# Carlitz operators and higher polylogarithm identities

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# CARLITZ OPERATORS AND HIGHER POLYLOGARITHM IDENTITIES

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Abstract. We study a higher dimension generalization of Carlitz's polynomials first introduced by Papanikolas and compute an $\infty$-adic limit of a sequence of normalizations, relating it to the exponential function of an Anderson module. We further describe some factorization properties generalizing recent results in the dimension one case, see [17, 4]. The factorizations we consider are in certain non-commutative algebras of operators. We present, as an application, how to deduce explicit non-trivial linear dependence relations connecting polylogarithms associated to the Carlitz's module and powers of the Carlitz fundamental period. While these identities are considered as well known and certainly central in this theory, the fact that they can be recovered from factorization properties of higher dimension generalization of Carlitz's polynomials is surprising.

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## 1. Introduction

Let $A$ be the ring of rational functions over a projective curve $X$ over $\mathbb{F}_q$ which are regular away from an $\mathbb{F}_q$-rational point $\infty$. Denote by $K$ its fraction field, $K_\infty$ its completion at the place corresponding to $\infty$, and by $L$ a separable closure of $K_\infty$. Later in this paper we will choose $X = \mathbb{P}^1$ with an $\mathbb{F}_q$-rational point $\infty$, but it is good to start the discussion without this restriction.

The equivalence of small categories, Drinfeld $A$-modules over $L$ on one side, and $A$-lattices of $L$, that is, projective finite rank and discrete $A$-modules of $L$ on the other side, is crucially founded over the existence of Weierstrass product expansions for the corresponding exponential functions. Given a Drinfeld module $\phi$, by the fact that its exponential function $\exp_\phi$ is an entire function in the rigid analytic sense, it is uniquely determined by a Weierstrass product expansion ([1])

$$\exp_\phi(z) = z \prod_{\lambda \in \Lambda_\phi}{}' \left(1 - \frac{z}{\lambda}\right)$$

running over its kernel $\Lambda_\phi$ which therefore carries a structure of projective finite rank $A$-module, the period $A$-lattice of $\phi$. This well known one-dimensional dynamic is very helpful in handling the so-called power sums, Carlitz zeta values, Thakur multiple zeta values, or Anderson-Thakur's polylogarithms at one, and Chang's multiple polylogarithms at one, and allied structures. Note that in the simplest case of $X = \mathbb{P}^1$ and $\infty = (1 : 0)$, when there exists a regular function $\theta$ such that $A = \mathbb{F}_q[\theta]$, multiple zeta values and multiple polylogarithms at one are two ways to expand the same elements, as one of the consequences of Ngo Dac's seminal paper [8].

Drinfeld modules are dimension one *Anderson $A$-modules*. A higher dimensional correspondence between Anderson $A$-modules and $A$-lattices embedded is suitable spaces being missing, there seem to be very few tools developed to factorize higher dimension exponential functions.

The present paper stems from the wish of factorizing the exponential function of the $n$-th tensor power $C^{\otimes n}$ of *Carlitz $A$-module* $C$ ([2]). Here $X = \mathbb{P}^1/\mathbb{F}_q$ and $\infty$ is an $\mathbb{F}_q$-rational point. Hence $A = \mathbb{F}_q[\theta]$ where $\theta$ is a regular function with a simple pole at $\infty$.

We recall that $C$ is a function field analogue of $\mathbb{G}_m$ and in this analogy $C^{\otimes n}$ becomes a very relevant object. In the case of Carlitz's module $C$ itself there is a type of 'non-commutative' factorization emerging (see the formula (1.2) below), that turns out, quite surprisingly, to partially generalize to this higher dimensional setting. This has also a generalization for Drinfeld $A$-modules of rank one such that $X$ has an $\mathbb{F}_q$-rational point $\infty$, in connection with [4]. We observed in [17, Proposition 4.4.9] that this 'non-commutative'

[1] the dash $'$ indicates that 0 is omitted.

[2] $C$ is the simplest of all Drinfeld modules.

product expansion can be obtained independently of the Weierstrass product expansion of Carlitz's exponential $\exp_C$ through a different process, as a limit of normalizations of *Carlitz's polynomials* (see [6, Definition 3.5.1]). A class of functional equations for the Carlitz polynomials $E_{C,k}$ (see [17, Proposition 4.4.8]) can be used in replacement of the Weierstrass product expansion

$$\exp_{\widetilde{C}}(z) = z \prod_{a \in A}{}' \Big(1 - \frac{z}{a}\Big), \tag{1.1}$$

for the exponential associated to a certain Drinfeld module of rank one $\widetilde{C}$ isomorphic to $C$ (normalization of $C$) over $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$, to obtain the factorization

$$\exp_{\widetilde{C}} = \cdots \Big(1 - \lambda_{C,k}^{q-1}\tau\Big) \cdots \Big(1 - \lambda_{C,1}^{q-1}\tau\Big)\Big(1 - \lambda_{C,0}^{q-1}\tau\Big). \tag{1.2}$$

The coefficients $\lambda_{C,k}$ are related to Carlitz's logarithm $\log_C$ in the following way:

$$\log_C = \sum_{k \geq 0} \lambda_{C,k}\tau^k \in K[[\tau]],$$

while $\exp_{\widetilde{C}}$ denotes the element of $K_\infty[[\tau]]$ corresponding to the exponential in (1.1). In this perspective, the family $(E_{C,k})_{k\in\mathbb{Z}}$ constituted by these Carlitz's polynomials can be viewed as a $\mathbb{Z}$-counterpart of the family $(C_a)_{a\in A}$ of the operators multiplication by $a$ for various $a \in A$, for the Carlitz module structure, emulating the dycotomy between Carlitz's zeta values, which are series over monic polynomials of $A$ on one side, and polylogarithms associated to Carlitz's module, which are series over positive elements of $\mathbb{Z}$, on the other side.

To give a concrete example of this phenomenology we compare the coefficients of $\tau$ in both sides of (1.2). In the left-hand side we find $\epsilon_{C,1}\widetilde{\pi}_C^{q-1}$ (with $\exp_C = \sum_i \epsilon_{C,i}\tau^i \in K_\infty[[\tau]]$). It is easy to compute $\epsilon_{C,1} = (\theta^q - \theta)^{-1}$. In the right-hand side we find the series $-\sum_{k\geq 0}\lambda_{C,k}^{q-1}$. This is minus one times the evaluation at one of a weight $q-1$ analogue of polylogarithm. It is well known that $\lambda_{C,k}^{q-1} = \sum_a a^{1-q}$, where the sum runs over the monic polynomials of $A = \mathbb{F}_q[\theta]$ of degree $k$ ($\theta$ is as described earlier). Therefore we obtain an identity involving a *Carlitz zeta value*, namely:

$$\zeta_A(q-1) := \sum_{\substack{a\in A\\ \text{monic}}} a^{1-q} = \sum_{k\geq 0}\lambda_{C,k}^{q-1} = \frac{\widetilde{\pi}_C^{q-1}}{\theta - \theta^q}. \tag{1.3}$$

This is Carlitz's analogue of Euler's famous 1734 identity $\zeta(2) = \frac{\pi^2}{6}$.

A higher dimension generalization of Carlitz polynomials, associated to Anderson's $A$-module $C^{\otimes n}$ has been introduced and discussed by Papanikolas in his monography [9], at the moment unpublished. This is our starting point.

1.1. **Results.** From now on we suppose that $A = \mathbb{F}_q[\theta]$. Let

$$\exp_{C^{\otimes n}} = \sum_{k\geq 0} \epsilon_{C^{\otimes n},k}\tau^k \in 1 + \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]\tau$$

be the exponential of $C^{\otimes n}$, with $\tau$ the $\mathbb{F}_q$-linear endomorphism that maps a point $x = {}^t(x_1,\dots,x_n)$ of $\mathbb{G}_a^n$ to ${}^t(x_1^q,\dots,x_n^q)$. We take the opportunity to point out a notational standard of this paper. The symbol 1 represents any multiplicative unit; notably the unit of $K$, the unit of $\mathrm{End}_L(\mathbb{G}_a^n)$ etc.

Fixing the standard basis of $\mathbb{G}_a^n$ the endomorphism $\epsilon_{C^{\otimes n},k}$ is represented by the left multiplication by the matrix $Q_k \in K^{n\times n}$ defined in [10, §2.1], for all $k \geq 0$. Let

$$\log_{C^{\otimes n}} = \sum_{k\geq 0} \lambda_{C^{\otimes n},k}\tau^k \in 1 + \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]\tau$$

be its local inverse near the origin of the metric vector space $L^{\oplus n}$. The matrix representing $\lambda_{C^{\otimes n},k}$ is denoted by $P_k \in K^{n\times n}$ in [10]. Papanikolas expands, for $X \in \mathrm{End}_L(\mathbb{G}_a^n)$:

$$\exp_{C^{\otimes n}} X\log_{C^{\otimes n}} = \sum_{k\geq 0} E_{C^{\otimes n},k}(X)\tau^k \in X + \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]\tau \tag{1.4}$$

(this can be done for every Anderson $A$-module) and this is his generalization of Carlitz polynomials $E_{C^{\otimes n},k}$, evaluated at $X$. Viewing Papanikolas' construction, it is also natural to identify $E_{C^{\otimes n},k}$ with an element of

$$\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[\boldsymbol{\tau}]$$

see our §2, with the caveat that now $\boldsymbol{\tau}$ is not an $\mathbb{F}_q$-linear endomorphism of $\mathbb{G}_a^n$, but rather an $\mathbb{F}_q$-linear endomorphism of a ring of endomorphisms. Note that the case $n = 1$ reduces to the well known Carlitz polynomials, see [6, Definition 3.5.1]. The next step is, following Carlitz, to study the convergence properties at the place $\infty$ of the sequence $(E_{C^{\otimes n},k})_{k\geq 0}$.

An explicit formula [9, (4.3.1)] obtained by Papanikolas for the matrix coefficients $P_k$ of $\log_{C^{\otimes n}}$ and reviewed in §7.2 is well suited to do this. It allows to show that for all $k$, $\lambda_{C^{\otimes n},k}$ is an $L$-automorphism of $\mathbb{G}_a^n$ (see Proposition 4.1). This allows to construct the normalization

$$\mathcal{E}_{C^{\otimes n},k} = P_k^{-1}E_{C^{\otimes n},k} \in 1 + \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[\boldsymbol{\tau}]\boldsymbol{\tau}$$

(see (4.4)) (so, in conformity with our conventions, 1 here denotes the identity endomorphism of $\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n)))$ and where $\boldsymbol{\tau}$ denotes the unique $\mathbb{F}_q$-linear endomorphism of $\mathrm{End}_L(\mathbb{G}_a^n)$ such that for all $f \in \mathrm{End}_L(\mathbb{G}_a^n)$, $\boldsymbol{\tau}(f) = \tau f\tau^{-1}$ (note that we distinguish $\boldsymbol{\tau}$ from $\tau$, which is an endomorphism of $\mathbb{G}_a^n$). We show (more details in Theorems 4.2 and 5.2):

**Theorem A.** *The sequence $(\mathcal{E}_{C^{\otimes n},k})_{k\geq 0}$ converges, $\infty$-adically, to a well defined element $\mathcal{E}_{C^{\otimes n}}$ of*

$$1 + \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[[\boldsymbol{\tau}]]\boldsymbol{\tau}.$$

*This element is the exponential function of a uniformizable Anderson $A$-module defined over $K_\infty$ which is the direct sum of $n$ summands, each one $K_\infty$-isomorphic to $C^{\otimes n}$. One of these*

*summands is the Anderson $A$-module $\widetilde{C^{\otimes n}}$, the period lattice of which is the normalization ([3]) of the period lattice of $C^{\otimes n}$.*

Nathan Green ([4]) independently found similar results with a different approach. He uses a pairing theory that he started to develop in [14] to construct another normalization for the operators $E_{\phi,k}$. More precisely, he normalizes the restriction to $\mathcal{Z}$ of the operators $E_{\phi,k}$ (while us, we restrict to $\mathcal{Z}$ the normalization of $E_{\phi,k}$, and the two ways to normalize are different so they give rise to different sequences of functions). Then, he applies a limit process similar to ours to deduce an explicit connection to a function different from $\mathcal{E}_\phi$, yet related to the exponential of $\phi$.

Our §5 contains a description of the structure of these Anderson modules. Up to our knowledge, ours is the first attempt of studying this kind of phenomena for higher dimension Anderson $A$-modules. The period lattice of the Anderson $A$-module $\widetilde{C^{\otimes n}}$ is the discrete $\mathbb{F}_q$-vector space:

$$\left\{ \begin{pmatrix} \mathcal{D}_{n-1}(a) \\ \vdots \\ \mathcal{D}_1(a) \\ a \end{pmatrix} : a \in A \right\} \subset \mathbb{G}_a^n(L). \tag{1.5}$$

Here we denote by $\mathcal{D}_j$ the $j$-th higher divided derivative, that is, the unique $\mathbb{F}_q$-linear endomorphism of $K_\infty$ determined by $\mathcal{D}_j(\theta^i) = \binom{i}{j}\theta^{i-j}$. One can see that the above $\mathbb{F}_q$-vector space carries the structure of an $A$-module of rank one generated by the last element of the standard basis of $\mathbb{G}_a^n$.

1.2. **Non-commutative factorizations.** The next question that we ask is if it is possible to extend the non-commutative product expansion (1.2) in the case $A = \mathbb{F}_q[\theta]$ for $\mathcal{E}_{C^{\otimes n}}$, $n > 1$ (at the moment of writing this paper there is no known Weierstrass', i.e. 'commutative' product expansion). Although not as simple as (1.2), we have results of this kind. To state them we first need to choose an embedding of $\mathrm{Lie}(C^{\otimes n})$ in $\mathrm{End}_L(\mathbb{G}_a^n)$. We can connect $\mathrm{Lie}(C^{\otimes n}) \cong \mathbb{G}_a^n$ (as $\mathbb{F}_q$-vector spaces), the source module of $\exp_\phi$, and $\mathrm{End}_L(\mathbb{G}_a^n)$, the source module of $\mathcal{E}_\phi$ by means of the following map

$$\mathbb{G}_a^n(L) \xrightarrow{h} \mathrm{End}_L(\mathbb{G}_a^n)$$

[3]'Normalizing an object' loosely means choosing another object in its isomorphism class so that certain canonical properties are satisfied. The 'normalization of an object', if it exists, is therefore a canonical element in its isomorphism class. For instance, normalizing an $A$-lattice $\Lambda$ is usually seen as a process that brings it, by applying an automorphism of the metric space $V$ in which it is embedded, to another lattice containing an isomorphic copy of the coefficient ring $A$, embedded, itself as a lattice, in the endomorphism ring of $V$. This is particularly meaningful when $\Lambda$ is the kernel of the exponential function of some commutative algebraic group, or of an Anderson $A$-module, when in parallel, one is also sometimes led to normalize these objects. We can also speak about normalization of exponential functions etc..

[4]Personal communication, April 2023.

defined by

$$h : \begin{pmatrix} x_{n-1} \\ \vdots \\ x_1 \\ x_0 \end{pmatrix} \mapsto \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ 0 & x_0 & \cdots & x_{n-3} & x_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x_0 & x_1 \\ 0 & 0 & \cdots & 0 & x_0 \end{pmatrix}. \tag{1.6}$$

Let $\mathcal{Z}$ be the image of the map $h$ defined in (1.6). It is a commutative $L$-algebra. We have (see Theorem 6.5):

**Theorem B.** *There exist two sequences $(\widetilde{\gamma}_k)_{k\geq 0}, (\widetilde{\delta}_k)_{k\geq 0}$ of elements of $\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))$ such that, identifying $\mathcal{E}_{C^{\otimes n}}$ with an element of $\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[[\boldsymbol{\tau}]]$, and then, comparing restrictions of evaluations over $\mathcal{Z}$:*

$$\mathcal{E}_{C^{\otimes n}}|_{\mathcal{Z}} = \cdots \left(\widetilde{\gamma}_k + \widetilde{\delta}_k \boldsymbol{\tau}\right) \cdots \left(\widetilde{\gamma}_0 + \widetilde{\delta}_0 \boldsymbol{\tau}\right)\Big|_{\mathcal{Z}}. \tag{1.7}$$

The factors $\widetilde{\gamma}_k + \widetilde{\delta}_k \boldsymbol{\tau}$ are explicitly described in the paper. Note that the two sequences of the Theorem are not uniquely determined. We have other factorization results in this paper (see Corollary 6.2) but the above is the more suitable for applications. We do not know if in general the factorization extends to the full space $\mathrm{End}_L(\mathbb{G}_a^n)$, outside the case $n = 1$.

The study of the evaluations of the operators $E_{C^{\otimes n},k}$ outside $\mathcal{Z}$ is rather difficult. We have another result, Theorem 6.1, where we consider a factorization of certain operators of

$$\mathrm{End}_L(\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n)))[\boldsymbol{\tau}]$$

that evaluate to $E_{C^{\otimes n},k}$. In the case $n = 1$, $\mathcal{Z} = \mathbb{G}_a$ and Theorem B reduces to (1.2) with $\widetilde{\gamma}_k = 1$ for all $k$.

Some properties of the operators $\widetilde{\gamma}_k$ and $\widetilde{\delta}_k$ are collected in Proposition 7.1. Our results contain a family of scalar identities with parameter $n \geq 1$, so that the case $n = 1$ corresponds to (1.3). In the above results, comparing the coefficients of powers of $\boldsymbol{\tau}$, we obtain endomorphisms and matrix identities. The case $n = 1$ is already considered in [17, Remark 4.4.11] but, in the case $n > 1$, to establish a connection with the theory of function field multiple polylogarithms and multiple zeta values, it is desirable to extract from these identities, scalar identities. By no means this is an easy task, and almost systematically, the computation of selected coefficients in these matrices yield challenging computational problems. As a consequence of an evaluation of a certain identity between elements of $\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))$, we can prove the following result (see Theorem 7.4); to present it we need a few notation.

The *polylogarithm of order $s \geq 1$ at $x \in L$ associated to Carlitz's module*, introduced and studied in [10, §2.1], is defined by the series

$$\mathrm{Li}_s(x) = \sum_{k\geq 0} l_k^{-s} x^{q^k} \in L, \quad x \in L,$$

where $l_k = (\theta - \theta^q)\cdots(\theta - \theta^{q^k}) \in A$. Convergence is ensured if $|x| < |\theta|^{\frac{sq}{q-1}}$ with $|\cdot|$ the multiplicative valuation that corresponds to the $\infty$-adic valuation of $K$, see [10, Proposition/Definition 2.4.3]. In particular, if $s = n(q-1)$ with $n \geq 1$, the series $\mathrm{Li}_s(\theta^{qi})$ converges for all $0 \leq i \leq n-1$.

**Theorem C.** *For all $n \geq 1$ there exists an explicitly computable sequence of elements $c_{n,0}, c_{n,1}, \ldots, c_{n,n-1}$ in $A$ such that*

$$\sum_{i=0}^{n-1} c_i \,\mathrm{Li}_{n(q-1)}(\theta^{qi}) = \frac{\widetilde{\pi}_C^{n(q-1)}}{\theta - \theta^q}.$$

Theorem 7.4 is much more precise. The sequence $c_{n,0}, c_{n,1}, \ldots, c_{n,n-1} \in A$ depends on the choice of $n$ and can be computed in non-recursive way (that is, directly). The recipe we give for the computation of these coefficients $c_{i,j}$ is in fact very simple and rests on the expansion of polynomials in three variables denoted by $S_n$ defined in (7.3). With a computer, tabulations of such relations can be described for quite large values of $n$, depending on how explicit are our expansions of the polynomials $S_n$. It is interesting that these polynomials 'do not depend on $q$' in the sense that they are just reductions modulo $p$, the characteristic of $\mathbb{F}_q$, of corresponding polynomials of $\mathbb{Z}[X, Y, Z]$.

This result is essentially derived from Theorem B by first comparing the evaluations of the coefficients of $\boldsymbol{\tau}$ (these coefficients are in $\mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))$) at the endomorphism $N^{n-1}$, where the endomorphism represented by $\theta + N$ gives the multiplication by $\theta$ in the $A$-module structure of $\mathrm{Lie}(C^{\otimes n})$. This can be expressed as an identity of two matrices, one of them with entries in $K(\Pi)$. Projecting on the uppermost left coefficient yields Theorem C. There is a connection between this result and recent results of Gazda and Maurischat's on the motivic cohomology of Carlitz twists, in a manuscript [13] that they shared with the author of the present paper.

