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Abstract. We consider a Boussinesq system of KdV-KdV type introduced by J. Bona, M. Chen and J.-C. Saut as a model for the motion of small amplitude long waves on the surface of an ideal fluid. This system of two equations can describe the propagation of waves in both directions, while the single KdV equation is limited to unidirectional waves. We are concerned here with the exact controllability of the Boussinesq system by using some boundary controls. By reducing the controllability problem to a spectral problem which is solved by using the Paley-Wiener method introduced by the third author for KdV, we determine explicitly all the critical lengths for which the exact controllability fails for the linearized system, and give a complete picture of the controllability results with one or two boundary controls of Dirichlet or Neumann type. The extension of the exact controllability to the full Boussinesq system is derived in the energy space in the case of a control of Neumann type. It is obtained by incorporating a boundary feedback in the control in order to ensure a global Kato smoothing effect.

1. Introduction

J. L. Boussinesq introduced in [5, 6] several simple nonlinear systems of PDEs (including the Korteweg-de Vries equation) to explain certain physical observations concerning the water waves, e.g. the emergence and stability of solitons. Unfortunately, several systems derived by Boussinesq proved to be ill-posed, so that there was a need to propose other systems similar to Boussinesq's ones but with better mathematical properties.

The four-parameter family of Boussinesq systems

$$\begin{cases} \eta_t + v_x + (\eta v)_x + a v_{xxx} - b\eta_{xxt} = 0, \\ v_t + \eta_x + v v_x + c\eta_{xxx} - d v_{xxt} = 0 \end{cases} \tag{1.1}$$

was introduced by J. J. Bona, M. Chen and J.-C. Saut in [3] to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations when the motion is sensibly two-dimensional. In (1.1), η is the elevation of the fluid surface from the equilibrium position, and $v = v_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid. The parameters a, b, c, d are required to fulfill the relations

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \tag{1.2}$$

[1]

where $\theta \in [0, 1]$ specifies which horizontal velocity the variable v represents. As it has been proved in [3], the initial value problem for the *linear system* associated with (1.1) is well posed

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[1]Note that $a + b + c + d = \frac{1}{3}$.

on $\mathbb{R}$ if and only if the parameters a, b, c, d fall in one of the following cases

$$\begin{aligned} &\text{(C1)} \qquad b, d \geq 0,\ a \leq 0,\ c \leq 0; \\ &\text{(C2)} \qquad b, d \geq 0,\ a = c > 0. \end{aligned}$$

The wellposedness of the full system (1.1) on the line ($x \in \mathbb{R}$) was investigated in [4].

Recently, a rather complete picture of the control properties of (1.1) on a periodic domain with a locally supported forcing term was given in [26]. According to the values of the four parameters a, b, c, d, the linearized system may be either controllable in any positive time, or solely in large time, or may not be controllable at all. These results were also extended in [26] to the generic nonlinear system (1.1); that is, when all the parameters are different from 0.

When $b = d = 0$ and (C2) is satisfied, then necessarily $a = c = 1/6$. Nevertheless, the scaling $x \to x/\sqrt{6}$, $t \to t/\sqrt{6}$ gives a system equivalent to (1.1) for which $a = c = 1$, namely

$$\begin{cases} \eta_t + v_x + (\eta v)_x + v_{xxx} = 0, \\ v_t + \eta_x + v v_x + \eta_{xxx} = 0. \end{cases} \tag{1.3}$$

The above system will be referred to as a *Boussinesq system of KdV-KdV type*, or as a *KdV-KdV system*.

The KdV–KdV system is expected to admit global solutions on $\mathbb{R}$ [4], and it also possesses good control properties on the torus [26].

The boundary stabilization of (1.3) on a bounded domain $(0, L)$ was investigated by two of the authors in [28]. They proved that if system (1.3) is supplemented with the following boundary conditions

$$\begin{cases} v(t,0) = v_{xx}(t,0) = 0 & \text{in } (0,T), \\ v_x(t,0) = \alpha_0 \eta_x(t,0) & \text{in } (0,T), \\ v(t,L) = \alpha_2 \eta(t,L) & \text{in } (0,T), \\ v_x(t,L) = -\alpha_1 \eta_x(t,L) & \text{in } (0,T), \\ v_{xx}(t,L) = -\alpha_2 \eta_{xx}(t,L) & \text{in } (0,T), \end{cases} \tag{1.4}$$

and the initial conditions

$$\eta(0,x) = \eta_0(x), \quad v(0,x) = v_0(x) \qquad \text{in } (0,L), \tag{1.5}$$

then the system is locally exponentially stable in the energy space $[L^2(0,L)]^2$ whenever the constants α_0, α_1 and α_2 satisfy

$$\alpha_0 \geq 0,\ \alpha_1 > 0, \text{ and } \alpha_2 = 1.$$

To our best knowledge, the boundary control of the Boussinesq system of KdV-KdV type on a bounded domain $(0, L)$ is completely open. The aim of this paper is to investigate the control properties of the following system

$$\begin{cases} \eta_t + v_x + (\eta v)_x + v_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ v_t + \eta_x + v v_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{1.6}$$

with the boundary conditions

$$\begin{cases} \eta(t,0) = h_0(t),\ \eta(t,L) = h_1(t),\ \eta_x(t,0) = h_2(t) & \text{in } (0,T), \\ v(t,0) = g_0(t),\ v(t,L) = g_1(t),\ v_x(t,L) = g_2(t) & \text{in } (0,T) \end{cases} \tag{1.7}$$

and the initial conditions

$$\eta(0,x)=\eta_0(x),\ \ v(0,x)=v_0(x) \qquad \text{in } (0,L). \tag{1.8}$$

It is of course desirable to obtain control results (or stabilization results) with a few number of controls inputs. Here, we will provide a complete picture of the exact controllability of the linearized system with one or two controls among $h_0, h_1, h_2, g_0, g_1, g_2$.

A similar study was performed for the Korteweg -de Vries (KdV) equation

$$y_t + y_{xxx} + y_x + yy_x = 0, \tag{1.9}$$

with the boundary conditions

$$y(t,L)=h_0(t),\ \ y(t,L)=h_1(t),\ \ y_x(t,L)=h_2(t). \tag{1.10}$$

More precisely, the exact controllability of (1.9)-(1.10) was established in [30] when $h_0=h_1=0$ (h_2 being the only effective control) for a length L which is not critical, in [8, 9, 13] for a length L which is critical, and in [19] when $h_0=h_2=0$ for a length L not critical. The null controllability of (1.9)-(1.10) was proved in [31] (see also [18]) when $h_1=h_2=0$ for any length $L>0$. Note that in that case there is no critical length, and that solely the null controllability holds (say in $L^2(0,L)$), because the terminal state $y(\cdot,T)$ is C^∞ smooth for $x>0$.

A length L is said to be *critical* when the linearized equation

$$y_t + y_{xxx} + y_x = 0, \tag{1.11}$$

fails to be controllable. This phenomenon was first noticed in [30]. It is due to the influence of the first order derivative ∂_x on the spectrum of the operator ∂_x^3 with the boundary conditions $y(0)=y(L)=y_x(L)=0$. If the high frequencies are asymptotically preserved, the low frequencies may be strongly modified, and some of the corresponding eigenfunctions may become uncontrollable for (1.11) for certain values of L (the critical ones).

The set of critical lengths for the linear control system

$$y_t + y_{xxx} + y_x = 0, \tag{1.12}$$

$$y(t,0)=y(t,L)=0, \qquad y_x(t,L)=h_2(t) \tag{1.13}$$

was found [30] to be

$$\mathcal{N} := \{2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l\in\mathbb{N}^*\}.$$

On the other hand, the set of critical lengths for the linear control system

$$y_t + y_{xxx} + y_x = 0, \tag{1.14}$$

$$y(t,0)=y_x(t,L)=0, \qquad y(t,L)=h_1(t) \tag{1.15}$$

was proved in [19] to be discrete, countable and given by

$$\tilde{\mathcal{N}} := \{L>0; \exists a\in\mathbb{C}, b\in\mathbb{C} \text{ with } ae^a=be^b=-(a+b)e^{-a-b},\ L^2=-(a^2+ab+b^2)\}.$$

The determination of the critical lengths for system (1.12)-(1.13) in [30] was based on a series of reductions (as in [1]), first to a unique continuation property for the adjoint system, next to a spectral problem with an extra condition:

$$-y'''-y'=\lambda y,\ \ y(0)=y(L)=y'(0)=0 \ \text{ and } \ y'(L)=0. \tag{1.16}$$

This spectral problem was then solved by extending the function y by 0 outside $(0, L)$, by taking its Fourier transform and by using Paley-Wiener theorem. The length L is then critical if and only if there exist some numbers $p \in \mathbb{C}$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that the function

$$f(\xi) = \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p} \tag{1.17}$$

is analytic on $\mathbb{C}$; that is, the roots of $\xi^3 - \xi + p$ are also roots of $\alpha - \beta e^{-iL\xi}$ with at least the same multiplicity. The determination of $\mathcal{N}$ follows then with some algebra.

By contrast, the critical lengths for (1.14)-(1.15), that is the elements of $\tilde{\mathcal{N}}$, are not explicitly known. This is likely due to the lack of symmetries in the corresponding spectral problem

$$-y''' - y' = \lambda y, \;\; y(0) = y(L) = y'(0) = 0 \;\; \text{and} \;\; y''(L) = 0. \tag{1.18}$$

Note that the boundary conditions in (1.16) are preserved by the transformation $x \to L - x$. This symmetry yields a more tractable function f in (1.17).

Boussinesq system is more convenient than KdV as a model for the propagation of water waves, for it is adapted to the propagation of waves in *both* directions, and it is still valid after bounces of waves at the boundary. It is striking that the control theory for Boussinesq system (exposed in this paper) is better understood than for KdV as far as the critical lengths are concerned: indeed, the critical lengths for Boussinesq system are *explicitly given* for any set of boundary controls (except in Case 3 where only g_0 is used, and in Case 12 where h_1 and g_0 are used, see below Table 1), which is not the case for KdV with a control as in (1.15). We believe that this is due to the numerous symmetries of Boussinesq system: for instance, $x = 0$ and $x = L$ (resp. η and v) play a symmetric role for the linearized Boussinesq system. The price to be paid is the lack of any Kato smoothing effect (the system being conservative), which makes the extension of the control results to the nonlinear Boussinesq system more delicate than for KdV. We refer the reader to [7, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 21, 27, 29, 30, 31, 32, 33] for the control and stabilization of KdV, and [14, 15, 17] for the critical lengths concerning some other dispersive equations.

To investigate the control properties of the linearized system, we proceed as in [30]. To prove the observability inequality, we use the reduction to a spectral problem with an extra condition and the Paley-Wiener method. Here, we have to see for which value of $L > 0$ two functions are entire for a set of parameters. For instance, in Case 1 where only g_2 is used, the two functions read

$$\begin{aligned} f(\xi) &= \frac{i}{\xi^3 - \xi + p}(\alpha + \beta i\xi + \gamma e^{-iL\xi}) \\ g(\xi) &= \frac{i}{\xi^3 - \xi + p}(\alpha' - \beta i\xi + \gamma' e^{iL\xi}) \end{aligned}$$

for some $p \in \mathbb{C}$ and $(\alpha, \beta, \gamma, \alpha', \gamma') \in \mathbb{C}^5 \setminus \{0\}$. Using the symmetries of the system, we can prove that the set of critical lengths is $\mathcal{N}$.

In the reduction to the Unique Continuation Property (UCP), we use in most cases an identity obtained by using Morawetz multiplier, namely (2.10) (see below). However, for Case 3 (see below Table 1), the identity (2.10) proved to be useless. We replace the observability inequality

by a weaker estimate

$$\|(\eta^0, v^0)\|^2_{[H^2(0,L)]^2} \leq C\left(\int_0^T |\eta_{xx}(t,L)|^2 dt + \|(\eta^0, v^0)\|^2_{[L^2(0,L)]^2}\right), \tag{1.19}$$

which in turn is established by performing a careful investigation of the spectral reduction of the *skewadjoint* operator $A(\eta, v) = (-v_x - v_{xxx}, -\eta_x - \eta_{xxx})$ with domain

$$D(A) = \{(\eta, v) \in [H^3(0,L) \cap H_0^1(0,L)]^2; \quad \eta_x(0) = v_x(L) = 0\} \subset [L^2(0,L)]^2.$$

We show that the space $[L^2(0,L)]^2$ admits an orthonormal basis constituted of eigenfunctions of A. To do that, we introduce the *selfadjoint* operator $(By)(x) = -y_{xxx}(L-x) - y_x(L-x)$ with domain

$$D(B) = \{y \in H^3(0,L) \cap H_0^1(0,L);\ y_x(L) = 0\} \subset L^2(0,L).$$

Then the spectral reduction of B yields at once those for A.

We give in Table 1 (see below) a complete picture of the exact controllability results for the linearized Boussinesq system. It is not difficult to extend those exact controllability results to the nonlinear Boussinesq system in spaces of sufficiently regular functions (e.g. some subspaces of $[H^2(0,L)]^2$), but here we will not do it for the sake of shortness. Rather, we shall explain how to obtain exact controllability/stabilization results in the *energy space* $[L^2(0,L)]^2$ by incorporating a feedback law in the control that yields a smoothing effect, as it was done in [22] for the Benjamin-Ono equation. We shall limit ourserves to the case of a single Neumann boundary control (Case 1), namely to the system

$$\begin{cases} \eta_t + v_x + (\eta v)_x + v_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ v_t + \eta_x + v v_x + \eta_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ \eta(t,0) = 0,\ \eta(t,L) = 0,\ \eta_x(t,0) = 0, & t \in (0,T), \\ v(t,0) = 0,\ v(t,L) = 0,\ v_x(t,L) = g_2(t), & t \in (0,T), \\ \eta(0,x) = \eta^0(x),\ v(0,x) = v^0(x), & x \in (0,L), \end{cases} \tag{1.20}$$

The first main result in this paper is a local exact controllability result for (1.20).

Theorem 1.1. *Let $T > 0$ and $L \in (0,+\infty) \setminus \mathcal{N}$ where $\mathcal{N} := \{\frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2};\quad k,l \in \mathbb{N}^*\}$. Then there exists some $\delta > 0$ such that for all states $(\eta^0, v^0), (\eta^1, v^1) \in [L^2(0,L)]^2$ with*

$$\|(\eta^0, v^0)\|_{[L^2(0,L)]^2} \leq \delta \quad \text{and} \quad \|(\eta^1, v^1)\|_{[L^2(0,L)]^2} \leq \delta,$$

one can find a control $g_2 \in L^2(0,T)$ and a solution $(\eta, v) \in C([0,T], [L^2(0,L)]^2) \cap L^2(0,T,[H^1(0,L)]^2)$ of (1.20) such that

$$\eta(T,x) = \eta^1(x),\ v(T,x) = v^1(x), \qquad x \in (0,L).$$

The second main result is a local exponential stability result for (1.20).

Theorem 1.2. *Let $T > 0$, $L \in (0,+\infty) \setminus \mathcal{N}$, and $\alpha > 0$. Then there exist some positive numbers δ, μ, C such that for all initial data $(\eta^0, v^0) \in [L^2(0,L)]^2$ with $\|(\eta^0, v^0)\|_{[L^2(0,L)]^2} \leq \delta$, the system (1.20) with the boundary feedback law $g_2(t) = -\alpha\eta_x(t,L)$ admits for all $T > 0$ a unique solution*

$$(\eta, v) \in C([0,T], [L^2(0,L)]^2) \cap L^2(0,T,[H^1(0,L)]^2),$$

and it holds

$$\|(\eta,v)(t)\|_{[L^2(0,L)]^2} \leq Ce^{-\mu t}\|(\eta_0,v_0)\|_{[L^2(0,L)]^2}, \quad \forall t\geq 0, \tag{1.21}$$

$$\|(\eta,v)(t)\|_{[H^1(0,L)]^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}}\|(\eta_0,v_0)\|_{[L^2(0,L)]^2}, \quad \forall t> 0. \tag{1.22}$$

The paper is outlined as follows. The wellposedness of the linearized Boussinesq system with the boundary conditions (1.7) is studied in Section 2. Section 3 is concerned with the exact controllability of the linearized Boussinesq system with one or two boundary control inputs. The proof of the main results, namely Theorems 1.1 and 1.2, is provided in Section 4. Finally, the proof of the weak observability estimate (1.19) based on some spectral reduction is given in appendix.

2. Wellposedness

2.1. **Wellposedness of the homogeneous problem.** Let $L>0$ be a fixed number. Introduce the spaces

$$X_0 := [L^2(0,L)]^2 = L^2(0,L)\times L^2(0,L) \tag{2.1}$$

$$X_3 := \{(\eta,v)\in [H^3(0,L)\cap H_0^1(0,L)]^2;\ \eta_x(0)=v_x(L)=0\}, \tag{2.2}$$

$$X_{3\theta} := [X_0,X_3]_{[\theta]}, \quad \text{for } 0<\theta<1, \tag{2.3}$$

where $[X_0,X_3]_{[\theta]}$ denotes the Banach space obtained by the complex interpolation method (see e.g. [2]). The space X_0 (resp. X_3) is endowed with the norm

$$\|(\eta,v)\|_{X_0} := \left(\int_0^L [\eta^2(x)+v^2(x)]dx\right)^{\frac{1}{2}}$$

(resp. with the norm $\|(\eta,v)\|_{X_3} := \|(\eta,v)\|_{X_0}+\|(v_x+v_{xxx},\eta_x+\eta_{xxx})\|_{X_0}$).

It is easily seen that

$$X_1 = H_0^1(0,L)\times H_0^1(0,L), \tag{2.4}$$

$$X_2 = \{(\eta,v)\in [H^2(0,L)\cap H_0^1(0,L)]^2;\ \eta_x(0)=v_x(L)=0\} \tag{2.5}$$

and that in the space X_1 (resp. X_2), the norm $\|(\eta,v)\|_{X_1}$ (resp. $\|(\eta,v)\|_{X_2}$) is equivalent to $\left(\int_0^L [\eta_x^2(x)+v_x^2(x)]dx\right)^{\frac{1}{2}}$ (resp. $\left(\int_0^L [\eta_{xx}^2(x)+v_{xx}^2(x)]dx\right)^{\frac{1}{2}}$). We shall use at some place the space

$$\begin{aligned} X_4 := \{(\eta,v)\in [H^4(0,L)\cap H_0^1(0,L)]^2;\ &\eta_x(0)=v_x(L)=\eta_{xxx}(0)=v_{xxx}(L)=0,\\ &\eta_{xxx}(L)+\eta_x(L)=v_{xxx}(0)+v_x(0)=0\}\end{aligned}$$

endowed with its natural norm, and for $s\in\{1,2\}$, the space $X_{-s}=(X_s)'$ which is the dual of X_s with respect to the pivot space $X_0=[L^2(0,L)]^2$. Note that

$$X_{-1}=H^{-1}(0,L)\times H^{-1}(0,L).$$

The bracket $\langle\cdot,\cdot\rangle_{X_{-s},X_s}$ stands for the duality bracket. We first investigate the wellposedness of the initial value problem

$$\begin{cases} \eta_t + v_x + v_{xxx} = 0, & t \geq 0,\ x \in (0,L), \\ v_t + \eta_x + \eta_{xxx} = 0, & t \geq 0,\ x \in (0,L), \\ \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0, & t \geq 0, \\ v(t,0) = v(t,L) = v_x(t,L) = 0, & t \geq 0, \\ \eta(0,x) = \eta^0(x),\ v(0,x) = v^0(x), & x \in (0,L). \end{cases} \tag{2.6}$$

We introduce the operator

$$A(\eta, v) := (-v_x - v_{xxx}, -\eta_x - \eta_{xxx})$$

with domain $D(A) := X_3 \subset X_0$.

