



HAL
open science

Optimal control for a two-sidedly degenerate aggregation equation

Mostafa Bendahmane, Fahd Karami, Elmahdi Erraji, Abdelghafour Atlas,
Abdelkbir Afraites

► **To cite this version:**

Mostafa Bendahmane, Fahd Karami, Elmahdi Erraji, Abdelghafour Atlas, Abdelkbir Afraites. Optimal control for a two-sidedly degenerate aggregation equation. *Nonlinear Analysis: Modelling and Control*, 2023, 10.15388/namc.2023.28.32395 . hal-04124303

HAL Id: hal-04124303

<https://hal.science/hal-04124303>

Submitted on 9 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Optimal control for a two-sidedly degenerate aggregation equation*

Mostafa Bendahmane^{1,a}, Fahd Karami^{2,b}, Elmahdi Erraji^{2,c}, Abdelghafour Atlas^{3,d}, Abdelkbir Afraites^{4,e}

¹Institut de mathématiques de Bordeaux, Université de Bordeaux, France
^amostafa.bendahmane@u-bordeaux.fr

²Ecole Supérieure de Technologie d'Essaouira, Université Cadi Ayyad, Morocco
^bfa.karami@uca.ma

^cerraji0elmahdi@gmail.com

³Ecole Nationale des Sciences Appliquées de Marrakech, Université Cadi Ayyad, Morocco
^da.atlas@uca.ac.ma

⁴EMI FST Béni-Mellal, Université Sultan Moulay Slimane, Morocco
^el.afraites@usms.ma

Received: date / Revised: date / Published online: data

Abstract

In this paper, we are concerned with the study of the mathematical analysis for an optimal control of a nonlocal degenerate aggregation model. This model describes the aggregation of organisms such as pedestrian movements, chemotaxis, animal swarming. We establish the well-posedness (existence and uniqueness) for the weak solution of the direct problem by means of an auxiliary nondegenerate aggregation equation, the Faedo-Galerkin method (for the existence result) and duality method (for the uniqueness). Moreover, for the adjoint problem, we prove the existence result of minimizers and first order necessary conditions. The main novelty of this work concern the presence of a control to our nonlocal degenerate aggregation model. Our results are complemented with some numerical simulations.

Keywords: Aggregation equation ; Nonlocal models ; Degenerate diffusion ; Finite volume ; Optimal control ; Adjoint problem.

1 Introduction

Nonlocal aggregation model has recently received great attention in biological applications. There exists a large variety of biological aggregation models such as the flocking of birds, aggregation of fish and the swarming insect. We refer the reader to a large number of works have focused on biological aggregation [6, 7, 10, 12, 16, 18, 19, 20, 22, 23].

*MB would like to thank the SPICY research project for supporting this work

In this paper, we consider the following nonlocal degenerate equation:

$$\begin{cases} \partial_t u - \operatorname{div}(a(u)\nabla u - u\nabla K \star u) = F(u, w) & \text{in } \Omega_T, \\ (a(u)\nabla u - u\nabla K \star u) \cdot \eta = 0 & \text{on } \Sigma_T, \\ u(x, 0) = u_0(x), & \text{in } \Omega. \end{cases} \quad (1)$$

Herein, $\Omega_T := \Omega \times (0, T)$, $\Sigma_T := \partial\Omega \times (0, T)$, $T > 0$ is a fixed time, and Ω is a bounded domain in \mathbb{R}^3 , with Lipschitz boundary $\partial\Omega$ and outer unit normal η . In the model above, the density of the population is represented by $u = u(x, t)$, $a(u)$ is a density-dependent diffusion coefficient. Furthermore, K is the sensing (interaction) kernel that models the long-range attraction. In the convolution term, u is extended by zero, outside of Ω . More precisely

$$\nabla K \star u(x) = \int_{\Omega} \nabla K(x - y)u(y)dy.$$

Note that system (1) arises in many models of biology and in particular in social organizations which is one of the fundamental aspects of animal behaviors.

In this paper, we assume that the density-dependent diffusion coefficient $a(u)$ degenerates for $u = 0$ and $u = \bar{u}$. This means that the diffusion vanishes when u approaches values close to the threshold \bar{u} and also in the absence of the population. This interpretation was proposed in [2] and in the references therein for the chemotaxis model.

To summarize, the following main assumptions are made:

$$\begin{aligned} & a \in C^1([0, 1]), a(0) = a(\bar{u}) = 0 \text{ and } a(s) > 0 \text{ for } 0 < s < \bar{u}, \\ & K \in C^2(\mathbb{R}^3) \text{ is a nonnegative radially nonincreasing function} \\ & \text{with } \|K\|_{C^2(\mathbb{R}^3)} < \infty \text{ and } \int_{\mathbb{R}^3} K(x) dx = 1. \end{aligned} \quad (2)$$

In addition, the reaction function F has the following form

$$F(u, w) = \alpha u - w u^2,$$

where, $\alpha > 0$ is the Malthusian growth coefficient and $w(\cdot)$ (the control) is a nonnegative function of the intraspecific competition.

Regarding the degeneracy of the diffusion coefficient, a typical example of a is $a(u) = u(\bar{u} - u)$. Note that the degeneracy of the diffusion coefficient and the nonlocal term are major concerns for the mathematical and numerical treatment of the equation (1).

To put this paper in the proper perspective, we mention that the nonlocal aggregation equation investigated analytically and numerically by many authors: [22, 6] for the study of the pure aggregation equation, i.e, $a(u) = 0$ and $F = 0$, [8] for the existence result, [3, 4] for the blowup of the solution and [13, 14] for the analysis of the numerical simulation.

Many studies have focused on the competition between the degenerate diffusion as a repellent force and nonlocal aggregation terms as attractive force (see for e.g. [15]). This competition is observed in many biological phenomenon from social pattern formation to microbiological dynamics under chemotaxis force [5]. From a mathematical perspective, we mention for example the work [17] where the author proposed and

proved the existence results of local and global solutions to a class of aggregation equations depending on attraction kernel regularity. In passing we want to mention that the authors in [5] considered and studied the model (1) with $F := 0$ and $a(u) = 0$ for $u = 0$.

In our study, we are concerned with the mathematical analysis and numerical simulations of an optimal control problem arising in the study of population dynamics. Our model is governed by a degenerate aggregation-diffusion equation. To this model, we introduce a notion of a weak solution for the direct problem and prove its well-posedness. Comparing to [5] (Equation (1) with $F := 0$ and $a(u) = 0$ for $u = 0$), in this paper we prove the existence of solutions by applying the Faedo-Galerkin method, deriving a priori estimates, and then passing to the limit in the approximate solutions using monotonicity and compactness arguments. The uniqueness of these weak solutions is guaranteed by using the duality method. For the analysis of our optimal control problem, we use the Lagrangian framework, in which the control problem is set as a constrained minimization problem. Note that, if there exists of a minimum to a suitable Lagrangian functional, it is a stationary point.

The numerical solution of our optimal control problem constrained by degenerate nonlocal aggregation model requires the proper discretization of the direct and the adjoint problems, and the treatment of an optimization problem. From the standpoint of our specific application, the main goal is to determine the control response to reduce the pattern formation generated by a nonlocal attraction term. More specifically, we are interested in determining the optimal intra species competition to insure a minimal pattern formation due the attraction force.

The structure of the paper is organized as follows: In Section 2, we present the main results and we prove the well-posedness (existence and uniqueness) result to our degenerate aggregation model. Section 3 will be devoted to the optimal control problem. We present our optimal control approach, introduce a functional useful for minimize, prove the existence of the control and we derive the adjoint-state problem. Finally, in Section 4, we introduce the numerical scheme for both direct and adjoint problem, present the optimal control procedure and we demonstrate various realizations showing the effect of the optimal control solution on the overcrowding of the population.

2 Existence and uniqueness of weak solution

2.1 Weak solutions for the nonlocal degenerate equation

Before stating our main results, we give the definition of a weak solution for the system (1).

Definition 1. *A weak solution of (1) is a nonnegative function u satisfying the following conditions:*

$$\begin{aligned} u &\in L^\infty(\Omega_T), \quad \mathcal{A}(u) := \int_0^u a(r) dr \in L^2(0, T; H^1(\Omega)), \\ u &\in C_w(0, T; L^2(\Omega)), \quad \partial_t u \in L^2(0, T; (H^1(\Omega))'), \quad u(0) = u_0, \end{aligned} \quad (3)$$

and, for all $\varphi \in L^2(0, T; (H^1(\Omega))')$

$$\int_0^T \langle \partial_t u, \varphi \rangle dt + \iint_{\Omega_T} a(u) \nabla u \cdot \nabla \varphi \, dx \, dt - \iint_{\Omega_T} u (\nabla K \star u) \cdot \nabla \varphi \, dx \, dt = \iint_{\Omega_T} F(u, w) \varphi \, dx \, dt, \quad (4)$$

where $C_w(0, T; L^2(\Omega))$ denotes the space of continuous functions with values in (a closed ball of) $L^2(\Omega)$ endowed with the weak topology, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$.

Our first result is the following well-posedness (existence and uniqueness) theorem for weak solutions.