It is easy to verify that, in the case $n = 1$, we are just rewriting (1.3), or more appropriately, in terms of polylogarithms (this theory does not see zeta values!),

$$\mathrm{Li}_{q-1}(1) = \frac{\widetilde{\pi}_C^{q-1}}{\theta - \theta^q}. \tag{1.8}$$

Of course, this implies the simplest Carlitz zeta value identity, analogue of Euler's $\zeta(2) = \mathrm{Li}_2(1) = \frac{\pi^2}{6}$.

The table (7.8) allows to describe more explicit formulas, let un analyze some of them. If $n = q^s$ with $s \geq 0$ (the case $s = 0$ is described in (1.8)) the relation of Theorem 7.4 is

$$(\theta - \theta^q)^{q^s - 1}\,\mathrm{Li}_{q^s(q-1)}(1) = \frac{\widetilde{\pi}_C^{q^s(q-1)}}{\theta - \theta^q}.$$

Dividing both sides by $(\theta - \theta^q)^{q^s-1}$ implies

$$\mathrm{Li}_{q^s(q-1)}(1) = \frac{\widetilde{\pi}_C^{q^s(q-1)}}{(\theta - \theta^q)^{q^s}}, \tag{1.9}$$

but this is again (1.8) after raising both sides to the power $q^s$ (sometimes, these relations obtained by simply raising both sides of an identity to a power $p^f$ where $p$ is the characteristic of $\mathbb{F}_q$ are called 'trivial relations').

If $n = 2$ we have (see table (7.8)) $S_2(\theta, Y, \theta^{(1)}) = \theta + \theta^q - 2Y$. Hence the relation of Theorem 7.4 becomes, in this case:

$$(\theta + \theta^q)\operatorname{Li}_{2(q-1)}(1) - 2\operatorname{Li}_{2(q-1)}(\theta^q) = \frac{\widetilde{\pi}_C^{2(q-1)}}{\theta - \theta^q}. \tag{1.10}$$

If $q$ is a power of 2 we are back to (1.8) after an appropriate exponentiation. Otherwise we have two non-vanishing terms in the left-hand side. The obtained identity is known and easily deducible, alternatively, from power sums identities and their generating series and Euler-Carlitz formulas for zeta values. To see this apply the formula of Thakur in [18, §3.4.3]. It implies that

$$\sum_{\substack{a\in A\\ \text{monic}\\ \deg_\theta(a)=d}} a^{-2(q-1)} = \frac{1}{l_d^{2(q-1)}}\left(1 - \frac{(q-2)(\theta^{q^d} - \theta)^q}{\theta^q - \theta}\right) = \frac{1}{l_d^{2(q-1)}}\left(\frac{\theta + \theta^q}{\theta - \theta^q} - 2\frac{\theta^{q^{d+1}}}{\theta - \theta^q}\right), d \geq 0.$$

Hence

$$\zeta_A(2(q-1)) = \sum_{\substack{a\in A\\ \text{monic}}} a^{-2(q-1)} = \frac{1}{\theta - \theta^q}\Big((\theta + \theta^q)\operatorname{Li}_{2(q-1)}(1) - 2\operatorname{Li}_{2(q-1)}(\theta^q)\Big).$$

In [16] a theory of $L$- and $\zeta$-values in Tate algebras has been started and the following formula

$$\zeta_A(1;\chi) := \sum_{\substack{a\in A\\ \text{monic}}} a^{-1}\chi_t(a) = \frac{\widetilde{\pi}_C}{(\theta - t)\omega(t)}$$

proved, where $\chi : A \to \mathbb{F}_q[\theta]$ is the unique $\mathbb{F}_q$-algebra map extending $\theta \mapsto t$, where

$$\omega := (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0}\left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \in \mathbb{T}^\times \cap L[[t]] \tag{1.11}$$

is *Anderson-Thakur's function* introduced in [10, Lemma 2.5.4] (the formula holds in a Tate algebra). This formula, twisting, and appropriate evaluation yields

$$\zeta_A(2(q-1)) = \frac{\widetilde{\pi}_C^{2(q-1)}}{(\theta - \theta^q)^2}$$

which matches with (1.10) (in fact one needs a variant of these functions in two variables). This is elementary and also follows from various elementary processes of obtaining an Euler-Carlitz zeta identity for $\zeta_A(n)$ with $q-1 \mid n$ but we described what we believe is the easiest way to obtain it. Although one can argue that these formulas are in principle well covered by corresponding Euler-Carlitz zeta identities it is undoubtedly surprising the way our method at once does not recognize any role to Carlitz zeta values, and yields rather

simple relations, essentially completely explicit, once the coefficients of the polynomials $S_n$ and a specialization are computed.

A simple consequence of Theorem C is the following, with $\psi = C^{\otimes n(q-1)}$, so that $\exp_\psi, \log_\psi$ are respectively the exponential and the logarithm of $\psi$, and $\psi_a$ denotes the multiplication by $a \in A$ for the module structure associated to $\psi$. We set

$$T_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta^{qi} \end{pmatrix}, \quad i = 0, \dots, n-1.$$

**Corollary D.** *The point $\sum_{i=0}^{n-1} \psi_{c_{n,i}}(T_i) \in \psi(A)$ is a $\theta - \theta^q$-torsion point.*

*Proof.* The deduction from Theorem C is rather standard thanks to Yu's analogue of Schneider-Lang criterion (see more specifically [19, Theorem 2.3]) but we prefer to give all details because in this way the reader can see how beautifully the module $C^{\otimes n}$ interfere with $C^{\otimes n(q-1)}$. The points $T_i$ are in the domain of convergence of $\log_\psi : \psi(L) \to \mathrm{Lie}(\psi)(L)$. Note that as $\mathbb{F}_q$-vector spaces, we have $\psi(L) \cong \mathrm{Lie}(\psi)(L) \cong \mathbb{G}_a^{n(q-1)}(L)$. Setting $t_i := \log_\psi(T_i)$ we can write

$$t_i = \begin{pmatrix} * \\ \vdots \\ * \\ \mathrm{Li}_{n(q-1)}(\theta^{qi}) \end{pmatrix}, \quad i = 0, \dots, n-1,$$

where the $*$ denote certain elements of $K_\infty$ that we do not try to compute (and that are likely to be arithmetically very interesting). Clearly $\exp_\psi(t_i) = T_i$ for all $i$. There exists an element $\lambda \in \mathrm{Ker}(\exp_\psi) \setminus \{0\}$ such that

$$\lambda = \begin{pmatrix} * \\ \vdots \\ * \\ \widetilde{\pi}^{n(q-1)} \end{pmatrix}.$$

Now with $d_\psi(a) \in \mathrm{End}_A(\mathbb{G}_a^{n(q-1)})$ representing the multiplication by $a \in A$ in $\mathrm{Lie}(\psi)(L)$, by Theorem C, the element

$$\gamma := d_\psi(\theta - \theta^q)\xi_n - \lambda$$

satisfies $\pi_n(\gamma) = 0$ with $\pi_n$ the projection on the last coordinate. But $\exp_\psi(\gamma) = \exp_\psi(d_\psi(\theta - \theta^q)\xi_n) \in \psi(A)$. If $\gamma \neq 0$, [19, Theorem 2.3] implies that $\pi_n(\gamma)$ is transcendental over $K$ which is false, as by construction, it is zero. Hence $\gamma = 0$. This means that $\exp_\psi(\xi_n)$ is a $(\theta - \theta^q)$-torsion point in $\psi(A) = C^{\otimes n(q-1)}(A)$. □

The simplest evidence of this phenomenon occurs when $q = 2$. Indeed $1 \in C(A) = C^{\otimes 1(2-1)}(A)$ is a point of $(\theta - \theta^2)$-torsion, read [10, Remark after Corollary 3.8.4]. Hopefully, forthcoming works of the author will tackle problems related to Eulerian multiple zeta

values and $v$-adic analogues as described in [12, 3]. The author owes these suggestions to Matt Papanikolas.

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**Remark 1.1.** In the following we will often use expressions like 'computing an endomorphism'. The author is aware of the fact that these locutions are not precise enough. Computing an endomorphism or a matrix loosely means finding an easy to handle expression in terms of the standard basis of $\mathrm{End}_L(\mathbb{G}_a^n)$. With an explicit choice of $n$, this is equivalent to give explicit coordinates in the standard basis. If $n$ is not fixed there is the need of discussing the issue of computability, as usual in this kind of problems. Also, for the sake of clarity, and by the fact that the construction of $C^{\otimes n}$ really depends on the choice of the standard basis.

## 2. Carlitz's operators

Papanikolas' generalization of Carlitz's polynomials for $\phi = C^{\otimes n}$ in [9] amounts to certain functions defined over the additive group $\mathcal{Z} \subset \mathrm{End}_L(\mathbb{G}_a^n) \cong \mathrm{Hom}_L(\mathbb{G}_a^n, \mathbb{G}_a)^{\oplus n}$ image of the map (1.6). We propose here an axiomatic viewpoint which can be of independent interest, after some elementary setup and some choice of terminology.

We call the elements of the $L$-algebra $\mathrm{End}_L(\mathbb{G}_a^n)$ *$L$-linear endomorphisms.* The $L$-vector space $\mathrm{End}_L(\mathbb{G}_a^n)$ is a left $\mathrm{End}_L(\mathbb{G}_a^n)$-module and a right $\mathrm{End}_L(\mathbb{G}_a^n)^\circ$-module where

$$\mathrm{End}_L(\mathbb{G}_a^n)^\circ \cong \mathrm{End}_L(\mathrm{Hom}_L(\mathbb{G}_a^n, \mathbb{G}_a))$$

is the opposite of $\mathrm{End}_L(\mathbb{G}_a^n)$, canonically anti-isomorphic to it via transposition. The images of $\mathrm{End}_L(\mathbb{G}_a^n)$ and $\mathrm{End}_L(\mathbb{G}_a^n)^\circ$ in $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n) = \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))$, corresponding to left and right multiplication, have the $L$-homotheties as intersection and, canonically, we have the well known isomorphism

$$\mathrm{End}_L^{(2)}(\mathbb{G}_a^n) \cong \mathrm{End}_L(\mathbb{G}_a^n) \otimes_L \mathrm{End}_L(\mathbb{G}_a^n)^\circ,$$

and the process can be iterated to construct $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$ and so on. We call the elements of $\mathrm{End}_L^{(m)}(\mathbb{G}_a^n)$ *$L$-linear operators* or $L$-linear *$m$-endomorphisms.* Note that in $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$, $(f \otimes g)(f' \otimes g') = (ff' \otimes g'g)$.

If $f \otimes g$ is in $\mathrm{End}_L(\mathbb{G}_a^n) \otimes_L \mathrm{End}_L(\mathbb{G}_a^n)^\circ$ and $X \in \mathrm{End}_L(\mathbb{G}_a^n)$, we write:

$$(f \otimes g)(X) := fXg \in \mathrm{End}_L(\mathbb{G}_a^n).$$

This defines the *evaluation map* of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$, defining a left $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$-module structure over $\mathrm{End}_L(\mathbb{G}_a^m)$. The space $\mathcal{Z}$ is the kernel of the evaluation map of $N \otimes 1 - 1 \otimes N$.

We can compose an element $X$ of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ with the $\mathbb{F}_q$-linear endomorphism $\tau$ of $\mathbb{G}_a^n$ both on the left and on the right. We denote these operations by $\tau \otimes 1$ and $1 \otimes \tau$ respectively, so that we have the evaluations $(\tau \otimes 1)(X) = \tau X$ and $(1 \otimes \tau)(X) = X\tau$. Since $\tau X = (\tau X \tau^{-1})\tau = X^{(1)}\tau$ we have

$$(\tau \otimes 1)(X) = X^{(1)}(1 \otimes \tau). \tag{2.1}$$

If $X \in \mathrm{End}_L(\mathbb{G}_a^n)$, $X^{(1)} \in \mathrm{End}_L(\mathbb{G}_a^n)$ and we can define, more generally, $X^{(k)}$ with $k \in \mathbb{Z}$. We denote the $\mathbb{F}_q$-linear map $X \mapsto X^{(k)}$, uniquely determined, by $\boldsymbol{\tau}^k$; it can be identified with an element of $\mathrm{End}_{\mathbb{F}_q}(\mathrm{End}_L(\mathbb{G}_a^n))$ (we must distinguish it from $\tau^k \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$). The formula (2.1) is equivalent to ([5]):

$$(\tau \otimes 1) = (1 \otimes \tau)\boldsymbol{\tau} = \boldsymbol{\tau}(1 \otimes \tau). \tag{2.2}$$

We have $\mathbb{F}_q[\boldsymbol{\tau}, 1 \otimes \tau] \cong \mathbb{F}_q[X, Y]$ and $\mathbb{F}_q[[\boldsymbol{\tau}, 1 \otimes \tau]] \cong \mathbb{F}_q[[X, Y]]$ with independent indeterminates $X, Y$. The operators $1 \otimes \tau, \boldsymbol{\tau}, \tau \otimes 1$ are also determined by the commutation rules

$$\begin{aligned} (1 \otimes \tau)(x \otimes y) &= (x \otimes y^{(-1)})(1 \otimes \tau), \\ \boldsymbol{\tau}(x \otimes y) &= (x^{(1)} \otimes y^{(1)})\boldsymbol{\tau}, \\ (\tau \otimes 1)(x \otimes y) &= (x^{(1)} \otimes y)(1 \otimes \tau). \end{aligned}$$

We can therefore identify

$$\mathrm{End}^{(2)}_{\mathbb{F}_q}(\mathbb{G}_a^n) = \mathrm{End}^{(2)}_L(\mathbb{G}_a^n)[\boldsymbol{\tau}, 1 \otimes \tau],$$

and there is a canonical isomorphism

$$\mathrm{End}^{(2)}_{\mathbb{F}_q}(\mathbb{G}_a^n) \cong \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n) \otimes_{L[\tau], \boldsymbol{\tau}} \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)^\circ, \tag{2.3}$$

where the tensor product $\otimes_{L[\tau], \boldsymbol{\tau}}$ encodes the identity (2.2). This allows to define the evaluation of an element of $\mathrm{End}_{\mathbb{F}_q}(\mathrm{End}_L(\mathbb{G}_a^n))$ at an $L$-linear endomorphism $X$. Explicitly, one simply sets, with $f \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ and $g \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)^\circ$,

$$(f \otimes g)(X) := fXg,$$

and extends linearly via the tensor product taking care of the commutation rule (2.2). This evaluation is not, in general, an element of $\mathrm{End}_L(\mathbb{G}_a^n)$. It just belongs to $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ (recall that the evaluation of $1 \otimes \tau$ at $X$ is $X\tau \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ which is not, in general, $L$-linear). The above evaluation induces an $\mathrm{End}^{(2)}_L(\mathbb{G}_a^n)[\boldsymbol{\tau}]$-module structure over $\mathrm{End}_L(\mathbb{G}_a^n)[\tau]$. The elements of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ are called *linear endomorphisms* (example: $\phi_\theta$). The elements of $\mathrm{End}^{(2)}_L(\mathbb{G}_a^n)[\boldsymbol{\tau}]$ are called *linear operators*. We are now going to revisit Papanikolas' construction of Carlitz's operators in [9] in the light of this formalism.

Let $\phi : A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ be an Anderson $A$-module defined over $L$. We have that $\phi_\theta = \phi(\theta) \in \mathrm{End}_L(\mathbb{G}_a^n)[\tau]$. Denote by $d : A \to \mathrm{End}_L(\mathrm{Lie}(\phi))$ the associated $A$-module structure over $\mathrm{Lie}(\phi)$ (the differentiation map in [2, §1.4]). There exists a unique formal series (the *exponential* of $\phi$) $\exp_\phi \in \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]$ such that $\exp_\phi d(\theta) = \phi_\theta \exp_\phi$ and $\exp_\phi - 1 \in \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]\tau$. There are several ways to construct $\exp_\phi$ ([1, §2.2] and [17, Theorem 4.3.7]). Let $\log_\phi \in \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]$ be the unique element (the *logarithm* of $\phi$) such that $\exp_\phi \log_\phi = \log_\phi \exp_\phi = 1$.

[5] Recall: what follows is a restatement of the identity $\tau(f(x)) = f^{(1)}(\tau(x))$, and $\boldsymbol{\tau}$ and $1 \otimes \tau$ commute (by associativity), as they represent respectively a left and a right action by operators.

We thus have the formal series

$$\exp_\phi \otimes 1, \quad 1 \otimes \log_\phi \in \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[[\boldsymbol{\tau}, 1 \otimes \tau]].$$

One sees immediately that their product $(\exp_\phi \otimes 1)(1 \otimes \log_\phi)$ is not the neutral element. Let us see its structure in more detail. Let us expand $\exp_\phi$ and $\log_\phi$:

$$\exp_\phi = \sum_{i \geq 0} \epsilon_{\phi,i} \tau^i, \quad \log_\phi = \sum_{j \geq 0} \epsilon_{\phi,j} \tau^j \in \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]].$$

Expanding in powers of $1 \otimes \tau$ and successively collecting the powers of $\boldsymbol{\tau}$ we get:

$$\begin{aligned}
\exp_\phi \otimes \log_\phi = (\exp_\phi \otimes 1)(1 \otimes \log_\phi) &= \sum_{k \geq 0} \sum_{i+j=k} (\epsilon_{\phi,i} \tau^i \otimes 1)(1 \otimes \lambda_{\phi,j} \tau^j) \\
&= \sum_{k \geq 0} \sum_{i+j=k} (\epsilon_{\phi,i} \otimes 1)(\tau^i \otimes 1)(1 \otimes \lambda_{\phi,j} \tau^j) \\
&= \sum_{k \geq 0} \Big( \sum_{i+j=k} (\epsilon_{\phi,i} \otimes 1)(1 \otimes \tau^i) \boldsymbol{\tau}^i (1 \otimes \tau^j)(1 \otimes \lambda_{\phi,j}) \Big) \\
&= \sum_{k \geq 0} (1 \otimes \tau^k) \Big( \sum_{i+j=k} (\epsilon_{\phi,i} \otimes \lambda_{\phi,j}^{(i)}) \boldsymbol{\tau}^i \Big).
\end{aligned}$$

We set

$$E_{\phi,k} := \sum_{i+j=k} (\epsilon_{\phi,i} \otimes \lambda_{\phi,j}^{(i)}) \boldsymbol{\tau}^i \in \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}] \subset \mathrm{End}_{\mathbb{F}_q}(\mathrm{End}_L(\mathbb{G}_a^n)). \tag{2.4}$$

We call it the $k$-th *Carlitz's linear operator associated to* $\phi$. In the case $n = 1$ and $\phi = C$ (Carlitz's module) this construction returns nothing but the classical polynomials of Carlitz in [11] viewed as elements of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a)$ (see Goss' book [6, Chapter 3]).

Just like $\exp_\phi$ determines an evaluation map $\mathbb{G}_a^n(L) \to \mathbb{G}_a^n(L)$, the product of formal series $\exp_\phi \otimes \log_\phi = (\exp_\phi \otimes 1)(1 \otimes \log_\phi)$ determines an $\mathbb{F}_q$-linear evaluation map $\mathrm{End}_L(\mathbb{G}_a^n) \to \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]]$ (remember that evaluating $1 \otimes \tau$ at an $L$-linear endomorphism returns this endomorphism multiplied by $\tau \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ on the right).

2.1. **Basic properties of Carlitz operators.** The properties that we examine here are at once formal and useful. We keep considering an Anderson $A$-module $\phi$ of dimension $n$.