Proposition 2.1. *The operator A is skew-adjoint in X_0, and thus it generates a group of isometries $(e^{tA})_{t\in\mathbb{R}}$ in X_0.*

Proof. We have to prove that $A^* = -A$. It is clear that we have $-A \subset A^*$ (i.e. $(\theta,u) \in D(A^*)$ and $A^*(\theta,u) = -A(\theta,u)$ for all $(\theta,u) \in D(A)$). Indeed, for any $(\eta,v),(\theta,u) \in D(A)$, we have after some integrations by parts

$$\begin{aligned} ((\theta,u), A(\eta,v))_{X_0} &= -\int_0^L [\theta(v_x + v_{xxx}) + u(\eta_x + \eta_{xxx})]dx \\ &= \int_0^L [v(\theta_x + \theta_{xxx}) + \eta(u_x + u_{xxx})]dx \\ &= -(A(\theta,u),(\eta,v))_{X_0}. \end{aligned}$$

Let us prove now that $A^* \subset -A$. Pick any $(\theta,u) \in D(A^*)$. Then, we have for some constant $C > 0$

$$|((\theta,u), A(\eta,v))_{X_0}| \leq C\|(\eta,v)\|_{X_0} \quad \forall (\eta,v) \in D(A),$$

i.e.

$$\left|\int_0^L [\theta(v_x + v_{xxx}) + u(\eta_x + \eta_{xxx})]dx\right| \leq C\left(\int_0^L [\eta^2 + v^2]dx\right)^{\frac{1}{2}}, \qquad \forall (\eta,v) \in D(A). \tag{2.7}$$

Picking $v = 0$ and $\eta \in C_c^\infty(0,L)$, we infer from (2.7) that $u_x + u_{xxx} \in L^2(0,L)$, and hence that $u \in H^3(0,L)$. Similarly, we obtain that $\theta \in H^3(0,L)$. Integrating by parts in the left hand side of (2.7), we obtain that

$$\begin{aligned} &|\theta(L)v_{xx}(L) - \theta(0)v_{xx}(0) + \theta_x(0)v_x(0) + u(L)\eta_{xx}(L) - u(0)\eta_{xx}(0) - u_x(L)\eta_x(L)| \\ &\quad \leq C\left(\int_0^L [\eta^2 + v^2]dx\right), \qquad \forall (\eta,v) \in D(A). \end{aligned}$$

It easily follows that

$$\theta(0) = \theta(L) = \theta_x(0) = u(0) = u(L) = u_x(L) = 0,$$

so that $(\theta,u) \in D(A) = D(-A)$. Thus $D(A^*) = D(-A)$ and $A^* = -A$. □

Corollary 2.2. *For any $(\eta^0, v^0) \in X_0$, system (2.6) admits a unique solution $(\eta, v) \in C(\mathbb{R}, X_0)$, which satisfies $\|(\eta(t), v(t))\|_{X_0} = \|(\eta^0, v^0)\|_{X_0}$ for all $t \in \mathbb{R}$. If, in addition, $(\eta^0, v^0) \in X_3$, then $(\eta, v) \in C(\mathbb{R}, X_3)$ with $\|(\eta, v)\|_{X_3} := \|(\eta, v)\|_{X_0} + \|A(\eta, v)\|_{X_0}$ constant.*

Using Corollary 2.2 combined with some interpolation argument between X_0 and X_3, we infer that for any $s \in (0, 3)$, there exists a constant $C_s > 0$ such that for any $(\eta^0, v^0) \in X_s$, the solution (η, v) of (2.6) satisfies $(\eta, v) \in C(\mathbb{R}, X_s)$ and

$$\|(\eta(t), v(t))\|_{X_s} \leq C_s \|(\eta^0, v^0)\|_{X_s}, \quad \forall t \in \mathbb{R}. \tag{2.8}$$

2.2. **Existence of traces.** For the solutions of system (2.6), we know that the traces

$$\eta(\cdot, 0), \eta(\cdot, L), \eta_x(\cdot, 0), v(\cdot, 0), v(\cdot, L), \text{ and } v_x(\cdot, L)$$

vanish. We have a look at the other traces $\eta_x(\cdot, L), v_x(\cdot, 0)$ and prove that they belong to $L^2_{loc}(\mathbb{R}_+)$ when $(\eta^0, v^0) \in X_1$.

Proposition 2.3. *Let $(\eta^0, v^0) \in X_1$ and let (η, v) denote the solution of (2.6). Pick any $T > 0$. Then $\eta_x(\cdot, L), v_x(\cdot, 0) \in L^2(0, T)$ with*

$$\int_0^T [|\eta_x(t, L)|^2 + |v_x(t, 0)|^2] dt \leq C \|(\eta^0, v^0)\|^2_{X_1} \tag{2.9}$$

for some constant $C = C(T)$.

Proof. Assume that $(\eta^0, v^0) \in X_3$, so that $(\eta, v) \in C([0, T], X_3) \cap C^1([0, T], X_0)$. We use Morawetz multipliers as in [30]. We multiply the first (resp. second) equation in (2.6) by xv (resp. $x\eta$), integrate by parts over $(0, T) \times (0, L)$, and add the two obtained equations to obtain

$$\frac{3}{2} \int_0^T \int_0^L [\eta_x^2 + v_x^2] \, dxdt - \frac{1}{2} \int_0^T \int_0^L [\eta^2 + v^2] \, dxdt + \left[\int_0^L [x\eta v] dx \right]_0^T - \frac{L}{2} \int_0^T \eta_x^2(t, L) \, dt = 0. \tag{2.10}$$

Since $\int_0^T \|(\eta, v)\|^2_{X_1} dt \leq C \|(\eta^0, v^0)\|^2_{X_1}$, this yields

$$\int_0^T \eta_x^2(t, L) \, dt \leq C \|(\eta^0, v^0)\|^2_{X_1}.$$

By symmetry, using now as multipliers $(L - x)v$ and $(L - x)\eta$, we infer that

$$\int_0^T v_x^2(t, 0) \, dt \leq C \|(\eta^0, v^0)\|^2_{X_1}.$$

Thus (2.9) is established when $(\eta^0, v^0) \in X_3$. Since X_1 is dense in X_3, the result holds as well for $(\eta^0, v^0) \in X_1$. □

We now turn our attention to the traces of order 2, namely $\eta_{xx}(\cdot, 0)$, $\eta_{xx}(\cdot, L)$, $v_{xx}(\cdot, 0)$, and $v_{xx}(\cdot, L)$.

Proposition 2.4. *Let $(\eta^0, v^0) \in X_2$ and let (η, v) denote the solution of (2.6). Pick any $T > 0$. Then $\eta_{xx}(\cdot, 0), \eta_{xx}(\cdot, L), v_{xx}(\cdot, 0), v_{xx}(\cdot, L) \in L^2(0, T)$ with*

$$\int_0^T [|\eta_{xx}(t, 0)|^2 + |\eta_{xx}(t, L)|^2 + |v_{xx}(t, 0)|^2 + |v_{xx}(t, L)|^2] dt \leq C \|(\eta^0, v^0)\|^2_{X_2} \tag{2.11}$$

for some constant $C = C(T)$. *Furthermore,* $\eta_x(\cdot, L), v_x(\cdot, 0) \in H^{\frac{1}{3}}(0,T)$ *with*

$$\|\eta_x(\cdot, L)\|^2_{H^{\frac{1}{3}}(0,T)} + \|v_x(\cdot, 0)\|^2_{H^{\frac{1}{3}}(0,T)} \leq C\|(\eta^0, v^0)\|^2_{X_2} \tag{2.12}$$

for some constant $C = C(T)$.

Proof. Assume that $(\eta^0, v^0) \in X_3$, so that $(\eta, v) \in C([0,T], X_3) \cap C^1([0,T], X_0)$. Pick $\rho \in C^\infty([0,L])$ with $\rho(x) = 0$ for $x \leq 1/4$ and $\rho(x) = 1$ for $x \geq 3/4$, and set $\tilde{\eta} = \rho\eta$, $\tilde{v} = \rho v$. Then we have

$$\tilde{\eta}_t + \tilde{v}_x + \tilde{v}_{xxx} = \rho_x v + 3\rho_x v_{xx} + 3\rho_{xx} v_x + \rho_{xxx} v =: \tilde{f}, \tag{2.13}$$

$$\tilde{v}_t + \tilde{\eta}_x + \tilde{\eta}_{xxx} = \rho_x \eta + 3\rho_x \eta_{xx} + 3\rho_{xx} \eta_x + \rho_{xxx} \eta =: \tilde{g}. \tag{2.14}$$

Note that $\tilde{f}, \tilde{g} \in C([0,T], L^2(0,L))$. Multiplying each term in (2.13) by $\tilde{v}_{xx}$ (resp. in (2.14) by $\tilde{\eta}_{xx}$) and integrating by parts, we arrive to

$$\frac{1}{2}\int_0^T [\tilde{\eta}^2_{xx}(t,L) + \tilde{v}^2_{xx}(t,L) + \tilde{\eta}^2_x(t,L)]\, dt = \left[\int_0^L \tilde{\eta}_x \tilde{v}_x\, dx\right]_0^T + \int_0^T\int_0^L [\tilde{f}\tilde{v}_{xx} + \tilde{g}\tilde{\eta}_{xx}]\, dxdt.$$

It follows that

$$\int_0^T [\eta^2_{xx}(t,L) + v^2_{xx}(t,L)]\, dt \leq C\left(\|(\eta^0, v^0)\|^2_{X_2} + \int_0^T \|(f,g)\|^2_{X_0} dt\right) \leq C\|(\eta^0, v^0)\|^2_{X_2},$$

and we can prove in a similar way that

$$\int_0^T [\eta^2_{xx}(t,0) + v^2_{xx}(t,0)]\, dt \leq C\|(\eta^0, v^0)\|^2_{X_2}.$$

Thus (2.11) is established when $(\eta^0, v^0) \in X_3$. Since X_3 is dense in X_2, the result holds as well for $(\eta^0, v^0) \in X_2$. Let us proceed with the proof of (2.12). If $(\eta^0, v^0) \in X_4$, then $(\eta, v) \in C(\mathbb{R}, X_4)$, so that $(\hat{\eta}, \hat{v}) := (\eta_t, v_t) = A(\eta, v)$ is in $C(\mathbb{R}, X_1)$ and it solves

$$(\hat{\eta}, \hat{v})_t = A(\hat{\eta}, \hat{v}), \quad (\hat{\eta}, \hat{v})(0) = A(\eta^0, v^0).$$

It follows from (2.9) that

$$\|\eta_x(\cdot, L)\|^2_{H^1(0,T)} + \|v_x(\cdot, 0)\|^2_{H^1(0,T)} \leq C\|(\eta^0, v^0)\|^2_{X_4}. \tag{2.15}$$

Since $X_2 = [X_1, X_4]_{\frac{1}{3}}$, we infer from (2.9) and (2.15) that

$$\|\eta_x(\cdot, L)\|^2_{H^{\frac{1}{3}}(0,T)} + \|v_x(\cdot, 0)\|^2_{H^{\frac{1}{3}}(0,T)} \leq C\|(\eta^0, v^0)\|^2_{X_2} \tag{2.16}$$

for some constant $C = C(T)$ and all $(\eta^0, v^0) \in X_2$. □

2.3. **Wellposedness of the nonhomogeneous problem.** We consider the nonhomogeneous system

$$\begin{cases} \eta_t + v_x + v_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ v_t + \eta_x + \eta_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ \eta(t,0) = h_0(t),\ \eta(t,L) = h_1(t),\ \eta_x(t,0) = h_2(t), & t \in (0,T), \\ v(t,0) = g_0(t),\ v(t,L) = g_1(t),\ v_x(t,L) = g_2(t), & t \in (0,T), \\ \eta(0,x) = \eta^0(x),\ v(0,x) = v^0(x), & x \in (0,L), \end{cases} \tag{2.17}$$

where the initial data (η^0, v^0) and the boundary data $(h_0, h_1, h_2, g_0, g_1, g_2)$ are given in appropriate spaces.

We also consider the homogenous system

$$\begin{cases} \theta_t + u_x + u_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ u_t + \theta_x + \theta_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ \theta(t,0) = \theta(t,L) = \theta_x(t,0) = 0, & t \in (0,T), \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & t \in (0,T), \\ \theta(0,x) = \theta^0(x),\ u(0,x) = u^0(x), & x \in (0,L). \end{cases} \tag{2.18}$$

If both (η, v) and (θ, u) are in $C([0,T],[H^3(0,L)]^2) \cap C^1([0,T],[L^2(0,L)]^2)$, which is the case when $(\eta^0, v^0), (\theta^0, u^0) \in X_3$ and $h_i, g_i \in C^2([0,T])$ with $h_i(0) = g_i(0) = 0$ for $i = 1,2,3$, then we obtain after multiplying the two first equations of (2.17) by u and θ respectively, and integrating by parts, that for all $S \in [0,T]$

$$\begin{aligned} \left[\int_0^L [\eta\theta + vu]dx\right]_0^S &= \int_0^S [\theta_x(t,L)g_2(t) - \theta_{xx}(t,L)g_1(t) + \theta_{xx}(t,0)g_0(t) \\ &\quad - u_x(t,0)h_2(t) - u_{xx}(t,L)h_1(t) + u_{xx}(t,0)h_0(t)]dt. \end{aligned} \tag{2.19}$$

Definition 2.5. *Let $(\eta^0, v^0) \in X_{-2}$, $g_0, g_1, h_0, h_1 \in L^2(0,T)$ and $g_2, h_2 \in H^{-\frac{1}{3}}(0,T)$. We say that $(\eta, v) \in C([0,T], X_{-2})$ is a solution of the nonhomogeneous problem (2.17) if we have that*

$$\langle (\eta(S), v(S)), (\theta(S), u(S))\rangle_{X_{-2},X_2} = L_S(\theta^0, u^0), \qquad \forall (\theta^0, u^0) \in X_2,\ \forall S \in [0,T], \tag{2.20}$$

where

$$\begin{aligned} L_S(\theta^0, u^0) &:= \langle (\eta^0, v^0), (\theta^0, u^0)\rangle_{X_{-2},X_2} + \langle g_2, \mathbf{1}_{(0,S)}\theta_x(\cdot, L)\rangle_{H^{-\frac{1}{3}}(0,T), H^{\frac{1}{3}}(0,T)} \\ &\quad + \int_0^S [-\theta_{xx}(t,L)g_1(t) + \theta_{xx}(t,0)g_0(t) - u_{xx}(t,L)h_1(t) + u_{xx}(t,0)h_0(t)]dt \\ &\quad - \langle h_2, \mathbf{1}_{(0,S)}u_x(\cdot,0)\rangle_{H^{-\frac{1}{3}}(0,T), H^{\frac{1}{3}}(0,T)}, \end{aligned} \tag{2.21}$$

(θ, u) denoting the solution of system (2.18).

Note that $L_S : X_2 \to \mathbb{R}$ is well defined for all $S \in [0,T]$, for $\theta_{xx}(\cdot,0), \theta_{xx}(\cdot,L), u_{xx}(\cdot,0)$, $u_{xx}(\cdot,L) \in L^2(0,T)$ and $\theta_x(\cdot,L), u_x(\cdot,0) \in H^{\frac{1}{3}}(0,T)$ by Proposition 2.4. The fact that $\mathbf{1}_{(0,S)}\theta_x(\cdot,L)$, $\mathbf{1}_{(0,S)}u_x(\cdot,0) \in H^{\frac{1}{3}}(0,T)$ for any $S \in [0,T]$ follows from [25, Théorème 11.4 p. 66].

The existence and uniqueness of a solution of system (2.17) is stated in the following result.

Proposition 2.6. *Let $(\eta^0, v^0) \in X_{-2}$, $g_0, g_1, h_0, h_1 \in L^2(0,T)$, and $g_2, h_2 \in H^{-\frac{1}{3}}(0,T)$. Then there exists a unique solution $(\eta, v) \in C([0,T], X_{-2})$ of the nonhomogeneous problem (2.17). Furthermore, we have*

$$\begin{aligned} \|(\eta, v)\|_{L^\infty(0,T,X_{-2})} &\le C\Big(\|(\eta^0, v^0)\|_{X_{-2}} + \|g_0\|_{L^2(0,T)} + \|h_0\|_{L^2(0,T)} \\ &\quad + \|g_1\|_{L^2(0,T)} + \|h_1\|_{L^2(0,T)} + \|g_2\|_{H^{-\frac{1}{3}}(0,T)} + \|h_2\|_{H^{-\frac{1}{3}}(0,T)}\Big) \end{aligned} \tag{2.22}$$

for some constant $C = C(T) > 0$. If, in addition, $(\eta^0, v^0) \in X_{-1}$, $g_0 = g_1 = h_0 = h_1 = 0$, and $g_2, h_2 \in L^2(0,T)$, then $(\eta, v) \in C([0,T], X_{-1})$ and we have for some constant $C' = C'(T)$

$$\|(\eta, v)\|_{L^\infty(0,T,X_{-1})} \leq C'\Big(\|(\eta^0, v^0)\|_{X_{-1}} + \|g_2\|_{L^2(0,T)} + \|h_2\|_{L^2(0,T)}\Big). \tag{2.23}$$

Proof. Let $(\eta^0, v^0) \in X_{-2}$, $g_0, g_1, h_0, h_1 \in L^2(0,T)$, and $g_2, h_2 \in H^{-\frac{1}{3}}(0,T)$. Pick any $(\theta^0, u^0) \in X_2$ and denote by (θ, u) the solution of the homogeneous system (2.18). Then, by Proposition 2.4, $\theta_{xx}(\cdot,0), \theta_{xx}(\cdot,L), u_{xx}(\cdot,0), u_{xx}(\cdot,L) \in L^2(0,T)$ and $\theta_x(\cdot,L), u_x(\cdot,0) \in H^{\frac{1}{3}}(0,T)$. It follows that for any $S \in [0,T]$, the linear map $L_S : X_{-2} \to \mathbb{R}$ defined in (2.21) is continuous. As the map $\Gamma_S : (\theta^0, u^0) \in X_2 \mapsto (\theta(S), u(S)) \in X_2$ is invertible, $(\eta(S), v(S))$ is defined in X_{-2} in a unique way by (2.20). Furthermore, as Γ_S and $\Gamma_S^{-1} = \Gamma_{-S}$ are uniformly bounded in $\mathcal{L}(X_2)$ for $S \in [0,T]$ (see (2.8)), we infer that $(\eta, v) \in L^\infty(0,T,X_{-2})$ and that (2.22) holds. On the other hand, if $(\eta^0, v^0), (\theta^0, u^0) \in X_3$ and $h_i, g_i \in C^2([0,T])$ with $h_i(0) = g_i(0) = 0$ for $i = 1,2,3$, the respective strong solutions $(\eta, v), (\theta, u) \in C([0,T], [H^3(0,L)]^2) \cap C^1([0,T], [L^2(0,L)]^2)$ of (2.17) and (2.18) satisfy (2.19), and thus (2.20), for X_3 is dense in X_2. Using (2.22) and a density argument, we infer that $(\theta, v) \in C([0,T], X_{-2})$ when we assume merely that $(\eta^0, v^0) \in X_{-2}$, $g_0, g_1, h_0, h_1 \in L^2(0,T)$, and $g_2, h_2 \in H^{-\frac{1}{3}}(0,T)$.

Assume finally that $(\eta^0, v^0) \in X_{-1}$, $g_0 = g_1 = h_0 = h_1 = 0$ and $g_2, h_2 \in L^2(0,T)$. Pick any $(\theta^0, u^0) \in X_2$. Then

$$\begin{aligned} L_S(\theta^0, u^0) &= \langle (\eta^0, v^0), (\theta^0, u^0) \rangle_{X_{-2}, X_2} + \langle g_2, \mathbf{1}_{(0,S)} \theta_x(\cdot, L) \rangle_{H^{-\frac{1}{3}}(0,T), H^{\frac{1}{3}}(0,T)} \\ &\quad - \langle h_2, \mathbf{1}_{(0,S)} u_x(\cdot, 0) \rangle_{H^{-\frac{1}{3}}(0,T), H^{\frac{1}{3}}(0,T)} \\ &= \langle (\eta^0, v^0), (\theta^0, u^0) \rangle_{X_{-1}, X_1} + \int_0^S [\theta_x(t,L) g_2(t) - u_x(t,0) h_2(t)] dt. \end{aligned} \tag{2.24}$$

Using Proposition 2.3 and (2.24), we see that L_S can be extended as a continuous linear map from X_1 to $\mathbb{R}$. It follows from (2.20), the density of X_2 in X_1, and the fact that the map $(\theta^0, u^0) \in X_1 \mapsto (\theta(S), u(S)) \in X_1$ is invertible with its norm and the norm of its inverse uniformly bounded for $S \in [0,T]$, that $(\eta, v) \in L^\infty(0,T,X_{-1})$ with (2.23) satisfied. The same argument as above shows that $(\eta, v) \in C([0,T], X_{-1})$. □

3. Controllability of the linearized system

In this section, we are interested in the exact controllability of the linear system (2.17) using at most two control inputs (the other control inputs being replaced by 0). The exact controllability in X_{-2} is defined as follows.

Definition 3.1. *We say that the system (2.17) is exactly controllable in X_{-2} (in small time) if for all $(\eta^0, v^0), (\eta^1, v^1) \in X_{-2}$ and all $T > 0$, one can find $g_0, g_1, h_0, h_1 \in L^2(0,T)$ and $g_2, h_2 \in H^{-\frac{1}{3}}(0,T)$ such that the solution (η, v) of (2.17) satisfies*

$$(\eta(T,\cdot), v(T,\cdot)) = (\eta^1, v^1). \tag{3.1}$$

Here, we will restrict to situations when at most two control inputs can be prescribed, the other being set to 0. Similarly, we say that the system (2.17) with $g_0 = g_1 = h_0 = h_1 = 0$ is exactly

controllable in X_{-1} (in small time) if for all $(\eta^0, v^0), (\eta^1, v^1) \in X_{-1}$ and all $T > 0$, one can find $g_2, h_2 \in L^2(0,T)$ such that the solution (η, v) of (2.17) satisfies (3.1).

As system (2.6) is reversible, the exact controllability of system (2.17) is equivalent to its null controllability (i.e., the above property with (η^0, v^0) arbitrary but $(\eta^1, v^1) = (0,0)$).

As it is well known, the exact controllability of a control system is equivalent to the observability of its adjoint system (see e.g. [12, 23, 24]). Here, using (2.20)-(2.21), we see that the exact controllability of (2.17) in X_{-2} with the six controls $g_0, g_1, h_0, h_1 \in L^2(0,T)$, $g_2, h_2 \in H^{-\frac{1}{3}}(0,T)$ is equivalent to the existence of a constant $C > 0$ such that for all $(\theta^0, u^0) \in X_2$,

$$\begin{aligned} \|(\theta^0, u^0)\|^2_{X_2} \leq\ & C\left(\int_0^T \left[|\theta_{xx}(t,0)|^2 + |\theta_{xx}(t,L)|^2 + |u_{xx}(t,0)|^2 + |u_{xx}(t,L)|^2\right] dt \right. \\ & \left. + \|\theta_x(\cdot, L)\|^2_{H^{\frac{1}{3}}(0,T)} + \|u_x(\cdot,0)\|^2_{H^{\frac{1}{3}}(0,T)}\right) \end{aligned} \tag{3.2}$$

where (θ, u) denotes the solution of (2.18).

