(t), \quad (9b)$$

$$f_\phi(t) = \frac{\sum_{k_1=0}^{n-3} \sum_{k_2=0}^{n-2} (P''_{k_1})^T Q P'_{k_2} B_{k_1, p-2}(t) B_{k_2, p-1}(t)}{g \sqrt{\sum_{k_1, k_2=0}^{n-2} (P'_{k_1})^T P'_{k_2} B_{k_1, p-1}(t) B_{k_2, p-1}(t)}}, \quad (9c)$$

where  $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $P'_k, P''_k$  are used as short-hands for the control points associated to the first and second order derivatives of the B-spline curve.

Using again the illustrative example from (8) we apply (2) thrice, once to approximate the nonlinear cost (8a), once to approximate the velocity in (8b) and once to approximate the roll angle in (8c). Correspondingly, the approximations are

$$\tilde{f}_{\{c,v,\phi\}}(t) = \sum_{k=0}^{\tilde{n}-1} f_{\{c,v,\phi\}}(\tilde{t}_k) \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t), \quad (10)$$

which allow to approximate:

- the cost from (8a) as

$$\int_0^T \tilde{f}_c(t) dt = \int_0^T \sum_{k=0}^{\tilde{n}-1} f_c(\tilde{t}_k) \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t) dt = \sum_{k=0}^{\tilde{n}-1} \left[ f_c(\tilde{t}_k) \int_0^T \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t) dt \right] \quad (11)$$

- the velocity inequalities  $\underline{v}^2 \leq f_v(t) \leq \bar{v}^2$  from (8b) by

$$\underline{v}^2 \leq \sum_{k=0}^{\tilde{n}-1} f_v(\tilde{t}_k) \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t) \leq \bar{v}^2, \quad (12)$$

- the roll angle inequalities  $\tan(\underline{\phi}) \leq f_\phi(t) \leq \tan(\bar{\phi})$  from (8c) by

$$\tan(\underline{\phi}) \leq \sum_{k=0}^{\tilde{n}-1} f_\phi(\tilde{t}_k) \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t) \leq \tan(\bar{\phi}). \quad (13)$$

Introducing approximations (11)–(13) and keeping the already linear equality (8d) leads to the approximated constrained-optimization problem [Marguet et al. 2023]

