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Rebellion in the ‘museum’: mechanisms for mathematical modelling at university level

Berta Barquero

Abstract. In this paper, we focus on the problem of what *mechanisms* can help mathematical modelling to be integrated into first-year courses of mathematics at university level. In front of the extended agreement about the necessity to foster functional teaching of mathematics as a modelling tool in different contexts, the research problem treated confronts some important difficulties that the more traditional university teaching must integrate modelling at the core of their teaching. Within the framework of the anthropological theory of the didactic (ATD), we have worked, since the last decades, on the design, implementation, and analysis of the *study and research paths* (SRP) as a teaching device persuading a double purpose: making students aware of the rationale of mathematical contents through the experience of modelling activities; and connecting these mathematical contents through a whole modelling process. Empirical results from the implementation of an SRP on population dynamics with first-year students at university level, and its adaptation and ‘migration’ to other university settings, are used to outline some useful mechanisms necessary for modelling at university.

Keywords. Mathematical modelling, Anthropological theory of the didactic, Study and research paths, Institutional constraint, Mechanisms.

Résumé. Dans cet article, nous nous concentrons sur le problème des *mécanismes* qui peuvent aider à intégrer la modélisation mathématique dans les cours de mathématiques de première année au niveau universitaire. Face à l’accord général sur la nécessité de favoriser l’enseignement fonctionnel des mathématiques comme outil de modélisation dans différents contextes, le problème de recherche abordé fait face à quelques contraintes que l’enseignement universitaire plus traditionnel doit intégrer la modélisation au cœur de leur enseignement. Dans le cadre de la théorie anthropologique du didactique (TAD), nous avons travaillé, depuis les dernières décennies, sur la conception, la mise en œuvre et l’analyse des *parcours d’étude et de recherche* (PER) en tant que dispositif d’enseignement ayant un double objectif : sensibiliser les étudiants à la logique des contenus mathématiques à travers l’expérience des activités de modélisation ; et relier ces contenus mathématiques à travers un processus de modélisation complet. Les résultats empiriques de la mise en œuvre du SRP sur la dynamique des populations avec des étudiants de première année à l’université, ainsi que son adaptation et sa ‘migration’ vers d’autres contextes universitaires, sont utilisés pour souligner certains mécanismes qui paraissent utiles à la modélisation à l’université.

Keywords. Modélisation mathématique, Théorie anthropologique du didactique, Parcours d’Études et de Recherche, Contrainte institutionnelle, Mécanismes.

1. Introduction

The starting point of the research is the problem of inquiring into the conditions that can help that mathematical modelling can be integrated and developed in the teaching and learning of mathematics into current educational systems. Researchers and practitioners agreed on the necessity of proposing alternative teaching practices, particularly at the university level, where the formal transmission of mathematical knowledge does not act alone to let mathematics be taught as a service subject becoming an essential modelling tool to enquire into the study of real problems (Holton, 2001; Niss, 2001; Blum, 2015).

This change requires moving from a more traditional pedagogical paradigm of transmission of knowledge, which mostly focuses on introducing students to already built mathematical knowledge, to a paradigm of inquiry where solving problematic questions leads to learning processes and motivates the study of new knowledge. This is what Y. Chevallard (2015) calls moving from the paradigm of ‘visiting works’, or of ‘visiting monuments’, where students are invited to visit mathematical knowledge muted from its rationale, towards the paradigm of ‘questioning the world’ where questions are placed at the core of knowledge construction and mathematics understood as a modelling tool to investigate questions to collectively provide answers.

“[In the paradigm of ‘visiting monuments’] Because these questions are usually hushed up—visiting a monument is no place to raise “What for?” or “So what?” questions—, students are reduced to almost mere spectators, even when educators passionately urge them to “enjoy” the pure spectacle of mathematical works” (Chevallard, 2015, p. 174)

In the research domain known as ‘Applications and modelling’, some big steps have been made to show how modelling activities can be successfully performed, under certain suitable conditions in different educational levels and curricular frames (Blum, 2015, Burkhardt, 2006 and 2018, Doerr & Lesh, 2011, among others). However, although school institutions and researchers agreed that modelling should play an important role for a change towards a new pedagogical paradigm, the real situation in school and university is not satisfactory (Stillman et al. 2013) and the dissemination and long-term survival of these teaching proposals based on modelling follows as a big challenge for mathematics education (Galbraith 2007, p. 79).

In the case of applications and modelling a shared excitement unites many who have been enthused about early experiences in the field, for example when students unleash latent power that for whatever reason had remained fettered in their previous mathematical life. However, this very exhilaration can work against further progress, both individually, and particularly at a system level, by creating a sense of adequate achievement that obscures the reality that there is so much more to do.

In this paper, we focus on first-year university courses in mathematics for non-specialists (González-Martín et al., 2021). In this context, in previous works (Barquero, Bosch, & Gascón, 2013), we analysed the more prevalent structure and way of teaching mathematics for natural sciences at university level. One of the main drawbacks underlined of such teaching organisation is that it regularly creates difficulties due to the disconnections between the taught contents and their poor motivation in relation to their uses and functionalities with the rest of scientific disciplines. It is also shown that even though mathematical models appear in the syllabi of most of the courses, teaching mathematical models often comes at the end of the process if there is time left for it. Then, the dominant ideology (that we characterize as ‘applicationism’) at the university level is that modelling represents in most cases a mere ‘application’ of some pre-established knowledge, leaving little room for the process of questioning, building, and validating mathematical models. Although it is not particular to university teaching, as Wozniak (2012, p. 28) states, when she analyses some teaching practices about a modelling activity experienced at the Primary school level, the only question about modelling that lives in Primary classrooms of the one-off choosing the coefficient values to apply a pre-given model by the teacher.

Our research, developed in the framework of the anthropological theory of the didactic (ATD henceforth), faces the question of how to change the institutional relation of university teaching about modelling by looking into what *mechanisms* could help modelling to have an explicit and central role in the teaching of mathematics. In other words:

How can we design instructional devices where mathematical modelling has an explicit and crucial role, emerging from generative questions and linking mathematical contents that appear as models to solve the questions? Which *mechanisms* have helped that modelling plays this role in mathematics teaching at university level?

To face these questions, we focus on the proposal of the *study and research paths* (SRP) that have been proposed, in the framework of the ATD, as epistemological and didactic model (Chevallard 2015; Bosch, 2018) to move toward a new pedagogical paradigm. In the next section, we present some of the main traits of the SRP, their design principles, and the conceptualisation of mathematical modelling into them. Our object of analysis is the design and implementation of modelling practices developed through the proposal of the SRP. In particular, in section 3-5, we take the case study of the design of an SRP about population dynamics, which were implemented during five consecutive years with first-year university students, as the case study of one of the first SRP implemented at the university level aiming to reform a traditional course on mathematics. The mathematical and didactic designs from this initial SRP were later adapted to migrate to other university institutions, starting from different initial questions and some variation on their modality of implementation. Then, we focus on how the a posteriori analysis of the successive transformation of the SRP to be implemented in three university contexts. We investigate the *mechanism* that can favour that modelling could be effectively integrated and developed.

2. The study and research paths and the role of mathematical modelling

Chevallard (2006, 2015) introduced the proposal of the *study and research paths* (SRP) as a general model for designing and analysing teaching and learning processes in asking for a change toward a pedagogical paradigm, from a ‘monumentalist’ approach to a paradigm of ‘questioning the world’. Previous investigations have shown how modelling can be integrated at different school levels through the SRP proposal (Bosch, 2018; Jessen, 2017; Florensa et al. 2018, among others). In the framework of the ATD most of mathematical activity can be described in terms of modelling (when it is proposed far from a ‘monumentalist’ approach). In any mathematical activity, there can be raised some questions to be addressed, which leads to the construction of mathematical models to provide possible answers. This looking for answers requires going through a modelling process, which can be characterised by different stages (Chevallard 1989), similarly as the description provided through the modelling cycles (e.g., Borromeo Ferri 2006), without decomposing too much its stages it includes:

1. Delimitation of the system (mathematical or extra-mathematical) where the questions emerge and the selection variables to study;
2. Formulation of hypothesis and construction of mathematical models;
3. Work with the mathematical models to provide answers;
4. Validation of models by contrasting them to the initial system.