Theorem 2. *Assume condition (2) holds and $u_0 \in L^\infty(\Omega)$ with $0 \leq u_0 \leq \bar{u}$, where \bar{u} is a positive constant in \mathbb{R} . Then there exists a unique weak solution to the nonlocal degenerate equation (1) in the sense of Definition 1.*

2.2 Existence of weak solution

Note that a major difficulty for the analysis of the equation (1) is the strong degeneracy of the diffusion term and the presence of the nonlocal term. To handle this difficulty, we replace the original diffusion term $a(u)$ by $a_\varepsilon(u) = a(u) + \varepsilon$ ($\varepsilon > 0$) and consider the following nonlocal non-degenerate equation

$$\begin{cases} \partial_t u - \operatorname{div}(a_\varepsilon(u)\nabla u - u\nabla K \star u) = F(u, w) & \text{in } \Omega_T, \\ (a_\varepsilon(u)\nabla u - u\nabla K \star u) \cdot \eta = 0 & \text{on } \Sigma_T, \\ u(x, 0) = u_0(x), & \text{in } \Omega. \end{cases} \quad (5)$$

To prove Theorem 2 we first prove existence of solutions to the non-degenerate problem (5) by using the Faedo-Galerkin method (in an appropriate functional setting). Convergence is achieved by means of *a priori* estimates and compactness arguments.

In what follows, we use a.e. to denote almost everywhere, which means that a property or condition holds for all points in a set except for a set of points that has measure zero.

2.2.1 The Faedo-Galerkin solution

To construct our Faedo-Galerkin approximation, we employ a classical Hilbert basis, which is orthonormal in L^2 and orthogonal in H^1 . We look for solutions to the problem obtained from the projection of (5) onto the finite dimensional subspace $S_n := \operatorname{Span}\{e_1, \dots, e_n\}$. The approximate solution takes the following form

$$u_n : [0, T] \rightarrow S_n, \quad u_n(t) = \sum_{l=1}^n c_{n,l}(t)e_l.$$

Herein, $\{e_l\}_{l=1}^\infty$ is an orthogonal basis in $H^1(\Omega)$ and an orthonormal basis in $L^2(\Omega)$. Our first goal is to determine the coefficients $c^n = \{c_{n,l}\}_{l=1}^n$ such that ($l = 1, \dots, n$)

$$\langle \partial_t u_n, e_l \rangle + \int_{\Omega} a_{\varepsilon_n}(u_n)\nabla u_n \cdot \nabla e_l \, dx - \int_{\Omega} u_n (\nabla K \star u_n) \cdot \nabla e_l \, dx = \int_{\Omega} (w u_n - u_n^2) e_l \, dx, \quad (6)$$

and, with reference to the initial condition

$$u_n(0) = u_{0,n} = \sum_{l=1}^n c_l^n(0)e_l, \quad c_l^n(0) := \int_{\Omega} u_{0,n}(x)e_l(x) \, dx.$$

Herein, $\varepsilon_n := 1/n$, $n > 0$. More explicitly we can write (6) as an equation of ordinary differential equation ($l = 1, \dots, n$)

$$\begin{aligned} c'_{n,l}(t) &= - \int_{\Omega} a_{\varepsilon_n}(u_n) \nabla u_n \cdot \nabla e_l \, dx + \int_{\Omega} u_n (\nabla K \star u_n) \cdot \nabla e_l \, dx + \int_{\Omega} (w u_n - u_n^2) e_l \, dx \\ &:= F_l(t, c_{n,1}(t), \dots, c_{n,n}(t)), \end{aligned} \quad (7)$$

where, we have used the orthonormality of the basis. Observe that F_l ($l = 1, \dots, n$) is a Carathéodory function. Therefore, using the standard ODE theory, there exists an absolutely continuous functions $\{c_{n,l}\}_{l=1}^n$ satisfying (7) for a. e. $t \in [0, t']$ for some $t' > 0$. The next is to show that the local solution constructed above can be extended to the whole time interval $[0, T]$ (independent of n) but this can be done as in [1], so we omit the details.

Observe that from (6), the Faedo-Galerkin solution satisfies the following weak formulation

$$\int_0^T \langle \partial_t u_n, \varphi \rangle \, dt + \iint_{\Omega_T} a_{\varepsilon_n}(u_n) \nabla u_n \cdot \nabla \varphi \, dx \, dt - \iint_{\Omega_T} u_n (\nabla K \star u_n) \cdot \nabla \varphi \, dx \, dt = \iint_{\Omega_T} F(u_n, w_n) \varphi \, dx \, dt, \quad (8)$$

for all test functions $\varphi \in L^2(0, T; H^1(\Omega))$.

2.2.2 Maximum principle

In this section, we prove that the solution of the nonlocal degenerate equation (4) satisfies the following version of maximum principle.

Lemma 3. *Assume that $0 \leq u_0 \leq \bar{u}$, then the solution u_n to the problem (5) satisfies*

$$0 \leq u_n(x, t) \leq e^{\lambda t} \bar{u}, \quad \text{for a.e. } (x, t) \in \Omega_T, \quad (9)$$

where $\lambda \in \mathbb{R}$ such that

$$\lambda \geq -\|w\|_{L^\infty(\Omega_T)} - \|\operatorname{div}(\nabla K \star u_n)\|_{L^\infty(\Omega_T)}. \quad (10)$$

Proof. For technical reasons, we need to extend the function $f(u) := \alpha u - w u^2$ so that it becomes measurable on Ω_T , continuous with respect to u . We do this by setting (recall that α and w are nonnegative)

$$\tilde{F}(u, w) = \begin{cases} F(u, w) & \text{if } u \geq 0, \\ 0 & \text{else.} \end{cases}$$

Next, we define the following new variable \tilde{u}_n by setting $u_n = e^{\lambda t} \tilde{u}_n$ where $\lambda > 0$ is defined in (10). It follows from (5) that \tilde{u}_n satisfies

$$\partial_t \tilde{u}_n - \operatorname{div} \left(a_m(e^{\lambda t} \tilde{u}_n) \nabla \tilde{u}_n \right) + e^{\lambda t} \operatorname{div} \left(\tilde{u}_n^+ \nabla K \star \tilde{u}_n \right) = -\lambda \tilde{u}_n + e^{-\lambda t} \tilde{F}(e^{\lambda t} \tilde{u}_n^+, w). \quad (11)$$

where $\tilde{u}_n^+ = \max\{\tilde{u}_n, 0\}$. Multiplying this equation by $u_n^- = \max\{-\tilde{u}_n, 0\}$ and integrating over Ω , the result is

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\tilde{u}_n^-|^2 dx - \int_{\Omega} a_{\varepsilon_n}(e^{\lambda t} \tilde{u}_n) \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n^- dx + e^{\lambda t} \int_{\Omega} \tilde{u}_n^+ \nabla K \star \tilde{u}_n \cdot \nabla \tilde{u}_n^- dx = \int_{\Omega} (\lambda \tilde{u}_n - e^{-\lambda t} \tilde{F}(e^{-\lambda t} \tilde{u}_n^+, w)) \tilde{u}_n^- dx. \quad (12)$$

Observe that

$$\begin{aligned} \int_{\Omega} a_{\varepsilon_n}(e^{\lambda t} \tilde{u}_n) \nabla \tilde{u}_n \cdot \nabla \tilde{u}_n^- dx &= - \int_{\Omega} a_{\varepsilon_n}(e^{\lambda t} \tilde{u}_n) |\nabla \tilde{u}_n^-|^2 dx \leq 0, \\ \int_{\Omega} \tilde{u}_n^+ \nabla K \star \tilde{u}_n \cdot \nabla \tilde{u}_n^- dx &= 0, \end{aligned}$$

and

$$\int_{\Omega} (\lambda \tilde{u}_n - e^{-\lambda t} F(e^{-\lambda t} \tilde{u}_n^+, w)) \tilde{u}_n^- dx = - \int_{\Omega} \lambda |\tilde{u}_n^-|^2 dx \leq 0.$$

Using this, we get easily from (12)

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\tilde{u}_n^-|^2 dx \leq 0.$$

Since the data u_0 is nonnegative, we deduce that $\tilde{u}_n^-(x, t) = 0$, for a.e. $(t, x) \in \Omega_T$.

In the following step, we show that $\tilde{u}_n(x, t) \leq \bar{u}$, for a.e. $(x, t) \in \Omega_T$. To do this, it suffices to prove that $(\tilde{u}_n - \bar{u})^+ = 0$. We multiply the equation (11) by $(\tilde{u}_n - \bar{u})^+$ and integrate over Ω , to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |(\tilde{u}_n - \bar{u})^+|^2 dx + \int_{\Omega} a_{\varepsilon_n}(e^{\lambda t} \tilde{u}_n) \nabla \tilde{u}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx - e^{\lambda t} \int_{\Omega} \tilde{u}_n \mathcal{K}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx \\ = \int_{\Omega} (-\lambda \tilde{u}_n + e^{-\lambda t} \tilde{F}(e^{\lambda t} \tilde{u}_n, w)) (\tilde{u}_n - \bar{u})^+ dx \\ \leq \int_{\Omega} (-\lambda \tilde{u}_n + \alpha \tilde{u}_n) (\tilde{u}_n - \bar{u})^+ dx \\ = \int_{\Omega} (-\lambda + \alpha) |(\tilde{u}_n - \bar{u})^+|^2 dx + \int_{\Omega} (-\lambda + \alpha) \bar{u} (\tilde{u}_n - \bar{u})^+ dx \end{aligned} \quad (13)$$

where $\mathcal{K}_n := \nabla K \star u_n$. Regarding the degenerate diffusion term, we have

$$\int_{\Omega} a_{\varepsilon_n}(e^{\lambda t} \tilde{u}_n) \nabla \tilde{u}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx = \int_{\Omega} a_{\varepsilon_n}(e^{\lambda t} \tilde{u}_n) |\nabla (\tilde{u}_n - \bar{u})^+|^2 dx \geq 0. \quad (14)$$