$$\min_{P_k} \epsilon^2 \quad (14a)$$

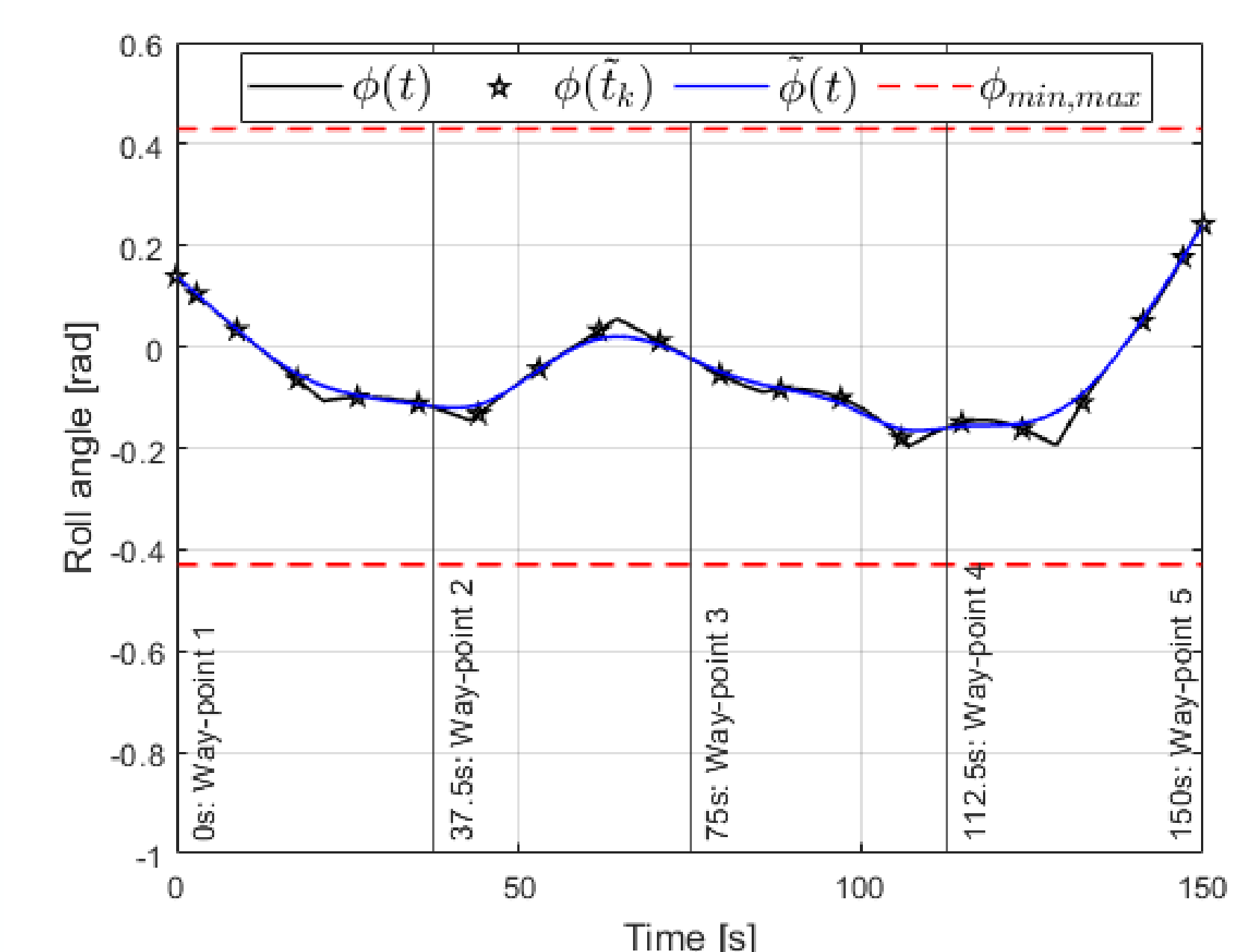
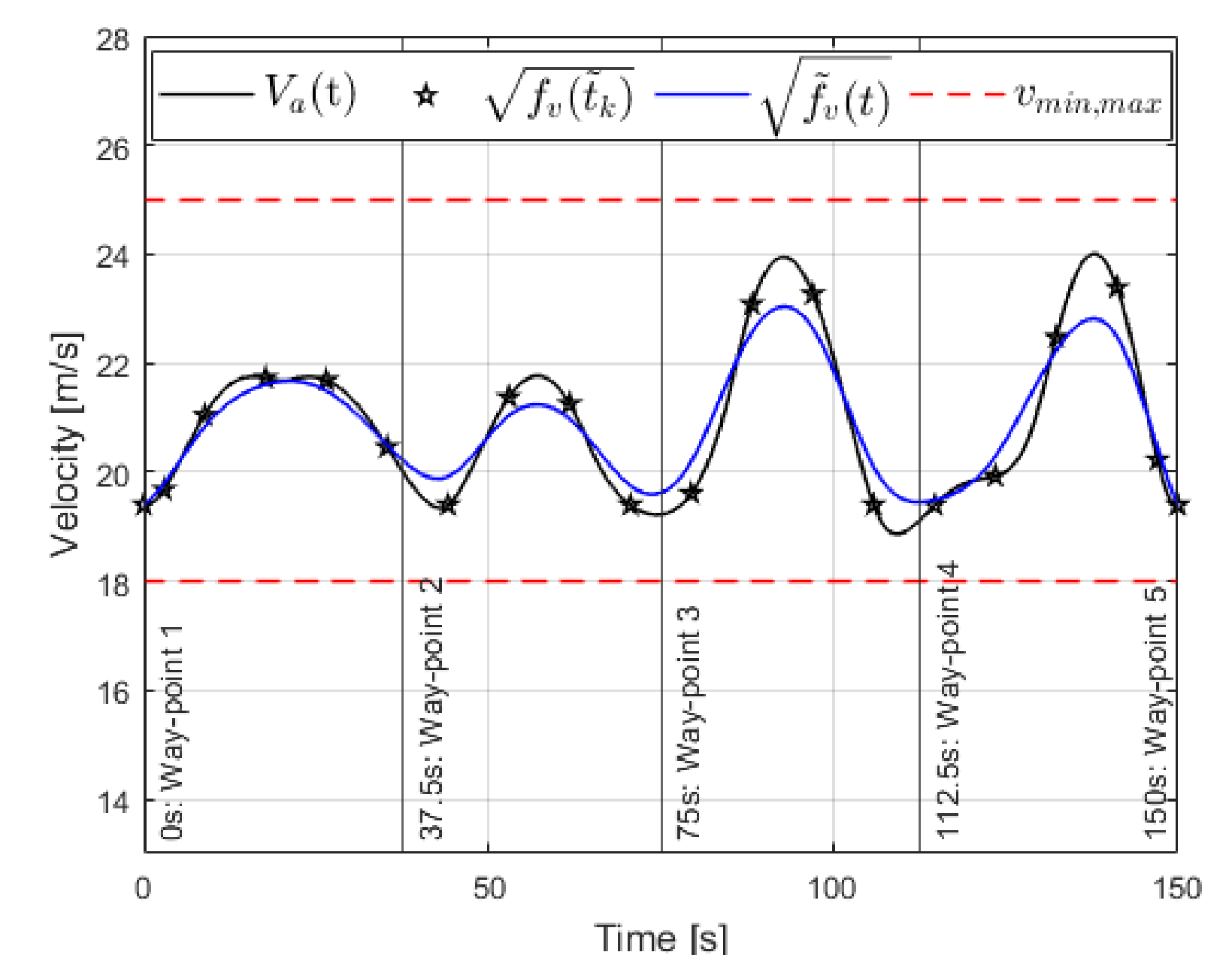
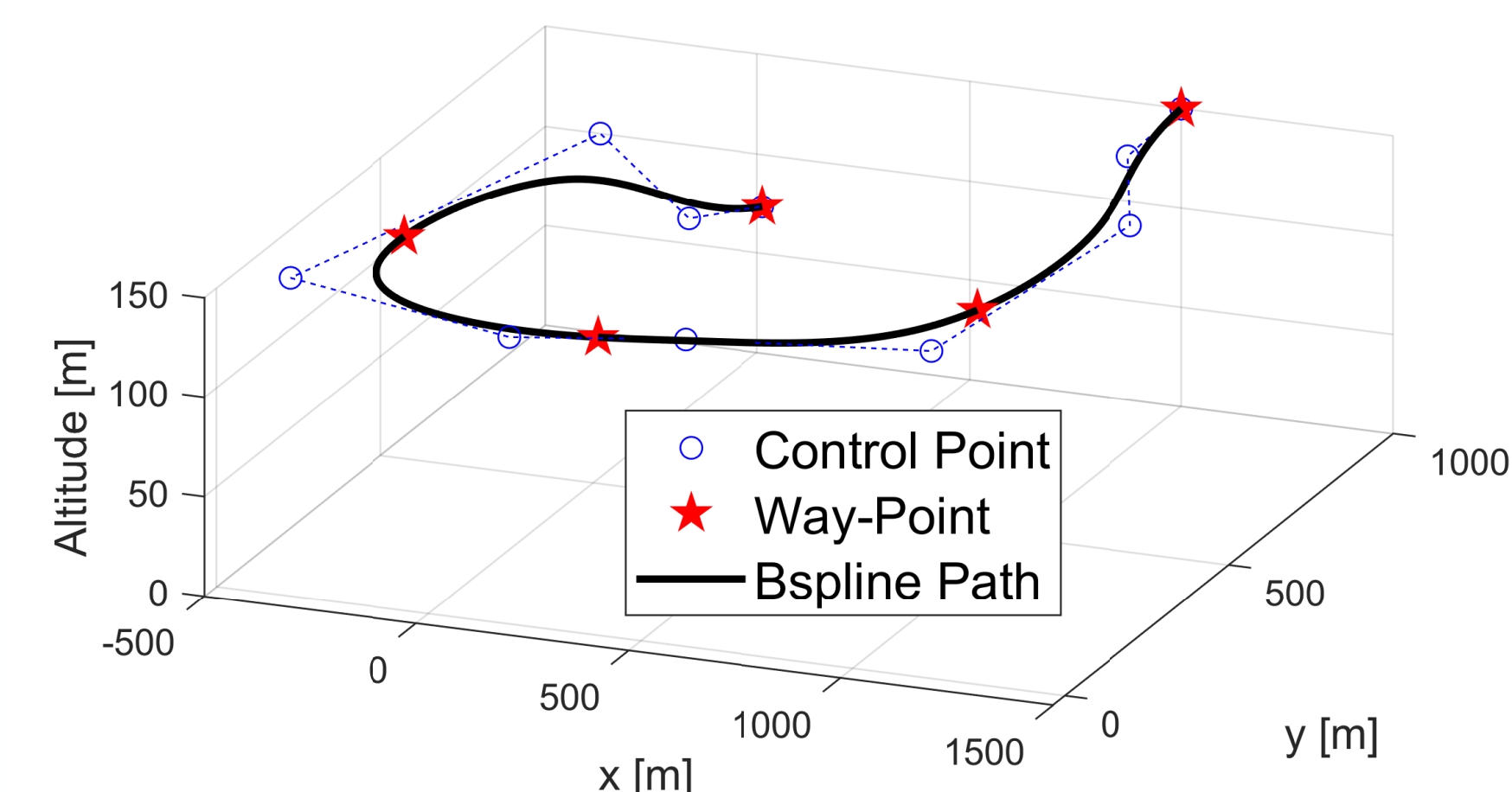
$$-\epsilon \leq \sum_{k=0}^{\tilde{n}-1} \left[ f_c(\tilde{t}_k) \int_0^T \tilde{B}_{k,\tilde{p},\tilde{\xi}}(t) dt \right] \leq \epsilon, \quad (14b)$$