In this process, many questions can appear which will lead to new modelling processes. Following the works of García et al. (2006), modelling activity can be reformulated as a process of construction and articulation of mathematical organisations of increasing complexity in a progressive and recursive process. In our research, we proposed to reformulate the modelling processes as processes of reconstructions and connection of *praxeologies* of increasing complexity (pinpoint – local – regional) that should emerge from questioning the rationale of the praxeologies that are to be reconstructed or connected (Ibid, p. 243). In this modelling process, what is central is to investigate how to make systems and models emerge, evolve, transform, and connect, at each step, in more complex and complete structures that allow mathematical knowledge to appear as functional (provide answers to something) and connected. With this aim, we characterise the proposal of the SRP and the role that modelling plays in them focusing on two critical questions: (1) How to enhance the dialectics between *posing questions* and *looking for answers* as the engine of the modelling activity? (2) What *milieu* is necessary for students (and lecturers) to facilitate an appropriate development of modelling?

2.1. *The generating question and the questions-answers dialectics*

One of the main traits of SRPs is that they start from a lively question of potential concern for the students and teachers called the *generating question*, expressed as Q_0 . When the community of students and teacher(s) decides to pursue it, Q_0 evolves by opening many other derived questions Q_i . The main purpose is that students, guided by the teacher, provide answer(s) to Q_0 together with the derived questions. The study of Q_i leads to looking for successive temporary answers A_i . Then, the structure of the SRP can be synthesized as a tree of linked questions and answers (Q_i, A_i), which traces the possible ‘paths’ to be followed in the effective experimentation of the SRP.

In the next sections, we use the questions-answers structure of the SRP to describe the *a priori* design and the *a posteriori* analysis of the SRP we focus on. Addressing Q_0 allows students to go through several modelling cycles in which they iteratively express-test-and-revise their answers and make them progress beyond first-iteration responses. In this process, the questions considered do not disappear, and neither the answers (hypothesis, models build, used, validated...), which take an integral part of the built knowledge.

This first layer of the design of the SRP allows us to deal with different aspects. On the one hand, to enquire into the potential of the generating questions Q_0 and to trace the possible path to be followed in the effective implementation of an SRP. That is, to foresee if Q_0 is ‘fertile’ enough and to sketch its ‘life expectation’ by describing the derived questions that can be opened and the extra- and intra-mathematical tools and models that the study of Q_0 and Q_i may ask for. On the other hand, to provide researchers and teachers with alternative epistemological models to describe mathematical activity, which is usually described following the logic of the mathematics contents, to be now described in terms of the interplay between questions, mathematical models, and possible answers.

The necessity of considering these alternative epistemological models for mathematical (or modelling) activity remains in the necessity of overcoming some important constraints linked to the dominant ‘museographic’ paradigm of visiting works, absent of the possible questions that can be at the origin and reason d’être for its consideration. Moreover, as expressed in the works of Orange (2005, 2007), there exists an important constraint about the few possibilities of integrating a real ‘problematising’ activity (posing questions) in classrooms. This supports the first working

assumption on the proposal of the SRP about the necessity of integrating rich *dialectics of questions-answers* or posing questions and discussing the answers as an engine for the SRP and for a rich and complete modelling activity. In section 6 we focus on some mechanisms that have helped with the first essential dialectic to be placed at the core of the SRP.

2.2. Progressive enrichment of the milieu

There are many possible ways to approach a generating question depending on the derived questions opened and the answers provided. But also, the SRP can significantly vary according to the different *means* that are made available during the teaching and learning process. Then, another dialectic, the one of *media-milieus* helps describe the dynamics of the SRP (Chevallard, 2011; Kidron, et al., 2014). Here, *media* is used in the same sense as any source of information (one person, textbooks, papers, lectures, websites, class notes, etc.) used to obtain previously established knowledge or answers that students consider relevant to solve the questions raised during the process. They provide pieces of knowledge that cannot always be used directly but should be validated and adapted by checking them against a milieu, that is, against the set of empirical objects and knowledge that is already available for the students and can act as a ‘piece of nature’ to them (Brousseau, 1997).

In front of the generating question, and to provide an answer A^\heartsuit to it, the class constructs a certain *milieu* (M) composed of different elements: derived questions (Q_i), answers that the class built (A_i) or pre-labeled answers already exist outside in different media (A_i^\diamond), mathematical work/objects (W_j) that help to evaluate the pertinence of certain answers, experimental data (D_l), etc. Chevallard (2011) synthesized the activity started in a didactic system $S(X; Y; Q_0)$, which is formed to study a certain question Q with a class X and a teacher Y who guides the study with the *Herbartian scheme*:

$$[S(X; Y; Q_0) \curvearrowright M] \curvearrowright A^\heartsuit$$

Where M can be composed of the elements mentioned before:

$$[S(X; Y; Q_0) \curvearrowright \{ Q_i, A_i, A_i^\diamond, W_j, D_l \}] \curvearrowright A^\heartsuit$$

By using the Herbartian scheme, one can analyse different teaching and learning practices, in particular, modelling teaching practices. For instance, Wozniak (2012) analyse several teaching practices with ‘Giant shoe’ modelling activity at Primary school level which show how underdeveloped is the *media-milieu dialectics*. In this activity, the teacher is the mean media, which provides students the questions and tools or models to be used (or to be apply), and where students’ answers are reduced to the application of pre-given models decided and provided by the teacher.

It thus appears that there can be no construction of models [...], without the implementation of a *dialectic of media-milieus* that allows for an explicitation of the problematization. If the hypotheses on which the model is based are not stated, or if they are stated without being questioned or legitimized, without their field of validity being explored, then we will consider that the modelling process has not been fully developed and the model building stage partially completed.

On the contrary, if the teachers propose to students to inquire beyond the pure application of pre-established models, there can appear questions and decisions concerning what hypothesis are better to consider, what models are proposed by experts in different media, how to validate and make

models evolve, and so on. Then, the *dialectics between the media and milieu* would be, at least, different, allowing a different degree of questioning and development of modelling for X and Y . In the last section, we focus on some mechanisms and levers that help to integrate this second essential dialectic at the core of the SRP.

2.3. *New responsibilities and their distribution in the collective construction of A*

The SRP aims to promote the role of the study community (X, Y) which, in contrast with the ‘dominant pedagogy’, with a prevailing role of ‘individual’ work under the guidelines of the teacher(s), the group of students may share a set of tasks and negotiate the responsibilities for each to assume.

This displacement going from the individual to the community has many important consequences to make the existence of mathematical modelling possible. On the one hand, the collective study of questions provides the opportunity of *defending responses* produced by the community, instead of accepting the *imposition of the official* answers. On the other hand, this work requires new distribution of responsibilities and, consequently, changes in the prevailing didactic and pedagogical contracts between the students and the teacher. The teacher thus had to assume a new role of acting like the leader of the study process, instead of lecturing the students. And students with the teacher(s) might share responsibilities on raising and agreeing on questions to address; formulating hypotheses; searching and discussing different models; collecting and selecting data; choosing the relevant mathematical tools to validate and reformulate models; writing and defending reports with partial or final answers, and so on. The teacher thus had to assume a new role of acting like the guide of the inquiry, instead of lecturing the students. It soon appeared that the teaching culture at university level does not offer a variety of teaching strategies for this purpose. This collective approach to modelling will be described when explaining the analysis of the SRPs.