For the nonlocal term, we use an integration by part to deduce

$$\begin{aligned} \int_{\Omega} \tilde{u}_n \mathcal{K}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx &= \int_{\Omega} (\tilde{u}_n - \bar{u})^+ \mathcal{K}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx + \int_{\Omega} \bar{u} \mathcal{K}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx \\ &= \frac{1}{2} \int_{\Omega} \mathcal{K}_n \cdot \nabla |(\tilde{u}_n - \bar{u})^+|^2 dx + \int_{\Omega} \bar{u} \mathcal{K}_n \cdot \nabla (\tilde{u}_n - \bar{u})^+ dx \\ &= -\frac{1}{2} \int_{\Omega} \operatorname{div}(\mathcal{K}_n) |(\tilde{u}_n - \bar{u})^+|^2 dx + \frac{1}{2} \int_{\partial\Omega} |(\tilde{u}_n - \bar{u})^+|^2 \mathcal{K}_n \cdot \eta d\sigma(x) \\ &\quad - \int_{\Omega} \bar{u} \operatorname{div}(\mathcal{K}_n) (\tilde{u}_n - \bar{u})^+ dx + \int_{\partial\Omega} \bar{u} (\tilde{u}_n - \bar{u})^+ \mathcal{K}_n \cdot \eta d\sigma(x) \\ &\leq -\frac{1}{2} \int_{\Omega} \operatorname{div}(\mathcal{K}_n) |(\tilde{u}_n - \bar{u})^+|^2 dx - \int_{\Omega} \bar{u} \operatorname{div}(\mathcal{K}_n) (\tilde{u}_n - \bar{u})^+ dx, \end{aligned} \quad (15)$$

where we have used (recall that in (2), K is radially nonincreasing)

$$\mathcal{K}_n \cdot \eta \leq 0 \quad \text{on } \Sigma_T.$$

Collecting the previous estimates (14) and (15), we readily conclude from (13)

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |(\tilde{u}_n - \bar{u})^+|^2 dx + \int_{\Omega} (\lambda - w + \frac{1}{2} \operatorname{div}(\mathcal{K}_n)) |(\tilde{u}_n - \bar{u})^+|^2 dx + \int_{\Omega} \bar{u} (\lambda - w + \operatorname{div}(\mathcal{K}_n)) (\tilde{u}_n - \bar{u})^+ dx \leq 0. \quad (16)$$

Now, by the choice of λ in (10), we deduce from (16)

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |(\tilde{u}_n - \bar{u})^+|^2 dx \leq 0.$$

Using that $u_0 \leq \bar{u}$, we conclude from this $u_n(t, \cdot) \leq e^{\lambda t} \bar{u}$ in Ω for $t \in (0, T)$. This concludes the proof of the lemma. \square

2.2.3 A priori estimates

First, we let the functions

$$A(r) = \int_0^r a(s) ds \quad \text{and} \quad \mathcal{A}(r) = \int_0^r A(s) ds.$$

To pass to the limit in (8) and prove the existence of the solution u , we need the following a priori estimates lemma.

Lemma 4. *The solution u_n to the problem (5) satisfies*

$$\begin{aligned} & \| \mathcal{A}(u_n(x, t)) \|_{L^\infty(0, T; L^1(\Omega))} + \| \nabla A(u_n) \|_{L^2(\Omega_T)} + \varepsilon_n \| u_n \|_{L^\infty(0, T; L^2(\Omega))} + \varepsilon_n \| \nabla u_n \|_{L^2(\Omega_T)} \leq C, \\ & \| \partial_t u_n \|_{L^2(0, T; (H^1(\Omega))')} \leq C, \end{aligned} \quad (17)$$

for some constant $C > 0$ not depending of n .

Proof. We substitute $\varphi = A_{\varepsilon_n}(u_n) := A(u_n) + \varepsilon_n u_n$ in (8) and we integrate over $(0, \tau)$ with $\tau < T$ to obtain

$$\begin{aligned} & \int_0^\tau \langle \partial_t u_n, A_{\varepsilon_n}(u_n) \rangle dt + \iint_{\Omega_\tau} |\nabla A(u_n)|^2 dx dt + \varepsilon_n \iint_{\Omega_\tau} |\nabla u_n|^2 dx dt \\ & - \iint_{\Omega_\tau} u_n (\nabla K \star u_n) \cdot \nabla A_{\varepsilon_n}(u_n) dx dt = \iint_{\Omega_\tau} F(u_n, w) A_{\varepsilon_n}(u_n) dx dt. \end{aligned} \quad (18)$$

Next, using Young inequality and Lemma 3, we obtain

$$\begin{aligned} \left| \iint_{\Omega_\tau} u_n (\nabla K \star u_n) \cdot \nabla A_{\varepsilon_n}(u_n) dx dt \right| & \leq \iint_{\Omega_\tau} |\nabla K \star u_n|^2 |u_n|^2 dx dt + \frac{1}{2} \iint_{\Omega_\tau} |\nabla A(u_n)|^2 dx dt \\ & + \frac{\varepsilon_n}{2} \iint_{\Omega_\tau} |\nabla u_n|^2 dx dt \\ & \leq C_1 + \frac{1}{2} \iint_{\Omega_\tau} |\nabla A(u_n)|^2 dx dt + \frac{\varepsilon_n}{2} \iint_{\Omega_\tau} |\nabla u_n|^2 dx dt \end{aligned} \quad (19)$$

and

$$\iint_{\Omega_\tau} |F(w, u_n) A_{\varepsilon_n}(u_n)| dx dt \leq C_2, \quad (20)$$

for some constants $C_1, C_2 > 0$. Now, exploiting (19) and (20), we deduce from (18)

$$\sup_{0 < \tau \leq T} \int_{\Omega} \mathcal{A}(u_n(\tau)) dx + \varepsilon_n \sup_{0 < \tau \leq T} \int_{\Omega} |u_n(\tau)|^2 dx + \frac{1}{2} \iint_{\Omega_T} |\nabla A(u_n)|^2 dx + \frac{\varepsilon_n}{2} \iint_{\Omega_T} |\nabla u_n|^2 dx dt \leq C_3, \quad (21)$$

for some constant $C_3 > 0$. This implies the desired first estimate of (17).

To prove the second estimate of (17), we take $\varphi \in L^2(0, T; H^1(\Omega))$ and we use the weak formulation (8) to obtain

$$\begin{aligned} \left| \int_0^T \langle \partial_t u_n, \varphi \rangle dt \right| &\leq \iint_{\Omega_T} |\nabla A_{\varepsilon_n}(u_n) \cdot \nabla \varphi| dx dt + \iint_{\Omega_T} |u_n (\nabla K \star u_n) \cdot \nabla \varphi| dx dt + \iint_{\Omega_T} |F(u_n, w) \varphi| dx dt, \\ &\leq \|\nabla A_{\varepsilon_n}(u_n)\|_{L^2(\Omega_T)} \|\nabla \varphi\|_{L^2(\Omega_T)} + \|\nabla K \star u_n\|_{L^\infty(\Omega_T)} \|u_n\|_{L^2(\Omega_T)} \|\nabla \varphi\|_{L^2(\Omega_T)} \\ &\quad + \|F(u_n, w)\|_{L^2(\Omega_T)} \|\varphi\|_{L^2(\Omega_T)} \\ &\leq C_4 \|\varphi\|_{L^2(0, T; H^1(\Omega))}, \end{aligned} \quad (22)$$

for some constant $C_4 > 0$, where we have used (21). This implies

$$\left\| \partial_t u_n \right\|_{L^2(0, T; (H^1(\Omega))')} \leq C_4. \quad (23)$$

□

2.2.4 Passing to the limit

Thanks to Lemma 4 and Aubin-Simon compactness theorem (see for e.g. [21]), we can extract subsequences, which we do not relabel, such that, as $n \rightarrow \infty$,

$$\begin{cases} u_n \rightharpoonup u \text{ weakly-}\star \text{ in } L^\infty(\Omega_T), \\ A(u_n) \rightharpoonup \bar{A} \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ \sqrt{\varepsilon_n} u_n \rightarrow 0 \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ \partial_t u_n \rightharpoonup \partial_t u \text{ weakly in } L^2(0, T; (H^1(\Omega))'). \end{cases} \quad (24)$$

Next, we use the compact embedding $L^\infty(\Omega) \subset (H^1(\Omega))'$ and Corollary 4 of [21], to deduce that u_n is a Cauchy sequence in $C(0, T; (H^1(\Omega))')$.