**Lemma 2.1.** *Let $X$ be an $L$-linear endomorphism. For all $k \geq 0$ we have the formulas*

$$\begin{aligned}
E_{\phi,k}(X \otimes 1) &= \sum_{i+j=k} (E_{\phi,i}(X) \otimes 1) E_{\phi,j}^{(i)} \boldsymbol{\tau}^i, \\
E_{\phi,k}(1 \otimes X) &= \sum_{i+j=k} (1 \otimes E_{\phi,i}(X)^{(j)}) E_{\phi,j}.
\end{aligned}$$

*Proof.* Since

$$\begin{aligned}\exp_\phi X \otimes \log_\phi &= \sum_{k\geq 0}(1\otimes\tau^k)E_{\phi,k}(X\otimes 1),\\ \exp_\phi \otimes X \log_\phi &= \sum_{k\geq 0}(1\otimes\tau^k)E_{\phi,k}(1\otimes X),\end{aligned}$$

it suffices to show that

$$\begin{aligned}\exp_\phi X \otimes \log_\phi &= \sum_{k\geq 0}(1\otimes\tau^k)\sum_{i+j=k}(E_{\phi,i}(X)\otimes 1)E_{\phi,j}^{(i)}\boldsymbol{\tau}^i\\ \exp_\phi \otimes X \log_\phi &= \sum_{k\geq 0}(1\otimes\tau^k)\sum_{i+j=k}(1\otimes E_{\phi,i}(X)^{(j)})E_{\phi,j}.\end{aligned}$$

This is just a matter of how one reorders the terms in sums. We clearly have

$$((\exp_\phi X\log_\phi)\otimes 1)(\exp_\phi\otimes\log_\phi) = \exp_\phi X\otimes\log_\phi\,.$$

But

$$\begin{aligned}((\exp_\phi X\log_\phi)\otimes 1)(\exp_\phi\otimes\log_\phi) &= \Big(\sum_{i\geq 0}E_{\phi,i}(X)\tau^i\otimes 1\Big)\Big(\sum_{j\geq 0}(1\otimes\tau^j)E_{\phi,j}\Big)\\ &= \sum_{i\geq 0}(1\otimes\tau^i)(E_{\phi,i}(X)\otimes 1)\sum_{j\geq 0}(1\otimes\tau^j)\boldsymbol{\tau}^iE_{\phi,j}\\ &= \sum_{k\geq 0}(1\otimes\tau^k)\sum_{i+j=k}(E_{\phi,i}(X)\otimes 1)E_{\phi,j}^{(i)}\boldsymbol{\tau}^i,\end{aligned}$$

which proves the first formula. For the second we proceed analogously. Indeed:

$$(1\otimes(\exp_\phi X\log_\phi))(\exp_\phi\otimes\log_\phi) = \exp_\phi\otimes X\log_\phi\,.$$

Moreover,

$$\begin{aligned}(1\otimes(\exp_\phi X\log_\phi))(\exp_\phi\otimes\log_\phi) &= \Big(1\otimes\sum_{i\geq 0}E_{\phi,i}(X)\tau^i\Big)\Big(\sum_{j\geq 0}(1\otimes\tau^j)E_{\phi,j}\Big)\\ &= \sum_{i\geq 0}(1\otimes\tau^i)(1\otimes E_{\phi,i}(X))\sum_{j\geq 0}(1\otimes\tau^j)E_{\phi,j}\\ &= \sum_{k\geq 0}(1\otimes\tau^k)\sum_{i+j=k}(1\otimes E_{\phi,i}(X)^{(j)})E_{\phi,j}.\end{aligned}$$

□

We can evaluate $E_{\phi,k}$ (for all $k$), at $X\in\mathrm{End}_L(\mathbb{G}_a^n)$ by means of the rules deduced from (2.3). The evaluation $E_{\phi,k}(X)$ of $E_{\phi,k}$ at $X$ is:

$$E_{\phi,k}(X) = \sum_{i=0}^{k}(\epsilon_{\phi,i}\otimes\lambda_{\phi,j}^{(i)})X^{(i)}\in\mathrm{End}_L(\mathbb{G}_a^n),$$

and the evaluation of $\exp_\phi \otimes \log_\phi$ at $X$ is

$$\exp_\phi X \log_\phi = \sum_{k\geq 0} E_{\phi,k}(X)\tau^k \in \mathrm{End}_L(\mathbb{G}_a^n)[[\tau]].$$

We get

**Corollary 2.2** (Twisted Leibniz rule)**.** *For all $X_1, X_2 \in \mathrm{End}_L(\mathbb{G}_a^n)$:*

$$E_{\phi,k}(X_1X_2) = \sum_{i+j=k} E_{\phi,i}(X_1)E_{\phi,j}(X_2)^{(i)}, \quad k \geq 0. \tag{2.5}$$

In the case $n = 1$, $\phi = C$, the identities we get are also contained in Carlitz's work [11].

The case $X = d(a)$ with $a \in A$ in Lemma 2.1 gives, in virtue of $\exp_\phi d(a) \log_\phi = \phi_a$, the following

**Corollary 2.3.** *For all $a \in A$, $\phi_a$ is the evaluation of $\exp_\phi \otimes \log_\phi$ at $d(a)$ so that*

$$\phi_a = \sum_{i\geq 0} E_{\phi,i}(d(a))\tau^i \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n) \tag{2.6}$$

*(the sum is finite). Moreover, for all $k \geq 0$:*

$$\begin{aligned} E_{\phi,k}(d(a) \otimes 1) &= \sum_{i+j=k} ((a)_i \otimes 1)\boldsymbol{\tau}^i E_j, \\ E_{\phi,k}(1 \otimes d(a)) &= \sum_{i+j=k} (1 \otimes (a)_i^{(j)})E_j, \end{aligned}$$

*where we have written $\phi_a = \sum_i (a)_i \tau^i$.*

## 3. Tensor powers of Carlitz's module

We now move to the more specific settings of tensor powers $C^{\otimes n}$ of Carlitz's module. We quickly present the tools we need. For the requested background on $C^{\otimes n}$ and Anderson $A$-modules we have at our disposal excellent references: [10, 2, 6, 15]. The $A$-module

$$\phi := C^{\otimes n}$$

can be defined as the unique injective $\mathbb{F}_q$-algebra map

$$A \xrightarrow{\phi} \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$$

which sends $\theta$ to the endomorphism $\phi_\theta \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^n)$ defined by

$$\phi_\theta = \theta + N + e_{n,1}\tau,$$

where $\tau$ is the $\mathbb{F}_q$-linear endomorphism of $\mathbb{G}_a^n$ defined by raising all the entries to the power $q$, $e_{i,j}$ is the element of $\mathrm{End}_L(\mathbb{G}_a^n)$ defined as the composition $\iota_i\pi_j$ of the projection $\pi_j$ to the $j$-th coordinate and the injection in the $i$-th component

$$\pi_j : \mathbb{G}_a^n \to \mathbb{G}_a, \quad \iota_i : \mathbb{G}_a \to \mathbb{G}_a^n,$$

and $N$ is the already encountered nilpotent endomorphism $e_{1,2} + e_{2,3} + \cdots + e_{n-1,n}$ (in all the following we set $N^0$ to be equal to the identity endomorphism 1). The elementary identities

$$e_{i,h}e_{k,j} = \delta_{h,k}e_{i,j} \tag{3.1}$$

hold where $\delta_{h,k}$ is Kronecker's delta. If $n = 1$, $N = 0, e_{n,1} = 1$ and we find Carlitz's module $C$. Unless otherwise specified in all the following $n$ denotes a fixed integer $\geq 1$ and $\phi = C^{\otimes n}$.

The exponential function $\exp_\phi$ defines a surjective $\mathbb{F}_q$-linear entire function

$$\mathbb{G}_a^n(L) \to \mathbb{G}_a^n(L)$$

because $L$ is a filtered union of local field extensions of $K_\infty$. Indeed, if $x \in L^n$, then $\exp_\phi(x) \in K_\infty[x]^n \subset L^n$ by the fact that the endomorphisms $\epsilon_{\phi,k}$ are defined over $K$. As $\mathbb{F}_q$-vector spaces, $\mathrm{Lie}(\phi)(L)$ and $\phi(L)$ are equal to $\mathbb{G}_a^n(L)$ and $\exp_\phi$ is an $A$-module map $\mathrm{Lie}(\phi)(L) \to \phi(L)$ (see [2, §1.4]). The $A$-module $\mathrm{Lie}(\phi)(L)$ also carries a structure of $L$-vector space of dimension $n$ that we review now. The scalar multiplication is defined in the following way (read [2, §1.4.5]). Recall the collection of higher divided derivatives $(\mathcal{D}_j)_{j\geq 0}$ of $K_\infty$, satisfying the Leibniz rule

$$\mathcal{D}_k(xy) = \sum_{i+j=k} \mathcal{D}_i(x)\mathcal{D}_j(y).$$

We call such a family of linear operators a *hyperderivative*. It can be uniquely extended to an hyperderivative of $L$ by results in Conrad's [5]. This can be easily seen directly in the following way. Any Galois extension of $K_\infty$ is contained in an iteration of finitely many Artin-Schreier extensions of a tamely ramified Kummer extension of $K_\infty$. There is no difficulty in seeing that any hyperderivative can be uniquely extended through tamely ramified Kummer extensions so that, to extend $(\mathcal{D}_k)_{k\geq 0}$ to $L$, it suffices to deal with Artin-Schreier extensions. Now, if $\xi^p - \xi = \alpha$ with $\alpha$ belonging to a subfield of $L$ over which $(\mathcal{D}_j)_{j\geq 0}$ extends, the Leibniz rule implies, for all $i \geq 0$,

$$\mathcal{D}_i(\xi) - \mathcal{D}_{\frac{i}{p}}(\xi)^p = \mathcal{D}_i(\alpha),$$

where $\mathcal{D}_{i/p}$ is set to be zero if $p \nmid i$. This allows, recursively, to extend $(\mathcal{D}_j)_{j\geq 0}$ uniquely to any Galois extension of $K_\infty$ and ultimately to $L$.

In particular the map

$$L \xrightarrow{d} \mathrm{End}_L(\mathbb{G}_a^n)(L)$$

defined, thanks to the map (1.6), by

$$d(x) = h\begin{pmatrix} \mathcal{D}_{n-1}(x) \\ \mathcal{D}_{n-2}(x) \\ \vdots \\ x \end{pmatrix}$$

is a ring homomorphism and this gives the requested $L$-vector space structure over $\mathrm{Lie}(\phi)$. The image $\mathcal{Z}$ of the map $h$ is the maximal commutative subring of $\mathrm{End}_L(\mathbb{G}_a^n)$ that contains

the endomorphism $d(\theta)$. It obviously also contains the image of $d$, so that $\mathcal{Z} = L[N]$. It can be proved (but it will not be needed in this paper) that $\mathcal{Z}$ also equals the analytic adherence of the image of $d$; in particular it coincides with its Zariski adherence. This means that, if $\mathbb{C}_\infty$ is the completion of $L$, any $n$-tuple $(x_1, \dots, x_n) \in \mathbb{C}_\infty^n$ is coefficientwise limit of a sequence $(y_i, \mathcal{D}_1(y_i), \dots, \mathcal{D}_{n-1}(y_i))_{i\geq 0}$ with $(y_i)_{i\geq 0} \subset L$.

Since the map $h$ is used often here, we also adopt some abridged notations. We also write

$$\widehat{Z} = h(Z), \quad Z \in \mathbb{G}_a^n(L).$$

Also, given $f \in \mathrm{End}_L(\mathbb{G}_a^n)(L)$, by the isomorphism $\mathrm{End}_L(\mathbb{G}_a^n) \cong \mathrm{Hom}_L(\mathbb{G}_a^n, \mathbb{G}_a)^{\oplus n}$ we can expand it in a unique way $f = \sum_{i,j} f_{i,j} e_{i,j}$, where we recall that $(e_{i,j})$ is the standard basis of $\mathrm{End}_L(\mathbb{G}_a^n)$. We write (extraction of the last column)

$$[f] = \begin{pmatrix} f_{1,n} \\ \vdots \\ f_{n,n} \end{pmatrix},$$

so that

$$[h(X)] = [\widehat{X}] = X.$$

Then we have, for elements $Z, W \in \mathbb{G}_a^n(L)$, the following commutation rule:

**Lemma 3.1.** $\widehat{Z}W = \widehat{W}Z$.

*Proof.* It follows from the elementary identities $[\widehat{Z}\widehat{W}] = \widehat{Z}[\widehat{W}] = \widehat{Z}W$ and the commutativity of $\mathcal{Z}$. □

Hence, with elements $Z_1, \dots, Z_m$ and $W$ in $\mathbb{G}_a^n(L)$ we have an identity $(\widehat{Z_1}W, \dots, \widehat{Z_m}W) = \widehat{W}(Z_1, \dots, Z_m)$.

3.1. **Certain $L$-linear operators.** We consider, for $X, Y$ independent indeterminates, the field $\mathbb{F}_q(X, Y)$. We consider the $\mathbb{F}_q(Y)$-algebra morphism $d_X : \mathbb{F}_q(X, Y) \to \mathcal{Z}(\mathbb{F}_q(X, Y))$ defined by $d_X(X) = X + N =: D_X$. We also define, analogously, $d_Y$ and $D_Y$. The endomorphism $\mu_{X,Y}$ of the space

$$\mathrm{End}_{\mathbb{F}_q(X,Y)}(\mathbb{G}_a^n(\mathbb{F}_q(X, Y)))$$

defined by

$$f \mapsto D_X f - f D_Y$$

is an automorphism and in this subsection we show how to compute, for any element of $\mathrm{End}_{\mathbb{F}_q(X,Y)}(\mathbb{G}_a^n(\mathbb{F}_q(X, Y)))$, its inverse image by $\mu_{X,Y}$. To do this we claim that it suffices to compute the inverse image of the endomorphism $d_X((X - Y)^n)e_{n,1}$.

The $\mathbb{F}_q$-vector space $A(<n)$, generated by the polynomials of $A$ that have degree $< n$, embeds in $\mathbb{F}_q(X, Y)$ by means of the maps $g_X, g_Y$ defined by $g_X(f(\theta)) = \partial_Y(f(Y))$ and $g_X(f(\theta)) = \partial_X(f(X))$, where $f$ is a polynomial of degree $\leq n - 1$ and where $\partial_X(f(X)) = {}^t(f, \mathcal{D}_{X,1}(f), \dots, \mathcal{D}_{X,n-1}(f))$, and $\partial_Y$ has the analogous meaning. By viewing the Taylor's

expansion of functions of the variable $X$ at $X = Y$ one easily sees that the left multiplication by the matrix

$$H_{X,Y} := \partial_X\Big((X-Y)^{n-1}, \dots, 1\Big) \in \mathrm{GL}_n(\mathbb{F}_q[X,Y]) \tag{3.2}$$

is uniquely determined by the following identity over the above space

$$H_{X,Y} g_Y = g_X, \tag{3.3}$$

so that $H_{X,Y}$ is the endomorphism of $\mathbb{G}_a^n(\mathbb{F}_q(X,Y))$ represented by the identity when one chooses the basis $\mathcal{X} := (X_i)_{0\le i\le n-1} = (\partial_X(X^i))_{0\le i\le n-1}$ in the target space, and the basis $\mathcal{Y} := (Y_i)_{0\le i\le n-1} = (\partial_Y(Y^i))_{0\le i\le n-1}$ in the source space or, in more figurative words, it realizes the canonical isomorphism 'replacement of the variable $Y$ with the variable $X$' between the $\mathbb{F}_q$-vector space of polynomials in $\mathbb{F}_q[Y]$ of degree $n-1$ and the analogous space of polynomials in $\mathbb{F}_q[X]$, when these spaces are both immersed in the common environment given by the $\mathbb{F}_q$-vector space $\mathbb{F}_q(X,Y)^n$, by means of the maps $\partial_Y, \partial_X$.

**Lemma 3.2.** *We have* $D_X H_{X,Y} - H_{X,Y} D_Y = d_X((X-Y)^n) H_{X,Y} e_{n,1} = d_X((X-Y)^n) e_{n,1}$.

*Proof.* Recall the bases $\mathcal{X} = (\partial_X(X^i))_{0\le i\le n-1}$ and $\mathcal{Y} = (\partial_Y(Y^i))_{0\le i\le n-1}$ of $\mathbb{G}_a^n(\mathbb{F}_q(X,Y))$ and the fact that with respect to them (for the target and source spaces), the endomorphism $H_{X,Y}$ is represented by the identity. Note that $D_Y Y_i = Y_{i+1}$ for $i = 0, \dots, n-2$ and $D_Y Y_{n-1} = \sum_{i=0}^{n-1} (-1)^{i+1} \binom{n}{i} Y^{n-i} Y_i$, and that we have similar formulas for $D_X X_i$. Then, if $0 \le i \le n-2$,

$$\begin{aligned} \Big(D_X H_{X,Y} - H_{X,Y} D_Y\Big) Y_i = \\ &= D_X X_i - H_{X,Y} Y_{i+1} \\ &= X_{i+1} - X_{i+1} = 0. \end{aligned}$$

Moreover, we have:

$$\begin{aligned} \Big(D_X H_{X,Y} - H_{X,Y} D_Y\Big) Y_{n-1} = \\ &= D_X X_{n-1} - H_{X,Y} \sum_{i=0}^{n-1} (-1)^{i+1} Y^{n-i} Y_i \\ &= \partial_X(X^n) + \sum_{i=0}^{n-1} (-1)^i Y^{n-i} H_{X,Y} Y_i \\ &= \partial_X((X-Y)^n). \end{aligned}$$

Note also that

$$d_X((X-Y)^n) H_{X,Y} e_{n,1} Y_i = 0, \quad i = 0, \dots, n-2,$$
$$d_X((X-Y)^n) H_{X,Y} e_{n,1} Y_{n-1} = \partial_X((X-Y)^n).$$

Hence $D_X H_{X,Y} - H_{X,Y} D_Y$ and $d_X((X-Y)^n) H_{X,Y} e_{n,1}$ are two different expressions of the same $L$-linear endomorphism. Additionally since $H_{X,Y} e_{n,1} = e_{n,1}$, it is clear that $d_X((X-Y)^n) H_{X,Y} e_{n,1} = d_X((X-Y)^n) e_{n,1}$. □

As a direct consequence of Lemma 3.2 we note that if we set

$$W_{X,Y} = \partial_X((X-Y)^{-1},\dots,(X-Y)^{-n}) = d_X((X-Y)^{-n})H_{X,Y} \in \mathrm{GL}_n(\mathbb{F}_q(X,Y)), \tag{3.4}$$

then

$$\mu_{X,Y}(W_{X,Y}) = e_{n,1}.$$

For this, observe that $\mu_{X,Y}$ is a left $\mathcal{Z}$-module homomorphism and a right ($\mathcal{Z} = \mathcal{Z}^\circ$)-module homomorphism ($(\cdot)^\circ$ is the opposite). From this, and by the elementary fact that

$$N^{n-i}e_{n,1}N^{j-1} = e_{i,j} \tag{3.5}$$

for all $1 \le i,j \le n$, implying $(\mathcal{Z} \otimes_L \mathcal{Z})(e_{n,1}) = \mathrm{End}_L(\mathbb{G}_a^n)$,

$$\mu_{X,Y}(N^{n-i}W_{X,Y}N^{j-1}) = N^{n-i}\mu_{X,Y}(W_{X,Y})N^{j-1} = e_{i,j}, \quad 1 \le i,j \le n.$$

We can formulate the following result that can be used to compute $\mu_{X,Y}^{-1}(f)$ for any endomorphism $f$.

**Lemma 3.3.** *For all $1 \le i,j \le n$, $\mu_{X,Y}(N^{n-i}W_{X,Y}N^{j-1}) = e_{i,j}$.*

3.2. **The module $\widetilde{C^{\otimes n}}$.** We describe the Anderson $A$-module that carries the normalized period lattice (1.5), in the class of isomorphism of $C^{\otimes n}$.

We recall the $\mathbb{F}_q$-linear entire Carlitz exponential function

$$L \xrightarrow{\exp_C} L$$

and the fact that there exists an element $\widetilde{\pi}_C \in L^\times$ such that the exact sequence of $\mathbb{F}_q$-vector spaces

$$0 \to \mathrm{Ker}(\exp_C) \to L \xrightarrow{\exp_C} L \to 0$$

is also an exact sequence of $A$-modules

$$0 \to \widetilde{\pi}_C A \to L \xrightarrow{\exp_C} C(L) \to 0.$$

The element $\widetilde{\pi}_C$, transcendental over $K$, can be explicitly computed through the following product expansion

$$\widetilde{\pi}_C \in \mathbb{F}_q^\times \theta(-\theta)^{\frac{1}{q-1}} \prod_{k\ge 1}\left(1-\frac{\theta}{\theta^{(k)}}\right)^{-1} \tag{3.6}$$

and is uniquely determined up to multiplication by an element of $\mathbb{F}_q^\times$ (it is therefore uniquely determined if $q=2$).