3. Research methodology and conditions for the design and testing of the SRP

The research methodology followed for the design, implementation, and analysis of the SRP is didactic engineering (Artigue, 2014), which includes four main stages. As explained by Barquero and Bosch (2015), the starting point of the research is the integration of mathematical modelling in first-year university courses of Mathematics—in natural science university degrees in Barquero et al. (2013) or business administration, in Serrano (2013)—. When analyzing what kind of mathematics is taught at this level, for the preliminary analysis of the existing conditions (and constraints), one could think (natural science, business administration) university degrees would offer favourable institutional conditions to teach mathematics as a modelling tool. However, this seems far away from reality: the dominant ideology is that modelling represents a mere application of some pre-established knowledge, leaving little room for the process of proposing, constructing, validating, and questioning mathematical models. The second stage is devoted to the *a priori* analysis of the teaching proposal, that is, to the mathematical and didactic design of the SRP. On the one hand, in the mathematical design, the SRP is described as a *map of questions and answers* that are derived from a generating question about population dynamics tracing the possible routes to be followed in the SRP implementation (Winsløw et al., 2013). In the following section, and in Appendix 1, we summarize the *a priori* analysis of the SRP regarding population dynamics. The didactic design of the SRP appears to complement the previous description with a more precise organisation of each session:

formulation of Q_i to address, set of data given, management of the possible responses and new questions, sharing of responsibilities between the students and instructors (as some the above mentioned), etc. The third stage of the process consists of the *in vivo* analysis with the implementation, observation, and data collection from the implemented SRP, which mainly came from students’ team and individual reports, the teacher’s written description of the work carried out during each session, worksheets given to the students, video recording of the workshop sessions, and brief questionnaire to the students at the end of the process. It constitutes the empirical base upon which rests the analysis *a posteriori* of the SRP’ testing, validation, and development of a priori design proposed by the two previous phases and reflects the role, participation, and dynamics generated by the students’ groups and by the teacher.

In this paper, we focus on an SRP that was initially implemented for five academic years (from 2005/06 to 2009/10), with first-year students of technical engineering degree at the Autonomous University of Barcelona. The testing took place within the one-year ‘Mathematical Foundations of Engineering’ course, with about 40 students. Their designs were afterwards adapted and migrated to other university institutions of Business Administration degrees (from 2006/07 to 2018/19). Table 1 summarizes the SRP topics, the degree and university institutions where they were implemented, the period, and some references to extend information about their analysis.

SRP	SRP topic	Subject, Level Degree, University	Period	References
SRP1	Population dynamics	<i>Mathematics</i> . 1 st year Chemical engineering, UAB	2005-2010	Barquero et al. (2011, 2013)
SRP2	Users of Lunatic world networks	<i>Mathematics</i> . 1 st year Business admi., IQS – URL	2006-2014	Serrano (2013) Serrano et al. (2010)
SRP3	Evolution of Facebook users	<i>Mathematics</i> . 1 st year Business administration, Tecnocampus	2015-2019	Barquero et al. (2018)

Table 1. List of experienced SRPs

The common conditions of SRP1-SRP2 were that the SRPs ran in a teaching device called the ‘mathematical modelling workshop’. The workshop run over 2-hour weekly sessions during the whole course. In the case of SRP3, the modelling workshop was implemented in the transition from the first to the second term, as independent of the regular course, but to bridge to the content of the regular course (1st term: one-variable sequences, functions, derivatives) with their use and applicability. The instructors of the workshop were researchers in Didactics, also lecturers of the regular course. In the next section, we describe the initially designed SRP on population dynamics, which will be used then to highlight the adaptation into the other university contexts where it was adapted (SRP2, SRP3). These three cases can be considered the first SRP implemented at the university level, after them our research groups have been designing SRP for other subjects (such as, mechanical engineering, statistics, and accountability, among others), where the instructors were not only researchers in didactics (see Barquero et al., 2022, for an overview of the SRP implemented at university level).

4. The study and research paths and the role of mathematical modelling

4.1. The *a priori* analysis of the SRP on population dynamics: structure and divisions

The starting point of the SRP is the study of the population evolution and how to predict their evolution, which gives to the following generating question Q_0 about: Given the size of a population over some periods, can we predict its size after n periods? Is it always possible to predict the long-term behaviour of the population size? What sort of assumptions about the population and its surroundings should be made? How can one create forecasts and test them?

In front of this question, we can initially assume that time t is measured in discrete units, and that x_t depends, among other factors, on past states $x_{t-1}, x_{t-2}, \dots, x_{t-d}$ ($0 < d \leq t$). Facing question Q_0 leads us to consider two main types of models: (1) when x_t only x_{t-1} depends on, that is a population with independent generations, where models based on recurrent sequences of order 1: $x_{t+1} = f(x_t)$ are considered, with f is a real valued function of one variable, or (2) when x_t depends on the $d > 1$ past generation $x_{t-1}, x_{t-2}, \dots, x_{t-d}$ (population with mixed generations), we study recurrent sequences of order $d > 1$, which can be expressed as vector recurrent sequences with $X_{t+1} = f(X_t)$ where $X_0 = (x_0, x_1, \dots, x_{d-1})$ is the vector of the d initial generation sizes and $X_i = (x_{id}, x_{id+1}, \dots, x_{id+(d-1)})$ is the i -th vector of d generations with $0 \leq i \leq n$. If we assume that time t is measured as a continuous variable, we study the continuous evolution of the population, which has an analogous structure to the situations described above.

From these first considerations, two main branches on the design of the SRP are opened. On the one hand, the first branch focused on the *discrete models* where the models to be built are based on *recurrent sequences*, with order 1 or bigger than 1, depending on if we consider populations with independent generation or populations with mixed generation. The first case covers the sector of sequences and their convergence and the one of one-variable calculus; the second case covers the sector of linear algebra. On the other hand, a second branch focused on the *continuous models* where, depending on if we consider independent or mixed generation, it allows us to consideration of models based on ordinary differential equations (ODE) of order 1 or bigger than 1. In the following section, we include the description of two parts of both branches, combined with inputs from the real implementation of the SRP. One may note that the *a priori* design of the SRP, in terms of questions-answers or in terms of modelling praxeologies, are dynamics structures, that is, with each implementation and its *a posteriori* analysis the initial design has been enriched and new questions, hypothesis, models, answers, etc. have been incorporated.

Before going into detail with the description of the modelling activity developed, its design shows different elements that collide with the dominant way of organising the teaching of mathematics at university level. First, the generating question is an open question without an immediate answer. It is expected to take a long time to be addressed, and it would imply changing the order of appearance of some contents of the regular course, following the logic of the questions opened in the workshop. Second, since the beginning students were informed that it was an open modelling activity where they, working in groups, could differ in their proposals and it was an aim to discuss and agree on the collective path to follow and to elaborate A^\heartsuit .

5. Implementation of the SRP on population dynamics

The initial workshop session of the SRP started presenting some data corresponding to the evolution of the size of a population of pheasants in an isolated island, over a period of 5 years (data extracted from Lack, 1967). The students were asked to analyse the data and provide an initial answer to the generative question Q_0 . In the five SRP experimentations, students took three main approaches to the problem: some groups tried finding the best polynomial interpolation to the data, other groups tried an exponential fit to the data and the rest tried using recurring sequences to model the population dynamics. In the class discussion, the three approaches were presented by the teams and the students, under the guidance of the instructor, decided that the discrete recurring sequences approach would be explored first, leaving the continuous approach for a later SRP (3rd branch of the SRP experienced during the 3rd term). This decision was taken mainly because when students were asked to explicit the different elements of the whole modelling praxeology: hypothesis assumed ($Q_{0.1}$), models build according to these hypotheses ($Q_{0.2}$), only the teams working on the discrete world and with models based on recurrent sequences could provide and justify $A_{0.1}$ and $A_{0.2}$ by formulating hypothesis over the absolute or relative rates of growth.