Observe that $A(u_n)$ is uniformly bounded in \mathcal{S} , where

$$\mathcal{S} = \{u \in L^2(0, T, H^1(\Omega)) : \partial_t u \in L^2(0, T; (H^1(\Omega))')\}.$$

From the compact imbedding $\mathcal{S} \subset L^2(\Omega_T)$, we deduce that there exists a subsequence of u_n such that

$$A(u_n) \rightarrow \bar{A} \text{ strongly in } L^2(\Omega_T).$$

Since A is monotone, we get $A(u) = \bar{A}$. Therefore,

$$A(u_n) \rightarrow A(u) \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Moreover, as A^{-1} is well-defined and continuous, we apply the dominated convergence theorem to $u_n = A^{-1}(A(u_n))$ to obtain

$$u_n \rightarrow u \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Using this and the weak- \star convergence of u_n to u in $L^\infty(\Omega_T)$, we obtain

$$u_n \rightarrow u \text{ strongly in } L^q(\Omega_T) \text{ for } 1 \leq q < \infty.$$

With the above convergences we are ready to identify the limit u as a weak solution of (1). Let $\varphi \in L^2(0, T; H^1(\Omega))$ be a test function in (8). By (24) it is clear that as $\varepsilon \rightarrow 0$

$$\int_0^T \langle \partial_t u_n, \varphi \rangle dt \rightarrow \int_0^T \langle \partial_t u, \varphi \rangle dt$$

and

$$\iint_{\Omega_T} a_{\varepsilon_n}(u_n) \nabla u_n \cdot \nabla \varphi dx dt \rightarrow \iint_{\Omega_T} a(u) \nabla u \cdot \nabla \varphi dx dt.$$

Since $u_n (\nabla K \star u_n)$ is bounded in $L^\infty(\Omega_T)$, we also have that, as $n \rightarrow \infty$,

$$\iint_{\Omega_T} u_n (\nabla K \star u_n) \cdot \nabla \varphi dx dt \rightarrow \iint_{\Omega_T} u (\nabla K \star u) \cdot \nabla \varphi dx dt,$$

Similarly, we have as $n \rightarrow \infty$,

$$\iint_{\Omega_T} F(u_n, w) \varphi dx dt \rightarrow \iint_{\Omega_T} F(u, w) \varphi dx dt.$$

We have finally identified u as a solution of (1).

2.3 Uniqueness of the weak solution

In this section we prove uniqueness of weak solutions to our nonlocal degenerate aggregation model, thereby completing the well-posedness analysis. The uniqueness proof of weak solutions is proved by using duality technique.

First, we consider u_1 and u_2 two solutions of the system (1). We set $U = u_1 - u_2$, then U satisfies (for $i = 1, 2$)

$$\begin{cases} \partial_t U - \Delta(A(u_1) - A(u_2)) + \operatorname{div}(u_1 \nabla K \star u_1 - u_2 \nabla K \star u_2) = F(u_1, w) - F(u_2, w) & \text{in } \Omega_T, \\ (\nabla A(u_i) - u_i \nabla K \star u_i) \cdot \eta = 0 & \text{on } \Sigma_T, \\ u_i(x, 0) = u_0(x), & \text{in } \Omega. \end{cases} \quad (25)$$

Now, we define the function φ solution of the problem

$$-\Delta \varphi(t, \cdot) = U(t, \cdot) \text{ in } \Omega \text{ and } \nabla \varphi(t, \cdot) \cdot \eta = 0 \text{ on } \partial \Omega, \quad (26)$$

for a.e. $t \in (0, T)$. Since u_1 and u_2 are bounded in L^∞ , then we get from the theory of linear elliptic equations, the existence, uniqueness and regularity of solution φ satisfying

$$\varphi \in C([0, T]; H^2(\Omega)) \text{ with } \int_{\Omega} \varphi(t, \cdot) dx = 0.$$

Note that from the boundary condition of φ in (26) and $U(0, \cdot) = 0$ we deduce that

$$\nabla \varphi(0, \cdot) = 0 \text{ in } L^2(\Omega). \quad (27)$$

Multiplying the second equation in (25) by $\psi \in L^2(0, T; H^1(\Omega))$ and integrating over $\Omega_t := (0, t) \times \Omega$, we get

$$\begin{aligned} \int_0^t \langle \partial_s U, \psi \rangle ds + \iint_{\Omega_t} \nabla(A(u_1) - A(u_2)) \cdot \nabla \psi dx ds &= \iint_{\Omega_t} (u_1 \nabla K \star u_1 - u_2 \nabla K \star u_2) \cdot \nabla \psi dx ds \\ &+ \iint_{\Omega_t} (F(u_1, w) - F(u_2, w)) \psi dx ds. \end{aligned} \quad (28)$$

Since $\varphi \in L^2(0, T; H^1(\Omega))$ we can take $\psi = \varphi$ in (28) and we obtain from (26) and (27)

$$\begin{aligned} 2 \int_0^t \langle \partial_s U, \varphi \rangle ds &= -2 \int_0^t \langle \partial_s \Delta \varphi, \varphi \rangle ds \\ &= \int_{\Omega} |\nabla \varphi(t, x)|^2 dx - \int_{\Omega} |\nabla \varphi(0, x)|^2 dx \\ &= \int_{\Omega} |\nabla \varphi(t, x)|^2 dx \end{aligned} \quad (29)$$

and

$$\begin{aligned} \int_0^t \langle \partial_s U, \varphi \rangle ds - \iint_{\Omega_t} (A(u_1) - A(u_2)) \Delta \varphi dx ds \\ = \iint_{\Omega_t} (u_1 \nabla K \star u_1 - u_2 \nabla K \star u_2) \cdot \nabla \varphi dx ds + \iint_{\Omega_t} (F(u_1, w) - F(u_2, w)) \varphi dx ds. \end{aligned} \quad (30)$$

Since u_1 and u_2 are bounded in L^∞ , then there exist constants $C_5, C_6 > 0$ such that

$$\begin{aligned} |F(u_1, w) - F(u_2, w)| &\leq C_5 |u_1 - u_2|, \\ |u_1 \nabla K \star u_1 - u_2 \nabla K \star u_2| &\leq C_6 |u_1 - u_2|. \end{aligned} \quad (31)$$

Using (26), (31), Hölder's, Young's, Sobolev Poincaré's inequalities yields from (30)

$$\begin{aligned} \int_0^t \langle \partial_s U, \varphi \rangle ds &\leq -C_a \iint_{\Omega_t} |U|^2 dx ds + \frac{C_a}{4} \iint_{\Omega_t} |U|^2 dx ds + C_7 \int_0^t \|\nabla \varphi\|_{L^2(\Omega)}^2 ds + \frac{C_a}{4} \iint_{\Omega_t} |U|^2 dx ds \\ &\quad + C_8 \int_0^t \|\varphi\|_{L^2(\Omega)}^2 ds \\ &\leq C_9 \int_0^t \|\nabla \varphi\|_{L^2(\Omega)}^2 ds, \end{aligned} \quad (32)$$

for some constants $C_7, C_8, C_9 > 0$. Using this and (29), we deduce

$$\begin{aligned} \int_{\Omega} |\nabla \varphi(t, x)|^2 dx &= 2 \int_0^t \langle \partial_s U, \varphi \rangle ds \\ &\leq 2 C_9 \int_0^t \|\nabla \varphi\|_{L^2(\Omega)}^2 ds. \end{aligned} \quad (33)$$

Finally, using Gronwall's lemma to conclude from (33)

$$\nabla \varphi = 0 \quad \text{a.e. in } \Omega_T,$$

ensuring the uniqueness of weak solutions.

3 The optimal control problem

In this subsection, we provide the existence of the solution for the optimal control problem of the nonlocal degenerate equation (1). We considered the following cost functional for the optimization of the population density and aggregation term

$$J(w, u) = \frac{\varepsilon_1}{2} \int_0^T \int_{\Omega_c} |w|^2 dx dt + \frac{\varepsilon_2}{2} \iint_{\Omega_T} |u(\nabla K \star u)|^2 dx dt, \quad (34)$$

where ε_1 and ε_2 denote regularization parameters. Herein, $\Omega_c \subset \Omega$ is the control subdomain. We define the set of admissible controls \mathcal{U} by

$$\mathcal{U} = \{w \in L^\infty(\Omega_T) : \underline{w} \leq w(t, x) \leq \bar{w}\}, \quad (35)$$

where $\underline{w} \in \mathbb{R}_+^*$ and $\bar{w} \in \mathbb{R}_+^*$ are the minimal and the maximal intraspecific competition rates, respectively. We consider the following minimization problem:

$$\min_w J(w, u) \quad \text{subject to (1)}. \quad (36)$$

3.1 Existence of the control

In this subsection, we show the existence of the optimal solution $w^* \in \mathcal{U}$ for the problem (36).

Lemma 5. *Assume that $u_0 \in L^\infty(\Omega)$. Then, there exists a solution $w^* \in \mathcal{U}$ of the optimal control problem (34).*

Proof. Let w_n be a minimizing sequence of J such that

$$\inf_{w \in \mathcal{U}} \{J\} \leq J(w_n) \leq \inf_{w \in \mathcal{U}} \{J\} + \frac{1}{n}.$$

Thanks to the definition of J , the sequence $(w_n)_n$ is bounded in $L^2(\Omega_T)$. This implies that w_n converge weakly to an w^* . Let u_n be a solution to the problem (1) with respect to the control w_n . Working exactly as in Section 2, we deduce the following convergence (upon a subsequence)

$$u_n \rightarrow u^* \text{ strongly in } L^q(\Omega_T) \text{ and a.e. in } \Omega_T \text{ for } 1 \leq q < \infty.$$

Note that, since the cost functional $J(\cdot, \cdot)$ is continuous and convex on $L^2(\Omega_T) \times \mathcal{U}$, it follows that $J(\cdot, \cdot)$ is weakly lower semicontinuous. Hence, by exploiting the strong convergence of u_n combined with the weak lower semi-continuity of J we arrive to

$$J(u^*, w^*) \leq \liminf_{n \rightarrow \infty} J(w_n) \leq \inf_{w \in \mathcal{U}} \{J(w)\} = J(u^*, w^*).$$