The exact controllabity in X_{-2} with less control inputs is equivalent to the observability inequality (3.2) in which we remove the traces corresponding to the missing controls. Similarly, the exact controllability of (2.17) in X_{-1} with the two controls $g_2, h_2 \in L^2(0,T)$ is equivalent to the existence of a constant $C > 0$ such that for all $(\theta^0, u^0) \in X_1$,

$$\|(\theta^0, u^0)\|^2_{X_1} \leq C \int_0^T \left[|\theta_x(t,L)|^2 + |u_x(t,0)|^2\right] dt. \tag{3.3}$$

The aim of this section is to provide a complete picture of the exact controllability results for system (2.17) with one or two control inputs among $g_0, g_1, g_2, h_0, h_1, h_2$.

We introduce some subsets of $(0, +\infty)$ that are needed to give a complete picture of the critical lengths for the control of the linearized KdV-KdV system:

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2};\quad k, l \in \mathbb{N}^* \right\},$$

$$\mathcal{N}_3 := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2};\quad k, l \in \mathbb{N}^*,\ 3|2k + l \right\} \subset \mathcal{N},$$

$$\mathcal{R} := \left\{ \pi \sqrt{(\frac{1}{2} + 2k)^2 + (\frac{1}{2} + 2l)^2 + (\frac{1}{2} + 2k)(\frac{1}{2} + 2l)};\quad k, l \in \mathbb{Z},\ k \neq l \right\},$$

$$\begin{aligned} \mathcal{G} := \Big\{ & L \in (0, +\infty);\ \exists a, b \in \mathbb{C},\ \frac{2a}{ie^a - 1} + a = \frac{2b}{ie^b - 1} + b = \frac{2(-a-b)}{ie^{-a-b} - 1} - a - b \neq 0, \\ & L^2 = -(a^2 + ab + b^2) \Big\}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}' := \Big\{ & L \in (0, +\infty);\ \exists a, b \in \mathbb{C},\ \frac{2a}{-ie^a - 1} + a = \frac{2b}{-ie^b - 1} + b = \frac{2(-a-b)}{-ie^{-a-b} - 1} - a - b \neq 0, \\ & L^2 = -(a^2 + ab + b^2) \Big\}. \end{aligned}$$

Then the control results for the linearized Boussinesq system with one or two boundary controls of Dirichlet or Neumann type are outlined in the following theorem.

Theorem 3.2. *Consider system (2.17) with one or two control inputs among $g_0, g_1, g_2, h_0, h_1, h_2$ (the other being set to 0). Then the exact controllability of (2.17) holds (in any time $T > 0$) if and only if L is not a critical length, a concept depending on the set of control inputs that are available. The precise results are reported in Table 1.*

	Controls							Properties	
Case	h_0	h_1	h_2	g_0	g_1	g_2	Control Inputs	State Space	Set of Critical Lengths
1	0	0	0	0	0	$\star$	$g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-1}$	$\mathcal{N}$
2	0	0	0	0	$\star$	0	$g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N} \cup \mathcal{R}$
3	0	0	0	$\star$	0	0	$g_0 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N} \cup \mathcal{G} \cup \mathcal{G}'$
4	0	0	$\star$	0	0	$\star$	$h_2, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-1}$	$\mathcal{N}$
5	0	$\star$	0	0	$\star$	0	$h_1, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\emptyset$
6	0	$\star$	0	0	0	$\star$	$h_1 \in L^2(0,T), g_2 \in H^{-\frac{1}{3}}(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N}$
7	0	0	0	0	$\star$	$\star$	$g_1 \in L^2(0,T), g_2 \in H^{-\frac{1}{3}}(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N}$
8	$\star$	0	0	0	0	$\star$	$h_0 \in L^2(0,T), g_2 \in H^{-\frac{1}{3}}(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N}$
9	0	0	0	$\star$	0	$\star$	$g_0 \in L^2(0,T), g_2 \in H^{-\frac{1}{3}}(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N}$
10	$\star$	0	0	0	$\star$	0	$h_0, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{R}$
11	0	0	0	$\star$	$\star$	0	$g_0, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{N}_3$
12	0	$\star$	0	$\star$	0	0	$h_1, g_0 \in L^2(0,T)$	$(\eta_0, w_0) \in X_{-2}$	$\mathcal{G} \cup \mathcal{G}'$

Table 1. Controllability results for the linearized system

Remark 3.3. *Using the symmetries, we notice that all the possibilities with one or two control inputs are really considered in Table 1: indeed, h_2 alone is similar to g_2 alone (case 1); h_0 alone is similar to g_1 alone (case 2); h_1 alone is similar to g_0 alone (case 3); the pair (h_0, g_0) is similar to the pair (h_1, g_1) (case 5); the pair (h_2, g_1) is similar to the pair (h_0, g_2) (case 8); the pair (h_1, h_2) is similar to the pair (g_0, g_2) (case 9); the pair (g_0, h_2) is similar to the pair (h_1, g_2) (case 6); the pair (h_0, h_2) is similar to the pair (g_1, g_2) (case 7); the pair (h_0, h_1) is similar to the pair (g_0, g_1) (case 11).*

The proof of Theorem 3.2 is detailed in a series of propositions or theorems displayed in several subsections.

3.1. **Neumann controls.** We consider first two controls (case 4) and next only one control (case 1). We shall use repeatedly the following results from [30].

Proposition 3.4. *([30, Proposition 3.3]) For all $L \in (0, +\infty) \setminus \mathcal{N}$ and $T > 0$, there exists a constant $C = C(L,T) > 0$ such that*

$$\int_0^L |y^0(x)|^2 \, dx \leq C \int_0^T |y_x(t,0)|^2 \, dt, \qquad \forall y^0 \in L^2(0,L), \tag{3.4}$$

where $y = y(t,x)$ denotes the solution to the linearized KdV system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y^0(x), & x \in (0,L). \end{cases} \tag{3.5}$$

Lemma 3.5. *([30, Lemma 3.5]) Let $L > 0$. Then $L \in \mathcal{N}$ if and only if there exist some numbers $p \in \mathbb{C}$ and $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that the map*

$$f(\xi) = \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p} \tag{3.6}$$

is an entire function, i.e. a complex analytic function on $\mathbb{C}$.

The controllability result for the case 4 is a consequence of the following observability inequality.

Proposition 3.6. *Let $L \notin \mathcal{N}$ and $T > 0$. Then there exists a constant $C = C(L,T) > 0$ such that*

$$\|(\theta^0,u^0)\|^2_{X_1} \leq C \int_0^T \left[|\theta_x(t,L)|^2 + |u_x(t,0)|^2\right] dt, \qquad \forall (\theta^0,u^0) \in X_1, \tag{3.7}$$

where (θ,u) denotes the solution of (2.18).

Proof. If (3.7) is not true, one can find two sequences $(\theta^0_n)_{n\geq 0}$ and $(u^0_n)_{n\geq 0}$ such that

$$1 = \|(\theta^0_n,u^0_n)\|^2_{X_1} > n \int_0^T [|\theta_{n,x}(t,L)|^2 + |u_{n,x}(t,0)|^2] dt. \tag{3.8}$$

Extracting a subsequence if necessary, one can assume that $(\theta^0_n,u^0_n) \to (\theta^0,u^0)$ weakly in X_1, and hence strongly in X_0. In particular, we have that

$$\begin{aligned} (\theta^0_n,u^0_n) &\to (\theta^0,u^0) && \text{strongly in } X_0, \\ (\theta_n,u_n) &\to (\theta,u) && \text{strongly in } C([0,T],X_0), \\ (\theta_n,u_n) &\to (\theta,u) && \text{weakly in } L^2(0,T,X_1), \\ \theta_{x,n}(\cdot,L) &\to \theta_x(\cdot,L) && \text{weakly in } L^2(0,T), \\ u_{x,n}(\cdot,0) &\to u_x(\cdot,0) && \text{weakly in } L^2(0,T). \end{aligned}$$

Using (3.8), we infer that $\theta_{x,n}(\cdot,L) \to 0$ and $u_{x,n}(\cdot,0) \to 0$ strongly in $L^2(0,T)$. Thus $\theta_x(\cdot,L) = u_x(\cdot,0) = 0$. It follows that $y(t,x) := \theta(t,x) + u(t,x)$ satisfies

$$\begin{cases} y_t + y_{xxx} + y_x = 0, & t \in (0,T),\ x \in (0,L), \\ y(t,0) = y(t,L) = y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = \theta^0(x) + u^0(x), & x \in (0,L) \end{cases}$$

together with $y_x(t,0) = 0$. We infer from Proposition 3.4 that $\theta^0 + u^0 = 0$. Similarly, considering the function $z(t,x) := \theta(t,L-x) - u(t,L-x)$, we can conclude that $\theta^0(L-x) - u^0(L-x) = 0$. Therefore, $\theta^0 = u^0 = 0$, and thus $(\theta_n,u_n) \to (0,0)$ strongly in $C([0,T],X_0)$. But it follows from the above convergences and from (2.10) that $(\theta_n,u_n) \to (0,0)$ strongly in $L^2(0,T,X_1)$, and with (2.8), that $(\theta^0_n,u^0_n) \to (0,0)$ strongly in X_1. This contradicts (3.8). □

Let us investigate the exact controllability of (2.17) in case 4. If $L \notin \mathcal{N}$, then by Proposition 3.6, system (2.17) is exactly controllable in X_{-1}. If $L \in \mathcal{N}$, then by [30, Remark 3.6 (i)] there exists a nontrivial solution y of (3.5) such that $y_x(\cdot, 0) = 0$. Then $(\theta(t,x), u(t,x)) := (y(t,x) + y(t, L-x), y(t,x) - y(t, L-x))$ is a nontrivial solution of (2.18) such that $\theta_x(\cdot, L) = u_x(\cdot, 0) = 0$. It follows that (2.17) is not exactly controllable in X_{-1} if $L \in \mathcal{N}$.

We now turn into the problem of the controllability of (2.17) with only one Neumann control (case 1). The following observability inequality improves Proposition 3.6, for only *one* boundary measurement is needed to estimate the initial data.

Theorem 3.7. *Let $L \notin \mathcal{N}$ and $T > 0$. Then there exists a constant $C = C(L,T) > 0$ such that*

$$\|(\theta^0, u^0)\|_{X_1}^2 \leq C \int_0^T |\theta_x(t, L)|^2 dt, \qquad \forall (\theta^0, u^0) \in X_1, \tag{3.9}$$

where (θ, u) denotes the solution of (2.18).

Proof. We follow closely [30]. In the first step, we transform the problem into a spectral problem. In the second step, we solve the spectral problem by using Paley-Wiener theorem combined with complex analysis. Introduce some notations. For any $T > 0$, let

$$\begin{aligned} N_T &:= \{(\theta^0, u^0) \in X_1;\ \text{ the solution } (\theta, u) \text{ of (2.18) satisfies } \theta_x(t, L) = 0 \text{ for a.e. } t \in (0,T)\}, \\ M_T &:= \{(\theta, u) = e^{tA}(\theta^0, u^0);\ (\theta^0, u^0) \in N_T,\ t \in [0,T]\} \subset C([0,T], X_1). \end{aligned}$$

Step 1. Reduction to a spectral problem.

Pick any $L \in (0, +\infty) \setminus \mathcal{N}$ and any $T > 0$. If (3.9) is not true, one can find a sequence $(\theta_n^0, u_n^0)_{n \geq 0}$ in X_1 such that

$$1 = \|(\theta_n^0, u_n^0)\|_{X_1}^2 > n \int_0^T |\theta_{n,x}(t, L)|^2 dt, \qquad \forall n \geq 0. \tag{3.10}$$

Extracting a subsequence if necessary, one can assume that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ weakly in X_1, and hence strongly in X_0. Let (θ_n, u_n) and (θ, u) denote the solutions of (2.18) associated with (θ_n^0, u_n^0) and (θ^0, u^0), respectively. Using (3.10), we infer that $\theta_{n,x}(\cdot, L) \to 0$ strongly in $L^2(0,T)$, and hence $\theta_x(\cdot, L) = 0$. Using (2.10), (3.10) and the fact that $(\theta_n, u_n) \to (\theta, u)$ in $C([0,T], X_0)$, we see that $(\theta_n, u_n) \to (\theta, u)$ strongly in $L^2(0,T,X_1)$, so that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ strongly in X_1, and hence $\|(\theta^0, u^0)\|_{X_1} = 1$. Thus $(\theta^0, u^0) \in N_T \setminus \{(0,0)\}$. To obtain a contradiction, it is sufficient to prove the following

Claim 1. $N_T = \{(0,0)\}$ for all $T > 0$.

It is clear that $T < T' \Rightarrow N_{T'} \subset N_T$. On the other hand, the vector space N_T has a finite dimension, for its unit ball is compact. Indeed, the same argument as above shows that any sequence $(\theta_n^0, u_n^0)_{n \in \mathbb{N}}$ in the unit ball of N_T has a convergent subsequence for the norm $\|\cdot\|_{X_1}$. If Claim 1 is not true, one may find $T' > 0$ such that $\dim N_{T'} > 0$. Since the map $T \in (0, +\infty) \mapsto \dim N_T \in \mathbb{N}$ is nonincreasing, one may pick $T \in (0, T')$ and $\varepsilon > 0$ such that $T + \varepsilon < T'$ and $\dim N_T = \dim N_{T+\varepsilon} \geq \dim N_{T'} > 0$. It follows that $N_t = N_T$ for $T \leq t \leq T + \varepsilon$. Let

$(\theta^0, u^0) \in N_T$, and let (θ, u) denote the solution of (2.18). By the semigroup property, we have that

$$\frac{1}{t}\left((\theta(t,\cdot), u(t,\cdot)) - (\theta^0, u^0)\right) \in N_T, \quad \forall t \in (0, \varepsilon),$$

and hence

$$\frac{1}{t}\left((\theta(t+\cdot,\cdot), u(t+\cdot,\cdot)) - (\theta, u)\right) \in M_T, \quad \forall t \in (0, \varepsilon).$$

Since $(\theta, u) \in H^1(0, T+\varepsilon, [H^{-2}(0,L)]^2)$, we have that

$$\lim_{t\to 0^+} \frac{1}{t}\left((\theta(t+\cdot,\cdot), u(t+\cdot,\cdot)) - (\theta, u)\right) = (\theta_t, u_t) \quad \text{ in } L^2(0, T, [H^{-2}(0,L)]^2).$$

But the vector space M_T is closed in $L^2(0, T, [H^{-2}(0,L)]^2)$, for it is finite-dimensional. It follows that $(\theta_t, u_t) \in M_T \subset C([0,T], X_1)$, so that $(\theta, u) \in C^1([0,T], X_1)$. This implies that

$$(\theta^0, u^0) \in D(A) = X_3, \quad A(\theta^0, u^0) = (\theta_t(0,.), u_t(0,.)) \in N_T \quad \text{ and } \theta_x(\cdot, L) \in C([0,T]).$$

In particular

$$\frac{d\theta^0}{dx}(L) = \theta_x(0, L) = 0.$$

Let $N_T^{\mathbb{C}}$ denote the complexification of N_T, and let $' = d/dx$. Since it was assumed that $\dim N_T \neq 0$, we infer that the linear map $(\theta^0, u^0) \in N_T^{\mathbb{C}} \mapsto A(\theta^0, u^0) \in N_T^{\mathbb{C}}$ has at least one eigenvalue; that is, there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ [2] solution of the spectral problem

$$-u' - u''' = \lambda\theta, \qquad x \in (0, L), \tag{3.11}$$

$$-\theta' - \theta''' = \lambda u, \qquad x \in (0, L), \tag{3.12}$$

$$\theta(0) = \theta(L) = \theta'(0) = \theta'(L) = 0, \tag{3.13}$$

$$u(0) = u(L) = u'(L) = 0. \tag{3.14}$$

We show in the next step that (3.11)-(3.14) has no nontrivial solution when $L \notin \mathcal{N}$.

Step 2. Study of the spectral problem.

We shall prove the following

Claim 2. For $L \notin \mathcal{N}$, if $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3$ satisfy (3.11)-(3.14), then $\theta = u = 0$.

Let (θ, u) be as in Claim 2, and extend θ and u to $\mathbb{R}$ by setting $\theta(x) = u(x) = 0$ for $x \notin [0, L]$. Then we have in $\mathcal{S}'(\mathbb{R})$

$$\begin{aligned} \lambda\theta + u' + u''' &= u'(0)\delta_0' + u''(0)\delta_0 - u''(L)\delta_L, \\ \lambda u + \theta' + \theta''' &= \theta''(0)\delta_0 - \theta''(L)\delta_L, \end{aligned}$$

where δ_ζ denotes the Dirac measure at $x = \zeta$ and the derivatives $u'(0), u''(0), u''(L), \theta''(0), \theta''(L)$ are those of the functions u and θ when restricted to $[0, L]$. Taking the Fourier transform of

[2] We removed the superscript 0 to simplify the notations. We still used the notation X_3 to denote the space of *complex-valued functions* in (2.2).

each term in the above system, we obtain

$$\begin{aligned}\lambda\hat{\theta}(\xi)+i\xi\hat{u}(\xi)+(i\xi)^3\hat{u}(\xi) &= u'(0)i\xi+u''(0)-u''(L)e^{-iL\xi},\\ \lambda\hat{u}(\xi)+i\xi\hat{\theta}(\xi)+(i\xi)^3\hat{\theta}(\xi) &= \theta''(0)-\theta''(L)e^{-iL\xi}.\end{aligned}$$

Setting $\lambda=-ip$, $f(\xi):=\hat{\theta}(\xi)+\hat{u}(\xi)$, and $g(\xi):=\hat{\theta}(\xi)-\hat{u}(\xi)$, we arrive to

$$\begin{aligned}f(\xi) &= \frac{i}{\xi^3-\xi+p}(\alpha+\beta i\xi+\gamma e^{-iL\xi}),\\ g(-\xi) &= \frac{i}{\xi^3-\xi+p}(\alpha'-\beta i\xi+\gamma' e^{iL\xi}),\end{aligned}$$

where $\alpha:=u''(0)+\theta''(0)$, $\alpha':=u''(0)-\theta''(0)$, $\beta:=u'(0)$, $\gamma:=-u''(L)-\theta''(L)$, and $\gamma':=-u''(L)+\theta''(L)$. Since both θ and u have a compact support, it follows from the Paley-Wiener theorem that the functions f and g have to be entire (i.e. holomorphic in the whole plane $\mathbb{C}$). The same is true for the function

$$h(\xi):=f(\xi)+g(-\xi)=\frac{ie^{-iL\xi}}{\xi^3-\xi+p}(\gamma' e^{2iL\xi}+(\alpha+\alpha')e^{iL\xi}+\gamma).$$

Introduce the polynomial functions $P(z):=\gamma' z^2+(\alpha+\alpha')z+\gamma$ and $Q(\xi):=\xi^3-\xi+p$, and let μ_k, $k=0,1,2$, denote the three roots of Q. Then $P(e^{iL\mu_k})=0$ for $k=0,1,2$, since h is entire. Let

$$z_\pm=\frac{-(\alpha+\alpha')\pm\sqrt{(\alpha+\alpha')^2-4\gamma\gamma'}}{2\gamma'}$$

denote the roots of the polynomial function P.
(i) If $\beta=0$, then applying Lemma 3.5 to $f(\xi)$ and $g(-\xi)$, we infer that $\alpha=\gamma=\alpha'=\gamma'=0$, so that

$$u''(0)=u''(L)=u'(0)=\theta''(0)=\theta''(L)=0.$$

It follows then that $\theta=u=0$ in $[0,L]$, as desired.
(ii) If $\beta\neq 0$, then since $g(-\xi)$ is entire, each root μ_k of Q has to satisfy $\alpha'-\beta i\mu_k+\gamma' e^{iL\mu_k}=0$, i.e. $\mu_k=(\alpha'+\gamma' e^{iL\mu_k})/(i\beta)$. Since $e^{iL\mu_k}$ is a root of P, we arrive to the conclusion that

$$\mu_k\in\left\{\frac{\alpha'+\gamma' z_+}{i\beta},\frac{\alpha'+\gamma' z_-}{i\beta}\right\}.$$

We infer that Q cannot have three distinct roots.
Assume that Q has a root of order 3, i.e. $\mu_0=\mu_1=\mu_2$. Then we have that $Q(\mu_0)=Q'(\mu_0)=Q''(\mu_0)=0$. Since $Q''(\xi)=6\xi$, we conclude that $\mu_0=0$, hence $p=0$, so that $Q(\xi)=\xi(\xi-1)(\xi+1)$ has three distinct roots, a contradiction.

We arrive to the conclusion that Q has a double root and a simple root. We can assume that $\mu_0=\mu_1\neq\mu_2$. Then, we infer from $Q(\mu_0)=Q'(\mu_0)=0$ and the fact that $Q'(\xi)=3\xi^2-1$ that $\mu_0=\pm\frac{1}{\sqrt{3}}$ and $p=\pm\frac{2}{3\sqrt{3}}$.
$\star$ Assume that $\mu_0=\frac{1}{\sqrt{3}}$. Then $Q(\xi)=\xi^3-\xi+\frac{2}{3\sqrt{3}}=(\xi-\frac{1}{\sqrt{3}})^2(\xi+\frac{2}{\sqrt{3}})$. On the other hand, since f is entire, $\mu_2=-\frac{2}{\sqrt{3}}$ (resp. μ_0) has to be a root (resp. a root of order at least 2) of the

numerator of f, i.e.

$$\alpha + \frac{i}{\sqrt{3}}\beta + \gamma e^{-i\frac{L}{\sqrt{3}}} = 0, \tag{3.15}$$

$$i\beta - iL\gamma e^{-i\frac{L}{\sqrt{3}}} = 0, \tag{3.16}$$

$$\alpha - \frac{2i}{\sqrt{3}}\beta + \gamma e^{2i\frac{L}{\sqrt{3}}} = 0. \tag{3.17}$$

From (3.16), we infer that $\beta = L\gamma e^{-iL/\sqrt{3}}$. Substituting this value of β in (3.15) and (3.17) and taking the difference of the two obtained equations, we obtain that

$$(i\sqrt{3}L\gamma + \gamma)e^{-i\frac{L}{\sqrt{3}}} = \gamma e^{2i\frac{L}{\sqrt{3}}}. \tag{3.18}$$

- If $\gamma = 0$, we obtain $\beta = 0$ which contradicts the assumption in (ii).
- If $\gamma \neq 0$, then taking the module of both terms in (3.18) yields $L = 0$, which is impossible.