$Q_{0.1}$: What hypothesis about rates of growth can be formulated? According to them, which mathematical models can be considered to fit and forecast data? ($Q_{0.2}$)

From here, during the following two months, we experienced the first branch of the SRP. It ran over 8 weekly sessions of 2 hours each. The instructor with the students agreed to retake the proposals of using continuous models (mainly based on exponential functions), during the 2nd semester, after experiencing this 1st branch, and when the regular course started with the block of derivatives and differential calculus.

5.1. From the Malthusian model to the reformulation of the hypothesis

In the second workshop session, the class agreed on the notation to be used and the instructor introduced some requirements on the population to simplify the study, such as considering independent generations. x_n was defined as the size of the n -th generation of population X and the study of the population evolution was thus characterized by the study of the sequence $(x_n)_{n \in \mathbb{N}}$. The assumption of independent generations leads to the class to consider several indicators of the population growth, there appeared the *relative rate of growth* of X between consecutive generations: $r_n = (x_n - x_{n-1}) / x_{n-1}$ or the net production rate: $i_{n+1} = x_{n+1} / x_n$. In all the experimentations some of the groups began by proposing one of the easiest assumptions about the growth of X (see figure 2):

$H_{1.1}$: The rate of growth of the population is constant: $r_n = r$, $r \in \mathbb{R}$ and $Q_{1.1}$: How does a population with constant rate of growth evolve over time?

This assumption leads to the construction to the first model ($M_{1.1}$) which is equivalent to the first-order recurrent equation: $x_{n+1} = r \cdot x_n + x_n = (1+r) \cdot x_n = \alpha x_n$, with $\alpha = 1 + r$. Given the initial population size $x_0 = c > 0$ and assuming that the relation is valid for any consecutive generation, we can approach x_n using the equivalence: $x_{n+1} = \alpha \cdot x_n \Leftrightarrow x_n = \alpha^n \cdot c$. The exploratory study of M_1 allowed the students to provide a temporary answer A_1 to initial question Q_0 depending on the value of α : (1)

If $\alpha < 1$, the population is wiped out; (2) if $\alpha = 1$, the population size remains constant independently of the time elapsed; and, finally, (3) if $\alpha > 1$, the population grows indefinitely.

La primera hipòtesi vàlida ho estat trobar

$$\frac{H(u+1)}{H(u)} = c, \quad \forall c \in \mathbb{R}, \text{ on } c = \text{cte}$$

Així en veure adonar que si trobàvem una c vàlida podríem definir la funció $H(u+1) = c \cdot H(u)$ útil per tots els casos.

-1r cas : per una $H(0) = 150$ i una $c = 20$
 tenim que: (podem plantejar la fórmula d'una altra manera per veure-ho més clar: $H(u+1) = c^u \cdot H(0)$)
 Aquesta fórmula surt de

u	H(u+1)
0	150
1	3000
2	60000
3	1200000
4	24000000

$H(u+1) = c \cdot H(u) = c \cdot c \cdot H(u-1) = c \cdot c \cdot c \cdot H(u-2) = \dots = c^u \cdot H(0)$

Per tant, en aquest cas, com la $c > 1$, la població creix infinitament, com en el cas del nostre model. Matemàticament, per $c > 1$, $\lim_{u \rightarrow \infty} H(u) = \infty$

Figure 1. Formulation of H1 about the relative rate of growth

Figure 2. Numerical simulation of M1 and general conclusions about population dynamics

Students quickly realized that this first model (M_1) presented a clear limitation: the fact that the population grows indefinitely (case $\alpha > 1$) falls in the biologic paradox of assuming the existence of infinite resources (see figure 3). In front of this, $Q_{1.1.1}$ was formulated as *how could we overcome this unrealistic fact?* At this point, most of students came to the conclusion that it was more coherent to assume that r_n decreases depending on x_n , which was finally formalise as:

H_{1.2}: The size of the population cannot exceed a given maximum value K . Therefore, the rate of growth must decrease when the population size approaches this maximum value. For example, we can assume the simplest case of a rate of growth decreasing linearly with size. **Q_{1.2}:** What are the dynamics of a population under $H_{1.2}$ conditions?

At some stages of the first phase, it was necessary that the instructor of the workshop introduced new mathematical knowledge to help the students' progress in the study of $Q_{1.1}$ and $Q_{1.2}$, that is to extend students' milieu to make them evolve in the SRP. As mentioned before, the first term of the regular course was mainly devoted to the study of elementary functions and their properties, so there were several important tools, such as the different kind of rates of growth, the definition of what a recurrent sequence is, how to numerically simulate sequences (through which experimental mean), etc. that were necessary to be formalized thanks to the work initiated throughout the workshop. Then, both instructors of the course met and agreed on where and how it was better to instruct on these necessities, for instance in the following workshop or in a lecture session by being the main media or by using external media and providing them some book or paper references. Furthermore, since the beginning, students were asked to look outside for possible external pre-existing answers to the questions they were dealing with. In several implementation, they brought to the class some information about the Malthusian or logistic models they had found in external resources. Thanks to this, we gave name to the models and several new questions emerged from looking and contracting the information contained into these external answers with the ones developed in the SRP.

5.2. From the logistic discrete model to a general functional model

One of the simplest models that satisfy $H_{1.2}$ is summarized by the recurrent equation $x_{n+1} = \alpha x_n(1-x_n/K)$, known as the *logistic equation* (discrete) or *Verhulst model*. Once this second model $M_{1.2}$ was built, students tried to look for a closed-form equation, i.e., with a general formula $x_n = f(n)$ which, in contrast with the Malthusian model ($M_{1.1}$), the logistic equation does not admit. The numerical simulation, for some values of the coefficients α and K , constitutes the first *experimental mean* to simulate the population dynamics modelled by $M_{1.2}$. This exploratory study provided the following experimental answer $A_{1.2}$: (1) If $\alpha < 1$, the population is wiped out; (2) if $1 < \alpha < 3$, the population size grows to an equilibrium situation; (3) if $\alpha > 3$, one finds cases which are difficult to analyse (for instance, there are cases where the population growth oscillates between several values (2, 4, 8, etc., or cases where the long-term behaviour is completely unpredictable). The institutionalization of these conjectures from different students’ teams was pertinent to reveal the appearance of new derived questions, such as:

$Q_{1.2.1}$: What does the well-definition and convergence of x_n depend on, in the logistic model case? and $Q_{1.2.2}$: What does the speed of convergence of x_n depend on?

At this point, it appeared necessary to consider a new model, including the previous ones and, able to re-formulate and face the questions that had remained open (such as $Q_{1.2.1}$ and $Q_{1.2.2}$). The work performed so far can be described by $Q_{1.3}$ which led to the consideration of more general model based on a recurrent relationships of the form $x_{n+1} = f(x_n)$ where f represents the functional relationship between two consecutive generations of the population. In particular, if f is a linear or quadratic function we have the previously analysed models.

$Q_{1.3}$: What are the dynamics of the sequence x_n generated by the relationship $x_{n+1} = f(x_n)$ where f can be any C^1 function?

In the SRP implementation, after being introduced to the graphical techniques of sequence iteration (particularly, to the cobwebbing techniques), the teams began by considering f as a linear function (the case of the Malthusian discrete model, $M_{1.1}$) to then face the case of f a quadratic function (the case of the logistic discrete model, $M_{1.2}$). Students used this new experimental milieu provided by the graphical simulation to check the conjectures and answers they had suspected with the numerical simulation of these models.