This implies the existence result of our optimal control solution (36). □

3.2 Optimal conditions and dual problem

In this subsection, we derive the optimality conditions based on the Lagrangian formulation. We introduce the Lagrange functional L defined by (recall that $A(u) = \int_0^u a(s) ds$):

$$\begin{aligned}
L(u, w, p) &= J(w, u) + \iint_{\Omega_T} \left(u_t - \operatorname{div} (a(u)\nabla u - u(\nabla K \star u)) - F(u, w) \right) p \, dx \, dt \\
&= \frac{\varepsilon_1}{2} \int_0^T \int_{\Omega_c} |w|^2 \, dx \, dt + \frac{\varepsilon_2}{2} \iint_{\Omega_T} |u(\nabla K \star u)|^2 \, dx \, dt \\
&\quad + \iint_{\Omega_T} \left(u_t - \operatorname{div} (a(u)\nabla u - u(\nabla K \star u)) - F(u, w) \right) p \, dx \, dt \\
&= \frac{\varepsilon_1}{2} \int_0^T \int_{\Omega_c} |w|^2 \, dx \, dt + \frac{\varepsilon_2}{2} \iint_{\Omega_T} |u(\nabla K \star u)|^2 \, dx \, dt - \iint_{\Omega_T} \partial_t u p \, dx \, dt \\
&\quad + \iint_{\Omega_T} \left(\nabla A(u) - u(\nabla K \star u) \right) \nabla p \, dx \, dt - \iint_{\Omega_T} F(u, w) p \, dx \, dt
\end{aligned} \tag{37}$$

$$\begin{aligned}
&= \frac{\varepsilon_1}{2} \int_0^T \int_{\Omega_c} |w|^2 \, dx \, dt + \frac{\varepsilon_2}{2} \iint_{\Omega_T} |u(\nabla K \star u)|^2 \, dx \, dt - \iint_{\Omega_T} \partial_t u p \, dx \, dt \\
&\quad - \iint_{\Omega_T} A(u) \Delta p \, dx \, dt + \iint_{\Sigma_T} A(u) \nabla p \cdot \eta \, d\sigma(x) \, dt - \iint_{\Omega_T} u(\nabla K \star u) \cdot \nabla p \, dx \, dt - \iint_{\Omega_T} F(u, w) p \, dx \, dt.
\end{aligned} \tag{38}$$

The first order optimality system characterizing the adjoint variables, is given by the Lagrange multipliers which result from equating the partial derivative of L with respect to u equal to zero:

$$\begin{aligned}
\left(\frac{\partial L(u, w, p)}{\partial u}, \delta u \right) &= \varepsilon_2 \iint_{\Omega_T} u(\nabla K \star u) \left((\nabla K \star u) \delta u + u(\nabla K \star \delta u) \right) \, dx \, dt - \iint_{\Omega_T} \partial_t p \delta u - \iint_{\Omega_T} a(u) \Delta p \delta u \, dx \, dt \\
&\quad - \iint_{\Omega_T} \left(\delta u(\nabla K \star u) + u(\nabla K \star \delta u) \right) \nabla p \, dx \, dt - \iint_{\Omega_T} \partial_u F(u, w) p \delta u \, dx \, dt \\
&= \varepsilon_2 \iint_{\Omega_T} u |\nabla K \star u|^2 \delta u \, dx \, dt + \varepsilon_2 \iint_{\Omega_T} |u|^2 (\nabla K \star u) (\nabla K \star \delta u) \, dx \, dt - \iint_{\Omega_T} \partial_t p \delta u \\
&\quad - \iint_{\Omega_T} a(u) \Delta p \delta u \, dx \, dt - \iint_{\Omega_T} (\nabla K \star u) \nabla p \delta u \, dx \, dt - \iint_{\Omega_T} u(\nabla K \star \delta u) \nabla p \, dx \, dt \\
&\quad + \iint_{\Omega_T} \partial_u F(u, w) p \delta u \, dx \, dt \\
&= \iint_{\Omega_T} \left[-\partial_t p - a(u) \Delta p - (\nabla K \star u) \nabla p + \varepsilon_2 |u|^2 \nabla K \star u - \partial_u F(u, w) p \right] \delta u \, dx \, dt \\
&\quad - \iint_{\Omega_T} \left[u \nabla p - \varepsilon_2 |u|^2 (\nabla K \star u) \right] \cdot (\nabla K \star \delta u) \, dx \, dt
\end{aligned} \tag{39}$$

Observe that

$$\begin{aligned}
\mathcal{B} &:= \iint_{\Omega_T} \left(u \nabla p - \varepsilon_2 |u|^2 (\nabla K \star u) \right) (x) \cdot (\nabla K \star \delta u) (x) dx dt \\
&= \int_0^T \int_{\Omega \times \Omega} \left(u \nabla p - \varepsilon_2 |u|^2 (\nabla K \star u) \right) (x) \cdot \nabla K(x-y) \delta u(y) dy dx dt \\
&= - \int_0^T \int_{\Omega \times \Omega} \left(u \nabla p - \varepsilon_2 |u|^2 (\nabla K \star u) \right) (x) \cdot \nabla K(y-x) \delta u(y) dy dx dt \\
&= - \iint_{\Omega_T} \left(u \nabla p - \varepsilon_2 |u|^2 (\nabla K \star u) \right) \star \nabla K \delta u dx dt.
\end{aligned} \tag{40}$$

Next, we exploit (39) and (40) to deduce the adjoint equation of the nonlocal degenerate aggregation model (1)

$$\begin{cases} -\partial_t p - a(u) \Delta p - (\nabla K \star u) \nabla p + (u \nabla p) \star \nabla K = \partial_u F(u, w) p + \mathcal{F}_K(u) & \text{in } \Omega_T, \\ \nabla p \cdot \eta = 0 & \text{on } \Sigma_T, \\ p(x, T) = p_T(x) = 0, & \text{in } \Omega. \end{cases} \tag{41}$$

where

$$\partial_u F(u, w) = \alpha - 2w u \text{ and } \mathcal{F}_K(u) := \varepsilon_2 (|u|^2 \nabla K \star u) \star \nabla K - \varepsilon_2 u |\nabla K \star u|^2.$$

To find the optimal conditions, we calculate the gradient of the functional $J(w, u)$:

$$\left(\frac{\partial L}{\partial w}, \delta w \right) = \varepsilon_1 \int_0^T \int_{\Omega_c} w \delta w dx dt - \iint_{\Omega_c} u^2 p \delta w dx dt \quad \text{and} \quad \nabla J(w, u) = \frac{\partial L}{\partial w}. \tag{42}$$

Therefore, the optimality condition can be written as follows

$$\int_0^T \int_{\Omega_c} (\varepsilon_1 w + u^2 p) dx dt = 0.$$

Remark 6. Note that in aggregation equations, it is common to use an even kernel ($K(-x) = K(x)$). We mention that the gradient of an even function became odd (i.e. $\nabla K(-x) = -\nabla K(x)$.)

4 Numerical Discretization

In this section, we present numerical methods to solve the nonlocal aggregation problem (1). We propose a numerical scheme to approximate the solution of the associated adjoint problem (41) and we implement the optimal control solver of the minimization problem (34). To approximate the solution of the direct problem (1) and adjoint problem (41), we will use the numerical scheme introduced in [9]. First, let us consider a Cartesian mesh with the step h_i in the direction $i \in \{1, \dots, d\}$ and $h = \max_i h_i$. Denoting by $(C_J)_{J \in \mathbb{Z}^d}$ the space cells, where each cell C_J has a center $x_J := (x_1, \dots, x_d)$ with $x_i = J_i h_i$ for $i \in \{1, \dots, d\}$. Next, we let e_i the canonical basis of \mathbb{Z}^d and we denote $(u_J^n)_{J \in \mathbb{Z}^d}$ the approximation of cell average of $u(t, \cdot)$ at a given time $t_n = n\tau$.

We propose the following numerical approximation for the direct problem (1) (recall that $(s)^+ = \max\{0, s\}$ and

$(s)^- = \max\{0, -s\}$ for a real number s

$$\begin{aligned}
u_J^{n+1} = u_J^n &+ \sum_{i=1}^d \frac{\tau}{h_i} \left((\mathcal{B}_{i,J}^n)^+ a(u_J^n) - (\mathcal{B}_{i,J+e_i}^n)^- a(u_{J+e_i}^n) - (\mathcal{B}_{i,J-e_i}^n)^+ a(u_{J-e_i}^n) + (\mathcal{B}_{i,J}^n)^- a(u_J^n) \right) \\
&- \sum_{i=1}^d \frac{\tau}{h_i} \left((\mathcal{A}_{i,J}^n)^+ u_J^n - (\mathcal{A}_{i,J+e_i}^n)^- u_{J+e_i}^n - (\mathcal{A}_{i,J-e_i}^n)^+ u_{J-e_i}^n + (\mathcal{A}_{i,J}^n)^- u_J^n \right) + \tau F(u_J^n, w_J^n).
\end{aligned} \tag{43}$$

The numerical discrete aggregation and diffusion velocities are defined respectively by

$$\mathcal{A}_{i,J}^n := - \sum_{L \in \mathbb{Z}^d} u_L^n D_i K_J^L \quad \text{and} \quad \mathcal{B}_{i,J}^n := \frac{u_{J+e_i}^n - u_J^n}{h_i},$$

where $D_i K_J^L = \partial_{x_i} K(x_J - x_L)$ for a pointy potential K .