$\star$ Assume now that $\mu_0 = -\frac{1}{\sqrt{3}}$. Then, proceeding as above, we obtain that $\beta = L\gamma e^{iL/\sqrt{3}}$ and that

$$(-i\sqrt{3}L\gamma + \gamma)e^{i\frac{L}{\sqrt{3}}} = \gamma e^{-2i\frac{L}{\sqrt{3}}}.$$

Again, we see that we obtain a contradiction for $\gamma = 0$ or for $\gamma \neq 0$. The proof of Theorem 3.7 is complete. □

With Theorem 3.7 at hand, we deduce that (2.17) is exactly controllable in X_{-1} with g_2 as unique control (case 1) when $L \notin \mathcal{N}$. On the other hand, the exact controllability fails when $L \in \mathcal{N}$ according to case 4.

Let us turn our attention to the cases 6, 7, 8 and 9 for which we have added one Dirichlet control to the Neumann control g_2. First, we notice that in case 1, we have for $L \notin \mathcal{N}$ the exact controllability in X_{-2} by picking the controls g_2 in $H^{-\frac{1}{3}}(0,T)$. Indeed, from (3.9), we have for $(\theta^0, u^0) \in X_4$

$$\|A(\theta^0, u^0)\|^2_{X_1} \leq C \int_0^T |\theta_{xt}(t, L)|^2 dt$$

and hence

$$\|(\theta^0, u^0)\|^2_{X_4} \leq C\|\theta_x(\cdot, L)\|^2_{H^1(0,T)}. \tag{3.19}$$

By interpolation between (3.9) and (3.19), we infer that for all $(\theta^0, u^0) \in X_2$,

$$\|(\theta^0, u^0)\|^2_{X_2} \leq C\|\theta_x(\cdot, L)\|^2_{H^{\frac{1}{3}}(0,T)}. \tag{3.20}$$

The observability inequality (3.20) implies the exact controllability in X_{-2} in case 1 (with $g_2 \in H^{-\frac{1}{3}}(0,T)$) as well as in cases 6 to 9 for $L \notin \mathcal{N}$. It remains to show that any $L \in \mathcal{N}$ is still a critical length in cases 6 to 9. We claim that in each of those cases, the corresponding spectral problem has for $L \in \mathcal{N}$ a nontrivial solution (θ, u), so that the corresponding observability inequality fails for the exponential solution $\mathrm{Re}[e^{\lambda t}(\theta, u)]$.

Proposition 3.8. *Let $L \in \mathcal{N}$. Then*

1. *(case 6) there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ satisfying (3.11)-(3.14) and $u''(L) = 0$.*
2. *(case 7) there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ satisfying (3.11)-(3.14) and $\theta''(L) = 0$.*

3. (case 8) there exist $\lambda \in \mathbb{C}$ *and* $(\theta, u) \in X_3 \setminus \{(0,0)\}$ *satisfying* (3.11)-(3.14) *and* $u''(0) = 0$.
4. (case 9) there exist $\lambda \in \mathbb{C}$ *and* $(\theta, u) \in X_3 \setminus \{(0,0)\}$ *satisfying* (3.11)-(3.14) *and* $\theta''(0) = 0$.

Proof. Let $L \in \mathcal{N}$. Then we can write $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ with $k, l \in \mathbb{N}^*$. Following [30], we introduce the numbers

$$\mu_0 = -\frac{1}{3}(2k+l)\frac{2\pi}{L}, \quad \mu_1 := \mu_0 + k\frac{2\pi}{L}, \quad \mu_2 = \mu_1 + l\frac{2\pi}{L}, \quad p := i\lambda = -\mu_0\mu_1\mu_2.$$

We use again the notations in the proof of Theorem 3.7 (Claim 2). Let

$$\begin{aligned} f(\xi) &= \frac{i}{\xi^3 - \xi + p}(\alpha + \beta i\xi + \gamma e^{-iL\xi}), \\ g(-\xi) &= \frac{i}{\xi^3 - \xi + p}(\alpha' - \beta i\xi + \gamma' e^{iL\xi}), \end{aligned}$$

where $\alpha := u''(0) + \theta''(0)$, $\alpha' := u''(0) - \theta''(0)$, $\beta := u'(0)$, $\gamma := -u''(L) - \theta''(L)$, and $\gamma' := -u''(L) + \theta''(L)$. We know that we can find coefficients $\alpha, \beta, \gamma, \alpha', \gamma'$ not all zero so that the two functions $f(\xi)$ and $g(-\xi)$ are entire, the roots of $Q(\xi) = \xi^3 - \xi + p$ being μ_0, μ_1, μ_2. Furthermore, it follows from Paley-Wiener theorem (see [30]) that the spectral problem (3.11)-(3.14) has indeed a nontrivial solution. Our concern is to prove that we can as well impose the addition condition in each case.

From our choice of the μ_k's, we have that the quantity $e^{-iL\mu_k}$ is independent of k. Set

$$C := e^{-iL\mu_0} = e^{-iL\mu_1} = e^{-iL\mu_2}.$$

We shall pick $\beta = 0$ in all the cases.
1. The additional condition $u''(L) = 0$ is equivalent to $\gamma' = -\gamma$. We can pick $\gamma = 1$, $\gamma' = -1$, $\alpha = -C$, and $\alpha' = \overline{C}$.
2. The additional condition $\theta''(L) = 0$ is equivalent to $\gamma' = \gamma$. We can pick $\gamma = \gamma' = 1$, $\alpha = -C$, and $\alpha' = -\overline{C}$.
3. The additional condition $u''(0) = 0$ is equivalent to $\alpha' = -\alpha$. We can pick $\alpha = 1$ and $\alpha' = -1$, $\gamma = -\overline{C}$, and $\gamma' = C$.
4. The additional condition $\theta''(0) = 0$ is equivalent to $\alpha' = \alpha$. We can pick $\alpha = \alpha' = 1$, $\gamma = -\overline{C}$, and $\gamma' = -C$. □

3.2. **Dirichlet controls.** We consider the cases in which only Dirichlet controls are involved. We start with a preparatory result which is an observability inequality with the measurement of three traces.

Proposition 3.9. *For all* $L > 0$ *and* $T > 0$, *there is a constant* $C = C(L,T) > 0$ *such that*

$$\|(\theta^0, u^0)\|_{X_2}^2 \le C\int_0^T \left[|\theta_{xx}(t,L)|^2 + |u_{xx}(t,L)|^2 + |\theta_x(t,L)|^2\right] dt, \qquad \forall (\theta^0, u^0) \in X_2. \tag{3.21}$$

Proof. First, we introduce the space

$$Y := L^2(0,T,[H^{\frac{7}{4}}(0,L)]^2) \cap C([0,T],[H^1(0,L)]^2)$$

which is a Banach space when endowed with the norm

$$\|(\theta,u)\|_Y^2=\int_0^T[\|\theta(t,\cdot)\|_{H^{\frac{7}{4}}}^2+\|u(t,\cdot)\|_{H^{\frac{7}{4}}}^2]dt+\sup_{0\leq t\leq T}[\|\theta(t,\cdot)\|_{H^1}^2+\|u(t,\cdot)\|_{H^1}^2].$$

Pick any $(\theta^0,u^0)\in X_4$ and let (θ,u) denote the solution of (2.18). Taking the derivative w.r.t. x of each term in the two first equations of (2.18) results in

$$\theta_{tx}+u_{xx}+u_{xxxx}=0,\qquad t\in(0,T),\ x\in(0,L),\tag{3.22}$$

$$u_{tx}+\theta_{xx}+\theta_{xxxx}=0,\qquad t\in(0,T),\ x\in(0,L).\tag{3.23}$$

Multiplying (3.22) by xu_x, (3.23) by $x\theta_x$, integrating by parts over $(0,T)\times(0,L)$ and adding the two equations, we arrive to

$$\begin{aligned}&\left[\int_0^L x\theta_x u_x\,dx\right]_0^T-\frac{1}{2}\int_0^T\int_0^L[\theta_x^2+u_x^2]dxdt+\frac{3}{2}\int_0^T\int_0^L[\theta_{xx}^2+u_{xx}^2]\,dxdt\\&\quad-\int_0^T\left[\frac{x}{2}(\theta_x^2+u_x^2)+\theta_x\theta_{xx}+u_xu_{xx}+\frac{x}{2}(\theta_{xx}^2+u_{xx}^2)\right]_0^L dt=0.\end{aligned}\tag{3.24}$$

Combining (3.24) with the forward/backward wellposedness in X_2 and Sobolev embedding, this yields

$$\begin{aligned}\|(\theta^0,u^0)\|_{X_2}^2&\leq C\int_0^T\|(\theta,u)\|_{X_2}^2dt\\&\leq C\int_0^T\int_0^L[\theta_{xx}^2+u_{xx}^2]\,dxdt\\&\leq C\left(\|(\theta,u)\|_Y^2+\int_0^T[\theta_{xx}^2(t,L)+u_{xx}^2(t,L)+u_{xx}^2(t,0)]dt\right).\end{aligned}\tag{3.25}$$

Multiplying the two first equations in (2.18) by u_{xx} and θ_{xx} respectively, integrating by parts over $(0,T)\times(0,L)$ and adding the two equations, we obtain

$$\frac{1}{2}\int_0^T[\theta_{xx}^2+u_{xx}^2]_0^L dt-\left[\int_0^L\theta_xu_x\,dx\right]_0^T+\frac{1}{2}\int_0^T\theta_x^2(t,L)dt-\frac{1}{2}\int_0^T u_x^2(t,0)dt=0,\tag{3.26}$$

and hence

$$\int_0^T[\theta_{xx}^2(t,0)+u_{xx}^2(t,0)]dt\leq C\left(\|(\theta,u)\|_Y^2+\int_0^T[\theta_{xx}^2(t,L)+u_{xx}^2(t,L)]dt\right).\tag{3.27}$$

Combined with (3.25), this gives

$$\|(\theta^0,u^0)\|_{X_2}^2\leq C\left(\|(\theta,u)\|_Y^2+\int_0^T[\theta_{xx}^2(t,L)+u_{xx}^2(t,L)]dt\right).\tag{3.28}$$

The result is also true for $(\theta^0,u^0)\in X_2$, by density of X_4 in X_2. It remains to "remove" the term $\|(\theta,u)\|_Y^2$ in (3.28). Assuming that (3.21) is not true, we can find a sequence $(\theta_n^0,u_n^0)_{n\in\mathbb{N}}$ such that

$$1=\|(\theta_n^0,u_n^0)\|_{X_2}^2>n\int_0^T\left[|\theta_{xx}^n(t,L)|^2+|u_{xx}^n(t,L)|^2+|\theta_x^n(t,L)|^2\right]dt.\tag{3.29}$$

Extracting a subsequence if needed, we can assume that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ weakly in X_2 (hence strongly in X_1). As the corresponding solution (θ_n, u_n) of (2.18) is bounded in $C([0,T], X_2)$ and in $H^1(0,T,X_{-1})$, and since the first embedding in $X_2 \subset [H^{\frac{7}{4}}(0,L) \cap H_0^1(0,L)]^2 \subset X_1$ is compact, we infer from Aubin-Lions' lemma (see e.g. [34, Corollary 4]) that the sequence $(\theta_n, u_n)_{n\geq 0}$ admits a subsequence, still denoted $(\theta_n, u_n)_{n\geq 0}$, which is strongly convergent in $C([0,T], [H^{\frac{7}{4}}(0,L) \cap H_0^1(0,L)]^2)$, hence in Y. It follows then from (3.28) that the sequence $(\theta_n^0, u_n^0)_{n\geq 0}$ is strongly convergent in X_2. Thus its limit (θ^0, u^0) is such that $\|(\theta^0, u^0)\|_{X_2} = 1$ and the corresponding solution of (2.18) satisfies

$$\theta_{xx}(t,L) = u_{xx}(t,L) = \theta_x(t,L) = 0.$$

We infer that the function $y(t,x) := \theta(t,x) + u(t,x)$ (resp. $y(t,x) := \theta(t,L-x) - u(t,L-x)$) solves the linearized KdV equation $y_t + y_x + y_{xxx} = 0$ and satisfies the boundary conditions $y(t,L) = y_x(t,L) = y_{xx}(t,L) = 0$ (resp. $y(t,0) = y_x(t,0) = y_{xx}(t,0) = 0$), and hence it vanishes in $(0,T)\times(0,L)$ by Holmgren's theorem. We conclude that $\theta = u = 0$ and this contradicts the fact that $\|(\theta^0, u^0)\|_{X_2} = 1$. □

We are in a position to investigate the case 5. The following observability inequality improves those in Proposition 3.9.

Corollary 3.10. *For all $L > 0$ and $T > 0$, there is a constant $C = C(L,T) > 0$ such that*

$$\|(\theta^0, u^0)\|_{X_2}^2 \leq C \int_0^T \left[|\theta_{xx}(t,L)|^2 + |u_{xx}(t,L)|^2\right] dt, \qquad \forall (\theta^0, u^0) \in X_2. \tag{3.30}$$

Proof. We proceed in two steps as in the proof of Theorem 3.7.

Step 1. Reduction to a spectral problem.
If (3.30) is false, then one can pick a sequence $(\theta_n^0, u_n^0)_{n\in\mathbb{N}}$ such that

$$1 = \|(\theta_n^0, u_n^0)\|_{X_2}^2 > n \int_0^T \left[|\theta_{n,xx}(t,L)|^2 + |u_{n,xx}(t,L)|^2\right] dt. \tag{3.31}$$

Extracting a subsequence if needed, we can assume that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ weakly in X_2. Denote by (θ_n, u_n) (resp. (θ, u)) the solution of (2.18) with initial data (θ_n^0, u_n^0) (resp. (θ^0, u^0)). Extracting again a subsequence, we can assume that $(\theta_n, u_n) \to (\theta, u)$ in Y. By (3.28) and (3.31), we have that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ strongly in X_2. Thus $\|(\theta^0, u^0)\|_{X_2} = 1$ and $\theta_{xx}(\cdot, L) = u_{xx}(\cdot, L) = 0$ a.e. in $(0,L)$. The same argument as the one in Step 1 of the proof of Theorem 3.7 shows that there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ (superscript 0 dropped for simplicity) which solve the spectral problem:

$$-u' - u''' = \lambda\theta, \qquad x \in (0,L), \tag{3.32}$$

$$-\theta' - \theta''' = \lambda u, \qquad x \in (0,L), \tag{3.33}$$

$$\theta(0) = \theta(L) = \theta'(0) = \theta''(L) = 0, \tag{3.34}$$

$$u(0) = u(L) = u'(L) = u''(L) = 0. \tag{3.35}$$

We show in the next step that system (3.32)-(3.35) has no nontrivial solution for any $L > 0$.

Step 2. Study of the spectral problem.

Claim 3. Let $L>0$. If $\lambda\in\mathbb{C}$ and $(\theta,u)\in X_3$ satisfy (3.32)-(3.35), then $\theta=u=0$.

Let us prove Claim 3. Pick (λ,θ,u) as in Claim 3, and extend θ and u to $\mathbb{R}$ by setting $\theta(x)=u(x)=0$ for $x\notin[0,L]$. Then we have

$$\begin{aligned}\lambda\theta+u'+u'''&=u'(0)\delta_0'+u''(0)\delta_0,\\ \lambda u+\theta'+\theta'''&=-\theta'(L)\delta_L'+\theta''(0)\delta_0,\end{aligned}$$

where δ_ζ denotes the Dirac measure at $x=\zeta$, and the derivatives $u'(0),u''(0),\theta'(L),\theta''(0)$ are those of the functions u and θ when restricted to $[0,L]$. Taking the Fourier transform of each term in the above system, we obtain

$$\begin{aligned}\lambda\hat{\theta}(\xi)+i\xi\hat{u}(\xi)+(i\xi)^3\hat{u}(\xi)&=u'(0)i\xi+u''(0),\\ \lambda\hat{u}(\xi)+i\xi\hat{\theta}(\xi)+(i\xi)^3\hat{\theta}(\xi)&=-\theta'(L)(i\xi)e^{-iL\xi}+\theta''(0).\end{aligned}$$

Setting $\lambda=-ip$, $f(\xi):=\hat{\theta}(\xi)+\hat{u}(\xi)$, and $g(\xi):=\hat{\theta}(\xi)-\hat{u}(\xi)$, we arrive to

$$\begin{aligned}f(\xi)&=\frac{i}{\xi^3-\xi+p}(\alpha+\beta i\xi+\gamma'(i\xi)e^{-iL\xi})\\ g(-\xi)&=\frac{i}{\xi^3-\xi+p}(\alpha'-\beta i\xi+\gamma'(i\xi)e^{iL\xi})\end{aligned}$$

where $\alpha:=u''(0)+\theta''(0)$, $\alpha':=u''(0)-\theta''(0)$, $\beta:=u'(0)$, and $\gamma':=-\theta'(L)$. It follows that

$$f(\xi)+g(-\xi)=\frac{i}{\xi^3-\xi+p}(\alpha+\alpha'+2\gamma'(i\xi)\cos(L\xi)) \tag{3.36}$$

$$f(\xi)-g(-\xi)=\frac{i}{\xi^3-\xi+p}(\alpha-\alpha'+2\beta i\xi-2i\gamma'(i\xi)\sin(L\xi)). \tag{3.37}$$

Let $Q(\xi)=\xi^3-\xi+p$ and let μ_0,μ_1,μ_2 be the roots of Q.
1. Assume that the three roots μ_0,μ_1,μ_1 are simple. If $\xi\in\{\mu_0,\mu_1,\mu_2\}$, then ξ must be a root of the numerators of $f(\xi)+g(-\xi)$ and $f(\xi)-g(-\xi)$, so that

$$2\gamma' i\xi\cos(L\xi)=-\alpha-\alpha', \tag{3.38}$$

$$2\gamma' i\xi\sin(L\xi)=\frac{\alpha-\alpha'}{i}+2\beta\xi. \tag{3.39}$$

Taking the square in both equations and summing, we obtain

$$-4\gamma'^2\xi^2=(\alpha+\alpha')^2-(\alpha-\alpha')^2+4\beta^2\xi^2+\frac{4\beta\xi}{i}(\alpha-\alpha')$$

and hence

$$(\beta^2+\gamma'^2)\xi^2+\frac{\beta\xi}{i}(\alpha-\alpha')+\alpha\alpha'=0. \tag{3.40}$$

As the three numbers μ_0,μ_1,μ_2 satisfy (3.40), we infer that all the coefficients in (3.40) are null; that is

$$\beta^2+\gamma'^2=\beta(\alpha-\alpha')=\alpha\alpha'=0.$$

If $\gamma' = 0$, then the solution (θ, u) of the system (3.32)-(3.33) of order 3 satisfies (3.34)-(3.35) together with $\theta'(L) = 0$ and is therefore null, by Cauchy-Lipschitz theorem (Cauchy data taken at $x = L$).

If $\gamma' \neq 0$, then $\beta = \pm i\gamma' \neq 0$ and thus $\alpha = \alpha' = 0$. Assume that $\beta = i\gamma'$. From (3.38)-(3.39), we infer that

$$\xi \cos(L\xi) = 0, \quad \xi(\sin(L\xi) - 1) = 0 \quad \text{for } \xi \in \{\mu_0, \mu_1, \mu_2\}.$$

If $0 \in \{\mu_0, \mu_1, \mu_2\}$, say $\mu_0 = 0$, then $\sin(L\mu_j) = 1$ for $j = 1, 2$, and hence

$$L\mu_j = \frac{\pi}{2} + 2k_j\pi, \quad k_j \in \mathbb{Z}.$$

Therefore $L(\mu_0 + \mu_1 + \mu_2) = \pi(1 + 2(k_1 + k_2))$, but this contradicts the fact that the sum of the roots of Q is 0.

If $0 \notin \{\mu_0, \mu_1, \mu_2\}$, then for $j = 1, 2, 3$ we have $\sin(L\mu_j) = 1$ and hence

$$L\mu_j = \frac{\pi}{2} + 2k_j\pi, \quad k_j \in \mathbb{Z}.$$

Then $L(\mu_0 + \mu_1 + \mu_2) = \pi(\frac{3}{2} + 2(k_1 + k_2 + k_3)) \neq 0$, a contradiction. Similarly, we obtain also a contradiction when $\beta = -i\gamma'$.