For $x \in L$ we write

$$\partial(x) := \begin{pmatrix} \mathcal{D}_{n-1}(x) \\ \mathcal{D}_{n-2}(x) \\ \vdots \\ x \end{pmatrix}.$$

The image of $A$ by $\partial$ is (1.5). Lemma 3.1 and Leibniz rule imply

$$d(a)\partial(b) = \partial(ab) = \partial(ba) = d(b)\partial(a), \quad a,b \in L,$$

identities previously noticed by Papanikolas in [9, Proposition 2.5.8] which can be reformulated by saying that $\partial : L \to \mathbb{G}_a^n(L)$ is an $L$-module map through the $L$-module structure induced by the injective ring map $d : L \to \mathrm{End}_L(\mathbb{G}_a^n)$. The kernel $\Lambda_\phi$ of $\exp_\phi$, the period lattice of $\phi$, is an $A$-module of rank one in $\mathrm{Lie}(C^{\otimes n})(L)$. By [10, Proposition 2.5.5] there exists a unique $\Pi \in L^n$ such that

$$\Lambda_\phi = \{d(a)\Pi : a \in A\} = \left\{\widehat{\Pi}\partial(a) : a \in A\right\} \subset \mathcal{Z}$$

so that, writing

$$\Pi = \begin{pmatrix} \pi_{n-1} \\ \vdots \\ \pi_0 \end{pmatrix} \in L^n, \tag{3.7}$$

we have

$$\pi_0 = (-\widetilde{\pi}_C)^n \tag{3.8}$$

(see also [7, Lemma 8.3]).

In particular, the endomorphism $\widehat{\Pi} \in \mathcal{Z}$ is an automorphism. The normalization of $\Lambda_\phi$ is the rank one $A$-module (1.5), that is:

$$\{\partial(a) : a \in A\} = \{d(a)\partial(1) : a \in A\} \subset \mathrm{Lie}(\phi)(L),$$

the $A$-module generated by $\partial(1)$.

The Anderson $A$-module $\widetilde{\phi}$ determined by

$$\widetilde{\phi}_\theta = I_n\theta + N + \widehat{\Pi}^{-1}e_{n,1}\widehat{\Pi}^{(1)}\tau,$$

is uniformized by the exponential

$$\exp_{\widetilde{\phi}} = \widehat{\Pi}^{-1}\exp_\phi \widehat{\Pi} \in \mathrm{End}_L(\mathbb{G}_a^n)(K_\infty)[[\tau]].$$

## 4. $\infty$-adic limit of $E_{\phi,k}$

The main result of this section is Theorem 4.2, where we establish a relation between the limit of the title, and the exponential $\exp_\phi$. To start, we introduce the following formal series

$$\mathbf{exp}_\phi := \sum_{k\geq 0}(\epsilon_{\phi,k} \otimes 1)\boldsymbol{\tau}^k \in \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[[\boldsymbol{\tau}]].$$

It is easily seen that $\mathbf{exp}_\phi$ gives rise to a series converging over $\mathrm{End}_L(\mathbb{G}_a^n)$ (the evaluation of $\boldsymbol{\tau}$ at $f \in \mathrm{End}_L(\mathbb{G}_a^n)$ is $f^{(1)} \in \mathrm{End}_L(\mathbb{G}_a^n)$) so there is an associated $\mathbb{F}_q$-linear evaluation map

$$\mathbf{exp}_\phi : \mathrm{End}_L(\mathbb{G}_a^n) \to \mathrm{End}_L(\mathbb{G}_a^n).$$

Fixing a basis of $\mathrm{End}_L(\mathbb{G}_a^n)$ and choosing a valuation structure compatible with that of $L$, one sees easily that this evaluation map is also an $\mathbb{F}_q$-linear rigid analytic entire map. This is the exponential that uniformizes the Anderson $A$-module

$$A \xrightarrow{\boldsymbol{\phi}} \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[\boldsymbol{\tau}]$$

defined by

$$\boldsymbol{\phi}_\theta = d(\theta) \otimes 1 + (e_{n,1} \otimes 1)\boldsymbol{\tau}. \tag{4.1}$$

The $A$-module structure of $\mathrm{Lie}(\boldsymbol{\phi})$ is given by the map $d \otimes 1$. Moreover, $\boldsymbol{\phi}$ has dimension $n^2$ and contains $\phi$ as an $A$-submodule. This comes from the map $h$ as it is easy to see that:

$$h\boldsymbol{\phi} = \phi h$$

and we deduce that, in the standard basis $(e_{i,j})_{i,j}$ of $\mathrm{End}_L(\mathbb{G}_a^n)$ we have an isomorphism

$$\boldsymbol{\phi} \xrightarrow{\cong} \phi^{\oplus n}$$

sending $\boldsymbol{\tau}$ to $\tau^{\oplus n}$, so the rank of $\boldsymbol{\phi}$ is $n$.

4.1. **The sequence $(\lambda_{\phi,k})_k$.** The following proposition presents the property of the coefficients of the logarithm of $\phi$ crucial to define the normalized operators $\mathcal{E}_{\phi,k}$. The proof is an immediate consequence of an unpublished formula (4.3) below, from Papanikolas' [9]. In §7.2 the reader can find more comments. In the next result, $\mathrm{Aut}_L(\mathbb{G}_a^n)$ denotes the automorphism group of $\mathbb{G}_a^n$.

**Proposition 4.1.** *For all $k \geq 0$ we have that $\lambda_{\phi,k} \in \mathrm{Aut}_L(\mathbb{G}_a^n)$.*

*Proof.* We fix the standard basis of $\mathbb{G}_a^n$. In this way we can identify linear endomorphisms of $\mathbb{G}_a^n$ with left multiplication by matrices representing them. In the standard basis, for all $k$, the endomorphism $\lambda_{\phi,k}$ is represented with the right multiplication by the matrix $P_k \in K^{n\times n}$ and the endomorphism $\epsilon_{\phi,k}$ is represented with the right multiplication by the matrix $Q_k \in K^{n\times n}$. These sequences $(P_k)_k, (Q_k)_k$ are uniquely defined inductively by the relations

$$\mu_k(P_k) = P_{k-1}e_{n,1}, \quad \mu_k(Q_k) = -e_{n,1}Q_{k-1}^{(1)}, \quad k \geq 1, \tag{4.2}$$

where

$$\mu_k := d(\theta) \otimes 1 - 1 \otimes d(\theta)^{(k)} \in \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n)).$$

Indeed note that, with $F$ any field, given nilpotent endomorphisms $\nu, \nu' \in \mathrm{End}_F(\mathbb{G}_a^n)$, the operator $f = \nu \otimes 1 - 1 \otimes \nu'$ is itself nilpotent (one sees easily that $f^{2n-1} = 0$) so that for all $\lambda \in L^\times$ the operator $\lambda + f \in \mathrm{End}_F(\mathrm{End}_F(\mathbb{G}_a^n))$ is an isomorphism and so is $\mu_k$, with $k > 0$, and we recover [10, (2.1.3)]. These formulas characterize the sequences $(P_k)_k$ and $(Q_k)_k$ uniquely but do not give direct information on the ranks in their dependence on $k$. To show that the determinant of $P_k$ is non zero for all $k$ we use a formula of Papanikolas in [9] that we describe now.

In parallel with the hyperderivative $(\mathcal{D}_k)_{k\geq 0}$ and the maps $d, \partial$ we already defined we have, over the Tate algebra $\mathbb{T} = \widehat{L[t]}_{\|\cdot\|}$ completion of $L[t]$ for the Gauss' norm that uniquely extends $|\cdot|$ (where $t$ is an independent variable), the $L$-linear hyperderivative $(\mathcal{D}_{t,k})_{k\geq 0}$ uniquely determined by $\mathcal{D}_{t,k}(t^m) = \binom{m}{k} t^{m-k}$. We can thus define maps $d_t : \mathbb{T} \to \mathrm{End}_{\mathbb{T}}(\mathbb{G}_a^n)$

and $\partial_t : \mathbb{T} \to (\mathbb{G}_a^n(\mathbb{T}))$ by

$$d_t(f) := \begin{pmatrix} f & \mathcal{D}_{t,1}(f) & \cdots & \mathcal{D}_{t,n-1}(f) \\ 0 & f & \cdots & \mathcal{D}_{t,n-2}(f) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}, \quad \partial_t(g) := \begin{pmatrix} \mathcal{D}_{t,n-1}(g) \\ \vdots \\ \mathcal{D}_{t,1}(g) \\ g \end{pmatrix}, \quad f, g \in \mathbb{T}.$$

By Lemma 3.1 we have that $d_t$ is an injective algebra morphism and $\partial_t$ a $\mathbb{T}$-module morphism via $d_t$ (this has been already noticed by Papanikolas):

$$d_t(g)\partial_t(g) = \partial_t(fg) = \partial_t(gf) = d_t(g)\partial_t(f).$$

If $g = (g_1, \dots, g_m)$ is a row matrix of elements of $\mathbb{T}$ we denote by $\partial_t(g)$ the matrix $(\partial_t(g_1), \dots, \partial_t(g_m)) \in \mathbb{T}^{n\times m}$. We set

$$b_k = (t-\theta)\cdots(t-\theta^{q^{k-1}})$$

for $k > 0$ and $b_0 = 1$. This is a sequence of polynomials in $L[t] \subset \mathbb{T}$. Extend $\tau : L \to L$ $\mathbb{F}_q[t]$-linearly to $\mathbb{T}$ (we also write $f^{(k)} = \tau^k(f)$).

In [9, Proposition 4.3.6] Papanikolas shows that

$$P_i = \Big(\tau\big(d_t(b_i^{-n})\big)\tau^i\Big(\partial_t((t-\theta)^{n-1}, \dots, 1)\Big)\Big)_{t=\theta}, \quad i \geq 0 \tag{4.3}$$

(Papanikolas also shows an analogue formula for the coefficients $Q_i$ of $\exp_\phi$). This is the product of an upper triangular matrix followed by a lower triangular matrix. We immediately get $\det(\lambda_{\phi,i}) = \det(P_i) = \det\Big(\tau\big(d_t(b_i^{-n})\big)\Big)_{t=\theta} = \tau(b_i)_{t=\theta}^{-n^2} \neq 0$. □

By Proposition 4.1 we have a well defined sequence of operators:

$$\mathcal{E}_{\phi,k} = (1 \otimes \lambda_{\phi,k}^{-1})E_{\phi,k} \in 1 + \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[\boldsymbol{\tau}]\boldsymbol{\tau}, \quad k \geq 0. \tag{4.4}$$

For example, for $X \in \mathcal{Z}$,

$$\mathcal{E}_1(X) = 1 + (\epsilon_{\phi,1} \otimes \lambda_{\phi,1}^{-1})\boldsymbol{\tau}.$$

We shall show the next result, where

$$\mu_\phi = \widehat{\Pi}\partial(\theta^{n-1}, \dots, 1) \in \mathrm{Aut}_L(\mathbb{G}_a^n), \tag{4.5}$$

where $\Pi$ is introduced in (3.7) and where

$$\partial(\theta^{n-1}, \dots, 1) = (\partial(\theta^{n-1}), \dots, \partial(1)) \in \mathrm{Aut}_K(\mathbb{G}_a^n).$$

**Theorem 4.2.** *The sequence $(\mathcal{E}_{\phi,k})_{k\geq 0}$ converges to*

$$\mathcal{E}_\phi := (1 \otimes \mu_\phi^{-1}) \cdot \mathbf{exp}_\phi \cdot (1 \otimes \mu_\phi) \in 1 + \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[[\boldsymbol{\tau}]]\boldsymbol{\tau}.$$

**Remark 4.3.** By Theorem 4.2 the Anderson $A$-module

$$\widetilde{\boldsymbol{\phi}} = (1 \otimes \mu_\phi^{-1})\boldsymbol{\phi}(1 \otimes \mu_\phi),$$

uniformized by the exponential $\mathcal{E}_\phi$ is isomorphic to $\boldsymbol{\phi}$ but the isomorphism is not induced by an element of $\mathrm{End}_L(\mathbb{G}_a^n)$ (left or right action).

4.2. **Proof of Theorem 4.2.** The next result is the main tool.

**Proposition 4.4.** *The sequence* $(P_{k-1}^{(1)}P_k^{-1})_{k\geq 1}$ *converges to an element* $c_\phi \in \mathrm{GL}_n(L)$.

*Proof.* If $H$ is a matrix with entries in $\mathbb{T}$ we denote by $H^{[i]}$ the matrix resulting from $H$ after raising all the entries to the power $q^i$.

We first show that the limit

$$\lim_{i\to\infty} d_t(b_i^{-n})\partial_t\Big((t-\theta^{q^{i-1}})^{n-1},\dots,1\Big)\Big(\partial_t\Big((t-\theta^{q^{i-2}})^{n-1},\dots,1\Big)^{-1}\Big)^{[1]} d_t(b_{i-1}^n)^{[1]} \tag{4.6}$$

exists in the completion of $L[t]^{n\times n}$ for all the spectral valuations associated to disks $\{t\in\mathbb{C}_\infty : |t|<c\}$ where $c\in|L|$ is such that $c<|\theta|$. We claim that for all $i$, the product

$$W := \partial_t\Big((t-\theta^{q^{i-1}})^{n},\dots,1\Big)\Big(\partial_t\Big((t-\theta^{q^{i-2}})^{n},\dots,1\Big)^{-1}\Big)^{[1]}$$

belongs to $\mathbb{F}_q[t]^{n\times n}$ and is independent on $i$. To see this, for $u,v$ in some $\mathbb{F}_q$-algebra $R$, we define (with $H_{X,Y}$ introduced in (3.2))

$$H_{u,v} = H_{X,Y}|_{X=u,Y=v} \in \mathrm{GL}_n(\mathbb{F}_q[u,v]) \subset \mathrm{GL}_n(R).$$

By (3.3) for $u,v,w\in R$,

$$H_{u,w} = H_{u,v}H_{v,w}, \quad H_{u,u}=1.$$

In particular

$$\partial_t\Big((t-\theta^{q^{i-1}})^{n-1},\dots,1\Big) = H_{u,v}$$

with $u=t$ and $w=\theta^{q^{i-1}}$, and

$$\Big[\partial_t\Big((t-\theta^{q^{i-2}})^{n-1},\dots,1\Big)^{-1}\Big]^{[1]} = H_{v,w}$$

where $v=t^q$ and again $w=\theta^{q^{i-1}}$. Therefore the product $W$ equals $H_{u,v}=H_{u,w}H_{v,w}^{-1}$ and does not depend on $\theta$. Note that $W=H_{t,t^q}=H_{t,0}H_{0,t^q}=H_{t,0}H_{t^q,0}^{-1}=V(V^{[1]})^{-1}$ where $V=H_{t,0}$.

Now observe that

$$d_t(b_i^{-n}) = (-1)^{in}\theta^{-n\frac{1-q^i}{1-q}}(-\theta)^{-\frac{n}{q-1}} d_t\left((-\theta)^{\frac{n}{q-1}}\prod_{j=0}^{i-1}\Big(1-\frac{t}{\theta^{q^j}}\Big)^{-n}\right). \tag{4.7}$$

Similarly,

$$d_t(b_{i-1}^n)^{[1]} = (-1)^{(i-1)n}\theta^{nq\frac{1-q^{i-1}}{1-q}}(-\theta)^{\frac{qn}{q-1}} d_t\left((-\theta)^{-\frac{n}{q-1}}\prod_{j=0}^{i-2}\Big(1-\frac{t}{\theta^{q^j}}\Big)^{n}\right)^{[1]}.$$

This implies that the limits (4.6) exist and equal

$$d_t(\omega^{-n})V\Big((d_t(\omega^{-n})V)^{[1]}\Big)^{-1}$$

(dependence in Anderson-Thakur's function). We therefore obtain that the limit matrix $c_\phi$ is well defined in $\mathrm{GL}_n(\mathbb{C}_\infty)$, where $\mathbb{C}_\infty$ is the completion of $L$. Observe now that

$$\mu'_\phi := d_t\Big(\tau(\omega^n)\Big)_{t=\theta}\partial\Big(\theta^{n-1},\dots,1\Big) \tag{4.8}$$

satisfies ${\mu'_\phi}^{(1)}{\mu'_\phi}^{-1} = c_\phi$. By (4.8), $\mu'_\phi \in \mathrm{GL}_n(L)$ and therefore $c_\phi$ does too. □

**Remark 4.5.** An aside remark is that for $n > 1$ we have realized in $\mathrm{GL}_n(R)$ the groupoid given by $R$ as set of objects, with the property that for any $u, v \in R$ there is one and only one isomorphism $u \to v$. This arrow is sent to the matrix $H_{u,v}$. If $n = 1$ the automorphisms $\lambda_k$ commute each other and can be identified with elements of $K^\times$. The above result gives a well known limit first obtained by Carlitz

$$\lim_{k\to\infty} \lambda^q_{C,k-1}\lambda^{-1}_{C,k} = \widetilde{\pi}^{q-1}_C.$$

*End of proof of Theorem 4.2.* We identify the left action by $\mu'_\phi$ in the standard basis with an element of $\mathrm{Aut}_L(\mathbb{G}^n_a)$. We note that, for all $k \geq i \geq 0$,

$$P^{(i)}_{k-i}P^{-1}_k = \Big(P^{(1)}_{k-i}P^{-1}_{k-i+1}\Big)^{(i-1)}\Big(P^{(1)}_{k-i+1}P^{-1}_{k-i+2}\Big)^{(i-2)}\cdots P^{(1)}_{k-1}P^{-1}_k$$

and the sequence converges for all given $i$ to $c^{(i-1)}_\phi c^{(i-2)}_\phi \cdots c_\phi = \mu^{(i)}_\phi \mu^{-1}_\phi$ as $k \to \infty$ by Proposition 4.4. In the standard basis of $\mathbb{G}^n_a$, $(1 \otimes \lambda^{-1}_{\phi,k})E_{\phi,k}(X)$ is represented by

$$\sum_{i=0}^{k} Q_i X^{(i)} P^{(i)}_{k-i}P^{-1}_k.$$

Therefore the limit series $\mathcal{E}_\phi$ is well defined and equals

$$(1 \otimes {\mu'_\phi}^{-1}) \cdot \mathbf{exp}_\phi \cdot (1 \otimes \mu'_\phi). \tag{4.9}$$

It remains to show that

$$\mu'_\phi = \mu_\phi$$

where $\mu_\phi$ is defined in (4.5). By (4.9), we observe that $\mathrm{Ker}(\mathcal{E}_\phi) = h(\Lambda_\phi)^{\oplus n}{\mu'_\phi}^{-1}$. In other words, $\lambda \in \mathrm{Ker}(\mathcal{E}_\phi)$ if and only if $\lambda = \widehat{\Pi}\partial(a_1,\dots,a_n){\mu'_\phi}^{-1}$ for some $a_1,\dots,a_n \in A$ (applying Lemma 3.1). We deduce that $d(A) \subset \mathrm{Ker}(\mathcal{E}_\phi)$. The conclusion follows from the next:

**Lemma 4.6.** *The following identity holds:*

$$\mu'_\phi = \widehat{\Pi}\partial\Big(\theta^{n-1},\dots,1\Big).$$

*Proof.* It suffices to show that

$$d_t(\tau(\omega^n))_{t=\theta} = \widehat{\Pi}. \tag{4.10}$$

Identity (4.10) is known, see for example Maurischat's [7]. We can prove it alternatively as follows. We have seen that $\mathcal{E}_\phi$ vanishes on $d(A)$. Hence, for all $a \in A$, $\mathbf{exp}_\phi(d(a)\mu'_\phi) = 0$.

By (4.9) and the fact that $\mathrm{Ker}(\exp_\phi) = d(A)\Pi$ in $\mathbb{G}_a^n(L)$, for all $a \in A \setminus \{0\}$, there exist $a_1, \dots, a_n \in A \setminus \{0\}$, depending on $a$, such that

$$d(a)\mu'_\phi = \Big(d(a_1)\Pi, \dots, d(a_n)\Pi\Big).$$

Comparing the last rows, we obtain the identities

$$\mu'_{n,i} = \frac{a_i}{a}\pi_0, \quad i = 1, \dots, n,$$

where $\mu'_\phi = (\mu'_{i,j})_{i,j}$ and $\pi_0 = (-\widetilde{\pi}_C)^n$ is the bottom coefficient of $\Pi$. Then we can set $\alpha_i := \frac{a_i}{a}$ for all $i$, independently of the choice of $a$. If we choose $a = 1$ we get, combining with (4.8) and Lemma 3.1:

$$\mu'_\phi = d_t\Big(\tau(\omega^n)\Big)_{t=\theta}\partial\Big(\theta^{n-1}, \dots, 1\Big) = \Big(d(\alpha_1)\Pi, \dots, d(\alpha_n)\Pi\Big) = \widehat{\Pi}\partial\Big(\alpha_1, \dots, \alpha_n\Big).$$

But the lower-right coefficient of $d_t(\tau(\omega^n))_{t=\theta}$ equals $(-\widetilde{\pi}_C)^n = \pi_0$. This implies that $\alpha_i = \theta^{n-i}$ for $i = 1, \dots, n$. Hence (4.10) holds and the identity $\mu_\phi = \mu'_\phi$ follows. □

We deduce that

$$\mathrm{Ker}(\mathcal{E}_\phi) = \{\widehat{\Pi}\partial(a_1, \dots, a_n)\partial(\theta^{n-1}, \dots, 1)^{-1}\widehat{\Pi}^{-1} : a_1, \dots, a_n \in A\} \tag{4.11}$$

and $\mathrm{Ker}(\mathcal{E}_\phi) \cap \mathcal{Z} = d(A)$.