Now, for the solution of adjoint problem (41), we consider $(p_J^n)_{J \in \mathbb{Z}}$ the finite volume approximation of cell average of $p(t, \cdot)$ at a given time $t = n\tau$. We use the following numerical approximation of the adjoint problem (41)

$$\begin{aligned}
p_J^{n-1} = p_J^n &+ \sum_{i=1}^d \frac{\tau}{h_i} \left((\overline{\mathcal{B}}_{i,J}^n)^+ a(u_J^n) - (\overline{\mathcal{B}}_{i,J+e_i}^n)^- a(u_{J+e_i}^n) - (\overline{\mathcal{B}}_{i,J-e_i}^n)^+ a(u_{J-e_i}^n) + (\overline{\mathcal{B}}_{i,J}^n)^- a(u_J^n) \right) \\
&- \sum_{i=1}^d \frac{\tau}{h_i} \left((\overline{\mathcal{A}}_{i,J}^n)^+ u_J^n - (\overline{\mathcal{A}}_{i,J+e_i}^n)^- u_{J+e_i}^n - (\overline{\mathcal{A}}_{i,J-e_i}^n)^+ u_{J-e_i}^n + (\overline{\mathcal{A}}_{i,J}^n)^- u_J^n \right) \\
&- \sum_{i=1}^d \frac{\tau}{h_i} (\mathcal{F}_{i,J}^n - \mathcal{F}_{i,J+e_i}^n) + \tau (\mathcal{F}_K(u_J^n) + \partial_u F(u_J^n, w_J^n) p_J^n),
\end{aligned} \tag{44}$$

where

$$\overline{\mathcal{B}}_{i,J}^n := \frac{p_{J+e_i}^n - p_J^n}{h_i}, \quad \text{and} \quad \overline{\mathcal{A}}_{i,J}^n := \sum_{i=1}^d \sum_{L \in \mathbb{Z}^d} u_L^n \frac{p_{L+e_i}^n - p_{L-e_i}^n}{2h_i} D_i K_J^L.$$

The term $\mathcal{F}_{i,J}^n$ can be computed as

$$\mathcal{F}_{i,J}^n := \psi_{i,J}^n u_{J-e_i}^n + \phi^n(r_{i,J}^n) \psi_{i,J}^n \left(1 - \frac{\tau}{h_i} \psi_{i,J}^n \right) [u_J^n - u_{J-e_i}^n]$$

where the convection velocity is given by

$$\psi_{i,J}^n := a'(u_J^n) \frac{u_J^n - u_{J-e_i}^n}{h_i} - \sum_{L \in \mathbb{Z}} u_L^n D_i K_J^L \quad \text{and} \quad r_{i,J}^n := \frac{u_{J-e_i}^n - u_{J-2e_i}^n}{u_J^n - u_{J-e_i}^n}.$$

Following the establishment of the essential discretization of the direct and adjoint problems. We must develop a numerical approach to minimize the specified cost function(34). It is common knowledge that the basic gradient descent approach does not ensure global convergence relative to the initial guest. Therefore, we implemented the nonlinear conjugate gradient technique [11] to achieve global convergence performance (see Algorithm 1). However, this class of methods has many limitations in terms of convergence to a global minimum.

Algorithm 1 The optimal control solver.

```

1: Input:  $u_0, err \leftarrow 1$ 
2: Initialize:  $w^0, \alpha, tol, k \leftarrow 0$ 
3:
4: while  $\|\nabla J(w^k)\| > tol$  do
5:   for  $t = t^1, \dots, t^{final}$  do
6:     Giving  $\tau_2^k$  Compute  $u^h$  from the direct problem;
7:   end for
8:   Compute the cost function  $J(w^k, u^h)$ 
9:   for  $t = t^{final}, \dots, t^0$  do
10:    Giving  $w^k, u^h$ , compute  $p^h$  by solving the adjoint problem;
11:   end for
12:   Compute the gradient  $g_{k+1} = \nabla J(w^k, p^h)$ ;
13:   Compute  $y_k = g_{k+1} - g_k$ 
14:   Compute step length  $\alpha_k$ 
15:   Update the values of  $w$   $w^{k+1} = w^k + \alpha_k d_k$ ;
16:   Compute  $\beta^k = (y_k - 2d_k \frac{\|y_k\|^2}{d_k^T y_k})^T \frac{g_{k+1}}{d_k^T y_k}$ 
17:    $d_k = -g_k + \beta^k d_{k-1}$ ;
18:   Update the direction  $d_k = g_k + \beta^k d_{k-1}$ 
19:    $k \leftarrow k + 1$ 
20: end while

```

5 Numerical simulations

In this section, we present an efficient implementation of the proposed numerical schemes (43)-(44) with the optimal control Algorithm 1 to simulate the population dynamics under attractive forces. We focus our simulations on the effect of the optimal control (34) on the pattern formation induced by attractive forces under several initial conditions. To compute u_j^n for direct (43) and p_j^n for adjoint (44) problems, we choose the computational domain $\Omega := (-4, 4) \times (-4, 4)$, the diffusion function $a(u) = u(\bar{u} - u)$, the time step $\tau = 0.001$ and $h_1 = h_2 = 0.1$. For the aggregation kernel, we choose the Gaussian distribution :

$$K(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{\|x\|^2}{\sigma^2}\right) \text{ for all } x \in \mathbb{R}^2 \quad (45)$$

where $\sigma > 0$ is a given parameter.

In the next subsections we will present various tests and simulations. The first test is devoted to examine the effect of the aggregation and degenerate diffusion terms in the absence of the reaction term. Afterward, we will investigate the effect of the reaction term under various parameters. The final subsection focuses on evaluating the efficiency of the optimal control algorithm by considering two different initial conditions, and examining the impact of the resulting optimal control on the aggregation dynamics in each case.

5.1 Aggregation diffusion equation

In order to show the effect of the attractive force under a degenerate diffusion coefficient, we run different numerical simulations under different initial conditions.

Figure 1 presents the dynamic of a three group under attractive force. In Figure 2, we depict the evolution of a randomly distributed initial population density.

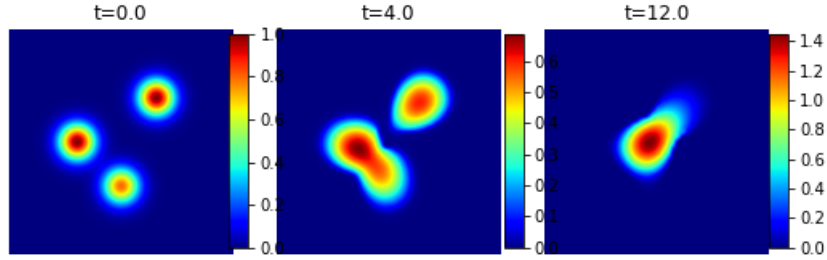


Figure 1: Evolution of the population dynamics using Gaussian attraction (with $\sigma = 0.8$), the initial condition $u_0 = \exp(c((X + 1)^2 + Y^2)) + 0.8 \exp(c(X^2 + (Y - 1)^2)) + \exp(c((X - 0.8)^2 + (Y + 1)^2))$ where $c = -1$.

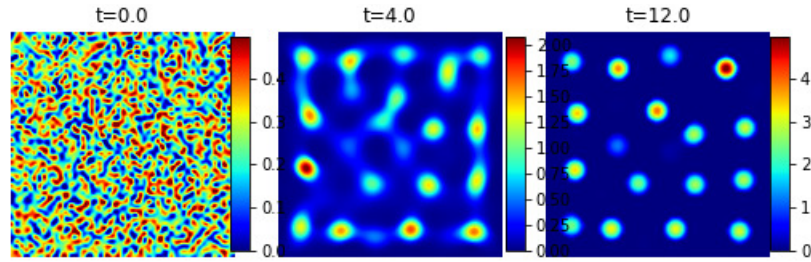


Figure 2: Evolution of the population dynamics using Gaussian attraction (with $\sigma = 0.2$) and a random initial population density u_0 .

It is well known that a diffusion process drives individuals of a given population towards lower densities according to gradient direction. In the other hand, the attraction force forms a velocity field that drives individuals from lower density groups to higher density groups. These two effects drives the solution to various forms of steady states. For example, in Figure 1, we notice that a single group is formed from three different population densities. Starting from a randomly distributed initial condition with a sharper attraction kernel (i.e. $\sigma = 0.2$), Figure 2 shows the formation of several groups due to attraction forces.

5.2 Aggregation equation with nonlinear interaction term

To lessen the effect of over-crowding phenomenon, a degenerate diffusion plays a counter effect role. In more realistic phenomenon inspired from nature, the overcrowding effect comes with costs on the population. The mortality rate of the population rises due to some limited resources. This can be modeled by using logistic reaction term $F(u, w) := \alpha u - w u^2$.

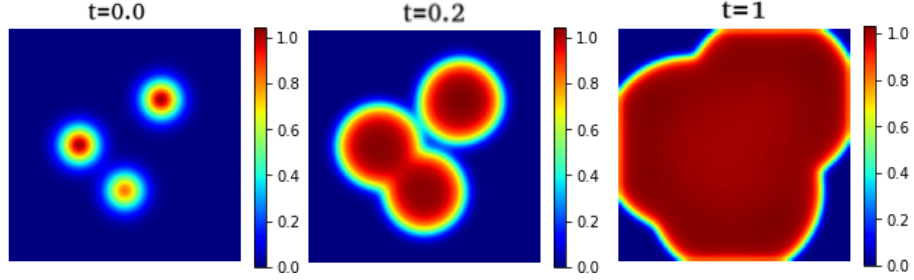


Figure 3: Evolution of the population dynamics using the logistic source term F (with $\alpha = 0.25$ and $w = 0.25$), a nonlocal pointy Gaussian potential (with $\sigma = 0.8$) and the initial population $u_0 = \exp(c((X + 1)^2 + Y^2)) + 0.8 \exp(c(X^2 + (Y - 1)^2)) + \exp(c((X - 0.8)^2 + (Y + 1)^2))$ where $c = -1$.