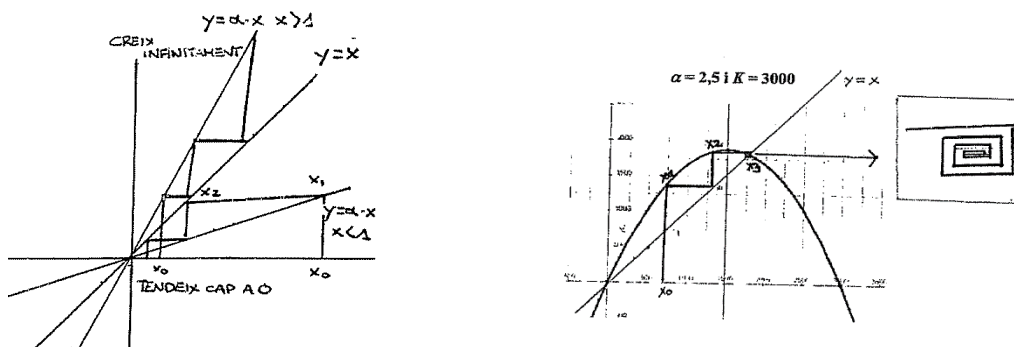


Figure 3 – Formulation of H_1 about the relative rate of growth

Through this phase, more advanced mathematical concepts (C^1 -functions, graphical techniques of simulation, analysis of derivatives, etc.) were necessary to analyse the behaviour of $x_{n+1} = f(x_n)$. They were out of the course programme fixed in advanced, but the instructors decided to introduce them in the workshop or lectures sessions to be sure students could use and understand properly. This induced a huge enlargement of the types of problems that could be considered, for instance, by extending the kind of functions to consider. In any of the five experimentations, we could not go further due to time restrictions. Although on several occasions some teams propose to consider other decreasing functions, which would lead us to taking into account new models, for instance, the Ricker model by considering that $f(x_n)$ is an exponential decreasing function or the Beverton-Holt model which considers that $f(x_n)$ is a rational function. These proposals help to extend the initial a priori design of the SRP, although they have never been experienced.

5.3. From the discrete to the continuous modelling processes

As mentioned before, it was in the second semester when the whole second full branch of the SRP about continuous models was developed. This third branch was focused on Q_2 about how and which continuous models can be used to fit data and to provide forecasts about population dynamics. Here it was used two kinds of yeast populations, *Saccharomyces cerevisiae* and *Saccharomyces kéfir* populations (Carlson, 1913) living in independent containers. In the second semester, students had overcome the initial resistance and progressively accepted a lot of new responsibilities they were asked to take on: defending their reports, posing questions, looking for pre-existing answers outside the classroom frame, etc. We cannot go into detail about this SRP development, but we want to underline some important aspects. First, thanks to the parallel structure between the 1st and 2nd branch of the SRP, the students' autonomy increased significantly in the different steps of the modelling process. For instance, the way they formulate hypotheses about population growth, or how they build, simulate, and test models and provide answers through this modelling work were too much facilitated by the work developed in the first SRP. Second, thanks to the parallel structure of both branches (see Figure 5) and the students' fluency showed in the modelling process at this stage, we could face a lot of new and crucial questions, not foreseen in the initial design of the SRP. It appeared interesting questions about likely relationships between the hypothesis assumed, models built, and answers reached, between the discrete and continuous world. For instance, we devoted the first sessions of the workshop and of the lectures sessions dealing with questions like Q_{1vs2} about the relation between the relative rate of growth and the derivative, or $Q_{1vs2.2}$ about if we reach similar conclusions with the discrete or continuous Malthusian or logistic models, or about, the meaning of the coefficients defining these models, among others. Figure 5 wants to show this parallel structure and the new questions that emerged from contrasting answers obtained in both worlds (the discrete and the continuous).

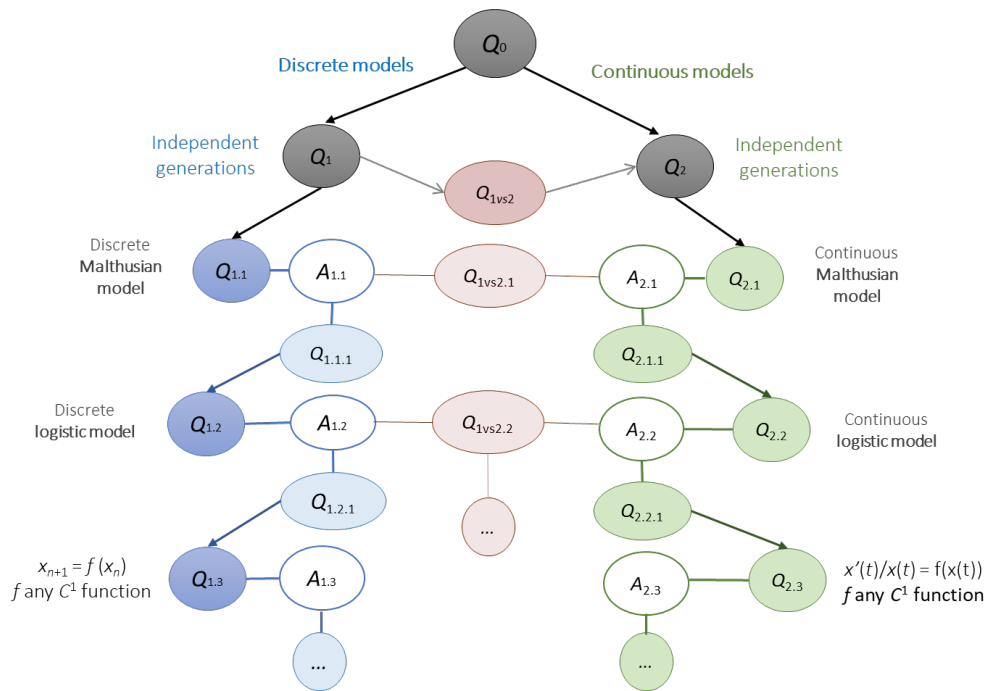


Figure 4 – QA map of the 1st and 2nd branches of the SRP with independent generation

Q_2 : If we consider time as a continuous magnitude, what assumptions about the rates of growth can we formulate? What mathematical models would appear?

Q_{1vs2} : What relation do exist between the relative rate of growth and the derivative?

$Q_{2.1}$: If $r(t) = p'(t)/p(t) = r$ is constant, how will the network users evolve over time?

$A_{2.1}$: Construction of the Malthusian continuous model

$Q_{1vs2.1}$: Do we obtain the same conclusions from the discrete and the continuous Malthusian model? Do the constant coefficient r has the same meaning and effect on the population evolution?

$Q_{2.2}$: If $r(t) = p'(t)/p(t)$ decrease linearly, how will the population evolve over time?

$A_{2.1}$: Construction of the logistic continuous model

$Q_{1vs2.2}$: Do we obtain the same conclusions from the discrete and the continuous logistic model? Do the coefficients (K and α) have the same meaning and effect?

5.3. The ‘migration’ of the SRP about population dynamics in another university setting

Since the academic year 2005/06, the research group started implementing SRP with first-year university students of business and administration degree (4-year programme) in IQS School of Management of Universitat Ramon Llull in Barcelona (Spain). A ‘mathematical modelling workshop’ was introduced in the general organisation of the mathematical course. It consists in 90-minute weekly sessions covering one-third of classroom time for students, and more than half of their personal work outside of the classroom. The instructor of the course is also responsible of the workshop sessions. These ran in parallel to the three-hour weekly lecture sessions, which included problem-solving activities. In the general organisation of a workshop, students work in teams of 3 or 4 members, under the supervision of the instructor responsible for the course and, if possible, of a

researcher who acts as an observer. The workshop focuses on a single initial problematic question Q , which was broken into three, one for each term.