## 5. The module underlying $\mathcal{E}_\phi$

In this section we study the exponential $\mathcal{E}_\phi$ and the module $\widetilde{\boldsymbol{\phi}}$ uniformized by it. Let $T$ be the automorphism of $\mathbb{G}_a^n(L)$ that cyclically permutes the coordinates sending the first to the last, the second to the first etc. If $f$ is a linear endomorphism of $\mathbb{G}_a^n$, then $(fT)(T^{-1}((x)) = x$ defining a right action of the finite $\mathbb{F}_q$-algebra $\mathbb{F}_q[T] \cong \mathbb{F}_q[X]/(X^n - 1)$ on $\mathrm{End}_L(\mathbb{G}_a^n)$. We also have the left action of the $L$-algebra $\mathcal{Z}$, and an action of the commutative $L$-algebra

$$\mathcal{Z}[T] := \mathcal{Z} \otimes_{\mathbb{F}_q} \mathbb{F}_q[T]$$

over $\mathrm{End}_L(\mathbb{G}_a^n)$ (not to be confused with the non-commutative subring of $\mathrm{End}_L(\mathbb{G}_a^n)$ generated by the elements of $\mathcal{Z}$ and $T$).

**Lemma 5.1.** $\mathrm{End}_L(\mathbb{G}_a^n)(L)$ *is free of rank one over* $\mathcal{Z}[T]$*, generated by the identity.*

*Proof.* Since $\mathcal{Z}[T] = \mathcal{Z} \otimes 1 + \mathcal{Z} \otimes T + \cdots + \mathcal{Z} \otimes T^{n-1}$ and $\dim_L(\mathcal{Z}) = n$, it suffices to show that the above sum is direct. Consider $Z_0, \dots, Z_{n-1} \in \mathbb{G}_a^n(L)$ and set $U := \sum_{i=0}^{n-1} \widehat{Z}_i \otimes T^i$. It is easy to show the elementary identity (in the standard basis of $\mathbb{G}_a^n$)

$$(U \otimes T - N \otimes U)(1) = UT - NU = \Big(Z_0, \dots, Z_{n-1}\Big) \in \mathrm{End}_L(\mathbb{G}_a^n). \tag{5.1}$$

Hence if the $L$-linear operator $U \in \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))$ is zero then $Z_0 = \cdots = Z_{n-1} = 0$. □

We extend $\boldsymbol{\phi}$ defined in (4.1) to an injective $A[T]$-algebra morphism, where $A[T] = A \otimes_{\mathbb{F}_q} \mathbb{F}_q[T]$,

$$A[T] \to \mathrm{End}_L(\mathrm{End}_L(\mathbb{G}_a^n))[\boldsymbol{\tau}]$$

by $\boldsymbol{\phi}(a \otimes b) = \boldsymbol{\phi}_a \otimes b$, and $\mathbf{exp}_\phi$ uniformizes the $A[T]$-module $\boldsymbol{\phi}$ over $L$.

Set

$$V := \mu_\phi T \mu_\phi^{-1} = \widehat{\Pi}\partial\Big(\theta^{n-1}, \dots, 1\Big) T \partial\Big(\theta^{n-1}, \dots, 1\Big)^{-1} \widehat{\Pi}^{-1} \in \mathrm{End}_{K_\infty}(\mathbb{G}_a^n).$$

The commutative $\mathbb{F}_q$-algebra $L[V]$ acts on $\mathrm{End}_L(\mathbb{G}_a^n)$ on the right so we have an action of $\mathcal{Z}[V] = \mathcal{Z} \otimes_L L[V] \cong \mathcal{Z}[X]/(X^n - 1)$ and the $\mathcal{Z}$-module $\mathrm{End}_L(\mathbb{G}_a^n)$ is free of rank $n$, while as a $\mathcal{Z}[V]$-module it is free of rank one generated by the identity.

We recall the Anderson $A$-module

$$\widetilde{\boldsymbol{\phi}} : A \to \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$$

defined by

$$\widetilde{\boldsymbol{\phi}}_\theta = d(\theta) \otimes 1 + (1 \otimes \mu_\phi^{-1})(e_{n,1} \otimes 1)\boldsymbol{\tau}(1 \otimes \mu_\phi) \tag{5.2}$$

that is, the module $(1 \otimes \mu_\phi^{-1})\boldsymbol{\phi}(1 \otimes \mu_\phi)$ where $\boldsymbol{\phi}$ is defined in (4.1). By Theorem 4.2, $\widetilde{\boldsymbol{\phi}}$ is uniformized by $\mathcal{E}_\phi$. Moreover, $\mathrm{End}_L(\mathbb{G}_a^n)$ acquires a structure of $A[V]$-module $\widetilde{\boldsymbol{\phi}}(\mathrm{End}_L(\mathbb{G}_a^n))$ (with $V$ that acts on the right), and $\mathcal{E}_\phi$ is also an $A[V]$-module morphism (it is easily seen that $\mathcal{E}_\phi(XV) = \mathcal{E}_\phi(X)V$). The $A[V]$-module structure of $\mathrm{Lie}(\widetilde{\boldsymbol{\phi}})$ is given by $(a \otimes V)(X) = d(a)XV$ for $a \in A$, $X \in \mathrm{End}_L(\mathbb{G}_a^n)$. The next result is the synthesis of the main properties of the module $\widetilde{\boldsymbol{\phi}}$.

**Theorem 5.2.** *The affine algebraic $\mathbb{F}_q$-vector space scheme $\mathrm{End}_L(\mathbb{G}_a^n)$, endowed with the structure of Anderson $A$-module $\widetilde{\boldsymbol{\phi}}$, has rank $n$ and dimension $n^2$. It is defined over $K_\infty$ and is $K_\infty$-isomorphic to $\phi^{\oplus n}$. Additionally it is an $A[V]$-module and its period lattice, seen as an $A[V]$-module, is free of rank one generated by the identity.*

*Proof.* We set, for $i \in \{0, \dots, n-1\}$, $\mathcal{Z}_i = \mathcal{Z}V^i$. By Lemma 5.1 we have the equality of left $\mathcal{Z}$-modules $\mathrm{End}_L(\mathbb{G}_a^n) = \oplus_i \mathcal{Z}_i$. By (4.11) we see that

$$\mathcal{Z}_i \cap \mathrm{Ker}(\mathcal{E}_\phi) = d_\phi(A)V^i,$$

a left $A$-module which is free of rank one, generated by $V^i$. Hence $\mathrm{Ker}(\mathcal{E}_\phi)$ splits over $K_\infty$ as a direct sum of free $A$-modules of rank one: $\mathrm{Ker}(\mathcal{E}_\phi) = \oplus_i d_\phi(A)V^i$. It is itself a free $A[V]$-module of rank one, generated by the identity. We set

$$\Phi(L) = \mathcal{E}_\phi(\mathcal{Z}).$$

For $Z \in \mathbb{G}_a^n(L)$ we have $\exp_\phi^{\oplus n}(\widehat{Z}\mu_\phi) = \exp_\phi^{\oplus n}(\widehat{Z}\widehat{\Pi}\partial(\theta^{n-1}, \dots, 1))$. We set $\widehat{W} = \widehat{Z}\widehat{\Pi} \in \mathcal{Z}$, for some $W \in \mathbb{G}_a^n(L)$. By Lemma 3.1

$$\widehat{W}\partial\big(\theta^{n-1}, \dots, 1\Big) = \Big(d(\theta^{n-1})W, \dots, d(1)W\Big).$$

Hence we get

$$\exp_\phi\left(\widehat{Z}\mu_\phi\right) = \Big(\phi_{\theta^{n-1}}(X), \dots, \phi_\theta(X), X\Big),$$

where $X = \exp_\phi(W)$. We deduce that $\Phi(L)$ equals the image of the map $\mathbb{G}_a^n(L) \to \mathrm{End}_L(\mathbb{G}_a^n)$ defined by

$$X \mapsto \Big(\phi_{\theta^{n-1}}(X), \dots, \phi_\theta(X), X\Big)\mu_\phi^{-1},$$

which is an affine algebraic $\mathbb{F}_q$-vector space scheme of dimension $n$ and carries, via $\widetilde{\boldsymbol{\phi}}$, a structure of Anderson $A$-module $K_\infty$-isomorphic to $\phi$. Since $\mathcal{E}_\phi(\mathcal{Z}_i) = \mathcal{E}_\phi(\mathcal{Z}V) = \mathcal{E}_\phi(\mathcal{Z})V$ we deduce that $\mathrm{End}_L(\mathbb{G}_a^n) = \oplus_i \Phi_i(L)$ where $\Phi_i = \Phi V^i$. □

To restate things differently, we have the exact sequence of $A[V]$-modules

$$0 \to \big(d(A)\Pi\big)^{\oplus n}\mu_\phi^{-1} \to \mathrm{End}_L(\mathbb{G}_a^n) \xrightarrow{\mathcal{E}_\phi} \widetilde{\boldsymbol{\phi}}(\mathrm{End}_L(\mathbb{G}_a^n)) \to 0$$

that splits into the direct sum of the exact sequences of $A$-modules

$$0 \to d_\phi(A)V^i \to \mathcal{Z}_i \xrightarrow{\mathcal{E}_\phi} \Phi_i(L) \to 0, \tag{5.3}$$

for $i = 0, \dots, n-1$, a set of sequences that is cyclically permuted by right action of $V$.

## 6. Factorization of Carlitz's operators

We examine the problem of factorizing the operators $E_{\phi,k}$ in order to obtain a higher dimensional partial generalization of the formula (1.2). There are two types of factorization (we do not expect to have uniqueness). The first type is described in §6.1, the factors are in $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$. The second one, apparently more suitable for applications, is described in §6.2 and has factors in $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$, but seems to hold only after evaluation on $\mathcal{Z}$. Both factorizations are rooted in an identity that we describe now.

For all $k \geq 0$ we subtract the two identities of Lemma 2.1. We get, for $X \in \mathrm{End}_L(\mathbb{G}_a^n)$ (this holds for any $\phi$):

$$E_{\phi,k}(1 \otimes X - X \otimes 1) = \sum_{i+j=k} \Big((1 \otimes E_{\phi,i}(X)^{(j)}) - (E_{\phi,i}(X) \otimes 1)\boldsymbol{\tau}^i\Big)E_{\phi,j}. \tag{6.1}$$

We choose $X = d(a)$ with $a \in A(\leq n)$ and we work with $\phi = C^{\otimes n}$. By some essentially well known calculations (see Papanikolas [9]) we can write ([6])

$$\phi_a = d(a) + \Delta_a\tau$$

for an endomorphism $\Delta_a \in \mathrm{End}_L(\mathbb{G}_a^n)$ and we get $E_{\phi,0}(d(a)) = d(a)$ (this holds for any $\phi$ as $E_{\phi,0}$ is just the identity), $E_{\phi,1}(d(a)) = \Delta_a$ and $E_{\phi,k}(d(\theta)) = 0$ for all $k \geq 2$ (compare with Corollary 2.3). Formula (6.1) becomes:

$$E_{\phi,k}(1\otimes d(a) - d(a)\otimes 1) = \Big((1\otimes d(a)^{(k)}) - (d(a)\otimes 1)\Big)E_{\phi,k} + \Big((1\otimes \Delta_a^{(k-1)}) - (\Delta_a \otimes 1)\boldsymbol{\tau}\Big)E_{\phi,k-1}.$$

[6] We anticipate that this gives the shape of the factors of our factorizations as combinations of 1 and $\boldsymbol{\tau}$.

The simplest choice is $a = \theta$ and from this point on, we restrict to this case. Other interesting cases are those determined by $a \in A \setminus \{0\}$ with $\deg_\theta(a) \leq n$ but to avoid a too long discussion we only focus on this one. Noting that in this case $d(\theta) = \theta + N$ and $\Delta_a = e_{n,1}$, we can rewrite the above identity as:

$$(\theta - \theta^{(k)})E_{\phi,k} + ((N \otimes 1) - (1 \otimes N))E_{\phi,k} - E_{\phi,k}((N \otimes 1) - (1 \otimes N)) = \\ = \Big((1 \otimes e_{n,1}) - (e_{n,1} \otimes 1)\boldsymbol{\tau}\Big)E_{\phi,k-1}. \tag{6.2}$$

It is not difficult to see that these relations determine the sequence of linear operators $(E_{\phi,k})_{k\geq 0} \subset \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$ in a unique way.

6.1. **Non-commutative product expansions in** $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$**.** The main results of this subsection are Theorem 6.1 and Corollary 6.2. For the reader: it seems that these results do not yield particularly interesting scalar identities and the next section is more interesting. The reader can therefore skip to §6.2 in a first reading.

The identity (6.2) holds in $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$ but there is a canonical way to interpret the left-hand side as an evaluation (or "left-evaluation") by some linear operator with coefficients in $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$. Obviously

$$\mathrm{End}_L^{(3)}(\mathbb{G}_a^n) = \mathrm{End}_L^{(2)}(\mathbb{G}_a^n) \otimes_L \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)^\circ = \\ = \mathrm{End}_L(\mathbb{G}_a^n) \otimes_L \mathrm{End}_L(\mathbb{G}_a^n)^\circ \otimes_L \mathrm{End}_L(\mathbb{G}_a^n)^\circ \otimes_L \mathrm{End}_L(\mathbb{G}_a^n)^{\circ\circ}.$$

so that the multiplication rule is defined by

$$(a \otimes b \otimes c \otimes d)(a' \otimes b' \otimes c' \otimes d') = (aa' \otimes b'b \otimes c'c \otimes dd').$$

There is a left-evaluation map (or evaluation) that sends, once an element of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ is chosen, an operator of $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$ to an element of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$. It is determined by the rule

$$(a \otimes b \otimes c \otimes d)(x \otimes y) = axc \otimes dyb.$$

This makes $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ into a left $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$-module, and the picture could be generalized even further. Let $\boldsymbol{\tau}$ be the $\mathbb{F}_q$-linear operator such that $\boldsymbol{\tau}X = \boldsymbol{\tau}(X)\boldsymbol{\tau} = X^{(1)}\boldsymbol{\tau}$ for $X \in \mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$. We define the evaluation $\boldsymbol{\tau}^i(\boldsymbol{\tau}^j)$ of $\boldsymbol{\tau}^i$ at $\boldsymbol{\tau}^j$ to be $\boldsymbol{\tau}^{i+j}$. In this way we have extended the above evaluation map to an evaluation map of operators of $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)[\tau]$ at operators of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}](= \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}])$ making $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$ into a left $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\tau]$-module. So we have three $\tau$'s:

| | |
|---|---|
| $\tau$ | over $\mathbb{G}_a^n$ |
| $\boldsymbol{\tau}$ | over $\mathrm{End}_L(\mathbb{G}_a^n)$ |
| $\boldsymbol{\tau}$ | over $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ |

Expanding $E_{\phi,k} = \sum_{i=0}^{k} (\epsilon_{\phi,i} \otimes \lambda_{\phi,k-i}^{(i)}) \boldsymbol{\tau}^i$ and expanding

$$(\theta - \theta^{(k)})E_{\phi,k} + (N \otimes 1 - 1 \otimes N)E_{\phi,k} + E_{\phi,k}(1 \otimes N - N \otimes 1) = \sum_{i=0}^{k} c_i \boldsymbol{\tau}^i \in \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$$

we can view, by the fact that $\tau N = N\tau$, $c_i$ as an evaluation:

$$c_i = \nu_k\big(\epsilon_{\phi,i} \otimes \lambda_{\phi,k-i}^{(i)}\big), \quad 0 \le i \le k,$$

where $\nu_k$ is the element of $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$ defined by:

$$\begin{aligned} \nu_k &= d(\theta) \otimes 1 \otimes 1 \otimes 1 - 1 \otimes d(\theta)^{(k)} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes d(\theta) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes d(\theta) \\ &= (\theta - \theta^{(k)}) + N \otimes 1 \otimes 1 \otimes 1 - 1 \otimes N \otimes 1 \otimes 1 - 1 \otimes 1 \otimes N \otimes 1 + 1 \otimes 1 \otimes 1 \otimes N \\ &= (\theta - \theta^{(k)}) - \boldsymbol{N}, \end{aligned}$$

having set

$$\boldsymbol{N} = -N \otimes 1 \otimes 1 \otimes 1 + 1 \otimes N \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes N + 1 \otimes 1 \otimes N \otimes 1.$$

The element $\boldsymbol{N}$ is nilpotent in $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$. More precisely, one sees that $\boldsymbol{N}^{4(n-1)} \neq 0$ and $\boldsymbol{N}^{4(n-1)+1} = 0$. Therefore for $k > 0$, $\nu_k$ is an automorphism and we have an explicit formula for the inverse:

$$\nu_k^{-1} = \sum_{i=0}^{4(n-1)} (\theta - \theta^{(k)})^{-i-1} \boldsymbol{N}^i, \tag{6.3}$$

which justifies again that the sequence $(E_{\phi,k})_k$ is uniquely determined by the condition that $E_{\phi,0}$ is the identity and (6.2). We can rewrite (6.2) as:

$$E_{\phi,k} = \Big(\nu_k^{-1}\big((1 \otimes e_{n,1} \otimes 1 \otimes 1) - (e_{n,1} \otimes 1 \otimes 1 \otimes 1)\tau\big)\Big)(E_{\phi,k-1}), \quad k \ge 1$$

(evaluation).

In terms of the normalizations $\mathcal{E}_{\phi,k}$ we have, by (4.4), the equivalent formulation

$$\begin{aligned} &\mathcal{E}_{\phi,k} = \\ &= \Big((1 \otimes \lambda_{\phi,k}^{-1} \otimes 1 \otimes 1)\nu_k^{-1}\big((1 \otimes e_{n,1} \otimes 1 \otimes 1) - (e_{n,1} \otimes 1 \otimes 1 \otimes 1)\tau\big) \times \\ &\quad \times (1 \otimes \lambda_{\phi,k-1} \otimes 1 \otimes 1)\Big)(\mathcal{E}_{\phi,k-1}). \end{aligned}$$

Setting for $k \ge 1$

$$\begin{aligned} \gamma_k &= (1 \otimes \lambda_{\phi,k}^{-1} \otimes 1 \otimes 1)\nu_k^{-1}(1 \otimes e_{n,1} \otimes 1 \otimes 1)(1 \otimes \lambda_{\phi,k-1} \otimes 1 \otimes 1), \\ \delta_k &= -(1 \otimes \lambda_{\phi,k}^{-1} \otimes 1 \otimes 1)\nu_k^{-1}(e_{n,1} \otimes 1 \otimes 1 \otimes 1)(1 \otimes \lambda_{\phi,k-1}^{(1)} \otimes 1 \otimes 1), \end{aligned}$$

we have reached the next result.