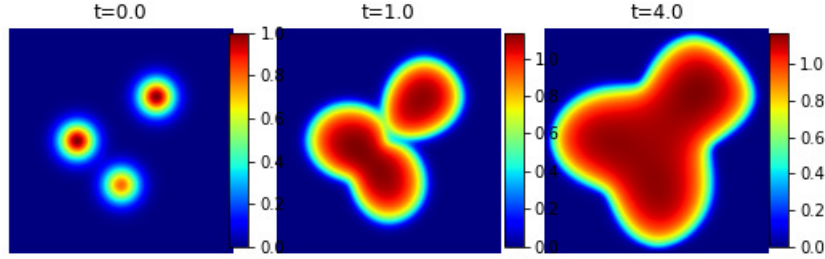


Figure 4: Evolution of the population dynamics using the logistic source term F (with $\alpha = 0.1$ and $w = 0.1$), the nonlocal pointy Gaussian potential (with $\sigma = 0.8$) and the initial population $u_0 = \exp(c((X + 1)^2 + Y^2)) + 0.8 \exp(c(X^2 + (Y - 1)^2)) + \exp(c((X - 0.8)^2 + (Y + 1)^2))$ where $c = -1$.

Note that the logistic reaction term F (with $\alpha = 0.25$ and $w = 0.25$) eliminates the aggregation phenomenon in Figure 3. Moreover, the solution achieves in short time range a constant steady-state solution $u \equiv 1$. When $\alpha = 0.1$ and $w = 0.1$, we observe that the evolution of the population is closer to the aggregation dynamics (see Figure 4). In the last experiment of the direct problem (consult Figure 5), we notice that the reaction term drives a random distributed initial condition to a more regular steady-state solution comparing to Figure 2. We conclude that, according to the value of α , we can control the dominance of the logistic reaction term.

In the next subsection, we study and control the aggregation effect acting on the intraspecific competition rate w .

5.3 Optimal control simulation of the degenerate aggregation model

In this subsection, we implement several tests showing the efficiency of the proposed optimal control procedure to eliminate the hoarding effect. In each test, we plot the comparison between the controlled and the uncontrolled dynamics. In Figure 6 and 8, we present a comparison between the controlled and uncontrolled dynamics of a given initial population density and a given attraction kernel. In the first and second rows of each figure, we illustrate the effects of an uncontrolled and controlled dynamic, respectively and the third row illustrates the development of the optimal solution w .

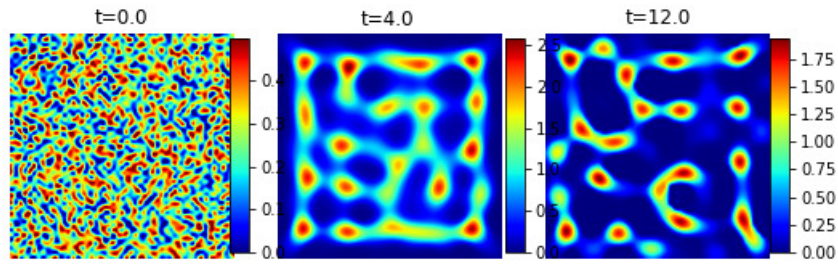


Figure 5: Evolution of the population dynamics using the logistic source term F (with $\alpha = 0.01$ and $w = 0.01$), the nonlocal pointy Gaussian potential (with $\sigma = 0.2$) and a random initial population density.

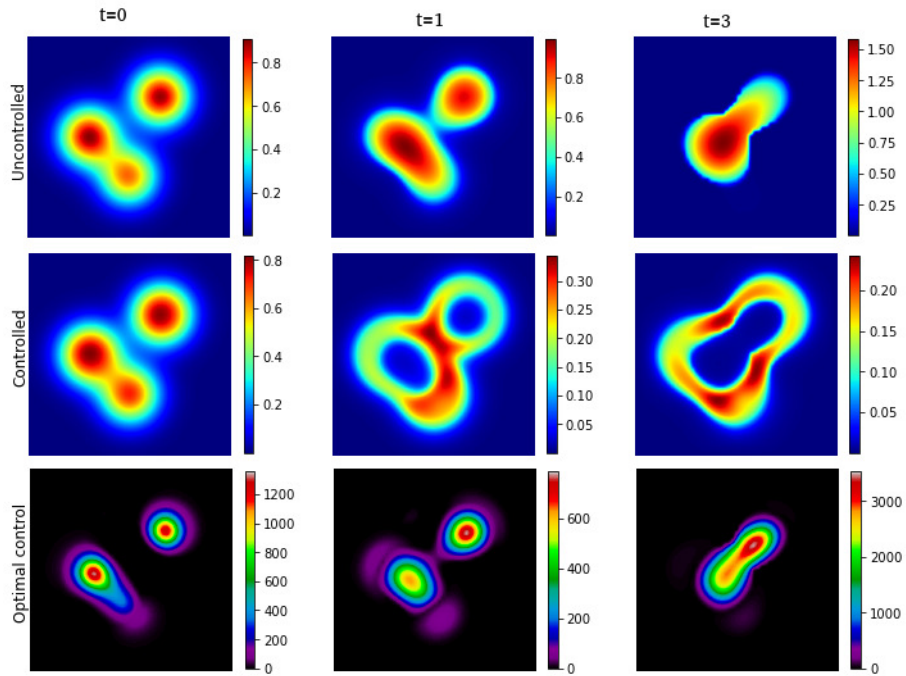


Figure 6: A comparison between controlled and uncontrolled dynamics acting on competition coefficient w where $a(u) = u(\bar{u} - u)$, $\sigma = 0.5$, $T = 3$, $c = -1$, $u_0(x, y) = 0.8 \exp(c((X + 1)^2 + Y^2)) + 0.64 \exp(c(X^2 + (Y - 1)^2)) + 0.8 \exp((X - 0.8)^2 + (Y + 1)^2)$.

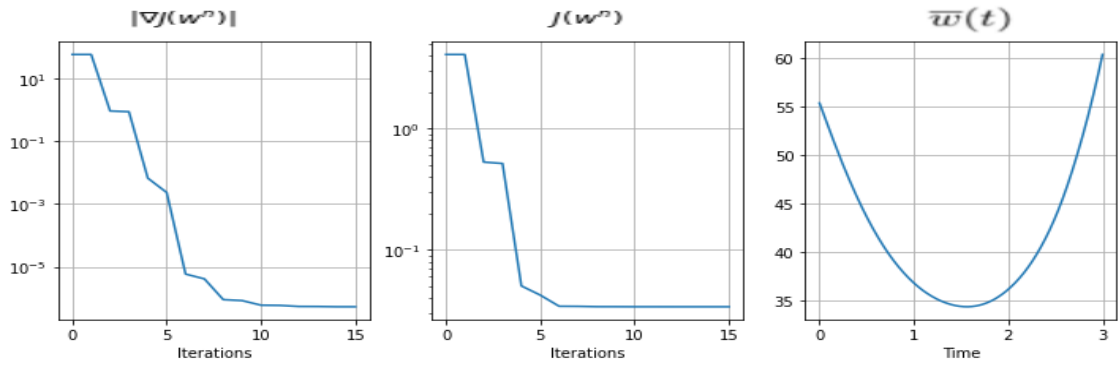


Figure 7: The outputs of the algorithm 1 with respect to Figure 6 with $\varepsilon_1 = 1$ and $\varepsilon_2 = 10^{-8}$. The L^2 -norm of the gradient of the cost functional at the left, the minimization values of the cost functional $J(w, u)$ at the middle and the average of the optimal control $\bar{w}(t) = \int_{\Omega} w(x, t) dx$ at the right.

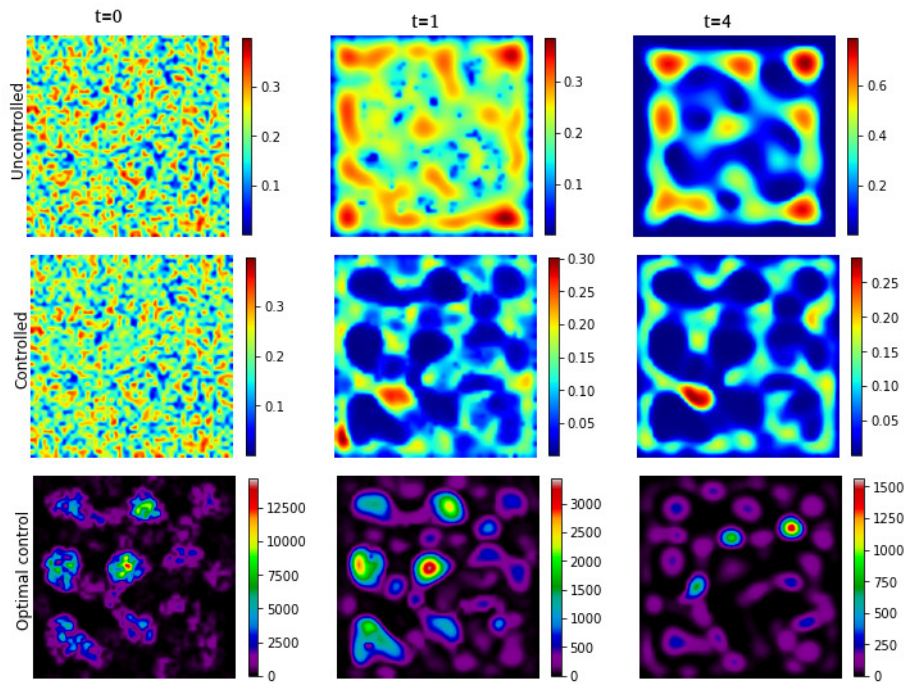


Figure 8: A comparison between controlled and uncontrolled dynamics acting on competition coefficient w where $\sigma = 0.25$, $T = 4$, $\alpha = 0.005$. The initial condition $u_0(x, y)$ is a uniform random distribution.