2. Assume now that Q has a double root $\mu_0 = \mu_1$ and a simple root $\mu_2 \neq \mu_0$. Then $(\mu_0, \mu_2) = \pm(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$. We shall consider the case $(\mu_0, \mu_2) = (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$, the case $(\mu_0, \mu_2) = (-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ being similar. As μ_0 is a double root of Q, it should be a root of the numerators of the functions $f(\xi) + g(-\xi)$ and $f(\xi) - g(-\xi)$ (see (3.36) and (3.37)) and of their first derivatives, so that

$$2\gamma' \frac{i}{\sqrt{3}} \cos(\frac{L}{\sqrt{3}}) = -(\alpha + \alpha'), \tag{3.41}$$

$$2\gamma' \frac{i}{\sqrt{3}} \sin(\frac{L}{\sqrt{3}}) = \frac{\alpha - \alpha'}{i} + 2\frac{\beta}{\sqrt{3}}, \tag{3.42}$$

$$2\gamma' i \cos(\frac{L}{\sqrt{3}}) + 2\gamma' \frac{i}{\sqrt{3}}(-L \sin \frac{L}{\sqrt{3}}) = 0, \tag{3.43}$$

$$2\beta i + 2\gamma' \sin(\frac{L}{\sqrt{3}}) + 2\gamma' \frac{L}{\sqrt{3}} \cos(\frac{L}{\sqrt{3}}) = 0. \tag{3.44}$$

On the other hand, the number μ_2 is a simple root of Q, and hence it should be a root of the numerators of the functions $f(\xi) + g(-\xi)$ and $f(\xi) - g(-\xi)$, so that

$$-4\gamma' \frac{i}{\sqrt{3}} \cos(\frac{2L}{\sqrt{3}}) = -(\alpha + \alpha'), \tag{3.45}$$

$$-4\gamma' \frac{i}{\sqrt{3}} \sin(-2\frac{L}{\sqrt{3}}) = \frac{\alpha - \alpha'}{i} - 4\frac{\beta}{\sqrt{3}}. \tag{3.46}$$

Note that (3.43)-(3.44) can be rewritten as

$$\begin{pmatrix} \cos(\frac{L}{\sqrt{3}}) & -\sin(\frac{L}{\sqrt{3}}) \\ \sin(\frac{L}{\sqrt{3}}) & \cos(\frac{L}{\sqrt{3}}) \end{pmatrix} \begin{pmatrix} \gamma' \\ \gamma' \frac{L}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ -\beta i \end{pmatrix}.$$

This yields

$$\gamma' = -i\beta \sin(\frac{L}{\sqrt{3}}), \tag{3.47}$$

$$\gamma' \frac{L}{\sqrt{3}} = -i\beta \cos(\frac{L}{\sqrt{3}})\cdot \tag{3.48}$$

If $\gamma' = 0$, then we infer as above that $(\theta^0, u^0) = (0, 0)$.
Assume that $\gamma' \neq 0$, so that $\beta \neq 0$. Without loss of generality, we can assume by linearity that $\beta = 1$. It follows from (3.47) and (3.48) that

$$\gamma' = \pm \frac{i}{\sqrt{1 + \frac{L^2}{3}}}\cdot \tag{3.49}$$

Let $X := \cos(L/\sqrt{3})$. Then we infer from (3.48) and (3.49) that $L^2/(3X^2) = 1 + (L^2/3)$, so that

$$X = \cos(\frac{L}{\sqrt{3}}) = \cos(\frac{1}{\sqrt{X^{-2} - 1}})\cdot$$

On the other hand, from (3.41) and (3.45), we obtain that $\cos(L/\sqrt{3}) = -2\cos(2L/\sqrt{3})$, and hence X satisfies $4X^2 + X - 2 = 0$. We denote the two roots of this equation by

$$X_+ := \frac{-1 + \sqrt{33}}{8} \approx 0.5931, \qquad X_- := \frac{-1 - \sqrt{33}}{8} \approx -0.8431.$$

Then we obtain by using a numerical computation that

$$\cos(\frac{1}{\sqrt{X_+^{-2} - 1}}) \approx 0.7408 \neq 0.5931, \qquad \cos(\frac{1}{\sqrt{X_-^{-2} - 1}}) \approx 0.0032 \neq -0.8431.$$

This proves that the assumption $\gamma' \neq 0$ leads to a contradiction.
3. The polynomial function Q cannot have a triple root $\mu_0 = \mu_1 = \mu_2$, otherwise $0 = Q''(\mu_0) = 6\mu_0$ yields $\mu_0 = 0$, $p = 0$, and $Q(\xi) = \xi(\xi - 1)(\xi + 1)$, a contradiction. □

We now turn our attention to cases 2 and 3. We need the following estimate, whose (long) proof is postponed in an appendix.

Theorem 3.11. *For all $L \in (0, +\infty) \setminus 2\pi\mathbb{Z}$ and all $T > 0$, there is a constant $C = C(L, T) > 0$ such that for all $(\theta^0, u^0) \in X_2$, we have*

$$\|(\theta^0, u^0)\|_{X_2}^2 \leq C \left(\int_0^T |\theta_{xx}(t, L)|^2 \, dt + \|(\theta^0, u^0)\|_{X_0}^2 \right), \tag{3.50}$$

$$\|(\theta^0, u^0)\|_{X_2}^2 \leq C \left(\int_0^T |\theta_{xx}(t, 0)|^2 \, dt + \|(\theta^0, u^0)\|_{X_0}^2 \right). \tag{3.51}$$

We first consider case 2.

Corollary 3.12. *For all $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$ and all $T > 0$, there exists a constant $C = C(L, T) > 0$ such that*

$$\|(\theta^0, u^0)\|_{X_2}^2 \leq C \int_0^T |\theta_{xx}(t, L)|^2 \, dt. \tag{3.52}$$

Proof of Corollary 3.12: We proceed in two steps as in the proof of Theorem 3.7.

Step 1. Reduction to a spectral problem.

If (3.52) is false, then one can pick a sequence $(\theta_n^0, u_n^0)_{n\in\mathbb{N}}$ such that

$$1 = \|(\theta_n^0, u_n^0)\|_{X_2}^2 > n \int_0^T |\theta_{n,xx}(t, L)|^2 \, dt. \tag{3.53}$$

Extracting a subsequence if needed, we can assume that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ weakly in X_2. Denote by (θ_n, u_n) (resp. (θ, u)) the solution of (2.18) with initial data (θ_n^0, u_n^0) (resp. (θ^0, u^0)). Extracting again a subsequence, we can assume that $(\theta_n, u_n) \to (\theta, u)$ in X_1. By (3.50) and (3.53), we have that $(\theta_n^0, u_n^0) \to (\theta^0, u^0)$ strongly in X_2. Thus $\|(\theta^0, u^0)\|_{X_2} = 1$ and $\theta_{xx}(\cdot, L) = 0$ a.e. in $(0, T)$. The same argument as the one in Step 1 of the proof of Theorem 3.7 shows that there exists $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0, 0)\}$ (superscript 0 dropped for simplicity) solution of the spectral problem

$$-u' - u''' = \lambda\theta, \qquad x \in (0, L), \tag{3.54}$$

$$-\theta' - \theta''' = \lambda u, \qquad x \in (0, L), \tag{3.55}$$

$$\theta(0) = \theta(L) = \theta'(0) = \theta''(L) = 0, \tag{3.56}$$

$$u(0) = u(L) = u'(L) = 0. \tag{3.57}$$

We show in the next step that (3.54)-(3.57) has no nontrivial solution for any $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$.

Step 2. Study of the spectral problem.

Proposition 3.13. *Let $L > 0$. Then there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0, 0)\}$ solution of (3.54)-(3.57) if and only if $L \in \mathcal{N} \cup \mathcal{R}$.*

Proof. For θ and $u \in H^3(0, L)$, we still denote by θ and u their extension by 0 on $\mathbb{R}$. Then

$$u''' = \mathbf{1}_{(0,L)} u''' + u''(0)\delta_0 - u''(L)\delta_L + u'(0)\delta_0' - u'(L)\delta_L' + u(0)\delta_0'' - u(L)\delta_L'' \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

where the derivatives of u at 0 or L stand for the derivatives of $u_{|(0,L)}$ viewed as traces and $\mathbf{1}_{(0,L)}$ denotes the characteristic function of the interval $(0, L)$. Assume that

$$\exists \lambda \in \mathbb{C}, \ \exists (\theta, u) \in X_3 \setminus \{(0, 0)\} \quad \text{such that (3.54)-(3.57) hold.} \tag{3.58}$$

Note that (3.54)-(3.55) yields

$$\lambda\theta + u' + u''' = u''(0)\delta_0 - u''(L)\delta_L + u'(0)\delta_0' \quad \text{in } \mathcal{S}'(\mathbb{R}), \tag{3.59}$$

$$\lambda u + \theta' + \theta''' = \theta''(0)\delta_0 - \theta'(L)\delta_L' \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{3.60}$$

Conversely, if a pair (θ, u) in $L^2(\mathbb{R})^2$ with $\text{Supp}\, \theta \cup \text{Supp}\, u \subset [0, L]$ satisfies (3.59)-(3.60) for some $\lambda \in \mathbb{C}$ and some complex coefficients (instead of $u''(0), u''(L)$, etc.) in the r.h.s. of (3.59)-(3.60), then $\theta, u \in H^3(0, L)$ and (3.54)-(3.57) hold.

Let $\hat{\theta}(\xi) = \int_{\mathbb{R}} \theta(x) e^{-i\xi x} dx$ denote the Fourier transform of θ. Introduce $f(\xi) := \hat{\theta}(\xi) + \hat{u}(\xi)$ and $g(\xi) := \hat{\theta}(\xi) - \hat{u}(\xi)$. Then (3.59)-(3.60) give

$$\lambda\hat{\theta}(\xi) + (i\xi + (i\xi)^3)\hat{u}(\xi) = u'(0)i\xi + u''(0) - u''(L)e^{-iL\xi}, \tag{3.61}$$

$$\lambda\hat{u}(\xi) + (i\xi + (i\xi)^3)\hat{\theta}(\xi) = -i\xi\theta'(L)e^{-iL\xi} + \theta''(0), \tag{3.62}$$

and hence

$$f(\xi) = \frac{1}{-i\xi^3 + i\xi + \lambda}\left(u''(0) + \theta''(0) + u'(0)i\xi - u''(L)e^{-iL\xi} - i\xi\theta'(L)e^{-iL\xi}\right), \tag{3.63}$$

$$g(\xi) = \frac{1}{i\xi^3 - i\xi + \lambda}\left(u''(0) - \theta''(0) + u'(0)i\xi - u''(L)e^{-iL\xi} + i\xi\theta'(L)e^{-iL\xi}\right). \tag{3.64}$$

Set $\lambda = -ip$. By Paley-Wiener theorem, we conclude that (3.58) is equivalent to the existence of numbers $p \in \mathbb{C}$ and $(\alpha, \alpha', \beta, \gamma, \gamma') \in \mathbb{C}^5 \setminus \{(0,0,0,0,0)\}$ such that the two functions defined for $\xi \in \mathbb{R}$ by

$$f(\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha + \beta i\xi + \gamma e^{-iL\xi} + \gamma'(i\xi)e^{-iL\xi}\right), \tag{3.65}$$

$$g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha' - \beta i\xi + \gamma e^{iL\xi} + \gamma'(i\xi)e^{iL\xi}\right) \tag{3.66}$$

fulfill the conditions

$$f \text{ and } g \text{ are entire;} \tag{3.67}$$

$$(f, g) \in L^2(\mathbb{R})^2; \tag{3.68}$$

$$\exists (C, N) \in (0, +\infty) \times \mathbb{N},\ \forall \xi \in \mathbb{C} \quad |f(\xi)| + |g(\xi)| \leq C(1 + |\xi|)^N e^{L|\mathrm{Im}\ \xi|}. \tag{3.69}$$

(We have set $\alpha := u''(0) + \theta''(0)$, $\alpha' := u''(0) - \theta''(0)$, $\beta := u'(0)$, $\gamma := -u''(L)$ and $\gamma' := -\theta'(L)$.) It is clear that for f and g given by (3.65)-(3.66), (3.67) implies both (3.68) and (3.69) (with $N = 1$). Clearly, (3.67) holds if and only if the two following functions

$$f(\xi) + g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha + \alpha' + 2(\gamma + \gamma' i\xi)\cos(L\xi)\right), \tag{3.70}$$

$$f(\xi) - g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha - \alpha' + 2\beta i\xi - 2i(\gamma + \gamma' i\xi)\sin(L\xi)\right) \tag{3.71}$$

are entire. Let $Q(\xi) = \xi^3 - \xi + p$ and let μ_0, μ_1, μ_2 denote its roots. The polynomial function Q cannot have a triple root (see the proof of Corollary 3.10), so either the roots μ_k, $k = 0, 1, 2$, are simple, or there are a double root $\mu_0 = \mu_1$ and a simple root $\mu_2 \neq \mu_0$.

1. Assume that the roots μ_0, μ_1, μ_2 of Q are simple. If $\xi \in \{\mu_0, \mu_1, \mu_2\}$, then from the fact that the functions in (3.70)-(3.71) are entire, we infer that

$$2(\gamma + \gamma' i\xi)\cos(L\xi) = -\alpha - \alpha', \tag{3.72}$$

$$2(\gamma + \gamma' i\xi)\sin(L\xi) = \frac{\alpha - \alpha'}{i} + 2\beta\xi, \tag{3.73}$$

and hence

$$4(\gamma + \gamma' i\xi)^2 = (\alpha + \alpha')^2 - (\alpha - \alpha')^2 + 4\beta^2\xi^2 + 4\frac{\beta\xi}{i}(\alpha - \alpha').$$

The polynomial functions in the two sides of the above equation are of degree two and they take the same values on the numbers μ_k, $k = 0, 1, 2$. Therefore, they must have the same coefficients; that is,

$$\beta^2 = -\gamma'^2, \tag{3.74}$$
$$\alpha\alpha' = \gamma^2, \tag{3.75}$$
$$\beta(\alpha - \alpha') = -2\gamma\gamma'. \tag{3.76}$$

(a) Assume that $\beta = i\gamma'$, so that (3.76) becomes

$$i\gamma'(\alpha - \alpha') = -2\gamma\gamma'. \tag{3.77}$$

(i) If $\gamma' = 0$, then $\beta = 0$ and (3.65)-(3.66) read

$$f(\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha + \gamma e^{-iL\xi}\right), \tag{3.78}$$
$$g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha' + \gamma e^{iL\xi}\right). \tag{3.79}$$

It follows from Lemma 3.5 that there exist $p \in \mathbb{C}$ and $(\alpha, \gamma) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that the function f defined in (3.78) is entire if and only if $L \in \mathcal{N}$. If it is the case, then the roots μ_0, μ_1, μ_2 of Q are simple and real, and $g(-\xi)$ defined in (3.79) is also an entire function if we pick $\alpha' = \overline{\alpha}\gamma/\overline{\gamma}$.
(ii) Assume now that $\gamma' \neq 0$. From (3.75) and (3.77), we infer that

$$\alpha' = -i\gamma, \quad \alpha = i\gamma.$$

Then (3.70)-(3.71) become

$$f(\xi) + g(-\xi) = \frac{2i}{\xi^3 - \xi + p}(\gamma + \gamma' i\xi)\cos(L\xi), \tag{3.80}$$
$$f(\xi) - g(-\xi) = -\frac{2}{\xi^3 - \xi + p}(\gamma + \gamma' i\xi)(1 - \sin(L\xi)). \tag{3.81}$$

• Assume that $-\frac{\gamma}{i\gamma'} \notin \{\mu_0, \mu_1, \mu_2\}$, then each root μ_j of Q should also be a root of $1 - \sin(L\xi)$ by (3.81), and hence it could be written as

$$L\mu_j = \frac{\pi}{2} + 2k_j\pi, \quad k_j \in \mathbb{Z}.$$

We arrive to the conclusion that

$$0 = L(\mu_0 + \mu_1 + \mu_2) = \pi(\frac{3}{2} + 2(k_0 + k_1 + k_2)),$$

which is impossible, for $2(k_0 + k_1 + k_2) \in 2\mathbb{Z}$.
• Assume that $-\frac{\gamma}{i\gamma'} \in \{\mu_0, \mu_1, \mu_2\}$, say $\mu_0 = -\gamma/(i\gamma')$. Then both $f(\xi) + g(-\xi)$ and $f(\xi) - g(-\xi)$ are entire if and only if the other roots μ_1 and μ_2 of Q satisfy both equations

$$\cos(L\xi) = 0 \text{ and } 1 - \sin(L\xi) = 0.$$

We arrive to the system

$$\mu_0 = -\frac{\gamma}{i\gamma'}, \tag{3.82}$$

$$L\mu_1 = \frac{\pi}{2} + 2k_1\pi, \quad k_1 \in \mathbb{Z}, \tag{3.83}$$

$$L\mu_2 = \frac{\pi}{2} + 2k_2\pi, \quad k_2 \in \mathbb{Z}. \tag{3.84}$$

With those expressions, μ_0, μ_1, μ_2 are roots of Q if and only if

$$\xi^3 - (\mu_0+\mu_1+\mu_2)\xi^2 + (\mu_0\mu_1+\mu_0\mu_2+\mu_1\mu_2)\xi - \mu_0\mu_1\mu_2 = \xi^3 - \xi + p, \quad \forall \xi \in \mathbb{C},$$

or

$$\mu_0+\mu_1+\mu_2 = 0, \tag{3.85}$$

$$\mu_0\mu_1+\mu_0\mu_2+\mu_1\mu_2 = -1, \tag{3.86}$$

$$-\mu_0\mu_1\mu_2 = p. \tag{3.87}$$

We note that μ_0 (and hence γ, if we pick $\gamma' = 1$) is defined in terms of k_1 and k_2 by (3.83)-(3.85), while p is defined in terms of k_1 and k_2 by (3.83)-(3.85) and (3.87). Replacing μ_0 by $-(\mu_1+\mu_2)$ in (3.86) results in

$$\mu_1^2 + \mu_1\mu_2 + \mu_2^2 = 1. \tag{3.88}$$

Substituting the expressions of μ_1 and μ_2 given in (3.83)-(3.84) in (3.88), we obtain the critical length

$$L = \pi\left((\frac{1}{2}+2k_1)^2 + (\frac{1}{2}+2k_2)^2 + (\frac{1}{2}+2k_1)(\frac{1}{2}+2k_2)\right)^{\frac{1}{2}}. \tag{3.89}$$

Finally, the roots μ_0, μ_1, μ_2 are pairwise distinct if and only if $k_1 \neq k_2$. Indeed, if $k_1 \neq k_2$, it is clear that $\mu_1 \neq \mu_2$, while

$$\mu_0 = -\frac{\pi}{L}(1+2(k_1+k_2)) \neq \mu_j = \frac{\pi}{L}(\frac{1}{2}+2k_j), \quad j = 1, 2.$$

(b) Assume now that $\beta = -i\gamma'$. The analysis is completely similar to the one in case (a), so that we only sketch the main facts. Here $i\gamma'(\alpha - \alpha') = 2\gamma\gamma'$.
If $\gamma' = 0$, we obtain again that the existence of a nontrivial solution $p \in \mathbb{C}$, $(\alpha, \alpha', \gamma) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ of the spectral problem is equivalent to $L \in \mathcal{N}$.
If $\gamma' \neq 0$, then we obtain that $\alpha' = i\gamma$, $\alpha = -i\gamma$, and

$$f(\xi) + g(-\xi) = \frac{2i}{\xi^3 - \xi + p}(\gamma + \gamma' i\xi)\cos(L\xi),$$

$$f(\xi) - g(-\xi) = \frac{2}{\xi^3 - \xi + p}(\gamma + \gamma' i\xi)(1 + \sin(L\xi)).$$

As in case (a), we can see that the assumption $-\frac{\gamma}{i\gamma'} \notin \{\mu_0, \mu_1, \mu_2\}$ leads to a contradiction. Assume thus that $-\frac{\gamma}{i\gamma'} \in \{\mu_0, \mu_1, \mu_2\}$, say $\mu_0 = -\gamma/(i\gamma')$. The other roots μ_1 and μ_2 should solve the equations

$$\cos(L\xi) = 0, \quad \sin(L\xi) = -1.$$

Writing

$$L\mu_1 = -\frac{\pi}{2} + 2k_1\pi,\ L\mu_2 = -\frac{\pi}{2} + 2k_2\pi, \qquad k_1 \neq k_2 \in \mathbb{Z},$$

we obtain as critical length

$$L = \pi\left((2k_1 - \frac{1}{2})^2 + (2k_2 - \frac{1}{2})^2 + (2k_1 - \frac{1}{2})(2k_2 - \frac{1}{2})\right)^{\frac{1}{2}},$$

and we notice that $L \in \mathcal{R}$.

2. Assume that Q has a double root $\mu_0 = \mu_1$ and a simple root $\mu_2 \neq \mu_0$. Then $(\mu_0, \mu_2) = \pm(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$. We shall consider the case $(\mu_0, \mu_2) = (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$, the case $(\mu_0, \mu_2) = (-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ being similar. As μ_0 is a double root of Q, it should be root of the numerators of the functions $f(\xi) + g(\xi)$ and $f(\xi) - g(-\xi)$ (see (3.70) and (3.71)) and of their first derivatives, so that

$$2(\gamma + \gamma'\frac{i}{\sqrt{3}})\cos(\frac{L}{\sqrt{3}}) = -(\alpha + \alpha'), \tag{3.90}$$

$$2i(\gamma + \gamma'\frac{i}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) = \alpha - \alpha' + 2\frac{\beta i}{\sqrt{3}}, \tag{3.91}$$

$$2\gamma' i\cos(\frac{L}{\sqrt{3}}) + 2(\gamma + i\frac{\gamma'}{\sqrt{3}})(-L\sin\frac{L}{\sqrt{3}}) = 0, \tag{3.92}$$

$$2\beta i + 2\gamma'\sin(\frac{L}{\sqrt{3}}) - 2i(\gamma + i\frac{\gamma'}{\sqrt{3}})L\cos(\frac{L}{\sqrt{3}}) = 0. \tag{3.93}$$

As μ_2 is a simple root of Q, it has to be a root of the numerators of the functions $f(\xi) + g(-\xi)$ and $f(\xi) - g(-\xi)$, so that

$$2(\gamma - \gamma'\frac{2i}{\sqrt{3}})\cos(\frac{2L}{\sqrt{3}}) = -(\alpha + \alpha'), \tag{3.94}$$

$$2i(\gamma - \gamma'\frac{2i}{\sqrt{3}})\sin(-2\frac{L}{\sqrt{3}}) = \alpha - \alpha' - 4\frac{\beta i}{\sqrt{3}}. \tag{3.95}$$

Note that (3.92)-(3.93) can be rewritten as

$$\begin{pmatrix} \cos(\frac{L}{\sqrt{3}}) & -\sin(\frac{L}{\sqrt{3}}) \\ \sin(\frac{L}{\sqrt{3}}) & \cos(\frac{L}{\sqrt{3}}) \end{pmatrix} \begin{pmatrix} i\gamma' \\ L(\gamma + i\frac{\gamma'}{\sqrt{3}}) \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}.$$

Picking $\beta = 1$, this yields

$$i\gamma' = \sin(\frac{L}{\sqrt{3}}), \tag{3.96}$$

$$L(\gamma + i\frac{\gamma'}{\sqrt{3}}) = \cos(\frac{L}{\sqrt{3}}). \tag{3.97}$$

In particular, $(i\gamma', \gamma) \in \mathbb{R}^2$. From (3.90) and (3.94), we infer that

$$(\gamma + \gamma'\frac{i}{\sqrt{3}})\cos(\frac{L}{\sqrt{3}}) = (\gamma - \gamma'\frac{2i}{\sqrt{3}})\cos(\frac{2L}{\sqrt{3}}).$$

Combined with (3.96) and (3.97), this yields

$$\cos^2(\frac{L}{\sqrt{3}}) = \left(\cos(\frac{L}{\sqrt{3}}) - \sqrt{3}L\sin(\frac{L}{\sqrt{3}})\right)\cos(\frac{2L}{\sqrt{3}}). \tag{3.98}$$

As the set of solutions of (3.98) cannot have a limit point, we conclude that on any segment $[0, R]$ there are at most finitely many L satisfying (3.98).