The *a priori* analysis of the SRP about population dynamics was adapted to this university setting. The first important change for the SRP2 was about generating question Q_0 that was about the evolution of the number of users of a social network. This network, called *LunaticWorld*, was supposed to open in 2004 with 18 users, with quick growing (from 18 users in 2004 to 3143, in 2009). Similarly, in the case of population dynamics, Q_0 was about what assumption and what models could help to predict the size evolution of the users, what predictions one could make and how to test them. The first kind of models that emerged used to correspond to the 1st branch of the previously described SRP (Q_1) with models based on recurrent sequences of order 1: $x_{t+1} = f(x_t)$. It was followed, during the 2nd term, by a new development of the SRP that focus on the use of elementary functions as models to fit data and discuss criteria for its fitting (Serrano, Bosch, & Gascón, 2013). What was not develop, in this SRP2, was the branch corresponding to differential equations and systems of differential equations (as in the case of the dynamic population), as in part, it was not included in the official syllabus of the course, and it was not proposed by the students. What was implemented in the 3rd term was the branch corresponding to the recurrent sequences of order $d > 1$, when x_t depends on the $d > 1$ past generation $x_{t-1}, x_{t-2}, \dots, x_{t-d}$, which can be expressed as vector recurrent sequences with $X_{t+1} = f(X_t)$ with $X_0 = (x_0, x_1, \dots, x_{d-1})$, which it could be alternatively described as a matrix model $X_{t+1} = M \cdot X_t$. Figure 5 shows the situation that was presented to students, where the group of users of the social network was proposed to be distributed into three groups: *Basic*, *Medium* and *Premium*.

MATHEMATICAL MODELLING WORKSHOP – 3rd term

The social network *Lunatic World* has recently introduced important changes. From now on, all users will be distributed in three different groups: **Basic**, **Medium** and **Premium**. After one month, a user can remain in the same group, be promoted to another group or leave the network. Moreover, the only type of users that are allowed to invite other users to join the following month are 'Medium' and 'Premium' groups.

CASE 1 [One of the 16 cases that were distributed to the different teams]:

- 15% of 'Basic' users remains as 'Basic' one month later and 75% of 'Basic' changes to 'Medium';
- 25% of 'Medium' changes to 'Premium' the following month and 50% remains as 'Medium'. Each 'Medium' user invites on average 4 new users that enter as 'Basic' users the following month;
- 90% of 'Premium' remains in the same group. Each 'Premium' user invites on average 3 new 'Basic' users.

Figure 5 – Introductory worksheet to the 3rd branch of the SRP:

Some examples of the kinds of questions that were address in this 3rd branch are the following, which can extend the initially presented QA map (see Figure 6).

$Q_{3.1}$: How can we describe the evolution of the distribution of users in groups under the new conditions of *Lunatic World network*? How to use models based on recurrent equations to address this change in the hypothesis?

$Q_{3.2}$: Is it possible to transform the recurrent sequences of order $d > 1$ into a matrix model? What characteristics does this matrix $\{L\}$ have? How to iterate $X(n+1) = L \cdot X(n)$ to forecast the future distribution of the users after some periods of time? [...]

$Q_{3.3}$: What are the main properties of L^n ? What can we say about $\lim_{n \rightarrow \infty} \{L^n\}$?

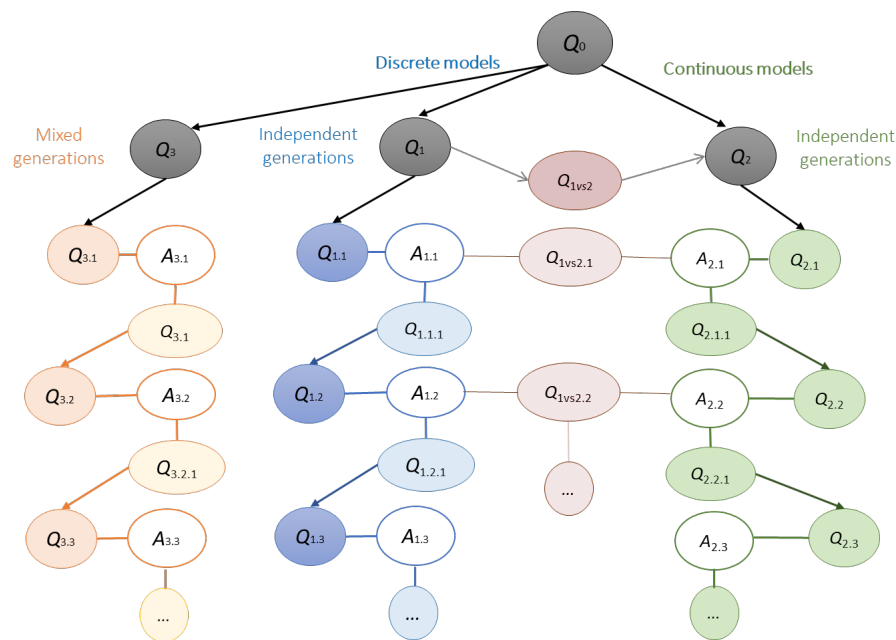


Figure 6 – QA map with the 3rd branch with mixed groups/generations about the social network users

The third case is the SRP3 on *comparing forecasts against reality in the case of Facebook users’ evolution*. At this occasion, the conditions for the implementation conditioned its adaptation, as we could not count on the whole course. On the contrary, it was planned to take place in the transition from the 1st to the 2nd term during Mathematics with first-year students of the Business Administration degree and the Marketing and Digital Communities degree at the Tecnocampus-UPF university. Again, this SRP3 was implemented in the new teaching, the ‘modelling workshop’, created for this implementation, which is offered to students as a voluntary activity outside the regular schedule of the course, adding an extra point to the final grade of the subject. The workshop run over seven sessions of 1h30 and asked for some amount of work that students need to do outside the classroom.

The initial situation begins with the presentation of some selected news about a research project developed by Princeton University, which anticipated that Facebook would lose 80% of their users before 2017. The generating question Q_0 was about: *Can these forecasts be true? How can we model real data about the evolution of Facebook users and forecast the short- and long-term evolution of the social network? How can we validate Princeton conclusions?* In Barquero et al. (2018), it is largely explained the design and implementation of SRP3, which presents three interconnected phases linked to Q_0 . The first phase focuses on the data exploration of data about Facebook users; the second one focused on which mathematical models (based on elementary functions, in this implementation) could provide a good fit for Facebook users’ data and discussion on how to determine and interpret models’ parameters. The third phase focused on how to decide on the ‘best’ and most reliable model fitting data and how to use it or them to provide short-, medium- and long-term forecasts of the development in Facebook user numbers. Although it will not be in this paper further developed, it might be noticed that SRP3 can help to integrate a new branch to the QA map presented in Figures 4 and 6, with models based on regression and interpolation. Although the modelling work developed

in the implementation of SRP3 was mainly based on the use of elementary functions and their derivatives (as it was an official context of the mathematics subject), it does not mean that when analyzing and modelling complex data sets, as the ones of Facebook users, a common path to follow would be towards regression and interpolation. In fact, in other experiences with secondary school teacher education or university teachers' education most of the times the modelling trajectory followed is the ones here considered.