**Theorem 6.1.** *For all $k \geq 1$, we can express $\mathcal{E}_{\phi,k}$ as the evaluation at $1 \otimes 1$ of an operator in $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)[\tau]$:*

$$\mathcal{E}_{\phi,k} = \Big((\gamma_k + \delta_k\tau)(\gamma_{k-1} + \delta_{k-1}\tau)\cdots(\gamma_1 + \delta_1\tau)\Big)(1 \otimes 1).$$

We can combine this result with Theorem 4.2. We set

$$f_{k,i} := \sum_{k \geq d_1 > \cdots > d_i > 0} \gamma_k \cdots \gamma_{d_1+1}\delta_{d_1}\gamma_{d_1-1}^{(1)} \cdots \gamma_{d_2+1}^{(1)}\delta_{d_2}^{(1)}\gamma_{d_2-1}^{(2)} \cdots \gamma_{d_i+1}^{(i-1)}\delta_{d_i}^{(i-1)}\gamma_{d_i+1}^{(i)} \cdots \gamma_1^{(i)}$$

so that $f_{k,i} \in \mathrm{End}_L^{(3)}(\mathbb{G}_a^n)$ and

$$\Big(\gamma_k + \delta_k\tau\Big) \cdots \Big(\gamma_1 + \delta_1\tau\Big) = \sum_{i=0}^{k} f_{k,i}\tau^i \in \mathrm{End}_L^{(3)}(\mathbb{G}_a^n)[\tau].$$

For example:

$$\begin{aligned} f_{k,0} &= \gamma_k\gamma_{k-1}\cdots\gamma_1, \\ f_{k,1} &= \sum_{d=1}^{k} \gamma_k \cdots \gamma_{d+1}\delta_d\gamma_{d-1}^{(1)} \cdots \gamma_1^{(1)}, \\ f_{k,k} &= \delta_k\delta_{k-1}^{(1)} \cdots \delta_1^{(k-1)}. \end{aligned} \tag{6.4}$$

We deduce, from Theorem 6.1 and Theorem 4.2:

**Corollary 6.2.** *For all $i \geq 0$ the limit $\lim_{k\to\infty} f_{k,i}(1 \otimes 1)$ exists in $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ and equals $\epsilon_{\phi,i} \otimes \mu_\phi^{(i)}\mu_\phi^{-1}$. Moreover, the limit $\lim_{k\to\infty}(\gamma_k + \delta_k\tau)\cdots(\gamma_1 + \delta_1\tau)(1 \otimes 1)$ exists and equals $\mathcal{E}_\phi$.*

The limit above is for the metric of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ induced by $L$, seeing $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ as a vector space of dimension $n^4$ over $L$, with the standard basis given by $(e_{i_1,j_1} \otimes e_{i_2,j_2})$. We do not know if the left-infinite product $\cdots(\gamma_k + \delta_k\tau)\cdots(\gamma_1 + \delta_1\tau)$ converges to an operator in $\mathrm{End}_L^{(3)}(\mathbb{G}_a^n)[[\tau]]$. Therefore, we cannot conclude that the formulas of Corollary 6.2 are equivalent to a non-commutative factorization of some operator, unless $n = 1$.

6.2. **Evaluation over $\mathcal{Z}$.** We come back to (6.2). We can evaluate both sides at $X \in \mathcal{Z}$, contained in the kernel of the operator $N \otimes 1 - 1 \otimes N$. Recall that $\mu_k = d(\theta) \otimes 1 - 1 \otimes d(\theta)^{(k)}$. We have the evaluation formula

$$\mu_k(E_{\phi,k}(X)) = \big((1 \otimes e_{n,1}) - (e_{n,1} \otimes 1)\boldsymbol{\tau}\big)(E_{\phi,k-1}(X)), \quad k \geq 1. \tag{6.5}$$

Since $\mu_k = (\theta - \theta^{(k)}) + N \otimes 1 - 1 \otimes N$, $\mu_k$ is an automorphism, and the sequence $(E_{\phi,k})_k$ is uniquely determined by the condition $E_{\phi,0} = 1 \otimes 1$.

**Remark 6.3.** The following consequence of (6.5) gives some new information on the structure of the kernels of these operators. We recall that $A(\leq s)$ denotes the $\mathbb{F}_q$-vector space of all polynomials of degree $\leq s$.

**Corollary 6.4.** *For $k \geq 1$, $\mathrm{Ker}(E_{\phi,k}) \cap \mathcal{Z} = d\Big(A(\leq (k-1)n)\Big)$.*

*Proof.* It is known that if $a \in A(\leq (k-1)n)$, then $E_{\phi,k}(d(a)) = 0$ (see Papanikolas' [9, Proposition 3.5.6]). One sees in fact that $\mathrm{Ker}(E_{\phi,k}) \cap d(A) = d(A(\leq (k-1)n))$. From (6.5) and the fact that $\mu_k$ is an automorphism we also see that $\mathrm{Ker}(E_{\phi,k-1}) \cap \mathcal{Z} \subset \mathrm{Ker}(E_{\phi,k}) \cap \mathcal{Z}$. But $\mathrm{Ker}(E_{\phi,k}) = \mathrm{Ker}(\mathcal{E}_{\phi,k})$ and $(\mathcal{E}_{\phi,k})_k$ converges to $\mathcal{E}_\phi$ (it uniformly converges on every bounded subset of $\mathrm{End}_L(\mathbb{G}_a^n)$). If $X \in \mathrm{Ker}(E_{\phi,k}) \cap \mathcal{Z}$ then $0 = \mathcal{E}_{\phi,i}(X)$ for all $i \geq k$ and by the fact that $\mathcal{E}_\phi(X) = \lim_i \mathcal{E}_{\phi,i}(X)$, we get $\mathcal{E}_\phi(X) = 0$. Hence $X = d(a)$ by (5.3). □

Iterating (6.5) we obtain the factorization, for $k \geq 1$:

$$\mathcal{E}_{\phi,k}\big|_{\mathcal{Z}} = (1 \otimes \lambda_{\phi,k}^{-1})\mu_k^{-1}((1 \otimes e_{n,1}) - (e_{n,1} \otimes 1)\boldsymbol{\tau}) \cdots \mu_1^{-1}((1 \otimes e_{n,1}) - (e_{n,1} \otimes 1)\boldsymbol{\tau})\big|_{\mathcal{Z}},$$

so that:

$$\mathcal{E}_{\phi,k}\big|_{\mathcal{Z}} = (\widetilde{\gamma}_k + \widetilde{\delta}_k\boldsymbol{\tau})(\widetilde{\gamma}_{k-1} + \widetilde{\delta}_{k-1}\boldsymbol{\tau}) \cdots (\widetilde{\gamma}_1 + \widetilde{\delta}_1\boldsymbol{\tau})\big|_{\mathcal{Z}},$$

where for all $k \geq 1$, $\widetilde{\gamma}_k, \widetilde{\delta}_k$ are the elements of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$:

$$\widetilde{\gamma}_k = (1 \otimes \lambda_{\phi,k}^{-1})\mu_k^{-1}(1 \otimes e_{n,1})(1 \otimes \lambda_{\phi,k-1}), \tag{6.6}$$

$$\widetilde{\delta}_k = (1 \otimes \lambda_{\phi,k}^{-1})\mu_k^{-1}(e_{n,1} \otimes 1)(1 \otimes \lambda_{\phi,k-1}^{(1)}). \tag{6.7}$$

Taking the limit $k \to \infty$ and combining with Theorem 4.2 we get:

**Theorem 6.5.** *For all $X \in \mathcal{Z}$,*

$$\lim_{k\to\infty} (\widetilde{\gamma}_k + \widetilde{\delta}_k\boldsymbol{\tau}) \cdots (\widetilde{\gamma}_1 + \widetilde{\delta}_1\boldsymbol{\tau})(X) = \big((1 \otimes \mu_\phi^{-1}))\mathbf{exp}_\phi(1 \otimes \mu_\phi^{-1})\big)(X).$$

We do not know if the product $\cdots (\widetilde{\gamma}_k + \widetilde{\delta}_k\boldsymbol{\tau}) \cdots (\widetilde{\gamma}_1 + \widetilde{\delta}_1\boldsymbol{\tau})$ converges to an operator of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$.

Setting:

$$\widetilde{f}_{k,i} := \sum_{k \geq d_1 > \cdots > d_i > 0} \widetilde{\gamma}_k \cdots \widetilde{\gamma}_{d_1+1}\widetilde{\delta}_{d_1}\widetilde{\gamma}_{d_1-1}^{(1)} \cdots \widetilde{\gamma}_{d_2+1}^{(1)}\widetilde{\delta}_{d_2}^{(1)}\widetilde{\gamma}_{d_2-1}^{(2)} \cdots \widetilde{\gamma}_{d_i+1}^{(i-1)}\widetilde{\delta}_{d_i}^{(i-1)}\widetilde{\gamma}_{d_i+1}^{(i)} \cdots \widetilde{\gamma}_1^{(i)}, \tag{6.8}$$

which is an element of $\mathrm{End}_K^{(2)}(\mathbb{G}_a^n)$, we also obtain:

**Theorem 6.6.** *The following identities hold:*

$$\widetilde{f}_{k,i}|_{\mathcal{Z}} = \epsilon_{\phi,i} \otimes \lambda_{\phi,k-i}^{(i)}\lambda_{\phi,k}^{-1}\big|_{\mathcal{Z}}, \quad k \geq i \geq 0. \tag{6.9}$$

*Moreover, for all $X \in \mathcal{Z}$, the limit*

$$\widetilde{f}_{\infty,i}(X) := \lim_{k\to\infty} \widetilde{f}_{k,i}(X)$$

*exists in $\mathrm{End}_L(\mathbb{G}_a^n)$. Its value is the evaluation $(\epsilon_{\phi,i} \otimes \mu_\phi^{(i)}\mu_\phi^{-1})(X) \in \mathrm{End}_L(\mathbb{G}_a^n)$.*

*Proof.* It suffices to check the following property. Assume that an element

$$f \in \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[[\boldsymbol{\tau}]], \quad f = \sum_i f_i \boldsymbol{\tau}^i$$

locally convergent at 0, is given. If, locally at 0 we have $f|_{\mathcal{Z}} = 0$ (the evaluation of $f$ at elements of $\mathcal{Z} = L[N]$ with small enough norms $|\cdot|$ is well defined), then $f_0|_{\mathcal{Z}} = f_1|_{\mathcal{Z}} = \cdots = 0$. To verify this property note that, for $X_{j,z} := zN^j \in \mathcal{Z}$ with $j = 0, \ldots, n-1$ and $z \in \mathbb{C}_\infty = \widehat{L}_{|\cdot|}$ (completion) with $|z| < c$ for some appropriately chosen $c \in |L^\times|$, having denoted $D := D_{\mathbb{C}_\infty}(0, c) = \{z \in \mathbb{C}_\infty : |z| < c\}$, the correspondence $z \mapsto f(X_{j,z})$ defines an $\mathbb{F}_q$-linear analytic function $D \to \mathrm{End}_L(\mathbb{G}_a^n)$ which is identically zero. This implies that $z \mapsto f_i(X_{j,z})$ vanishes identically for all $i$, so that $f_i|_{\mathcal{Z}} = 0$ for all $i$.

The above remark applied to elements of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)[\boldsymbol{\tau}]$ ensures that, for all $k \geq i \geq 0$, (6.6) holds (evaluations of coefficients agree). Taking the limit for $k \to \infty$ in the left-hand side of (6.6) we see that for all $X \in \mathcal{Z}$, the limit $\widetilde{f}_{\infty,i}(X) = \lim_{k\to\infty} \widetilde{f}_{k,i}(X)$ exists in $\mathrm{End}_L(\mathbb{G}_a^n)$. Its value is, by Theorem 4.2, the evaluation $(\epsilon_{\phi,i} \otimes \mu_\phi^{(i)} \mu_\phi^{-1})(X) \in \mathrm{End}_L(\mathbb{G}_a^n)$. □

It is important to notice that given integers $i \leq k$, identity (6.9) connects restrictions of endomorphisms to $\mathcal{Z}$.

## 7. A study of endomorphisms

The next task is to turn the evaluations of the functions $\widetilde{f}_{\infty,i}$ into some kind of higher variants of (multiple) polylogarithms. In the next proposition we describe enough properties of the operators $\widetilde{\gamma}_k, \widetilde{\delta}_k$ to simplify these expressions so that they become limits of finite expressions involving products and sums of elements of $\mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$. One important point is to recognize that the maps $\widetilde{\gamma}_k$ are projections on $\mathcal{Z}$.

**Proposition 7.1.** *For all $k \geq 1$, the map $\widetilde{\gamma}_k$ is a left $\mathcal{Z}$-module morphism. There is a direct sum decomposition $\mathrm{End}_L(\mathbb{G}_a^n) = (1 \otimes \lambda_{\phi,k-1}^{(1)}) \operatorname{Ker}(1 \otimes e_{n,1}) \oplus \mathcal{Z}$. With respect to this decomposition, $\widetilde{\gamma}_k = 0 \oplus 1$. Hence for all $k$, $\widetilde{\gamma}_k$ is a projection on $\mathcal{Z}$.*

*Proof.* The computation of the kernel of $\widetilde{\gamma}_k$ is obvious (the operator $(1 \otimes \lambda_{\phi,k}) \mu_k^{-1} (1 \otimes \lambda_{\phi,k-1})$ is an isomorphism), and we deduce also that $\widetilde{\gamma}_k$ has rank $n$ for all $k$. It is also clear from the definition that it is a left $\mathcal{Z}$-module morphism.

To compute the image of $\widetilde{\gamma}_k$ one tool we need is an explicit computation of $\mu_{X,Y}^{-1}(1 \otimes e_{n,1})$ ($\mu_{X,Y}$ is first used in Lemma 3.3) from which we deduce that $\widetilde{\gamma}_k$ is a projection. If $S \in \mathrm{End}_L(\mathbb{G}_a^n)$ we have

$$Se_{n,1} = \widehat{[S]} e_{n,1} = h([S]) e_{n,1} \tag{7.1}$$

(in the standard basis, the matrix is given by the left to right concatenation of the last column of $Y$ followed by $n-1$ vanishing columns). By the fact that the map $\mu_{X,Y}$ is a left $\mathcal{Z}$-module homomorphism and by Lemma 3.3 applied in the case $i = n, j = 1$,

$$\mu_{X,Y}^{-1}(Se_{n,1}) = \mu_{X,Y}^{-1}(h([S])e_{n,1}) = h([S])\mu_{X,Y}^{-1}(e_{n,1}) = h([S])W_{X,Y}$$

hence for all $X \in \mathrm{End}_L(\mathbb{G}_a^n)$,

$$\begin{aligned}\widetilde{\gamma}_k(X) &= (1\otimes\lambda_{\phi,k}^{-1})\mu_k^{-1}(1\otimes e_{n,1})(1\otimes\lambda_{\phi,k-1})(X)\\ &= (1\otimes\lambda_{\phi,k}^{-1})\mu_k^{-1}\big([X\lambda_{\phi,k-1}e_{n,1}]\big)\\ &= (1\otimes\lambda_{\phi,k}^{-1})h\big([X\lambda_{\phi,k-1}]\big)W_{\theta,\theta^{(k)}}\\ &= h\big([X\lambda_{\phi,k-1}]\big)W_{\theta,\theta^{(k)}}\lambda_{\phi,k}^{-1}.\end{aligned}$$

We claim that for all $k>0$,

$$W_{\theta,\theta^{(k)}}\lambda_{\phi,k}^{-1} = W_{\theta,\theta^{(k)}}P_k^{-1} = d_t(\tau(b_{k-1}))_{t=\theta}^n \in \mathcal{Z}.$$

Indeed we have, taking into account the identity (4.3), equivalent to the identity $P_k = d_t(\tau(b_k^{-n}))_{t=\theta}H_{\theta,\theta^{(k)}}$ and the definition of $W_{X,Y}$:

$$W_{\theta,\theta^{(k)}}P_k^{-1} = \Big(d_t((t-\theta^{(k)})^{-n})H_{\theta,\theta^{(k)}}H_{\theta,\theta^{(k)}}^{-1}d_t(\tau(b_k)^n)\Big)_{t=\theta} = \Big(d_t((t-\theta^{(k)})^{-n})d_t(\tau(b_k)^n)\Big)_{t=\theta}.$$

In particular we have obtained, in the standard basis:

$$\widetilde{\gamma}_k(X) = h([XP_{k-1}])d_t(\tau(b_{k-1})^n)_{t=\theta} \in \mathcal{Z}. \tag{7.2}$$

We have, by (7.2), $\widetilde{\gamma}_k(1) = d_t(\tau(b_{k-1})^n)_{t=\theta}h([P_{k-1}])$. Observe that

$$[P_{k-1}] = [d_t(\tau(b_{k-1})^{-n})H_{t,\theta^{q^{k-1}}}]_{t=\theta} = \partial_t(\tau(b_{k-1})^{-n})_{t=\theta}$$

(this follows from (4.3) and the fact that the last column of $H_{X,Y}$ is $e_n$ the last element of the standard basis), so that

$$h([P_{k-1}]) = d_t(\tau(b_{k-1})^{-n})_{t=\theta}.$$

Inserting this expression in the above we get the fundamental property that

$$\widetilde{\gamma}_k(1) = 1.$$

In particular, if $X \in \mathcal{Z}$, $\widetilde{\gamma}_k(X) = X\gamma_k(1) = X$ by the fact that for all $k$, $\widetilde{\gamma}_k$ is a left $\mathcal{Z}$-module morphism, and $\widetilde{\gamma}_k$ induces the identity on $\mathcal{Z}$. But $\widetilde{\gamma}_k$ has rank $n$ so that it is a projection over $\mathcal{Z}$ as expected. □

We have two corollaries.

**Corollary 7.2.** *If $X \in \mathrm{End}_L(\mathbb{G}_a^n)$ we have, for all $k > d \geq 0$:*

$$\gamma_k \cdots \gamma_{d+1}(X) = \gamma_{d+1}(X).$$

*Proof.* This is immediate. □

This gives a simplification in the expression of the sums $\widetilde{f}_{k,i}$ in (6.8) that makes them closer in aspect to partial multiple sums.

For the next corollary we introduce the map

$$\pi : \mathcal{Z} \to L$$

which sends an element $X$ to the element $\pi(X) \in L$ which is the unique element of $L$ that is repeated to form the diagonal of $X$.

7.1. **Polylogarithm identities.** One can extract multiple polylogarithm identities (evaluated at one), or multiple zeta values, from (1.2) by comparing the coefficients of the powers of $\tau$. The reader is directed to [4, 17] for details. If $n > 1$ it is considerably more difficult to obtain this kind of result. More tools are needed to handle repeated applications of the operators $\widetilde{\gamma}_k, \widetilde{\delta}_k$ and we hope to come back to these questions in another work. By the second part of Theorem 6.6, if $i$ is fixed, and $X \in \mathcal{Z}$, then we have the equalities of convergent limits:

$$(\epsilon_{\phi,i} \otimes \mu_\phi^{(i)} \mu_\phi^{-1})(X) = \lim_{k\to\infty} (\epsilon_{\phi,i} \otimes \lambda_{\phi,k-i}^{(i)} \lambda_{\phi,k}^{-1})(X) = \lim_{k\to\infty} \widetilde{f}_{k,i}(X) = \widetilde{f}_{\infty,i}(X).$$

We shall focus on the case $i = 1$. If $n = 1$ we know that this equality implies the classical Carlitz zeta identity (1.3). As an immediate consequence of Proposition 7.1 and Theorem 6.6 we get the following refinement of (7.1) in the case $i = 1$.

**Lemma 7.3.** *For all $X \in \mathcal{Z}$ we have*

$$\lim_{k\to\infty} \Big( \widetilde{\delta}_k(X) + \sum_{k>d\geq 1} \widetilde{\gamma}_{d+1}(\widetilde{\delta}_d(X)) \Big) = (\epsilon_{\phi,1} \otimes \mu_\phi^{(1)} \mu_\phi^{-1})(X).$$

We set, for $n \geq 1$:

$$S_n(X, Y, Z) := \sum_{i=0}^{n-1} (X - Y)^{n-1-i} (Z - Y)^i \in \mathbb{F}_q[X, Y, Z]. \tag{7.3}$$

**Theorem 7.4.** *For $n \geq 1$, we write $S_n(\theta, Y, \theta^{(1)}) = \sum_{i=0}^{n-1} c_{n,i} Y^i \in A[Y]$, so that the coefficients $c_{n,0}, \ldots, c_{n,n-1}$ belong to $A$. Then*

$$\sum_{i=0}^{n-1} c_{n,i} \operatorname{Li}_{n(q-1)}(\theta^{qi}) = \frac{\widetilde{\pi}_C^{n(q-1)}}{\theta - \theta^q}.$$

To prove Theorem 7.4 we consider evaluations of the formulas of Theorem 6.6 and more precisely of Lemma 7.3 at $X = N^i \in \mathcal{Z}$ with $i = 0, \ldots, n-1$, with the convention that $N^0 = 1$. The evaluation at $X = N^{n-1}$ is the one that delivers the result. Computations for other powers of $N$ seem to lead to intricate and hard to describe scalar formulas.