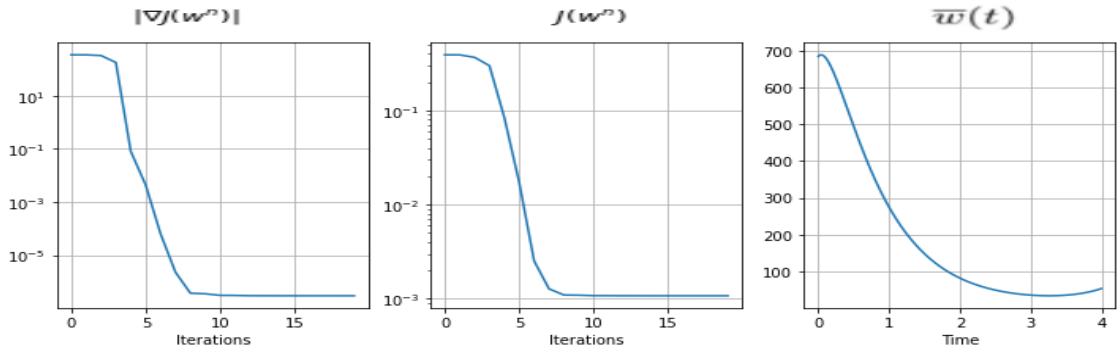


Figure 9: The outputs of the algorithm 1 with respect to Figure 8 with $\varepsilon_1 = 1$ and $\varepsilon_2 = 10^{-10}$. The L^2 -norm of the gradient of the cost functional at the left, the minimization values of the cost functional $J(w, u)$ at the middle and the average of the optimal control $\bar{w}(t) = \int_{\Omega} w(x, t) dx$ at the right.

The gradient of the functional, as depicted in Figures 7 and 9, is an important metric for evaluating the performance of the optimization process. A decrease in the gradient to a value less than 10^{-5} indicates a significant improvement in the minimization process. This decrease in gradient is accompanied by a significant reduction in the cost functional, which approaches values close to 0. This leads to a reduced crowding effect compared to the outcome of the direct problem, where the population dynamics is not guided by a control mechanism. Observations of the controlled solution in comparison to the uncontrolled solution in Figure 6 reveal that the optimal control, represented by the control variable w , is effective in minimizing the attraction force between groups of individuals. By targeting the centers of the groups, the optimal control minimizes the attraction force and prevents the formation of a single crowd at $t = 3.0$. This same effect is observed in Figure 8 with a random initial distribution, where the optimal control decreases the gradient and the resulting attractive force, preventing the formation of multiple crowds. These observations demonstrate the effectiveness of the optimal control approach in reducing crowding effects.

6 Conclusion

In this paper we dealt with an optimal control to a two-sidedly degenerate aggregation equation with logistic source term. We provided a rigorous analysis of the mathematical model. We have proposed an optimal control procedure to reduce the over-crowding and pattern formation. We derived the adjoint state equation with the corresponding explicit formulation of the gradient of the cost functional. We showed a numerical simulation of the natural dynamics of different initial population under different attraction forces. Moreover, we have computed the optimal carrying capacity that reduces the pattern formation and the over-crowding effect.

Finally, we want to mention that the well-posedness of the adjoint problem (41) will be the subject of a forthcoming paper.

Acknowledgment.

References

- [1] Mostafa Bendahmane, Kenneth H Karlsen, Convergence of a finite volume scheme for the bidomain model of cardiac tissue, *Applied numerical mathematics*, **59**(9):2266–2284, 2009, <https://doi.org/10.1016/j.apnum.2008.12.016>.
- [2] Mostafa Bendahmane, Kenneth H Karlsen, José Miguel Urbano, On a two-sidedly degenerate chemotaxis model with volume-filling effect, *Mathematical Models and Methods in Applied Sciences*, **17**(05):783–804, 2007, <https://doi.org/10.1142/S0218202507002108>.
- [3] Andrea L Bertozzi, José A Carrillo, Thomas Laurent, Blow-up in multidimensional aggregation equations with mildly singular interaction kernels, *Nonlinearity*, **22**(3):683, 2009, <https://doi.org/10.1088/0951-7715/22/3/009>.
- [4] Andrea L Bertozzi, Thomas Laurent, Finite-time blow-up of solutions of an aggregation equation in r^n , *Communications in mathematical physics*, **274**(3):717–735, 2007, <https://doi.org/10.1007/s00220-007-0288-1>.
- [5] Andrea L Bertozzi, Dejan Slepcev, Existence and uniqueness of solutions to an aggregation equation with degenerate diffusion, *Communications on Pure & Applied Analysis*, **9**(6):1617, 2010, <https://doi.org/10.3934/cpaa.2010.9.1617>.
- [6] Silvia Boi, Vincenzo Capasso, Daniela Morale, Modeling the aggregative behavior of ants of the species *polyergus rufescens*, *Nonlinear Analysis: Real World Applications*, **1**(1):163–176, 2000, [https://doi.org/10.1016/S0362-546X\(99\)00399-5](https://doi.org/10.1016/S0362-546X(99)00399-5).
- [7] Martin Burger, Vincenzo Capasso, Daniela Morale, On an aggregation model with long and short range interactions, *Nonlinear Analysis: Real World Applications*, **8**(3):939–958, 2007, <https://doi.org/10.1016/j.nonrwa.2006.04.002>.
- [8] José A Carrillo, Marco DiFrancesco, Alessio Figalli, Thomas Laurent, Dejan Slepčev, Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations, *Duke Mathematical Journal*, **156**(2):229–271, 2011, <https://doi.org/10.1215/00127094-2010-211>.
- [9] François Delarue, Frédéric Lagoutière, Nicolas Vauchelet, Convergence order of upwind type schemes for transport equations with discontinuous coefficients, *Journal de Mathématiques Pures et Appliquées*, **108**(6):918–951, 2017, <https://doi.org/10.1016/j.matpur.2017.05.012>.
- [10] Morton E Gurtin, Richard C MacCamy, On the diffusion of biological populations, *Mathematical biosciences*, **33**(1-2):35–49, 1977, [https://doi.org/10.1016/0025-5564\(77\)90062-1](https://doi.org/10.1016/0025-5564(77)90062-1).
- [11] William W Hager, Hongchao Zhang, A survey of nonlinear conjugate gradient methods, *Pacific journal of Optimization*, **2**(1):35–58, 2006.
- [12] Thomas Hillen, Kevin J Painter, A user’s guide to pde models for chemotaxis, *Journal of mathematical biology*, **58**(1):183–217, 2009, <https://doi.org/10.1007/s00285-008-0201-3>.
- [13] Yanghong Huang, Andrea L Bertozzi, Self-similar blowup solutions to an aggregation equation in r^n , *SIAM Journal on Applied Mathematics*, **70**(7):2582–2603, 2010, <https://doi.org/10.1137/090774495>.
- [14] Yanghong Huang, TP Witelski, Andrea L Bertozzi, Anomalous exponents of self-similar blow-up solutions to an aggregation equation in odd dimensions, *Applied Mathematics Letters*, **25**(12):2317–2321, 2012, <https://doi.org/10.1016/j.aml.2012.06.023>.
- [15] Evelyn F Keller, Lee A Segel, Initiation of slime mold aggregation viewed as an instability, *Journal of theoretical biology*, **26**(3):399–415, 1970, [https://doi.org/10.1016/0022-5193\(70\)90092-5](https://doi.org/10.1016/0022-5193(70)90092-5).
- [16] Evelyn F Keller, Lee A Segel, Model for chemotaxis, *Journal of theoretical biology*, **30**(2):225–234, 1971, [https://doi.org/10.1016/0022-5193\(71\)90050-6](https://doi.org/10.1016/0022-5193(71)90050-6).

- [17] Thomas Laurent, Local and global existence for an aggregation equation, *Communications in Partial Differential Equations*, **32**(12):1941–1964, 2007, <https://doi.org/10.1080/03605300701318955>.
- [18] Paul A Milewski, Xu Yang, A simple model for biological aggregation with asymmetric sensing, *Communications in Mathematical Sciences*, **6**(2):397–416, 2008, <https://doi.org/10.4310/CMS.2008.v6.n2.a7>.
- [19] Alexander Mogilner, Leah Edelstein-Keshet, A non-local model for a swarm, *Journal of mathematical biology*, **38**(6):534–570, 1999, <https://doi.org/10.1007/s002850050158>.
- [20] Daniela Morale, Vincenzo Capasso, Karl Oelschläger, An interacting particle system modelling aggregation behavior: from individuals to populations, *Journal of mathematical biology*, **50**(1):49–66, 2005, <https://doi.org/10.1007/s00285-004-0279-1>.
- [21] Jacques Simon, Compact sets in the space $l^p(0, t; b)$, *Annali di Matematica pura ed applicata*, **146**(1):65–96, 1986, <https://doi.org/10.1007/BF01762360>.
- [22] Chad M Topaz, Andrea L Bertozzi, Swarming patterns in a two-dimensional kinematic model for biological groups, *SIAM Journal on Applied Mathematics*, **65**(1):152–174, 2004, <https://doi.org/10.1137/S0036139903437424>.
- [23] Chad M Topaz, Andrea L Bertozzi, Mark A Lewis, A nonlocal continuum model for biological aggregation, *Bulletin of mathematical biology*, **68**(7):1601, 2006, <https://doi.org/10.1007/s11538-006-9088-6>.