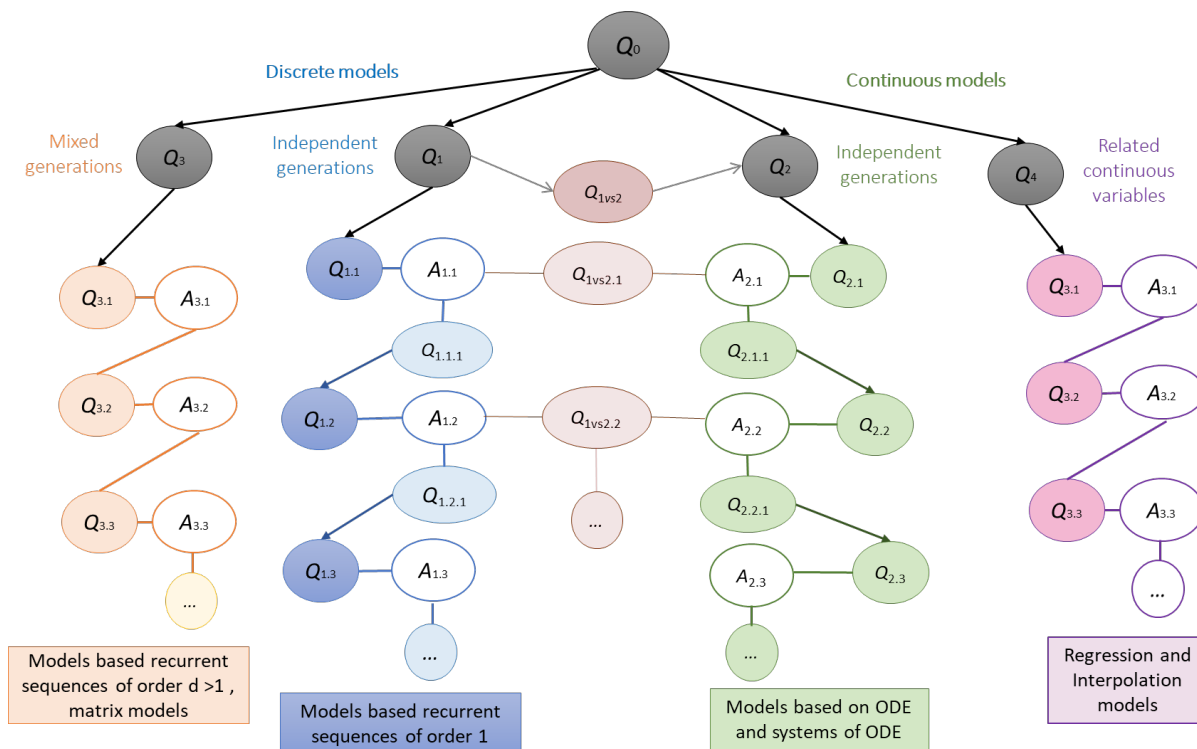


Figure 7 – QA map with the 4th branch with models based on regression and interpolation

6. Results concerning the mechanisms to foster the questions-answers dialectics

With the successive implementation of the SRP, we tested how the sequence of questions arising from the generating question Q_0 about population dynamics (and its variation about social networks). This led the students and the teachers to consider most of the main contents of the entire mathematics course, plus some additions (recurrent sequences, speed of convergence, graphical simulation techniques, relation between the rates of growth and derivatives, transition matrices, etc.). However, during the modelling workshop, these contents appeared in a very different structure from the traditional organisation proposed by the syllabus. Instead of the classical ‘logic of mathematical concepts’, the workshop was more guided by the ‘logic of the questions to deal with’ and ‘types of models to build’ that progressively appeared or temporary answers provided. After analysing these implementations, we can highlight different levers that could facilitate this questions-answers dialectics.

6.1. Devices helping to institutionalise the interplay between questions and answers

The officialization of the questions students dealt with and of the answers they reach was taken as a central task, since the beginning of the SRP. Different teaching devices facilitated this task. The

first one was the *report of the week*. It was a useful device where students explicitly asked to write and formalise the questions they dealt with, the answers obtained, and new questions to follow (according to its structure fixed in advance). After each workshop session the working teams had to write, deliver and (on some occasions) defend their team report. This report had a fixed structure provided by the instructor, in terms of (1) Questions dealt with; (2) Answers found in terms of the modelling process followed (hypothesis assumed, models build, answers obtained by simulation of models, techniques of validation, etc.); (3) New questions emerged from your work; and (4) Resources used. At the beginning of the SRP, students had many difficulties in writing and describing their activity in these terms but thanks to the workshop sessions guided by the instructor, having long courses to set up changes in the didactic contract and the role of the *secretary of the week*, these tasks became easier with the time. More concretely, most of the workshop sessions started by asking the secretary of the week to explain their team report. If necessary, the instructor helped them to formalise the questions and answers that sometimes remained in the shadow.

For instance, in the SRP1, the first sessions, students showed many difficulties in writing the report, they usually came with the final models used, the result of simulating the models, that is, they mainly focused on describing the answers they found if not only a small representation of them. Then, an interesting debate started when the instructor posed some new questions to facilitate the comparison of the different proposals: $Q_{0.1}$ about the hypothesis about rates of growth of the population or about the habitat capacity, or $Q_{0.2}$ about how models proposed derived from the hypothesis assumed, etc. Moreover, the secretary of the week was also responsible for asking other teams for their proposals and summarising them in the workshop. When necessary, other groups participated in the discussion and summarized the work done in the ‘workshop logbook’, which was also described in these terms. Moreover, with the final report, it was asked that each student summarized the whole modelling work developed in each term (or in each SRP branch) by collecting and organising the questions and answers (hypothesis, models, work with models, validation of models...) they had worked with.

In the SRP2 and SRP3, the teaching strategy was different and proposed to combine two kinds of workshop sessions every week: teamwork and presentations. In the first ones, each team had to look for ‘temporary’ responses to partial questions derived from Q and prepare a ‘partial’ report with these responses. Then, the reports were orally defended in the subsequent sessions by some selected working teams. A discussion followed to state what progress had been made and to agree on how to continue the study process. During the presentation sessions, the *secretary of the week* prepared a report containing the main points in the discussion and the new questions proposed to be studied in the following sessions.

6.2. *Terms to talk about, terms to question mathematical modelling*

Since the first workshops’ session, in all the SRPs, one important constraint appears around the necessity of an *ad-hoc* mathematical discourse available to talk about the modelling activity that has just taken place and the results obtained. It was necessary to build a ‘new’ discourse, which was absent for students. That was why it was so important, since the first sessions, that the instructor institutionalize some terms to refer to modelling, such as: system, variables to focus on, hypothesis assumed, models built, models’ simulation, validation of the model, and so on.

For instance, in the initial sessions of the workshop, students were able to present different models but none of them was able to define precisely the variables considered, the hypothesis under which they had built up the models, ways to validate models, the range of validity of the model, etc. The instructor started to set up new words around modelling, which allows students to develop a *didactic technology* (that is, a way to justify and evaluate) the modelling process itself. As already mentioned, one important didactic strategy used by the instructor consisted of selecting different groups and ask them to explain and compare their modelling progresses. This discussion usually started by the ‘secretary of the week’ and was followed by the rest of the working groups. As, in most of the occasions, the working groups come with different modelling processes and models’ proposals, there appeared several questions concerning how to compare their work. Sometimes the real data (of population or of social networks) acted as the main *media* to compare and validate the different proposals. But, on most occasions, this discussion led to questions about the modelling process itself, and students had to open and understand the different proposals according to the modelling process followed: hypothesis formulated, the definition and selection of the models’ coefficients, simulations generated by the models, and so on.