From (3.91) and (3.95), we infer that

$$2i(\gamma + \gamma'\frac{i}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) - 2\frac{\beta i}{\sqrt{3}} = 2i(\gamma - \gamma'\frac{2i}{\sqrt{3}})\sin(-2\frac{L}{\sqrt{3}}) + 4\frac{\beta i}{\sqrt{3}}.$$

Combined with (3.96) and (3.97), this yields (with $\beta = 1$)

$$L^{-1}\cos(\frac{L}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) = \left(L^{-1}\cos(\frac{L}{\sqrt{3}}) - \sqrt{3}\sin(\frac{L}{\sqrt{3}})\right)\sin(\frac{-2L}{\sqrt{3}}) + \sqrt{3}. \tag{3.99}$$

We claim that

$$\cos(\frac{L}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) \neq 0.$$

Indeed, otherwise we would have either $\cos(\frac{L}{\sqrt{3}}) = 0$ and then $\sin(\frac{L}{\sqrt{3}}) = \pm 1$, so that $L = 0$ by (3.98), which is impossible, or $\sin(\frac{L}{\sqrt{3}}) = 0$ which is impossible by (3.99).

We infer from (3.98) and (3.99) that

$$\cos(\frac{L}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) = \frac{\cos^2(\frac{L}{\sqrt{3}})}{\cos(\frac{2L}{\sqrt{3}})}\sin(\frac{-2L}{\sqrt{3}}) + \sqrt{3}L. \tag{3.100}$$

On the other hand, it follows from (3.98) that

$$\sqrt{3}L = \frac{1}{\sin(\frac{L}{\sqrt{3}})}\left(\cos(\frac{L}{\sqrt{3}}) - \frac{\cos^2(\frac{L}{\sqrt{3}})}{\cos(\frac{2L}{\sqrt{3}})}\right). \tag{3.101}$$

Replacing the last term in (3.100) by its expression in (3.101), multiplying the equation by $\cos(\frac{2L}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}})$ and simplifying by $\cos(\frac{L}{\sqrt{3}})$, we arrive to

$$\sin^2(\frac{L}{\sqrt{3}})\cos(\frac{2L}{\sqrt{3}}) = -2\cos^2(\frac{L}{\sqrt{3}})\sin^2(\frac{L}{\sqrt{3}}) + \cos(\frac{2L}{\sqrt{3}}) - \cos(\frac{L}{\sqrt{3}}),$$

or

$$\sin^2(\frac{L}{\sqrt{3}})\big(\cos(\frac{2L}{\sqrt{3}}) + 2\cos^2(\frac{L}{\sqrt{3}})\big) = -2\sin(\frac{3L}{2\sqrt{3}})\sin(\frac{L}{2\sqrt{3}}).$$

Letting $x := \frac{L}{2\sqrt{3}}$, we obtain that

$$\sin^2(2x)(4\cos^2(2x) - 1) = -2\sin(3x)\sin x = -2(3\sin x - 4\sin^3 x)\sin x.$$

Simplifying in the above equation by $\sin^2 x$ (which is different from 0 since $\sin(L/\sqrt{3})$ is), we arrive to

$$4(1 - \sin^2 x)(3 - 16\sin^2 x + 16\sin^4 x) = -2(3 - 4\sin^2 x).$$

Setting $y := \sin^2 x$, we obtain the polynomial equation

$$32y^3 - 64y^2 + 42y - 9 = 0$$

which has $y = 3/4$ as a double root and $y = 1/2$ as a single root. But $y = 1/2$ would give $\cos(L/\sqrt{3}) = \cos(2x) = 2\cos^2 x - 1 = 0$, which is impossible, and $y = 3/4$ would give $\cos(L/\sqrt{3}) = 1 - 2y = -1/2$, $\cos(2L/\sqrt{3}) = -1/2$, $\sin(L/\sqrt{3}) = \pm\sqrt{3}/2$. Plugging those values in (3.98) would give

$$\frac{1}{4} = \left(-\frac{1}{2} - \sqrt{3}L \cdot (\pm\frac{\sqrt{3}}{2})\right) \cdot (-\frac{1}{2})$$

and hence $L = 0$, which is impossible. The proof of Proposition 3.13 and of Corollary 3.12 is complete. □

Next, we investigate the spectral problems associated with cases 10 and 11.

Corollary 3.14. *Let $L > 0$. Then there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ solution of (3.54)-(3.57) together with $u''(0) = 0$ if and only if $L \in \mathcal{R}$.*

Proof. We use the same notations as in the proof of Proposition 3.13. The additional condition $u''(0) = 0$ is equivalent to the condition $\alpha' = -\alpha$. When $\gamma' \neq 0$, we obtained that $(\alpha, \alpha') = (i\gamma, -i\gamma)$ for $\beta = i\gamma'$ (case (a)), and that $(\alpha, \alpha') = (-i\gamma, i\gamma)$ for $\beta = -i\gamma'$ (case (b)). Thus the condition $\alpha' = -\alpha$ is automatically satisfied for the critical lengths $L \in \mathcal{R}$.

Consider now the case $\gamma' = 0$ with $\beta = \pm i\gamma' = 0$ corresponding to the critical length $L \in \mathcal{N}$. From [30, Lemma 3.5], we know that the three roots μ_0, μ_1, μ_2 of Q can be written as

$$\mu_0 = -\frac{1}{3}(2k + l)\frac{2\pi}{L}, \quad \mu_1 = \mu_0 + k\frac{2\pi}{L}, \quad \mu_2 = \mu_1 + l\frac{2\pi}{L}, \tag{3.102}$$

for some $k, l \in \mathbb{N}^*$. From (3.70), we know that the μ_j have to be zeros of the function $\gamma\cos(L\xi)$. If $\gamma = 0$, then we infer from (3.65)-(3.66) that $0 = \alpha = \alpha'$, and hence $(\theta^0, u^0) = (0, 0)$. If $\gamma \neq 0$, from $\cos(L\mu_0) = 0$, we infer the existence of some $p \in \mathbb{Z}$ such that $L\mu_0 = \pi(\frac{1}{2} + p)$, so that $\frac{1}{2} + p = -2(2k+l)/3$. This gives $3/2 = -3p - 2(2k+l)$ where $k, l, p \in \mathbb{Z}$, which is impossible. □

Corollary 3.15. *Let $L > 0$. Then there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ solution of (3.54)-(3.57) together with $\theta''(0) = 0$ if and only if $L \in \mathcal{N}_3$.*

Proof. The additional condition $\theta''(0) = 0$ is equivalent to $\alpha' = \alpha$.
If $\gamma' \neq 0$, then we have that $\alpha' = -\alpha = \pm i\gamma$, so that $\alpha = \alpha' = \gamma = 0$. If $\beta = i\gamma'$ and $0 = -\frac{\gamma}{i\gamma'} \in \{\mu_0, \mu_1, \mu_2\}$, say $\mu_0 = 0$, then by (3.82)-(3.84)

$$0 = -L\mu_0 = L(\mu_1 + \mu_2) = \pi(1 + 2(k_1 + k_2))$$

with $k_1, k_2 \in \mathbb{Z}$, which is impossible. If $\beta = -i\gamma'$ and $0 = -\frac{\gamma}{i\gamma'} = \mu_0$, we arrive to

$$0 = -L\mu_0 = L(\mu_1 + \mu_2) = \pi(-1 + 2(k_1 + k_2))$$

with $k_1, k_2 \in \mathbb{Z}$, which again is impossible.
Thus no $L \in \mathcal{R}$ is a critical length for (3.54)-(3.57) with the additional condition $\theta''(0) = 0$.
If $\gamma' = 0$, then $\beta = 0$ by (3.74), and (3.78)-(3.79) hold with $\alpha' = \alpha$. For each root $\xi \in \{\mu_0, \mu_1, \mu_2\}$ of Q, we have that

$$\alpha + \gamma e^{-iL\xi} = 0 = \alpha + \gamma e^{iL\xi}.$$

This is equivalent to (3.102) together with the the condition $2L\mu_j \in 2\pi\mathbb{Z}$ for $j = 0, 1, 2$. With (3.102), the latter condition is equivalent to $2L\mu_0 \in 2\pi\mathbb{Z}$, i.e. there exists $q \in \mathbb{Z}$ such that

$(-(2k+l)2\pi/(3L))2L = 2\pi q$, or $3q = -2(2k+l)$. This is possible if and only if $3|2k+l$, i.e. $L \in \mathcal{N}_3$. □

We now turn our attention to case 3.

Theorem 3.16. *For all $L \in (0,+\infty) \setminus (\mathcal{N} \cup \mathcal{G} \cup \mathcal{G}')$ and all $T > 0$, there exists a constant $C = C(L,T) > 0$ such that*

$$\|(\theta^0, u^0)\|^2_{X_2} \leq C \int_0^T |\theta_{xx}(t,0)|^2 \, dt. \tag{3.103}$$

Proof of Theorem 3.16: We proceed in two steps as in the proof of Theorem 3.7.

Step 1. Reduction to a spectral problem.

If (3.103) is false, then one can pick a sequence $(\theta^0_n, u^0_n)_{n\in\mathbb{N}}$ such that

$$1 = \|(\theta^0_n, u^0_n)\|^2_{X_2} > n \int_0^T |\theta_{n,xx}(t,0)|^2 \, dt. \tag{3.104}$$

Extracting a subsequence if needed, we can assume that $(\theta^0_n, u^0_n) \to (\theta^0, u^0)$ weakly in X_2. Denote by (θ_n, u_n) (resp. (θ, u)) the solution of (2.18) with initial data (θ^0_n, u^0_n) (resp. (θ^0, u^0)). Extracting again a subsequence, we can assume that $(\theta^0_n, u^0_n) \to (\theta^0, u^0)$ in X_1. By (3.51) and (3.104), we have that $(\theta^0_n, u^0_n) \to (\theta^0, u^0)$ strongly in X_2. Thus $\|(\theta^0, u^0)\|_{X_2} = 1$ and $\theta_{xx}(\cdot, 0) = 0$ a.e. in $(0,T)$. The same argument as the one in Step 1 of the proof of Theorem 3.7 shows that there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ (superscript 0 dropped for simplicity) which are solution of the spectral problem:

$$-u' - u''' = \lambda\theta, \qquad x \in (0,L), \tag{3.105}$$

$$-\theta' - \theta''' = \lambda u, \qquad x \in (0,L), \tag{3.106}$$

$$\theta(0) = \theta(L) = \theta'(0) = \theta''(0) = 0, \tag{3.107}$$

$$u(0) = u(L) = u'(L) = 0. \tag{3.108}$$

Proposition 3.17. *Let $L > 0$. Then there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ solution of (3.105)-(3.108) if and only if $L \in \mathcal{N} \cup \mathcal{G} \cup \mathcal{G}'$.*

Proof. We use the same notations as those in the proof of Proposition 3.13. Consider the property:

$$\exists \lambda \in \mathbb{C}, \ \exists (\theta, u) \in X_3 \setminus \{(0,0)\}, \quad \text{(3.105)-(3.108) hold.} \tag{3.109}$$

Note that (3.105)-(3.106) yields

$$\lambda\theta + u' + u''' = u''(0)\delta_0 - u''(L)\delta_L + u'(0)\delta_0' \quad \text{in } \mathcal{S}'(\mathbb{R}), \tag{3.110}$$

$$\lambda u + \theta' + \theta''' = -\theta''(L)\delta_L - \theta'(L)\delta_L' \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{3.111}$$

Conversely, if a pair (θ, u) in $L^2(\mathbb{R})^2$ with $\operatorname{Supp}\theta \cup \operatorname{Supp} u \subset [0,L]$ satisfies (3.110)-(3.111) for some $\lambda \in \mathbb{C}$ and some complex coefficients (instead of $u''(0), u''(L)$, etc.) in the r.h.s. of (3.110)-(3.111), then $\theta, u \in H^3(0,L)$ and (3.105)-(3.108) hold.

Let $f(\xi) := \hat{\theta}(\xi) + \hat{u}(\xi)$ and $g(\xi) := \hat{\theta}(\xi) - \hat{u}(\xi)$. Then (3.110)-(3.111) yield

$$\lambda\hat{\theta}(\xi) + (i\xi + (i\xi)^3)\hat{u}(\xi) = u''(0) - u''(L)e^{-iL\xi} + u'(0)i\xi, \tag{3.112}$$

$$\lambda\hat{u}(\xi) + (i\xi + (i\xi)^3)\hat{\theta}(\xi) = -\theta''(L)e^{-iL\xi} - i\xi\theta'(L)e^{-iL\xi}, \tag{3.113}$$

and hence

$$f(\xi) = \frac{1}{-i\xi^3 + i\xi + \lambda}\left(u''(0) - (u''(L) + \theta''(L))e^{-iL\xi} + u'(0)i\xi - i\xi\theta'(L)e^{-iL\xi}\right), \tag{3.114}$$

$$g(\xi) = \frac{1}{i\xi^3 - i\xi + \lambda}\left(u''(0) + (-u''(L) + \theta''(L))e^{-iL\xi} + u'(0)i\xi + i\xi\theta'(L)e^{-iL\xi}\right). \tag{3.115}$$

Set $\lambda = -ip$. By Paley-Wiener theorem, we conclude that (3.109) is equivalent to the existence of numbers $p \in \mathbb{C}$ and $(\alpha, \beta, \gamma_1, \gamma_2, \gamma') \in \mathbb{C}^5 \setminus \{(0,0,0,0,0)\}$ such that the two functions defined for $\xi \in \mathbb{R}$ by

$$f(\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha + \beta i\xi + (\gamma_1 + \gamma_2)e^{-iL\xi} + \gamma'(i\xi)e^{-iL\xi}\right), \tag{3.116}$$

$$g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha - \beta i\xi + (\gamma_1 - \gamma_2)e^{iL\xi} + \gamma'(i\xi)e^{iL\xi}\right) \tag{3.117}$$

fulfill the conditions

$$f \text{ and } g \text{ are entire;} \tag{3.118}$$

$$(f, g) \in L^2(\mathbb{R})^2; \tag{3.119}$$

$$\exists (C, N) \in (0, +\infty) \times \mathbb{N},\ \forall \xi \in \mathbb{C} \quad |f(\xi)| + |g(\xi)| \leq C(1 + |\xi|)^N e^{L|\mathrm{Im}\ \xi|}. \tag{3.120}$$

(We have set $\alpha := u''(0)$, $\beta := u'(0)$, $\gamma_1 := -u''(L)$, $\gamma_2 := -\theta''(L)$ and $\gamma' := -\theta'(L)$.) It is clear that for f and g as in (3.116)-(3.117), the condition (3.118) implies both (3.119) and (3.120) (with $N = 1$). Clearly, (3.118) holds if and only if the two following functions

$$f(\xi) + g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(2\alpha - 2\gamma_2 i\sin(L\xi) + 2(\gamma_1 + \gamma' i\xi)\cos(L\xi)\right), \tag{3.121}$$

$$f(\xi) - g(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(2\beta i\xi + 2\gamma_2\cos(L\xi) - 2i(\gamma_1 + \gamma' i\xi)\sin(L\xi)\right) \tag{3.122}$$

are entire. Let $Q(\xi) = \xi^3 - \xi + p$ and let μ_0, μ_1, μ_2 denote the roots of Q. The polynomial function Q cannot have a triple root, so either the roots μ_k, $k = 0, 1, 2$, are simple, or there are a double root $\mu_0 = \mu_1$ and a simple root $\mu_2 \neq \mu_0$.

1. Assume that Q has three simple roots. Let ξ be a root of Q. Setting

$$a := \gamma_1 + \gamma' i\xi, \quad b := i\gamma_2,$$

we infer from the fact that the numerators of the r.h.s. of (3.121) and (3.122) vanish at ξ that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \cos L\xi \\ \sin L\xi \end{pmatrix} = \begin{pmatrix} -\alpha \\ \beta\xi \end{pmatrix}.$$

This yields

$$\cos(L\xi) = \frac{-\alpha a + b\beta\xi}{a^2 + b^2}, \quad \sin(L\xi) = \frac{a\beta\xi + \alpha b}{a^2 + b^2}.$$

Expanding in the identity $\cos^2(L\xi)+\sin^2(L\xi)=1$ gives $a^2+b^2=\alpha^2+\beta^2\xi^2$, i.e.

$$-(\gamma')^2\xi^2+2\gamma_1\gamma' i\xi+\gamma_1^2-\gamma_2^2=\alpha^2+\beta^2\xi^2.$$

Since Q has three different roots, we arrive at

$$-(\gamma')^2 = \beta^2, \tag{3.123}$$
$$\gamma_1\gamma' = 0, \tag{3.124}$$
$$\gamma_1^2-\gamma_2^2 = \alpha^2. \tag{3.125}$$

a. If $\gamma'=0$, then $\beta=0$, and the existence of $p\in\mathbb{C}$ and of $(\alpha,\gamma_1,\gamma_2)\in\mathbb{C}^3\setminus\{(0,0,0)\}$ such that both $f(\xi)$ and $g(-\xi)$ are analytic in $\mathbb{C}$ is equivalent to the fact that $L\in\mathcal{N}$, by Lemma 3.5.
b. If $\gamma'\neq 0$, then $\gamma_1=0$, $\beta=\pm i\gamma'\neq 0$ and $\alpha=\pm i\gamma_2$.

Case 1. $\beta=i\gamma'$ and $\alpha=i\gamma_2$.
Then

$$f(\xi)=\frac{i}{\xi^3-\xi+p}(i+e^{-iL\xi})(\gamma_2+\gamma'(i\xi)),\quad g(-\xi)=\frac{i}{\xi^3-\xi+p}(i-e^{iL\xi})(\gamma_2-\gamma'(i\xi)).$$

If, each root μ_j $(0\leq j\leq 2)$ of Q is such that $\gamma_2+i\mu_j\gamma'\neq 0$, then

$$e^{-iL\mu_j}=-i=e^{-i\frac{\pi}{2}},$$

and hence $L\mu_j=\frac{\pi}{2}+2k_j\pi$, for some $k_j\in\mathbb{Z}$. This contradicts the fact that $\mu_0+\mu_1+\mu_2=0$. Therefore, there exists some root μ_j of Q, say μ_0, such that $\gamma_2+i\mu_0\gamma'=0$. In the same way, we can prove that some root μ_k of Q should satisfy $\gamma_2-i\mu_k\gamma'=0$. If $\mu_k=\mu_0$, then $\gamma_2=\mu_0=0$, and hence $p=0$ and $\{\mu_1,\mu_2\}=\{\pm 1\}$. Then we should have $i+e^{-iL\mu_1}=i+e^{-iL\mu_2}=0$, i.e $e^{iL}=e^{-iL}=-i$, and this is impossible. If $\mu_k\neq\mu_0$, say $\mu_k=\mu_1$, then $\mu_1=\gamma_2/(i\gamma')=-\mu_0$, and thus $\mu_2=-(\mu_0+\mu_1)=0$. But then $i+e^{-iL\mu_2}=i\neq 0$, and μ_2 is a pole of f, which yields a contradiction.

Case 2. $\beta=-i\gamma'$ and $\alpha=-i\gamma_2$.
It is similar to Case 1 and is impossible.

Case 3. $\beta=i\gamma'$ and $\alpha=-i\gamma_2$.
Then

$$f(\xi) = \frac{i}{\xi^3-\xi+p}\left(-i\gamma_2+i\gamma'(i\xi)+\gamma_2e^{-iL\xi}+\gamma'(i\xi)e^{-iL\xi}\right), \tag{3.126}$$
$$g(-\xi) = \frac{i}{\xi^3-\xi+p}\left(-i\gamma_2-i\gamma'(i\xi)-\gamma_2e^{iL\xi}+\gamma'(i\xi)e^{iL\xi}\right). \tag{3.127}$$

If $\gamma_2=0$, then

$$f(\xi)+g(-\xi)=\frac{-2\gamma'}{\xi^3-\xi+p}\xi\cos(L\xi),\quad f(\xi)-g(-\xi)=-\frac{2\gamma' i\xi}{\xi^3-\xi+p}(1-\sin(L\xi)).$$

If $p\neq 0$, then $0\notin\{\mu_0,\mu_1,\mu_2\}$ and $\cos(L\mu_j)=0$ yields $L\mu_j=(\pi/2)+2k_j\pi$ with $k_j\in\mathbb{Z}$ for $k=0,1,2$, contradicting the fact that $\mu_0+\mu_1+\mu_2=0$.
If $p=0$, then we have (after relabeling) $\mu_0=0$, $\mu_1=1$ and $\mu_2=-1$. Then both μ_1 and

$\mu_2 = -\mu_1$ should solve $1 - \sin(L\xi) = 0$, which is impossible. We conclude that $\gamma_2 \neq 0$ when $\beta = i\gamma' \neq 0$ and $\alpha = -i\gamma_2$.