We proceed in three steps. The first step, in §7.1.1, is a simple explicit computation of $(\epsilon_{\phi,1} \otimes \mu_\phi^{(1)} \mu_\phi^{-1})(N^i)$. One sees easily that the uppermost left coefficient of this evaluation in the case $i = n - 1$ and in the standard basis corresponds to the right-hand side of the identity of Theorem 7.4. The second step is the core of the proof and is divided in two parts. Viewing Lemma 7.3 it is desirable to split the limit on the left-hand side in two components, but this is not always possible. Fortunately, it can be done precisely in the case of the evaluation at $N^{n-1}$. In §7.1.2 we apply Proposition 7.1 to describe an explicit computation of $\widetilde{\gamma}_{k+1}\widetilde{\delta}_k$ and consequent explicit evaluation at $N^{n-1}$. This allows to show that the series $\sum_k \widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^{i-1})$ converges. Therefore, thanks to Lemma 7.3, the limit for $k \to \infty$ of $\widetilde{\delta}_k(N^{n-1})$ exists. In §7.1.3 we gather some explicit data to compute this limit, which is quite interesting in itself, and is non-zero. In particular, the uppermost left coefficient of this limit vanishes. But the uppermost left coefficient of $\sum_k \widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^{i-1})$

can be proved to be the left-hand side of the scalar identities of Theorem 7.4, and we reach the conclusion.

7.1.1. *Proof of Theorem 7.4: first step.* We compute the evaluation of $\epsilon_{\phi,1} \otimes \mu_\phi^{(1)} \mu_\phi^{-1}$ at $X = N^i \in \mathcal{Z}$. We get:

**Proposition 7.5.** *The evaluation of $\epsilon_{\phi,1} \otimes \mu_\phi^{(1)} \mu_\phi^{-1}$ at $X = N^i \in \mathcal{Z}$ with $i = 0, \dots, n-1$ is the rank $n-i$ matrix*

$$\epsilon_{\phi,1} N^i \widehat{\Pi}^{(1)} H_{\theta^q,\theta} \widehat{\Pi}^{-1}$$

*and in the case $i = n-1$, when $N^i = e_{n,1}$, this is not in $\mathcal{Z}$, the coefficient on the top left of the matrix representing the evaluation of $\epsilon_{\phi,1} \otimes \mu_\phi^{(1)} \mu_\phi^{-1}$ at $e_{1,n}$ equals*

$$\frac{\widetilde{\pi}_C^{n(q-1)}}{\theta - \theta^q}.$$

*Proof.* By the proof of Theorem 4.2,

$$c_\phi = \mu_\phi^{(1)} \mu_\phi^{-1} = \widehat{\Pi}^{(1)} H_{\theta^q,\theta} \widehat{\Pi}^{-1} \in \mathrm{End}_L(\mathbb{G}_a^n).$$

Denoting the last line of $c_\phi$ in the standard basis by $\kappa_\phi$, the evaluation of $\epsilon_{\phi,1} \otimes \mu_\phi^{(1)} \mu_\phi^{-1} = \epsilon_{\phi,1} \otimes c_\phi$ at $e_{1,n}$ is represented by the matrix obtained by left multiplication by $Q_1$ of the matrix having as the first row $\kappa_\phi$, and all the remaining coefficients 0. If $\epsilon_1$ denotes the first column of $\epsilon_{\phi,1}$, then the required evaluation is the product $\epsilon_1 \kappa_\phi$. It is not difficult to show that this represents a rank 1 element of $\mathrm{End}_L(\mathbb{G}_a^n)$ which does not belong to $\mathcal{Z}$. Indeed, writing $\kappa_\phi = (\kappa_{\phi,1}, \dots, \kappa_{\phi,n})$ one sees, with a direct computation and using (3.8), that

$$\kappa_{\phi,1} = (\theta^q - \theta)^{n-1} \widetilde{\pi}_C^{n(q-1)}.$$

In [9, (4.1.4)] Papanikolas determines an explicit formula for the matrix $P_1$ representing $\lambda_{\phi,1}$. This can also be described by using (4.3). Since $\epsilon_{\phi,1} = -\lambda_{\phi,1}$, we deduce that the projection of $(\epsilon_{\phi,1} \otimes c_\phi)(e_{1,n})$ on the subspace generated by $e_{1,1}$ equals

$$-(\theta^q - \theta)^{-n} \kappa_{\phi,1} = \frac{\widetilde{\pi}_C^{n(q-1)}}{\theta - \theta^q}.$$

□

7.1.2. *Proof of Theorem 7.4: second step.* For this we now describe a crucial simplification in the computation of $\widetilde{\gamma}_{k+1} \widetilde{\delta}_k$ for $k \geq 1$. We recall that, from the definitions (6.6) and (6.7),

$$\widetilde{\gamma}_{k+1} \widetilde{\delta}_k = (1 \otimes \lambda_{\phi,k+1}^{-1}) \mu_{k+1}^{-1} (1 \otimes e_{n,1}) \mu_k^{-1} (e_{n,1} \otimes 1) (1 \otimes \lambda_{\phi,k-1}^{(1)}).$$

We show:

**Lemma 7.6.** *$(1 \otimes e_{n,1}) \mu_{X,Y}^{-1} (e_{n,1} \otimes 1) = (W_{X,Y} \otimes e_{1,1}) \alpha$ where $\alpha \in \mathrm{End}_L^{(2)}(\mathbb{G}_a^n)$ is the automorphism defined by $\alpha(e_{i,j}) = e_{n-j+1,i}$.*

The action of the operator $\alpha$ can be easily described as follows. Applying $\alpha$ to a matrix $X$ amounts, in the standard basis, to perform a rotation on the coefficients centered in the center of the matrix, of angle $\pi/2$ counterclockwise (besides the empiric description this has a precise meaning, and the center is located at the same place of a coefficient that will remain invariant if $n$ is odd). In particular, if $\pi_{i,j}$ denotes the projection of $\mathrm{End}_L(\mathbb{G}_a^n)$ over the line $Le_{i,j}$ parallel to $\sum_{(i',j')\neq(i,j)} Le_{i',j'}$ so that, writing $X = (x_{i,j})_{i,j}$ we get $x_{i,j} = \pi_{i,j}(X)$ for all $1 \leq i,j \leq n$, we have $\pi_{i,j} \circ \alpha = \pi_{j,n-i+1}$. This is because, writing $X = \sum_{i,j} x_{i,j}e_{i,j}$, we have $\alpha(X) = \sum_{i,j} x_{i,j}\alpha(e_{i,j}) = \sum_{i,j} x_{i,j}e_{n-j+1,i}$, so that $\alpha(X)_{i,j} = x_{j,n-i+1}$, for all $1 \leq i,j \leq n$.

*Proof of Lemma 7.6.* It suffices to show that the members of the identity agree evaluating at each element of the standard basis $(e_{i,j})_{i,j}$ of $\mathrm{End}_L(\mathbb{G}_a^n)$. By (3.1) we have $(e_{n,1} \otimes 1)(e_{i,j}) = e_{n,1}e_{i,j} = \delta_{i,1}e_{n,j}$. Hence $\mu_{X,Y}^{-1}(e_{n,1} \otimes 1)(e_{i,j}) = \delta_{i,1}\mu_{X,Y}^{-1}(e_{n,j}) = \delta_{i,1}W_{X,Y}N^{j-1}$ thanks to Lemma 3.3. Therefore $(1 \otimes e_{n,1})\mu_k^{-1}(e_{n,1} \otimes 1)(e_{i,j}) = \delta_{i,1}W_{X,Y}N^{j-1}e_{n,1} = W_{X,Y}\delta_{i,1}e_{n-j+1,1} = W_{X,Y}e_{n-j+1,i}e_{1,1} = (W_{X,Y} \otimes e_{1,1})(e_{n-j+1,i})$ by (3.5) and (again) (3.1). □

Lemma 7.6 implies the formula

$$\widetilde{\gamma}_{k+1}\widetilde{\delta}_k = (1 \otimes \lambda_{\phi,k+1}^{-1})\mu_{k+1}^{-1}(W_{\theta,\theta^{(k)}} \otimes e_{1,1})\alpha(1 \otimes \lambda_{\phi,k-1}^{(1)}). \tag{7.4}$$

We are going to see that this formula allows to compute the evaluation of $\widetilde{\gamma}_{k+1}\widetilde{\delta}_k$ at elements of $\mathcal{Z}$. More precisely, we can give simple expressions for $\widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^i)$ for $i = 0, \ldots, n-1$ (recall that we set $N^0 = 1$). Writing $P_{i,j}^{\langle k\rangle}$ for the coefficient on the $i$-th row and $j$-th column of $P_k$ (we borrow notations from [9]), with

$$C_j^{\langle k\rangle} := \begin{pmatrix} C_{j,n}^{\langle k\rangle} \\ C_{j,n-1}^{\langle k\rangle} \\ \vdots \\ C_{j,1}^{\langle k\rangle} \end{pmatrix} := H_{\theta,\theta^{(k)}} \begin{pmatrix} (P_{j,n}^{\langle k-1\rangle})^{(1)} \\ (P_{j,n-1}^{\langle k-1\rangle})^{(1)} \\ \vdots \\ (P_{j,1}^{\langle k-1\rangle})^{(1)} \end{pmatrix} \tag{7.5}$$

the following result holds.

**Lemma 7.7.** *For all $k \geq 0$ we have*

$$\widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^{j-1}) = C_j^{\langle k\rangle} d_t(\tau(b_{k-1})^n)_{t=\theta} \in \mathcal{Z}, \quad j = 1, \ldots, n.$$

*Proof.* If $X = (x_{i,j})_{i,j} \in \mathrm{End}_L(\mathbb{G}_a^n)$,

$$\alpha(N^{j-1}X)e_{1,1} = \begin{pmatrix} x_{j,n} & 0 & \cdots & 0 \\ x_{j,n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_{j,1} & 0 & \cdots & 0 \end{pmatrix}, \quad j = 1, \ldots, n$$

(multiplying on the right by $e_{1,1}$ amounts in projecting on the first column parallel to the remaining columns). We deduce that

$$\alpha(N^{j-1}\lambda^{(1)}_{\phi,k-1})e_{1,1} = \begin{pmatrix} (P^{\langle k-1\rangle}_{j,n})^{(1)} & 0 & \cdots & 0 \\ (P^{\langle k-1\rangle}_{j,n-1})^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (P^{\langle k-1\rangle}_{j,1})^{(1)} & 0 & \cdots & 0 \end{pmatrix}.$$

By the definition (3.4) of $W_{\theta,\theta^{(k)}} = W_{X,Y}$ (with $X = \theta$ and $Y = \theta^{(k)}$) we have that

$$W_{\theta,\theta^{(k)}}\alpha(N^{j-1}\lambda^{(1)}_{\phi,k-1})e_{1,1} = d_t((t-\theta^{(k)})^{-n})_{t=\theta}(C^{\langle k\rangle}_j, 0, \dots, 0).$$

According to (7.4), to complete the computation of $\widetilde{\gamma}_{k+1}\widetilde{\delta}_{k+1}(N^{j-1})$ we must compute the endomorphism

$$(1\otimes\lambda^{-1}_{\phi,k+1})\mu^{-1}_{k+1}\Big(d_t(t-\theta^{(k)})^{-n}_{t=\theta}(C^{\langle k\rangle}_j, 0, \dots, 0)\Big).$$

Note that by (7.1)

$$(C^{\langle k\rangle}_j, 0, \dots, 0) = \widehat{C}^{\langle k\rangle}_j e_{n,1}. \tag{7.6}$$

We have:

$$\begin{aligned}
&(1\otimes\lambda^{-1}_{\phi,k+1})\mu^{-1}_{k+1}\Big(d_t(t-\theta^{(k)})^{-n}_{t=\theta}(C^{\langle k\rangle}_j, 0, \dots, 0)\Big) = \\
&\overset{(1)}{=} d_t(t-\theta^{(k)})^{-n}_{t=\theta}\mu^{-1}_{k+1}\Big((C^{\langle k\rangle}_j, 0, \dots, 0)\Big)\lambda^{-1}_{\phi,k+1} \\
&\overset{(2)}{=} d_t(t-\theta^{(k)})^{-n}_{t=\theta}\mu^{-1}_{k+1}(\widehat{C}^{\langle k\rangle}_j e_{n,1})\lambda^{-1}_{\phi,k+1} \\
&\overset{(3)}{=} d_t(t-\theta^{(k)})^{-n}_{t=\theta}\widehat{C}^{\langle k\rangle}_j\mu^{-1}_{k+1}(e_{n,1})\lambda^{-1}_{\phi,k+1} \\
&\overset{(4)}{=} d_t(t-\theta^{(k)})^{-n}_{t=\theta}\widehat{C}^{\langle k\rangle}_j W_{\theta,\theta^{(k+1)}}H_{\theta^{(k+1)},\theta}d_t(\tau(b_{k+1})^n)_{t=\theta} \\
&\overset{(5)}{=} d_t(t-\theta^{(k)})^{-n}_{t=\theta}\widehat{C}^{\langle k\rangle}_j d_t((t-\theta^{(k+1)})^{-n})_{t=\theta}H_{\theta,\theta^{(k+1)}}H_{\theta^{(k+1)},\theta}d_t(\tau(b_{k+1})^n)_{t=\theta} \\
&\overset{(6)}{=} \widehat{C}^{\langle k\rangle}_j d_t(\tau(b_{k-1})^n)_{t=\theta},
\end{aligned}$$

where in (1) and (3) we have used that $\mu_k^{-1}$ is left $\mathcal{Z}$-linear, in (2) we have used (7.6), in (4) we have applied (4.3) and used Lemma 3.3, in (5) we have used the definition of $W_{X,Y}$ and in (6) we used $(H_{X,Y})^{-1} = H_{Y,X}$. □

The last line $(P^{\langle k\rangle}_{n,1}, \dots, P^{\langle k\rangle}_{n,n})$ of $P_k$ is easy to compute by using Papanikolas' identity (4.3) (cf. [9, Corollary 4.1.5]) ([7]). We have:

$$(P^{\langle k\rangle}_{n,1}, \dots, P^{\langle k\rangle}_{n,n}) = l_k^{-n}\Big((\theta-\theta^{(k)})^{n-1}, \dots, \theta-\theta^{(k)}, 1\Big), \quad k \geq 0. \tag{7.7}$$

[7] In [9], Papanikolas recalls that the computation of the coefficient on the last row, last column of $P_k$ is due to Anderson and Thakur [10, Proposition 2.1.5].

Note that

$$H_{X,Y}\begin{pmatrix}1\\ Z-Y\\ \vdots\\ (Z-Y)^{n-1}\end{pmatrix}=\partial_X\big(S_n(X,Y,Z)\big),$$

where $S_n$ is the polynomial introduced in (7.3), with degree $\leq n-1$ in $Y$. From this we see that

$$\widehat{C}_n^{\langle k\rangle}=l_{k-1}^{-qn}d_t(S_n(t,\theta^{(k)},\theta^{(1)}))_{t=\theta}.$$

Combining with Lemma 7.7 we see that

$$\widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^{n-1})=l_{k-1}^{-nq}\big(d_t(\tau(b_{k-1})^n)d_t(S_n(t,\theta^{(k)},\theta^{(1)})\big)_{t=\theta}$$

It is easy to see that the matrix series ('upper triangular higher polylogarithms')

$$\mathcal{L}_{n(q-1)}(X):=\sum_{k\geq 1}l_{k-1}^{-nq}\big(d_t(\tau(b_{k-1})^n)_{t=\theta}X^{(k)}$$

converges for any $X\in\mathrm{End}_L(\mathbb{G}_a^n)$ with entries having the supremum of the norms $|\cdot|$ of the coefficients not exceeding $|\theta|^{n-1}$. We have reached the next result.

**Corollary 7.8.** *The series*

$$\sum_{k\geq 1}\widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^{n-1})$$

*converges to an endomorphism in $\mathcal{Z}(K_\infty)$. Moreover, there exist elements $a_0,\dots,a_l\in\mathcal{Z}(A)$, not all zero, (matrices with entries in $A$) and elements $X_0,\dots,X_l\in\mathcal{Z}(A)$, not all zero, whose entries have degrees $\leq n-1$ in $\theta$, such that a non-trivial linear dependence relation holds:*

$$\sum_{k\geq 1}\widetilde{\gamma}_{k+1}\widetilde{\delta}_k(N^{n-1})=\sum_{i=0}^{n-1}a_i\mathcal{L}_{n(q-1)}(X_i).$$

*The coefficients $a_i$ and the elements $X_i$ can be explicitly computed from the explicit computation of the coefficients in $Y$ of $d_t(S_n(t,Y,\theta^{(1)}))_{t=\theta}\in\mathcal{Z}(A[Y])$.*

We add a table of values for $S_n$ for small values of $n$ to help explicit computations and experiments.

(7.8)

| $n$ | $S_n(X,Y,Z)$ | $S_n(\theta,Y,\theta^{(1)})$ |
|---|---|---|
| 1 | 1 | 1 |
| $q^s$ | $\frac{(X-Z)^n}{X-Z}$ | $(\theta-\theta^q)^{n-1}$ |
| 2 | $X-2Y+Z$ | $\theta+\theta^q-2Y$ |
| $q+1$ | $\frac{(X-Z)^n}{X-Z}-(X-Z)^{n-1}Y-Y^q$ | $\frac{(\theta-\theta^q)^n}{\theta-\theta^q}-(\theta-\theta^q)^{n-1}Y-Y^q$ |

Combining with Lemma 7.3 we deduce:

**Corollary 7.9.** *The limit* $\lim_{k\to}\widetilde{\delta}_k(N^{n-1})$ *exists.*

Projecting on the diagonal via the map $\pi : \mathcal{Z} \to L$ yields the following:

**Corollary 7.10.** *Write* $S_n(\theta, Y, \theta^{(1)}) = \sum_{i=0}^{n-1} c_{n,i} Y^i \in A[Y]$, *with* $c_{n,0}, \dots, c_{n,n-1} \in A$. *Then*

$$\pi\Big( \sum_{k\geq 1} \widetilde{\gamma}_{k+1} \widetilde{\delta}_k(N^{n-1}) \Big) = \sum_{i=0}^{n-1} c_{n,i} \operatorname{Li}_{n(q-1)}(\theta^{qi}).$$

7.1.3. *The sequence* $\widetilde{\delta}_k(N^{n-1})$. It is difficult to study the asymptotic behavior of the sequence $(\widetilde{\delta}_k(N^i))_{k\geq 1}$ for all $i \in \{0, \dots, n-1\}$ unless $i = n-1$. In this case we have the next result.

**Proposition 7.11.** *The limit* $\lim_{k\to\infty} \widetilde{\delta}_k(N^{n-1})$ *exists and there exists a non-zero matrix* $\Xi_n \in K^{n\times n}$ *such that*

$$\lim_{k\to\infty} \widetilde{\delta}_k(N^{n-1}) = \Xi_n \widetilde{\pi}_C^{qn} \widehat{\Pi}^{-1}.$$

*The uppermost left corner coefficient of this limit vanishes but the matrix limit itself is non-zero if* $n > 1$.