$$\text{s.t. } \underline{v}^2 \leq f_v(\tilde{t}_k) \leq \bar{v}^2, \quad 0 \leq k \leq \tilde{n} - 1, \quad (14c)$$

$$\tan(\underline{\phi}) \leq f_\phi(\tilde{t}_k) \leq \tan(\bar{\phi}), \quad 0 \leq k \leq \tilde{n} - 1, \quad (14d)$$

$$p_j = \sum_{k=0}^{n-1} P_k B_{k,p}(t_j), \quad \forall j. \quad (14e)$$

## Simulation results



Simulation data	Numerical Value
Gravitational acceleration $g$	9.81 $m/s^2$
Velocity control input	$V_a \in [18 \ 25]$ $m/s$
Bank control input	$\phi \in [-0.43 \ 0.43]$ rad
Altitude	150 m
Simulation time $T$	150 s
Way-point list	$\mathbb{P} = \{(1000, 1000, 150), (1100, 250, 150), (400, 0, 150), (-250, 250, 150), (400, 500, 150)\}$
Number of control points $n$	10
Order of the B-spline curves $p$	4
Number of greville points $\tilde{n}$	20

Table 1. Numerical values of the simulations

Due to the approximation error, the velocity constraint was tightened from 40%, centered in the middle of its interval.

Simulation Results	Numerical Value
Number of inequality constraints of (8)	$\infty$
Computational time of (8)	—
Number of inequality constraints of (14)	60
Computational time of (14)	145 s
Number of iterations to solve (14)	608

Table 2. Computational time analysis for the approximated curves

## Future work

- Extend this algorithm to multiple UAVs, taking into account "inter-UAV" constraints like communication range among them and collision avoidance
- Apply this algorithm to other UAV models: Quadcopter Crazyflie that we are using in the Esisarum platform.
- Study the approximation error caused by the Schoenberg operator in order to obtain bounds to know a priori the constraints tightening.

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