Dividing in (3.126)-(3.127) by γ', we can assume that $\gamma' = 1$. Each root ξ of Q should satisfy

$$\gamma_2(-i + e^{-iL\xi}) + i\xi(i + e^{-iL\xi}) = 0, \tag{3.128}$$

$$\gamma_2(-i - e^{iL\xi}) + i\xi(-i + e^{iL\xi}) = 0. \tag{3.129}$$

Then system (3.128)-(3.129) is equivalent to

$$e^{-iL\xi} = \frac{i\gamma_2 + \xi}{\gamma_2 + i\xi}, \tag{3.130}$$

$$e^{iL\xi} = \frac{i\gamma_2 - \xi}{-\gamma_2 + i\xi}. \tag{3.131}$$

(Note that both $\gamma_2 + i\xi$ and $-\gamma_2 + i\xi$ are different from 0, otherwise we would infer from (3.128)-(3.129) that $\gamma_2 = 0$.) We notice that (3.130) and (3.131) are equivalent, so that we can focus on (3.131). Using basic algebra, we have that

$$e^{iL\xi} = \frac{i\gamma_2 - \xi}{-\gamma_2 + i\xi} \iff ie^{iL\xi} = 1 + \frac{2i\xi}{\gamma_2 - i\xi}$$
$$\iff \frac{2i\xi}{ie^{iL\xi} - 1} + i\xi = \gamma_2 = Const.$$

Thus, the existence of $p \in \mathbb{C}$, $\gamma_2 \in \mathbb{C}^*$ and μ_0, μ_1, μ_2 such that (3.85)-(3.87) hold together with

$$\frac{2i\mu_0}{ie^{iL\mu_0} - 1} + i\mu_0 = \frac{2i\mu_1}{ie^{iL\mu_1} - 1} + i\mu_1 = \frac{2i\mu_2}{ie^{iL\mu_2} - 1} + i\mu_2 = \gamma_2$$

is reduced, by letting $a := iL\mu_0$, $b := iL\mu_1$, to the existence of $a, b \in \mathbb{C}$ such that

$$\frac{2a}{ie^a - 1} + a = \frac{2b}{ie^b - 1} + b = \frac{2(-a-b)}{ie^{-a-b} - 1} - a - b \neq 0,$$
$$L^2 = -(a^2 + ab + b^2).$$

Thus, this is possible if and only if $L \in \mathcal{G}$.

Case 4. $\beta = -i\gamma'$ and $\alpha = i\gamma_2$.
It is similar to Case 3. We find, instead of (3.130)-(3.131), the system

$$-e^{-iL\xi} = \frac{i\gamma_2 + \xi}{\gamma_2 + i\xi},$$

$$-e^{iL\xi} = \frac{i\gamma_2 - \xi}{-\gamma_2 + i\xi}$$

which can be reduced to its second equation equivalent to

$$\frac{2i\xi}{-ie^{iL\xi} - 1} + i\xi = \gamma_2 = Const.$$

Introducing again $a := iL\mu_0$, $b = iL\mu_1$, we see that the existence of $p \in \mathbb{C}$, $\gamma_2 \in \mathbb{C}^*$ and μ_0, μ_1, μ_2 (the roots of Q) holds if and only if $L \in \mathcal{G}'$.

2. Assume that Q has a double root $\mu_0 = \mu_1$ and a simple root $\mu_2 \neq \mu_0$. Then $(\mu_0, \mu_2) = \pm(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$. We shall consider the case $(\mu_0, \mu_2) = (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$, the case $(\mu_0, \mu_2) = (-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ being similar. As μ_0 is a double root of Q, it should be a root of the numerators of the functions $f(\xi) + g(-\xi)$ and $f(\xi) - g(-\xi)$ (see (3.121) and (3.122)) and of their first derivatives, so that

$$-\gamma_2 i \sin(\frac{L}{\sqrt{3}}) + (\gamma_1 + \gamma' \frac{i}{\sqrt{3}}) \cos(\frac{L}{\sqrt{3}}) = -\alpha, \tag{3.132}$$

$$\gamma_2 \cos(\frac{L}{\sqrt{3}}) - i(\gamma_1 + \gamma' \frac{i}{\sqrt{3}}) \sin(\frac{L}{\sqrt{3}}) = -\beta \frac{i}{\sqrt{3}}, \tag{3.133}$$

$$(-\gamma_2 i L + \gamma' i) \cos(\frac{L}{\sqrt{3}}) - L(\gamma_1 + \frac{i\gamma'}{\sqrt{3}}) \sin(\frac{L}{\sqrt{3}}) = 0, \tag{3.134}$$

$$(\gamma' - \gamma_2 L) \sin(\frac{L}{\sqrt{3}}) - i(\gamma_1 + \frac{i\gamma'}{\sqrt{3}}) L \cos(\frac{L}{\sqrt{3}}) = -\beta i. \tag{3.135}$$

As μ_2 is a simple root of Q, it has to be a root of the numerators of the functions $f(\xi) + g(-\xi)$ and $f(\xi) - g(-\xi)$, so that

$$-\gamma_2 i \sin(-\frac{2L}{\sqrt{3}}) + (\gamma_1 - \gamma' \frac{2i}{\sqrt{3}}) \cos(\frac{2L}{\sqrt{3}}) = -\alpha, \tag{3.136}$$

$$\gamma_2 \cos(\frac{2L}{\sqrt{3}}) - i(\gamma_1 - \gamma' \frac{2i}{\sqrt{3}}) \sin(-2\frac{L}{\sqrt{3}}) = 2\frac{\beta i}{\sqrt{3}}. \tag{3.137}$$

Note that (3.134)-(3.135) can be rewritten as

$$\begin{pmatrix} \cos(\frac{L}{\sqrt{3}}) & -\sin(\frac{L}{\sqrt{3}}) \\ \sin(\frac{L}{\sqrt{3}}) & \cos(\frac{L}{\sqrt{3}}) \end{pmatrix} \begin{pmatrix} -\gamma_2 i L + i\gamma' \\ L(\gamma_1 + i\frac{\gamma'}{\sqrt{3}}) \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}.$$

If $\beta = 0$, then we obtain $-\gamma_2 i L + i\gamma' = \gamma_1 + i\frac{\gamma'}{\sqrt{3}} = 0$, and (3.133) gives $\gamma_2 \cos(L/\sqrt{3}) = 0$. Then either $\cos(L/\sqrt{3}) \neq 0$ and $\gamma_2 = 0$, or $\cos(L/\sqrt{3}) = 0$ and (3.137) gives again $\gamma_2 = 0$. Thus $0 = \gamma_2 = \gamma' = \gamma_1$, and (3.132) yields $\alpha = 0$, so that $(\theta, u) = (0, 0)$, which is impossible.

Assume now that $\beta \neq 0$. Picking $\beta = 1$, we obtain

$$-\gamma_2 i L + i\gamma' = \sin(\frac{L}{\sqrt{3}}), \tag{3.138}$$

$$L(\gamma_1 + i\frac{\gamma'}{\sqrt{3}}) = \cos(\frac{L}{\sqrt{3}}). \tag{3.139}$$

We can compute all the coefficients in (3.132)-(3.137) in terms of L. Indeed, we infer from (3.133) and (3.139) that $\cos(L/\sqrt{3}) \neq 0$ and

$$\gamma_2 = \frac{i}{L} \sin(\frac{L}{\sqrt{3}}) - \frac{i}{\sqrt{3} \cos(\frac{L}{\sqrt{3}})}.$$

Combined with (3.138), this yields

$$\gamma' = L\gamma_2 - i \sin(\frac{L}{\sqrt{3}}) = -\frac{iL}{\sqrt{3} \cos(\frac{L}{\sqrt{3}})}.$$

It follows that

$$\gamma_1 - \gamma' \frac{2i}{\sqrt{3}} = (\gamma_1 + \frac{i\gamma'}{\sqrt{3}}) - \sqrt{3} i \gamma' = \frac{1}{L}\cos(\frac{L}{\sqrt{3}}) - \frac{L}{\cos(\frac{L}{\sqrt{3}})}.$$

From (3.132) and (3.136), we infer that

$$-\gamma_2 i \sin(\frac{L}{\sqrt{3}}) + (\gamma_1 + \gamma' \frac{i}{\sqrt{3}})\cos(\frac{L}{\sqrt{3}}) = \gamma_2 i \sin(\frac{2L}{\sqrt{3}}) + (\gamma_1 - \gamma' \frac{2i}{\sqrt{3}})\cos(\frac{2L}{\sqrt{3}}). \tag{3.140}$$

Substituting the values of γ_2, $\gamma_1 + \gamma' \frac{i}{\sqrt{3}}$ and $\gamma_1 - \gamma' \frac{2i}{\sqrt{3}}$ in (3.140), we obtain the equation

$$i\left(\frac{i}{L}\sin(\frac{L}{\sqrt{3}}) - \frac{i}{\sqrt{3}\cos(\frac{L}{\sqrt{3}})}\right)\cdot\left(\sin(\frac{L}{\sqrt{3}}) + \sin(\frac{2L}{\sqrt{3}})\right) + \left(\frac{1}{L}\cos(\frac{L}{\sqrt{3}}) - \frac{L}{\cos(\frac{L}{\sqrt{3}})}\right)\cos(\frac{2L}{\sqrt{3}}) - \frac{1}{L}\cos^2(\frac{L}{\sqrt{3}}) = 0.$$

Simplifying the above equation, we arrive to

$$\underbrace{\frac{1}{L}\cos(\frac{L}{\sqrt{3}})\left(\cos(\frac{L}{\sqrt{3}})(1 - 4\sin^2(\frac{L}{\sqrt{3}})) - 1\right) - L\cos(\frac{2L}{\sqrt{3}}) + \frac{1}{\sqrt{3}}\left(\sin(\frac{L}{\sqrt{3}}) + \sin(\frac{2L}{\sqrt{3}})\right)}_{h(L)} = 0.$$

From (3.133) and (3.137), we infer that

$$-2\gamma_2\cos(\frac{L}{\sqrt{3}}) + 2i(\gamma_1 + \gamma' \frac{i}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) = \gamma_2\cos(\frac{2L}{\sqrt{3}}) + i(\gamma_1 - \gamma' \frac{2i}{\sqrt{3}})\sin(\frac{2L}{\sqrt{3}}). \tag{3.141}$$

Substituting the values of γ_2, $\gamma_1 + \gamma' \frac{i}{\sqrt{3}}$ and $\gamma_1 - \gamma' \frac{2i}{\sqrt{3}}$ in (3.141), we obtain the equation

$$\left(\frac{i}{L}\sin(\frac{L}{\sqrt{3}}) - \frac{i}{\sqrt{3}\cos(\frac{L}{\sqrt{3}})}\right)\cdot\left(\cos(\frac{2L}{\sqrt{3}}) + 2\cos(\frac{L}{\sqrt{3}})\right) + i\left(\frac{1}{L}\cos(\frac{L}{\sqrt{3}}) - \frac{L}{\cos(\frac{L}{\sqrt{3}})}\right)\sin(\frac{2L}{\sqrt{3}}) - \frac{2i}{L}\cos(\frac{L}{\sqrt{3}})\sin(\frac{L}{\sqrt{3}}) = 0.$$

After some simplifications, we arrive to

$$\underbrace{(\frac{1}{2L} + L)\sin(\frac{2L}{\sqrt{3}}) + \frac{1}{\sqrt{3}}\left(\cos(\frac{2L}{\sqrt{3}}) + 2\cos(\frac{L}{\sqrt{3}})\right)}_{k(L)} = 0.$$

Let $\zeta(L) := h(L)^2 + k(L)^2$. Then $\zeta(L) = L^2 + O(L)$ as $L \to +\infty$, so that the equation $\zeta(L) = 0$ has no root $L \gg 1$. Using a numerical computation, we see that $\inf_{L>0} \zeta(L) \approx 5.3333 > 0$. Thus the system

$$\begin{cases} h(L) = 0, \\ k(L) = 0 \end{cases}$$

has no solution $L > 0$. Therefore, there is no critical length in the case of a double root. The proof of Proposition 3.17 and of Theorem 3.16 is complete. □

Next, we investigate the spectral problem associated with case 12.

Corollary 3.18. *Let $L > 0$. Then there exist $\lambda \in \mathbb{C}$ and $(\theta, u) \in X_3 \setminus \{(0,0)\}$ solution of (3.105)-(3.108) together with $u''(L) = 0$ if and only if $L \in \mathcal{G} \cup \mathcal{G}'$.*

Proof. We use the same notations as in the proof of Proposition 3.17. The extra condition $u''(L) = 0$ means that $\gamma_1 = 0$. As for Proposition 3.17, the case when Q has multiple roots is impossible. We therefore assume that Q has three different roots.
a. If $\gamma' = 0$, then $\beta = 0$ and for each root ξ of Q, we have that

$$\alpha + \gamma_2 e^{-iL\xi} = 0 = \alpha - \gamma_2 e^{iL\xi}.$$

We assume $\gamma_2 \neq 0$ (otherwise all the coefficients are zero), and we arrive to $e^{2iL\xi} = -1$ and $2L\xi = \pi + 2\pi k$, $k \in \mathbb{Z}$. Since the sum of the three roots of Q is 0, we arrive to a contradiction.
b. If $\gamma' \neq 0$, then we have (still) $\gamma_1 = 0$, $\beta = \pm i\gamma' \neq 0$ and $\alpha = \pm i\gamma_2$, a case already considered in the proof of Proposition 3.17. The cases 1 and 2 have to be excluded, while the cases 3 and 4 are valid and they lead to $L \in \mathcal{G} \cup \mathcal{G}'$. □

4. Proof of Theorems 1.1 and 1.2.

We shall first investigate the wellposedness of system (1.20) when picking $g_2(t) = -\alpha\eta_x(t, L)$ as control input, with $\alpha > 0$. We first notice that a *global Kato smoothing effect holds*, thanks to which we can derive the wellposedness of the system in the energy space $X_0 = [L^2(0, L)]^2$.

Let us first pay our attention to the linearized system

$$\begin{cases} \eta_t + v_x + v_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ v_t + \eta_x + \eta_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ \eta(t,0) = 0,\ \eta(t,L) = 0,\ \eta_x(t,0) = 0, & t \in (0,T), \\ v(t,0) = 0,\ v(t,L) = 0,\ v_x(t,L) = -\alpha\eta_x(t,L), & t \in (0,T), \\ \eta(0,x) = \eta^0(x),\ v(0,x) = v^0(x), & x \in (0,L). \end{cases} \tag{4.1}$$

We introduce the operator

$$\tilde{A}(\eta, v) := (-v_x - v_{xxx}, -\eta_x - \eta_{xxx})$$

with domain

$$D(\tilde{A}) := \{(\eta, v) \in H^3(0,L)^2;\ \eta(0) = \eta(L) = \eta_x(0) = v(0) = v(L) = 0,\ v_x(L) = -\alpha\eta_x(L)\} \subset X_0.$$

Proposition 4.1. *Assume that $\alpha > 0$. Then the operator $\tilde{A}$ generates a semigroup of contractions $(e^{t\tilde{A}})_{t\geq 0}$ in X_0. Furthermore, for any $T > 0$ and any $(\eta^0, v^0) \in X_0$, the solution $(\eta, v) := e^{t\tilde{A}}(\eta^0, v^0)$ of (4.1) belongs to $L^2(0, T, [H^1(0,L)]^2)$ and we have*

$$\int_0^T \int_0^L (\eta_x^2 + v_x^2)\, dxdt \leq \frac{2}{3}\left(L + \frac{T}{2} + \frac{L(\alpha^2+1)}{4\alpha}\right) \|(\eta^0, v^0)\|_{X_0}^2. \tag{4.2}$$

Proof. It is clear that $D(\tilde{A})$ is dense in X_0 and that $\tilde{A}$ is closed. Let $(\eta, v) \in D(\tilde{A})$. Then we readily obtain that

$$(\tilde{A}(\eta, v), (\eta, v))_{X_0} = -\alpha\eta_x(L)^2 \leq 0,$$

so that $\tilde{A}$ is a dissipative operator. Introduce the operator $B(\theta,u) := (u_x+u_{xxx},\theta_x+\theta_{xxx})$ with domain

$$D(B) := \{(\theta,u)\in H^3(0,L)^2;\ \theta(0)=\theta(L)=\theta_x(0)=u(0)=u(L)=0,\ u_x(L)=\alpha\theta_x(L)\}\subset X_0.$$

We easily obtain that $(B(\theta,u),(\theta,u))_{X_0} = -\alpha\theta_x(L)^2 \le 0$, so that B is also a dissipative operator. We claim that $B=\tilde{A}^*$. First, for all $(\eta,v)\in D(\tilde{A})$ and all $(\theta,u)\in D(B)$, we have by integration by parts that

$$(\tilde{A}(\eta,v),(\theta,u))_{X_0} = ((\eta,v),B(\theta,u))_{X_0},$$

which shows that $B\subset\tilde{A}^*$.

Let us now check that $\tilde{A}^*\subset B$. Pick any $(\theta,u)\in D(\tilde{A}^*)$. Then, we have for some constant $C>0$ that

$$\left|((\theta,u),\tilde{A}(\eta,v))_{X_0}\right| \le C\|(\eta,v)\|_{X_0}\quad \forall(\eta,v)\in D(\tilde{A}),$$

i.e.

$$\left|\int_0^L[\theta(v_x+v_{xxx})+u(\eta_x+\eta_{xxx})]dx\right| \le C\left(\int_0^L[\eta^2+v^2]dx\right)^{\frac12}\quad \forall(\eta,v)\in D(\tilde{A}). \tag{4.3}$$

Picking $v=0$ and $\eta\in C_c^\infty(0,L)$, we infer from (2.7) that $u_x+u_{xxx}\in L^2(0,L)$, and hence that $u\in H^3(0,L)$. Similarly, we obtain that $\theta\in H^3(0,L)$. Integrating by parts in the left hand side of (4.3), we obtain that

$$\begin{aligned}&|\theta(L)v_{xx}(L)-\theta(0)v_{xx}(0)+\alpha\theta_x(L)\eta_x(L)+\theta_x(0)v_x(0)+u(L)\eta_{xx}(L)-u(0)\eta_{xx}(0)-u_x(L)\eta_x(L)|\\ &\quad\le C\left(\int_0^L[\eta^2+v^2]dx\right)\qquad\forall(\eta,v)\in D(\tilde{A}).\end{aligned}$$

It follows that

$$\theta(0)=\theta(L)=\theta_x(0)=u(0)=u(L)=0,\quad u_x(L)=\alpha\theta_x(L),$$

so that $(\theta,u)\in D(B)$ and $\tilde{A}^*=B$. We conclude that the operator A is $m-$dissipative in X_0 and that it generates a semigroup of contractions in X_0.

Pick any $(\eta^0,v^0)\in D(\tilde{A})$ and let $(\eta,v)(t) := e^{t\tilde{A}}(\eta^0,v^0)$. Scaling the two first equations in (4.1) by η and v respectively, and summing the two obtained equations we arrive to the energy identity

$$\left[\frac12\int_0^L[\eta^2+v^2]dx\right]_0^T+\alpha\int_0^T\eta_x(t,L)^2dt=0. \tag{4.4}$$

Scaling the two first equations in (4.1) by xv and $x\eta$, respectively, and summing the obtained equations, we arrive to

$$\left[\int_0^L(x\eta v)dx\right]_0^T-\frac12\int_0^T\int_0^L(\eta^2+v^2)dxdt+\frac32\int_0^T\int_0^L(\eta_x^2+v_x^2)dxdt-\frac{L}{2}(\alpha^2+1)\int_0^T\eta_x(t,L)^2dt=0.$$

Combined with (4.4), it yields

$$\begin{aligned}\frac{3}{2}\int_0^T\int_0^L(\eta_x^2+v_x^2)dxdt &= -\left[\int_0^L(x\eta v)dx\right]_0^T+\frac{1}{2}\int_0^T\int_0^L(\eta^2+v^2)dxdt+\frac{L}{2}(\alpha^2+1)\int_0^T\eta_x(t,L)^2dt\\ &\leq L\left(\|\eta(T)\|_{L^2(0,L)}\|v(T)\|_{L^2(0,L)}+\|\eta^0\|_{L^2(0,L)}\|v^0\|_{L^2(0,L)}\right)\\ &\quad+\left(\frac{T}{2}+\frac{L}{4\alpha}(\alpha^2+1)\right)\left(\|\eta^0\|_{L^2(0,L)}^2+\|v^0\|_{L^2(0,L)}^2\right)\\ &\leq \left(L+\frac{T}{2}+\frac{L}{4\alpha}(\alpha^2+1)\right)\|(\eta^0,v^0)\|_{X_0}^2,\end{aligned}$$

as desired. □

We search a solution of

$$\begin{cases}\eta_t+v_x+(\eta v)_x+v_{xxx}=0, & t\in(0,\infty),\ x\in(0,L),\\ v_t+\eta_x+vv_x+\eta_{xxx}=0, & t\in(0,\infty),\ x\in(0,L),\\ \eta(t,0)=0,\ \eta(t,L)=0,\ \eta_x(t,0)=0, & t\in(0,\infty),\\ v(t,0)=0,\ v(t,L)=0,\ v_x(t,L)=-\alpha\eta_x(t,L), & t\in(0,\infty),\\ \eta(0,x)=\eta^0(x),\ v(0,x)=v^0(x), & x\in(0,L),\end{cases}\tag{4.5}$$

as a solution of the integral equation

$$(\eta,v)(t)=e^{t\tilde{A}}(\eta^0,v^0)-\int_0^t e^{(t-\tau)\tilde{A}}((\eta v)_x(\tau),(vv_x)(\tau))d\tau,\quad t\geq 0.\tag{4.6}$$

Introducing the map Γ defined by

$$\Gamma(\eta,v)(t):=e^{t\tilde{A}}(\eta^0,v^0)-\int_0^t e^{(t-\tau)\tilde{A}}((\eta v)_x(\tau),(vv_x)(\tau))d\tau,$$

we will show that for any (sufficiently small) time $T>0$, the map Γ has a unique fixed-point in some closed ball $B(0,R)$ in the space

$$E_T:=C([0,T],X_0)\cap L^2(0,T,[H^1(0,L)]^2)$$

endowed with the norm

$$\|(\eta,v)\|_{E_T}:=\sup_{t\in[0,T]}\|(\eta(t),v(t))\|_{X_0}+\left(\int_0^T\|(\eta_x(t),v_x(t))\|_{X_0}^2\right)^{\frac{1}{2}}.$$

Such a fixed-point yields a local solution of (4.5).