7. Results concerning the mechanisms concerning the *media-milieus dialectics*

In the previous sections, we have already introduced many elements that progressively took part of the students’ *milieu*: the generating question Q_0 , the derived questions Q_i , temporary answers A_i , previous knowledge of students that appear as tools to progress on the SRP, etc. And, different *media* that helped that all these elements were progressively integrated in students’ *milieu*, such as the instructor of the workshop, the worksheets prepared, the students’ presentation, the ‘secretary of the week’, among others. But they were not the only ones, mainly because the workshop did not run out isolated of other university teaching devices, either because students were asked to work autonomously outside the workshop. In fact, when students and teachers experienced the SRP, there appeared different moments when the existent *milieu* was not rich enough and there appeared new necessities that required stopping and extending students’ *milieu* with new elements. As Kidron et al. (2014) underline “the media-milieus dialectics appears when considering the different kinds of general didactic gestures performed by students and teachers in the interaction with M (the *milieu*) to produce A^\heartsuit (the final answer)”. We focus now on underlying some basic didactic gestures that were developed during the implementation of the SRP that provoked the expected progressive extension of the *milieu*.

7.1. Integration of the SRP with other devices at university level

We could detect a rich interaction between the running SRP and the activities that were organised according to the necessities opened. Indeed, in the implementations of the different SRP, the workshop was year after year more and better integrated with the rest of the university teaching devices. Lectures and exercise-problems’ sessions were, on several occasions used, to provide students with some of the necessary tools to follow the work developed in the modelling workshop.

For instance, concerning the 1st branch of SRP1 about discrete models for populations with independent generations, the regular course usually started with a pre-established syllabus contents about the study of elementary functions (linear, quadratic, exponential, etc.), also with the resolution of equalities and inequalities by algebraic and graphical techniques. But very fast there appeared in

the workshop some questions that asked for the introduction of some mathematical knowledge not included in the official curricula. Then, the usual way to proceed was to stop and plan in the lectures and exercise-problems’ sessions some time to introduce this new knowledge. For instance, when there was not a clear definition of the different rates of growth or when it was necessary to the definition of recurrent sequences (in SRP1, 1st branch) or about matrix diagonalization (SRP2, when simulating and looking at the properties of L^n). Then, the instructor guiding the workshop took a break on the workshop to introduce and work on this needed knowledge and tools for students to be able to address these new questions, planning as many sessions as necessary of all the teaching devices.

In this sense, an important condition for modelling was to break the rigidity of the classical structure ‘lectures-problem sessions-exams’, based on the sequence introducing new contents-applying the contents. On the contrary, it was important to ensure that both lectures and problem sessions were considered during the workshop, and that they contribute to the evolution of the workshop in two directions. On the one hand, we could find the situation that a previously introduced answers A_i^\diamond , or a mathematical work/object (W_j) had been previously introduced to students in a more ‘transmissive’ way (for instance, in the lectures) and it is part of the milieu to be put into use during the SRP. On the other hand, along the SRP, some derived questions Q_i or answers A_i could in the workshop calling for the introduction of new A_i^\diamond or W_j . As a result, the ideal situation was to maintain a bidirectional relationship between all the existing didactic devices. In the several implementations of SRP1, SRP2 and SRP3, ‘lectures and problem sessions’ were used to provide students with some of the necessary tools to be able to follow with the workshop. And, vice versa, the workshop motivates and to shows the functionality of the main content of the course.

7.2. Making accessible external answers by enlarging the media

It was not only on the frame of the workshop itself or in the classroom university, from where elements for the *milieu* came from. Along the workshop, students were asked to look outside for possible external answers to the questions they were addressing and (if pertinent) bring them to the workshop. In several implementations, since the beginning of the workshops, they looked for what exist outside, what experts know and say about the phenomena (on dynamic population or in the evolution of social networks or of Facebook, in the SRP cases described in this paper). Students were asked to bring to the class some information about the phenomena we were analysing or about the models they were proposing. It was in fact an important part of the *weekly reports* and of the common debate between the working groups. They were asked to share what pre-existing answers students had found A_i^\diamond , which could seem useful for the SRP. Thanks to this, we could give name to the models (such as the Malthusian or logistic models, in SRP1, or to the transitions matrix models, in SRP2). This gesture needed to be followed by an accurate study about how to decompose and build up these external answers to be incorporated in the SRP dynamics.

Other important mechanism was the interaction of experts on other disciplines. In the case SRP3, as explained in Barquero et al. (2018) since the first phase of this SRP, students had to look for many data about Facebook users’ evolution (instead of being provided by the instructors). The main *media* used were their consultation in websites and the interaction with experts. For instance, at this occasion, a course called ‘Introduction to digital communities’ dedicated some classroom sessions, in their course, to letting students know what the main webpages were to consult for social

network data, different ways to organize data and about pre-existing models that experts had use to look at the evolution of networks.

We may mention some other important mechanisms that exist in all the SRP to enlarge the sources of information (media) accessible to students. An important step in modelling processes is the simulation of models to generate new data by using models to contrast with the real one. Here there was an important role of the simulators, which were made accessible to students, and for which a specific training was planned during the course. In SRP1 and SRP2 the main technological environment was Excel. In the case of SRP3, the design of new applets that could act as media for models' simulation or models contrast or validation, was of special importance at this occasion, as researchers were collaborating with technology developers (Cinderella, Geogebra, etc.). Hence, the lecturers suggested that the students could use other tools (such as a spreadsheet or specific mathematics and statistics software) to simulate and test their proposals.

Finally, as already mentioned, was of importance the (parts of or full) sessions that were devoted to the presentation of the working teams, with their weekly reports. All teams could act, as well as the instructor of the workshop, as media for the rest of the class. In this sense, the secretary of the week, had the rich but difficult task of reporting on her working teamwork and on the advances of the different teams, and to make accessible this information for all.

6.1. Conclusion and discussion

During the past decades, our research group have been working of the design and analysis of several SRP at university level to promote the teaching and learning of mathematical modelling (see for an overview in Barquero et al., 2022). Thanks to this research line, we have detected many constraints that hinder their implementation, but also many desirable conditions to step-by-step progress on introducing changes in the largely extended 'paradigm of visiting works' in the museum of university mathematics. This paper has selected some of the initially implemented SRP (from 2005/06 to the following years) at university level to show what *mechanisms* have been useful to favour mathematical modelling to be integrated in first-year mathematics courses and to play an important role to break the rigidity of mathematics programmes and its teaching organization.

After providing some characteristics of the SRP, and of the methodology for their design and analysis (sections 2 and 3), we have focus on the SRP on population dynamics to stress the importance and utility for researchers (and for lecturers) of the *a priori* analysis of the SRP. On the one hand, to enquire into the potential of the generating questions Q_0 and to trace the possible path to be followed in the effective implementation of an SRP. On the other hand, to provide researchers and teachers with alternative epistemological models to describe mathematical activity, which is usually described following the logic of the mathematics contents, to be now described in terms of the interplay between questions, mathematical models, and possible answers. This *a priori* design has led us to focus on two central dialectics for the SRP: the questions-answers and the media-milieus dialectics.

Furthermore, these initial designs have been selected not only for their consecutive implementation in one university context (the one of first-year mathematics courses for natural sciences degrees), but also for their 'migration' and adaptations into other university context (those of mathematics for business administration university degrees). After our experience with this SRP, that have migrated from one institution to another, we have been interested in the 'ecological'

invariants, that is, the conditions (also the constraints) that, independently of a change of institution, have brought light on some mechanisms that have helped that mathematical modelling take part of the regular university courses. Then, not only the designs could be of interest for possible future adaptation to other university context, but also the result about the mechanisms that have fostered questions-answers dialectics and the media-milieu dialectics. We expect to open the discussion in, at least, these two main directions. On the one hand, about the transferability of the mathematical-didactic infrastructures (as the ones synthesised in section 5). On the other hand, about the transferability of the different mechanisms to enhance the different dialectics for a full development the SRP. Both complementary aspects would contribute to the analysis of the ecology of the SRP and of the teaching and of mathematical modelling at university level.

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