We also recall that

$$\widetilde{\delta}_k = (1 \otimes \lambda_{\phi,k}^{-1}) \mu_k^{-1} (e_{n,1} \otimes 1)(1 \otimes \lambda_{\phi,k-1}^{(1)}).$$

We note that $N^{n-1} = e_{1,n}$. We want to compute the limit of the sequence $\widetilde{\delta}_k(e_{1,n})$. Since the left multiplication by $e_{n,n}$ corresponds to the projection on the last matrix line we see easily that $(e_{n,1} \otimes 1)(1 \otimes \lambda_{\phi,k-1}^{(1)})(e_{1,n}) = e_{n,1} e_{1,n} \lambda_{\phi,k-1}^{(1)} = e_{n,n} \lambda_{\phi,k-1}^{(1)}$ is represented by the matrix obtained from the matrix $P_{k-1}^{(1)}$ of $\lambda_{\phi,k-1}^{(1)}$ replacing all the lines but the last one by 0. By (7.7), $(e_{n,1} \otimes 1)(1 \otimes \lambda_{\phi,k-1}^{(1)})(e_{1,n})$ is also equal to $e_{n,1}$ multiplied on the right by the matrix $\widehat{E}_{k-1}^{(1)} \in \mathcal{Z}$ where

$$E_k = \begin{pmatrix} P_{n,n}^{\langle k \rangle} \\ \vdots \\ P_{n,1}^{\langle k \rangle} \end{pmatrix}.$$

Thus, by the fact that $\mu_k$ is a right $\mathcal{Z}$-module morphism,

$$\begin{aligned} \mu_k^{-1}(e_{n,1} \otimes 1)(1 \otimes \lambda_{k-1}^{(1)})(e_{1,n}) &= \mu_k^{-1}(e_{n,1} \widehat{E}_{k-1}^{(1)}) \\ &= \mu_k^{-1}(e_{n,1}) \widehat{E}_{k-1}^{(1)} \\ &= W_{\theta,\theta^{(k)}} \widehat{E}_{k-1}^{(1)}. \end{aligned}$$

We deduce that

$$\begin{aligned} \widetilde{\delta}_k(N^{n-1}) &= (1 \otimes \lambda_{\phi,k}^{-1}) W_{\theta,\theta^{(k)}} \widehat{E}_{k-1}^{(1)} \\ &= d_t((t - \theta^{(k)})^{-n})_{t=\theta} H_{\theta,\theta^{(k)}} \widehat{E}_{k-1}^{(1)} H_{\theta^{(k)},\theta} d_t(b_k^{(1)})_{t=\theta}^n. \end{aligned}$$

We recall the matrices $H_{X,Y}$, $H_{X,Z}$ etc. introduced in §3.1. We also set

$$U_{Z,Y} := \begin{pmatrix} (Z-Y)^{n-1} & (Z-Y)^{n-2} & \cdots & 1 \\ 0 & (Z-Y)^{n-1} & \cdots & Z-Y \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (Z-Y)^{n-1} \end{pmatrix} \in \mathcal{Z}(\mathbb{F}_q[Z,Y]).$$

We can write, thanks to (7.7):

$$\widetilde{\delta}_k(N^{n-1}) = \Big(d_X((X-Y)^{-n})H_{X,Y}U_{Z,Y}H_{Y,X}\Big)_{X=\theta,Y=\theta^{(k)},Z=\theta^q}\Big(d_t(b_k^{(1)})^n l_{k-1}^{-nq}\Big)_{t=\theta}. \tag{7.9}$$

To handle this expression and to study the limit for $k \to \infty$ The first tool we need is expressed by the following lemma.

**Lemma 7.12.** *The matrix sequence* $\Big(d_t(b_k^{(1)})^n l_{k-1}^{-nq}\Big)_{t=\theta}$ *converges to* $(-\widetilde{\pi}_C)^{qn}\widehat{\Pi}^{-1}$.

*Proof.* Note that $l_{k-1} = b_{k-1}(t)^{(1)}|_{t=\theta}$. The property of the Lemma follows directly from the fact that the sequence $(d_t(b_k^{(1)})(b_{k-1}^{(1)})^{-nq})$ converges, in $\mathrm{GL}_n(\mathbb{T})$, to $d_t(\omega^{(1)})^{-n}(\omega^{(1)})^{nq}$. This is a simple consequence of the well known factorization (1.11) of Anderson and Thakur omega function in $\mathbb{T}^\times$. Additionally, it is well known that $\omega_{t=\theta}^{(1)} = -\widetilde{\pi}_C$ and $d_t(\omega^{(1)})_{t=\theta}^n = \widehat{\Pi}$ follows from (4.10). □

In particular, the second parenthesized factor in (7.9) converges as $k \to \infty$. By Corollary 7.9, the limit

$$\lim_{k\to\infty}\Big(d_X((X-Y)^{-n})H_{X,Y}U_{Z,Y}H_{Y,X}\Big)_{X=\theta,Y=\theta^{(k)},Z=\theta^q}$$

exists. Since $H_{X,Y}U_{Z,Y}H_{Y,X}$ is a matrix with entries that are polynomials in $Y$, the convergence of the limit is equivalent to the validity of the property that all the entries of this matrix have degree $\leq n$ in $Y$. Additionally, since $H_{X,Y}U_{Z,Y}H_{Y,X}$ has entries in $\mathbb{F}_q[X,Y,Z]$, the first term in parentheses has a $K$-rational limit justifying the existence of the matrix $\Xi_n \in K^{n\times n}$ in Proposition 7.11. This proves the main part of Proposition 7.11 but we are not done yet. To complete the proof we study more carefully the degrees in $Y$ of the entries of $H_{X,Y}U_{Z,Y}H_{Y,X}$, see Lemma 7.13 below.

*A key lemma.* We shall consider here the set $B^{r\times s}$ of matrices with entries in $B := \mathbb{F}_p[X,Y,Z]$, with $p$ the characteristic of $\mathbb{F}_q$, and with $X,Y,Z$ indeterminates. Let $M = (m_{i,j})_{i,j}$ be an element of $B^{r\times s}$ and let us consider $\mathfrak{M} = (\mathfrak{m}_{i,j})_{i,j} \in (\mathbb{N}\cup\{-\infty\})^{r\times s}$. We write $\deg_Y(M) \leq \mathfrak{M}$ if for all $i,j$, $\deg_Y(m_{i,j}) \leq \mathfrak{m}_{i,j}$ (matrix inequality). If for some $i,j$ we have $\mathfrak{m}_{i,j} = -\infty$ this means that the corresponding coefficient $m_{i,j}$ is zero. It is plain

that

$$\deg_Y(H_{X,Y}), \deg_Y(H_{Y,X}) \leq \begin{pmatrix} 0 & -\infty & \cdots & -\infty \\ 1 & 0 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ n-1 & n-2 & \cdots & 0 \end{pmatrix},$$

$$\deg_Y(U_{X,Y}) \leq \begin{pmatrix} n-1 & n-2 & \cdots & 0 \\ -\infty & n-1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & n-1 \end{pmatrix}.$$

These matrix inequalities also satisfy elementary multiplication rules that can be easily derived from ultrametric inequalities. For example, the reader can easily check, writing

$$M := H_{X,Y} U_{Z,Y} H_{Y,X},$$

the matrix inequality

$$\deg_Y(M) \leq \begin{pmatrix} n-1 & n-2 & \cdots & 1 & 0 \\ n & n-1 & \cdots & 2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 2n-3 & 2n-4 & \cdots & n-1 & n-2 \\ 2n-2 & 2n-3 & \cdots & n & n-1 \end{pmatrix}$$

holds. The aim of the present subsection is to prove a sharper matrix inequality.

**Lemma 7.13.**

$$\deg_Y(M) \leq \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ n & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ n & n & \cdots & n-2 & n-2 \\ n & n & \cdots & n & n-1 \end{pmatrix}.$$

*Proof.* We shall proceed by induction on $n \geq 2$. The case $n = 2$ is a simple computation that we leave to the reader (it suffices to explicitly compute the product of three $2 \times 2$ matrices). For general $n$ we are not able to describe the structure of the matrix $M$ explicitly. In the following proof, we have been able to compute the first line only (and this computation suffices to complete the proof of Proposition 7.11). Furthermore, as we are considering the various matrices $H_{X,Y}, U_{Z,Y}, M$ etc. for various choices of $n$ (to apply induction hypothesis), we shall now write

$$H_{X,Y} = H_{X,Y}^{\langle n \rangle}, \quad U_{Z,Y} = U_{Z,Y}^{\langle n \rangle}, \quad M^{\langle n \rangle} = M$$

exhibiting a dependence on the size of the matrices. Then, we have obvious block decompositions:

$$H_{X,Y}^{\langle n\rangle} = \begin{pmatrix} 1 & 0\cdots 0 \\ h_n & H_{X,Y}^{\langle n-1\rangle} \end{pmatrix}, \quad H_{Y,X}^{\langle n\rangle} = \begin{pmatrix} 1 & 0\cdots 0 \\ h'_n & H_{Y,X}^{\langle n-1\rangle} \end{pmatrix},$$

$$U_{Z,Y}^{\langle n\rangle} = \begin{pmatrix} (Z-Y)^{n-1} & u_n \\ 0 & \\ \vdots & (Z-Y)U_{Z,Y}^{\langle n-1\rangle} \\ 0 & \end{pmatrix},$$

with the column and row matrices $h_n, h'_n, u_n$ having $n-1$ entries that can be easily computed from definitions. Recalling that $(H_{X,Y}^{\langle n\rangle})^{-1} = H_{Y,X}^{\langle n\rangle}$ we observe that $M^{\langle n\rangle}$ is the unique matrix in $B^{n\times n}$ such that

$$U_{Z,Y}^{\langle n\rangle} H_{Y,X}^{\langle n\rangle} = H_{Y,X}^{\langle n\rangle} M^{\langle n\rangle} \tag{7.10}$$

$$H_{X,Y}^{\langle n\rangle} U_{Z,Y}^{\langle n\rangle} = M^{\langle n\rangle} H_{X,Y}^{\langle n\rangle}. \tag{7.11}$$

Decomposing $M^{\langle n\rangle}$ in blocks

$$M^{\langle n\rangle} = \begin{pmatrix} \mathcal{X}_n & \mathcal{Y}_n \\ \mathcal{Z}_n & \mathcal{T}_n \end{pmatrix}, \quad \mathcal{T}_n \in B^{n-1\times n-1}$$

the identities (7.10) and (7.11) yield eight matrix identities three of which suffice to apply the inductive hypothesis and prove the lemma. We therefore proceed in three steps.
(1). We study the first lines in the identity (7.10). Observing that the first line of $U_{Z,Y}^{\langle n\rangle}$ equals the last line of $H_{Z,Y}^{\langle n\rangle}$, it is plain that the first line of $U_{Z,Y}^{\langle n\rangle} H_{Y,X}^{\langle n\rangle}$ equals the last line of $H_{Z,Y}^{\langle n\rangle} H_{Y,X}^{\langle n\rangle}$. But this matrix is equal to $H_{Z,X}^{\langle n\rangle}$ by (3.3). Combining with (7.10) we deduce that the first line $(\mathcal{X}_n, \mathcal{Y}_n)$ of $M^{\langle n\rangle}$ equals $((Z-X)^{n-1}, \dots, 1)$, independent of $Y$, so that $\deg_Y((\mathcal{X}_n, \mathcal{Y}_n)) \le (0, \dots, 0)$, in agreement with the first line of the matrix inequality of our lemma.
(2). We study the identity in $B^{n-1\times n-1}$ that one can derive from the $n-1\times n-1$ block identity in (7.11). This is

$$\mathcal{T}_n H_{X,Y}^{\langle n-1\rangle} = h_n u_n + (Z-Y) H_{X,Y}^{\langle n-1\rangle} U_{Z,Y}^{\langle n-1\rangle}.$$

Multiplying on the right by $H_{Y,X}^{\langle n-1\rangle}$ we get

$$\mathcal{T}_n = h_n u_n H_{Y,X}^{\langle n-1\rangle} + (Z-Y) H_{X,Y}^{\langle n-1\rangle} U_{Z,Y}^{\langle n-1\rangle} H_{Y,X}^{\langle n-1\rangle} = h_n u_n H_{Y,X}^{\langle n-1\rangle} + (Z-Y) M^{\langle n-1\rangle}, \tag{7.12}$$

where the last equality follows from the definition of $M^{\langle n-1\rangle}$. Note that $u_n$, a block of $U_{Z,y}^{\langle n\rangle}$, also equals the first row of $U_{Z,Y}^{\langle n-1\rangle}$. So the argument in the first step tells us that $u_n H_{Y,X}^{\langle n-1\rangle}$

is the last row of $H_{Z,Y}^{\langle n-1\rangle}H_{Y,X}^{\langle n-1\rangle}=H_{Z,X}^{\langle n-1\rangle}$, whose degree in $Y$ is $\leq(0,\dots,0)$ because it is independent of $Y$. Since

$$\deg_Y(h_n)\leq\begin{pmatrix}0\\1\\\vdots\\n-1\end{pmatrix},$$

the degree in $Y$ of the rank $\leq 1$ matrix $h_nu_nH_{Y,X}^{\langle n-1\rangle}$ is

$$\leq\begin{pmatrix}1&\cdots&1\\2&\cdots&2\\\vdots&&\vdots\\n-1&\cdots&n-1\end{pmatrix}.$$

Coming back to (7.12), the induction hypothesis implies the $n-1\times n-1$ matrix inequality

$$\deg_Y((Z-Y)M^{\langle n-1\rangle})\leq\begin{pmatrix}1&1&\cdots&1&1\\n&2&\cdots&2&2\\\vdots&\vdots&&\vdots&\vdots\\n&n&\cdots&n-2&n-2\\n&n&\cdots&n&n-1\end{pmatrix}.$$

Comparing with the previous matrix inequality we therefore proved that the above matrix with integer entries gives a matrix upper bound to $\deg_Y(\mathcal{T}_n)$ which is compatible with the statement of our lemma, that is

$$\deg_Y(\mathcal{T}_n)\leq\begin{pmatrix}1&1&\cdots&1&1\\n&2&\cdots&2&2\\\vdots&\vdots&&\vdots&\vdots\\n&n&\cdots&n-2&n-2\\n&n&\cdots&n&n-1\end{pmatrix}.$$

All we need to show, to complete the proof of the lemma, is to show that

$$\deg_Y(\mathcal{Z}_n)\leq\begin{pmatrix}n\\\vdots\\n\end{pmatrix}.$$

(3). We study the identity extracted from (7.10) by looking at the first column, after having deleted the uppermost left corner coefficient. This is the identity

$$(Z-Y)U_{Z,Y}^{\langle n-1\rangle}h_n'=(Z-X)^{n-1}h_n'+H_{Y,X}^{\langle n-1\rangle}\mathcal{Z}_n.$$

Multiplying on the left by $H_{X,Y}^{\langle n-1\rangle}$ we get the identity

$$\mathcal{Z}_n=(Z-Y)H_{X,Y}^{\langle n-1\rangle}U_{Z,Y}^{\langle n-1\rangle}h_n'-(Z-X)H_{X,Y}^{\langle n-1\rangle}h_n'.$$

A simple computation implies that

$$\deg_Y((Z-X)H_{X,Y}^{\langle n-1\rangle}h'_n) \leq \begin{pmatrix} 1 \\ \vdots \\ n-1 \end{pmatrix}.$$

Hence, to complete our proof, we still need to prove that

$$\deg_Y(H_{X,Y}^{\langle n-1\rangle}U_{Z,Y}^{\langle n-1\rangle}h'_n) \leq \begin{pmatrix} n-1 \\ \vdots \\ n-1 \end{pmatrix}.$$

This uses an explicit computation. Let $f_i^{\langle n-1\rangle}$ be the $i$-th coefficient of the column matrix $H_{X,Y}^{\langle n-1\rangle}U_{Z,Y}^{\langle n-1\rangle}h'_n \in B^{n-1\times 1}$. Developing the product, the reader can easily check that

$$f_i^{\langle n-1\rangle} = \sum_{j=1}^{i}\sum_{k=0}^{n-1-j}\binom{n-1-j}{n-1-i}\binom{n-1}{k}(X-Y)^{i-j}(Y-X)^{n-1-k}(Z-Y)^{k+j-1},$$
$$1 \leq i \leq n-1.$$

We extend these coefficients to all $i \in \mathbb{Z}$, with $n \geq 2$, by setting $f_i^{\langle n-1\rangle} = 0$ for $i \leq 0$ and $i \geq n$. The above formula implies that $f_\alpha^{\langle\alpha\rangle} = (Y-X)^\alpha(X-Y)^{\alpha-1}$ for $\alpha \geq 1$ and elementary identities of binomial coefficients yield the recursive relations

$$f_i^{\langle\alpha+1\rangle} = (Z-X)f_i^{\langle\alpha\rangle} + (Z-X)(X-Y)f_{i-1}^{\langle\alpha\rangle}, \quad \alpha \geq 1, \quad 1 \leq i \leq \alpha$$

that completely determines the double sequence of these coefficients. Since $f_1^{\langle 1\rangle} = Y-X$ we deduce, by induction, that $\deg_Y(f_i^{\langle n-1\rangle}) \leq n-1$ for all $i$. The proof of the lemma is complete. □

*End of proof of Proposition 7.11.* By Lemma 7.13 there is a matrix $\Xi_n \in K^{n\times n}$ such that $\lim_{k\to\infty}\widetilde{\delta}_k(e_{1,n}) = \Xi_n\widehat{\Pi}^{-1}\widetilde{\pi}_C^{nq}$. The fact that the matrix degree upper bound of Lemma 7.13 has integers $< n$ in the upper triangular part including the diagonal implies that $\Xi_n$ is lower triangular,

$$\Xi_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ * & * & \cdots & * & 0 \end{pmatrix}$$

where $*$ denotes an element of $K$. The identity of Lemma 7.3 becomes, for $X = N^{n-1} \in \mathcal{Z}$,

$$\sum_{d\geq 1}\widetilde{\gamma}_{d+1}(\widetilde{\delta}_d(N^{n-1})) = (\epsilon_{\phi,1}\otimes\mu_\phi^{(1)}\mu_\phi^{-1})(N^{n-1}) - \Xi_n\widehat{\Pi}^{-1}\widetilde{\pi}_C^{nq}.$$

The left-hand side is a convergent series in $\mathcal{Z}(K_\infty)$. In the right-hand side, it is easy to show that the rank one matrix $\epsilon_{\phi,1}e_{1,n}\widehat{\Pi}^{(1)}H_{\theta^q,\theta}\widehat{\Pi}^{-1}$ is not in $\mathcal{Z}$. Hence $\Xi_n$ is not in $\mathcal{Z}$ and therefore is non-zero. □

7.2. **Appendix. Comments on Papanikolas' formula.** This subsection is only a transcription of Papanikolas' proof of (4.3) that we give here for completeness. To show (4.3) we first observe that there exists a unique formal series

$$\mathcal{L} := \sum_{i\geq 0} \mathcal{P}_i \tau^i \in \mathrm{End}_{\mathbb{T}}(\mathbb{G}_a^n)[[\tau]]$$

such that

$$d_t(t)\mathcal{L} - \mathcal{L}\phi_\theta = d_t((t-\theta)^n)e_{n,1}. \tag{7.13}$$

Assume its existence. Then, $\mathcal{P}_k|_{t=\theta} = P_k$ for all $k$ because $\log_\phi = \sum_{k\geq 0} P_k\tau^k$ is uniquely determined by the conditions $P_0 = 1$ and $d(\theta)\log_\phi - \log_\phi \phi_\theta = 0$ which comes from (7.13) after evaluation at $t = \theta$ (note that $d_t((t-\theta)^n)$ vanishes at $t = \theta$). To verify existence and uniqueness of $\mathcal{L}$ observe that $d_t(t)\otimes 1 - 1\otimes\phi_\theta = (t-\theta) + \nu\otimes 1 + 1\otimes\nu' + e_{n,1}\tau$ with $\nu, \nu'$ nilpotent. Since $(t-\theta)+\nu\otimes 1+1\otimes\nu'$ is an isomorphism, the only compatible setting, that of $\mathcal{P}_0 = H_{t,\theta}$, determines a unique solution of (7.13). We construct the formal series $\mathcal{L}$ by using Lemma 3.2 in the following way. Consider the identity of the Lemma, and replace $X = t$ (central variable) and $Y = \theta$ so that:

$$d_t(t)H_{t,\theta} - H_{t,\theta}d(\theta) - d_t((t-\theta)^n)H_{t,\theta}e_{n,1} = 0. \tag{7.14}$$

Consider, on the other hand, the formal series $(1-\tau)^{-1} = \sum_{k\geq 0}\tau^k$, inverse of $1-\tau$ in $\mathbb{T}[[\tau]]$, and multiply both sides of the above identity on the left by the operator

$$\mathcal{M} := \Omega_n^{-1}(1-\tau)^{-1}\Omega_n = \sum_{k\geq 0} \tau\big(d_t(b_k^{-n})\big)\tau^k \in \mathrm{End}_{\mathbb{T}}(\mathbb{G}_a^n)[[\tau]],$$

where

$$\Omega_n := \tau(d_t(\omega^{-n})).$$

We note that $\mathcal{M}d_t((t-\theta)^n)H_{t,\theta}e_{n,1} = d_t((t-\theta)^n)e_{n,1} + \mathcal{M}H_{t,\theta}e_{n,1}\tau$. Hence, if we set

$$\mathcal{L}' = \mathcal{M}H_{t,\theta}$$

we get

$$d_t(t)\mathcal{L}' - \mathcal{L}'\phi_\theta = d_t((t-\theta)^n)e_{n,1},$$

so that $\mathcal{L}' = \mathcal{L}$. Papanikolas formula (4.3) follows. To instantly see that $P_k$ is invertible directly note that, by the fact that $\Omega_n$ is invertible, all the coefficients of $\mathcal{M}$ are invertible and we already know that $H_{t,\theta}$ represents an automorphism.

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