Proposition 4.2. *For any $(\eta^0,v^0)\in X_0$, there are some positive numbers T and R such that the system (4.5) has a unique (integral) solution $(\eta,v)\in B(0,R)\subset E_T$.*

Proof. The proof is very similar to those of the wellposedness of the KdV equation with the boundary conditions $u(t,0)=u(t,L)=u_x(t,L)=0$ (see [29, 30]). First, we can prove as in [30] that for some constant $C_1>0$, we have for all $T\in(0,1]$ and all $(f,g)\in L^1(0,T,X_0)$

$$\|\int_0^t e^{(t-\tau)\tilde{A}}(f(\tau),g(\tau))d\tau\|_{E_T}\leq C_1\|(f,g)\|_{L^1(0,T,X_0)}.\tag{4.7}$$

Following [29], one can find a constant $C(L)>0$ such that for all $T\in(0,1]$ and all $(\eta,v)\in E_T$, it holds

$$\begin{aligned}\int_0^T \|(\eta v)_x\|_{X_0}dt \;\leq\; CT^{\frac14}\Big(&\|\eta_x\|_{L^2(0,T,L^2(0,L))}\|v\|^{\frac12}_{L^\infty(0,T,L^2(0,L))}\|v_x\|^{\frac12}_{L^2(0,T,L^2(0,L))}\\ &+\|v_x\|_{L^2(0,T,L^2(0,L))}\|\eta\|^{\frac12}_{L^\infty(0,T,L^2(0,L))}\|\eta_x\|^{\frac12}_{L^2(0,T,L^2(0,L))}\Big).\end{aligned}\tag{4.8}$$

It follows that there are some positive constants $C_2(L),C_3(L)$ such that for all $T>0$ and all $(\eta^1,v^1),(\eta^2,v^2)\in E_T$ we have

$$\|\Gamma(\eta^1,v^1)\|_{E_T} \;\leq\; C_2\|(\eta^0,v^0)\|_{X_0}+C_3T^{\frac14}\|(\eta^1,v^1)\|^2_{E_T},\tag{4.9}$$

$$\|\Gamma(\eta^1,v^1)-\Gamma(\eta^2,v^2)\|_{E_T} \;\leq\; C_3T^{\frac14}(\|(\eta^1,v^1)\|_{E_T}+\|(\eta^2,v^2)\|_{E_T})\|(\eta^1-\eta^2,v^1-v^2)\|_{E_T}.\tag{4.10}$$

Picking $R=2C_2\|(\eta^0,v^0)\|_{X_0}$ and $T>0$ such that $2C_3T^{\frac14}R=1/2$, we see that the map Γ is a contraction in the closed ball $B(0,R)$ of E_T, and hence that it has a unique fixed point by the contraction mapping theorem. □

Let us now proceed to the proof of Theorems 1.1 and 1.2. Assume that $L\in(0,+\infty)\setminus\mathcal{N}$. First, we show that the semigroup $(e^{t\tilde{A}})_{t\geq0}$ is exponentially stable in X_0; that is, for some constants $C,\mu>0$ it holds

$$\|e^{t\tilde{A}}(\eta^0,v^0)\|_{X_0}\leq Ce^{-\mu t}\|(\eta^0,v^0)\|_{X_0}\qquad\forall t\in\mathbb{R}_+.\tag{4.11}$$

It is actually sufficient to prove that for some $T>0$ and some $C=C(T)>0$,

$$\|(\eta^0,v^0)\|^2_{X_0}\leq C\int_0^T\eta_x(t,L)^2dt.\tag{4.12}$$

Indeed, combining (4.12) with (4.4), we obtain $\|(\eta(T),v(T))\|^2_{X_0}\leq(1-2\alpha C^{-1})\|(\eta^0,v^0)\|^2_{X_0}$ which yields (4.11) by the semigroup property.

To prove (4.12), we proceed by contradiction. If (4.12) is not true, one can find a sequence $(\eta^{0,n},v^{0,n})_{n\geq0}$ in X_0 such that, denoting $(\eta^n,v^n)(t):=e^{t\tilde{A}}(\eta^{0,n},v^{0,n})$, we have

$$1=\|(\eta^{0,n},v^{0,n})\|^2_{X_0}>n\int_0^T\eta^n_x(t,L)^2dt.\tag{4.13}$$

Scaling the two first equations in (4.1) by $(T-t)\eta$ and $(T-t)v$ respectively, and summing the two obtained equations we obtain

$$\frac{T}{2}\|(\eta(0),v(0))\|^2_{X_0}=\frac12\int_0^T\int_0^L(\eta^2+v^2)dxdt+\alpha\int_0^T(T-t)\eta_x(t,L)^2dt.\tag{4.14}$$

Since by (4.1) and (4.2) the sequence (η^n,v^n) is bounded in $L^2(0,T,[H^1_0(0,L)]^2)\cap H^1(0,T,[H^{-2}(0,L)]^2)$, we infer from Aubin-Lions' lemma that a subsequence (η^{n_k},v^{n_k}) is convergent in $L^2(0,T,[L^2(0,L)]^2)$. Combined to (4.13)-(4.14), this yields that $(\eta^{0,n_k},v^{0,n_k})\to(\eta^0,v^0)$ strongly in X_0 for some $(\eta^0,v^0)\in X_0$ satisfying $\|(\eta^0,v^0)\|_{X_0}=1$ and $\eta_x(\cdot,L)\equiv0$. Taking into account (4.1), we infer that $v_x(\cdot,L)\equiv0$ as well. But since $L\notin\mathcal{N}$, this is impossible by Theorem 3.7.

Thus (4.12) is established. Proceeding in much the same way as for [28, Theorem 1.1], we can derive Theorem 1.2. (The proof is omitted for the sake of shortness.)

Let us now proceed to the proof of Theorem 1.1. We first notice that the linear system

$$\begin{cases} \eta_t + v_x + v_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ v_t + \eta_x + \eta_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ \eta(t,0) = 0,\ \eta(t,L) = 0,\ \eta_x(t,0) = 0, & t \in (0,T), \\ v(t,0) = 0,\ v(t,L) = 0,\ v_x(t,L) = -\alpha\eta_x(t,L) + h(t), & t \in (0,T), \\ \eta(0,x) = \eta^0(x),\ v(0,x) = v^0(x), & x \in (0,L), \end{cases} \tag{4.15}$$

is well posed for $(\eta^0, v^0) \in X_0$ and $h \in L^2(0,T)$. Clearly, the wellposedness can be derived for $h \in C^2([0,T])$ by performing the change of unknowns $\tilde{\eta}(t,x) := \eta(t,x),\ \tilde{v}(t,L) := v(t,L) + h(t)g(x)$, where the function $g \in C^\infty([0,L])$ is such that $g(0) = g(L) = 0$ and $g'(L) = -1$. To extend the result from $C^2([0,T])$ to $L^2(0,T)$, we need to derive some *a priori* estimates. Scaling in the first (resp. second) equation of (4.15) by η (resp. v), we obtain after some integrations by parts

$$\left[\frac{1}{2}\int_0^L [\eta^2 + v^2]dx\right]_0^T + \alpha\int_0^T \eta_x(t,L)^2 dt - \int_0^T \eta_x(t,L)h(t)dt = 0.$$

This yields for all $T \geq 0$

$$\|(\eta(T,.), v(T,.))\|_{X_0}^2 + \alpha\int_0^T \eta_x(t,L)^2 dt \leq \|(\eta^0, v^0)\|_{X_0}^2 + \frac{1}{\alpha}\int_0^T h(t)^2 dt, \tag{4.16}$$

and thus $(\eta, v) \in C([0,T], X_0)$ if $(\eta^0, v^0) \in X_0$ and $h \in L^2(0,T)$. Scaling now in the first (resp. second) equation of (4.15) by xv (resp. $x\eta$) yields for some constant $C = C(L,T) > 0$

$$\int_0^T\int_0^L (\eta_x^2 + v_x^2)dxdt \leq C\left(\|(\eta^0, v^0)\|_{X_0}^2 + \int_0^T h(t)^2 dt\right),$$

so that $(\eta, v) \in E_T$ if $(\eta^0, v^0) \in X_0$ and $h \in L^2(0,T)$.

The same computations as in the proof of Theorem 1.2 show that the operator

$$\hat{A}(\eta, v) = (-v_x - v_{xxx}, -\eta_x - \eta_{xxx})$$

with domain

$$D(\hat{A}) = \{(\eta, v) \in H^3(0,L)^2;\ \eta(0) = \eta(L) = \eta_x(L) = v(0) = v(L) = 0,\ \ v_x(0) = \alpha\eta_x(0)\} \subset X_0$$

generates a semigroup of contractions in X_0. Next, performing the change of variables $t \to T - t$ and $x \to L - x$, we infer that for any $(\theta^1, u^1) \in X_0$, the backward system

$$\begin{cases} \theta_t + u_x + u_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ u_t + \theta_x + \theta_{xxx} = 0, & t \in (0,T),\ x \in (0,L), \\ \theta(t,0) = 0,\ \theta(t,L) = 0,\ \theta_x(t,0) = 0, & t \in (0,T), \\ u(t,0) = 0,\ u(t,L) = 0,\ u_x(t,L) = \alpha\theta_x(t,L), & t \in (0,T), \\ \theta(T,x) = \theta^1(x),\ u(T,x) = u^1(x), & x \in (0,L), \end{cases} \tag{4.17}$$

has a unique solution $(\theta, u) \in C([0,T], X_0)$, which belongs to $C([0,T], [H^3(0,L)]^2) \cap C^1([0,T], X_0)$ if $(\theta^1(L - \cdot), u^1(L - \cdot)) \in D(\hat{A})$.

We now prove the exact controllability in X_0 of the linear system (4.15). Scaling in the first (resp. second) equation of (4.15) by θ (resp. u), we obtain after some integrations by parts

$$\left[\int_0^L [\eta\theta + vu]dx\right]_0^T = \int_0^T \theta_x(t,L)h(t)dt.$$

By the Hilbert Uniqueness Method (see [23, 24]), the exact controllability of (4.15) holds in X_0 with control inputs $h \in L^2(0,T)$ if and only if the following observability inequality holds

$$\|(\theta^1,u^1)\|_{X_0}^2 \le C\int_0^T |\theta_x(t,L)|^2dt \tag{4.18}$$

for the solutions of the *backward* system (4.17). Again the backward system enjoys the *global Kato smoothing property*:

$$\|(\theta_x,u_x)\|_{L^2(0,T,X_0)} \le C\|(\theta^1,u^1)\|_{X_0}.$$

Proceeding as in [30], we easily see that if (4.18) is false, then we can find a pair of data (θ^1,u^1) in X_0 such that $\|(\theta^1,u^1)\|_{X_0}=1$ and $\theta_x(\cdot,L)\equiv 0$. But this is impossible by Theorem 3.7, for $L\notin\mathcal{N}$.

We have established the exact controllability of the linear system (4.15). The (local) exact controllability of the nonlinear system (1.20) in X_0 follows at once by applying the contraction mapping theorem as in [30] for KdV. (Note that $\eta_x(\cdot,L)\in L^2(0,T)$ by (4.16), and hence $g_2 = -\alpha\eta_x(\cdot,L)+h\in L^2(0,T)$ as well.) We omit the details for the sake of shortness. The proof of Theorem 1.1 is complete.

Appendix: Proof of Theorem 3.11.

The proof of Theorem 3.11 is not based on the multiplier method, but on the analysis of the spectral properties of the operator A. More precisely, we estimate the asymptotic behavior of the eigenvalues of A and use it to establish the observability inequalities (3.50)-(3.51). The proof of Theorem 3.11 is outlined as follows. In Step 1, we introduce the operator $By = -y'''(L-x)-y'(L-x)$ with domain $D(B)=\{y\in H^3(0,L)\cap H_0^1(0,L);\ y'(L)=0\}$, which is closely related to the operator A (but more easy to handle). We prove that it is selfadjoint and that it has a compact resolvent, so that it can be diagonalized in an orthonormal basis in $L^2(0,L)$. In Step 2, we estimate the asymptotic behavior of the eigenvalues of B. Finally, in Step 3 we show that A can be diagonalized in an orthonormal basis of $[L^2(0,L)]^2$ and use the expansions of the solutions in terms of the eigenfunctions to prove (3.50) and (3.51).

Step 1 (Study of the operator B)
Let

$$(By)(x) := -y'''(L-x)-y'(L-x) \quad \text{for } y\in D(B) =: \{y\in H^3(0,L)\cap H_0^1(0,L);\ y'(L)=0\},$$

where $' = d/dx$. Then we have the following result.

Claim A.1 B is a selfadjoint operator in $L^2(0,L)$.

Picking any $y, z \in D(B)$, we readily obtain by integration by parts that

$$\int_0^L [-y'''(L-x) - y'(L-x)]z(x)dx = \int_0^L y(x)[-z'''(L-x) - z'(L-x)]dx,$$

i.e. $(By, z)_{L^2} = (y, Bz)_{L^2}$ where $(.,.)_{L^2}$ stands for the scalar product in $L^2(0, L)$. This means that $D(B) \subset D(B^*)$ and that $B^* = B$ on $D(B)$. Conversely, pick any $z \in D(B^*)$. Then

$$|(By, z)_{L^2}| = \left| \int_0^L [-y'''(L-x) - y'(L-x)]z(x)\, dx \right| \leq C\|y\|_{L^2} \quad \forall y \in D(B).$$

Setting $w(x) = z(L-x)$, we arrive to $|(y''' + y', w)_{L^2}| \leq C\|y\|_{L^2}$, or equivalently for $y \in \mathcal{D} = \mathcal{D}(0, L)$

$$\left|\langle w''' + w', y\rangle_{\mathcal{D}',\mathcal{D}}\right| \leq C\|y\|_{L^2}.$$

It follows that $w''' + w' \in L^2(0, L)$, and hence $w \in H^3(0, L)$ and $z \in H^3(0, L)$. Integrating by parts in $(By, z)_{L^2}$ and using the fact that $y''' + y' \in L^2(0, L)$, we arrive to

$$\left|-y''(L)z(0) + y''(0)z(L) + y'(0)z'(L)\right| \leq C'\|y\|_{L^2}, \quad \forall y \in D(B).$$

This yields $z(0) = z(L) = z'(L) = 0$. Thus $z \in D(B)$, and we infer that $D(B^*) = D(B)$ and that $B^* = B$.

Claim A.2 If $L \in (0, +\infty) \setminus \mathcal{N}$, then $B^{-1} : L^2(0, L) \to H^3(0, L)$ is a well-defined continuous operator.

Pick any $L \notin \mathcal{N}$ and any $z \in L^2(0, L)$. We search for $y \in D(B)$ solving the equation $By = z$, i.e.

$$\begin{aligned} -y'''(L-x) - y'(L-x) &= z(x), \quad x \in (0, L), \\ y(0) = y(L) = y'(L) &= 0. \end{aligned}$$

We search for the function y in the form $y(x) = y_1(x) + y_2(x)$ with

$$y_1(x) = \int_x^L [1 - \cos(x-s)]z(L-s)ds.$$

We see at once that $y_1 \in H^3(0, L)$ with $y_1'''(x) + y_1'(x) = -z(L-x)$ a.e. in $(0, L)$ and $y_1(L) = y_1'(L) = y_1''(L) = 0$. Thus, it remains to find $y_2 \in H^3(0, L)$ such that $y_2''' + y_2' = 0$ and $y_2(L) = y_2'(L) = 0$, $y_2(0) = -y_1(0)$. By linearity, there is no loss of generality in assuming that $y_2(0) = 1$. The three roots of the equation $r^3 + r = 0$ are $r_1 = i$, $r_2 = -i$ and $r_3 = 0$. We search for y_2 in the form

$$y_2(x) = \sum_{j=1}^{3} a_j[e^{r_j x} - ie^{r_j(L-x)}],$$

where the coefficients a_1, a_2, a_3 have still to be found. It is clear that y_2 solves $y_2''' + y_2' = 0$, and the boundary conditions $y_2(L) = y_2'(L) = 0$ and $y_2(0) = 1$ give the following conditions

$$\sum_{j=1}^{3} a_j(e^{r_j L} - i) = 0, \tag{4.19}$$

$$\sum_{j=1}^{3} a_j(1 - ie^{r_j L}) = 1, \tag{4.20}$$

$$\sum_{j=1}^{3} r_j a_j(e^{r_j L} + i) = 0. \tag{4.21}$$

Using (4.19), (4.20) and the precise values of r_1, r_2, r_3, we obtain that $a_3 = \frac{1}{2} - a_1 - a_2$ and

$$a_2 = \frac{1}{e^{-iL} - 1}\left(\frac{i-1}{2} - a_1(e^{iL} - 1)\right).$$

(Note that $L \notin 2\pi\mathbb{Z}$, for $L \notin \mathcal{N}$.) Plugging these expressions of a_2 and a_3 in (4.21), we arrive to

$$a_1\left(i(e^{iL} + i) - \frac{e^{iL} - 1}{e^{-iL} - 1}(-i)(e^{-iL} + i)\right) + (-i)(e^{-iL} + i)\frac{i-1}{2(e^{-iL} - 1)} = 0.$$

We readily see that the coefficient behind a_1 is not zero, for $L \notin 2\pi\mathbb{Z}$, so that the last equation for a_1 can be solved. Claim A.2 is proved.

Claim A.3 There is an orthonormal basis $(v_n)_{n\in\mathbb{N}}$ in $L^2(0, L)$ composed of eigenvectors of B: for all $n \in \mathbb{N}$, $v_n \in D(B)$ and $Bv_n = \lambda_n v_n$ for some $\lambda_n \in \mathbb{R}$.

It is a direct consequence of Claims A.1 and A.2, since B^{-1} is a bounded compact selfadjoint operator in $L^2(0, L)$. Thus B^{-1} is diagonalizable in an orthonormal basis in $L^2(0, L)$, and the same is true for B.

Step 2 (Asymptotics of the eigenvalues of B)

Claim A.4 Using a convenient relabeling, the sequence of eigenvalues of B can be written $(\lambda_n)_{n\in\mathbb{Z}}$, with $\lambda_n \leq \lambda_{n+1}$ for all $n \in \mathbb{Z}$ and

$$\lambda_n = \left(\frac{\frac{\pi}{6} + 2\pi(k_1 + n)}{L}\right)^3 + O(n) \quad \text{as} \quad n \to +\infty, \tag{4.22}$$

$$\lambda_n = -\left(\frac{\frac{7\pi}{6} + 2\pi(k_2 - n)}{L}\right)^3 + O(n) \quad \text{as} \quad n \to -\infty \tag{4.23}$$

for some numbers $k_1, k_2 \in \mathbb{Z}$.

The eigenvalues of B as given in Claim A.3 satisfy $|\lambda_n| \to +\infty$ as $n \to +\infty$. We shall show that they can be separated into two subsequences, one with the asymptotics (4.22) and another one with the asymptotics (4.23).

Assume that (v,λ) is a pair of eigenvector/eigenvalue for B; that is, $v\in D(B)$, $v\neq 0$, and $v'''(x)+v'(x)=-\lambda v(L-x)$. This yields

$$v^{(6)}+2v^{(4)}+v''=-\lambda^2 v. \tag{4.24}$$

The roots of the equation

$$(r^3+r)^2=-\lambda^2 \tag{4.25}$$

read $r_1,r_2,r_3,-r_1,-r_2,-r_3$, where r_1,r_2,r_3 denote the roots of

$$r^3+r=i\lambda. \tag{4.26}$$

Note that if the r_j are not pairwise distinct, then any multiple root r should also solve $3r^2+1=0$, so that $r=\pm i/\sqrt{3}$ and $\lambda=\pm 2/(3\sqrt{3})$. Thereafter, we assume that $|\lambda|>2/(2\sqrt{3})$ so that the roots r_j $(1\leq j\leq 3)$ of (4.26) are simple. Note that the roots $\pm r_j$ $(1\leq j\leq 3)$ of (4.25) are also simple, for $r_i=-r_j$ yields $i\lambda=r_i^3+r_i=-(r_j^3+r_j)=-i\lambda$ and $\lambda=0$. It follows that the exponential maps $e^{\pm r_j x}$ $(1\leq j\leq 3)$ are not linked. From (4.24), we can write

$$v(x)=\sum_{j=1}^{3}[a_j e^{r_j x}+b_j e^{-r_j x}]$$

for some constants $a_j,b_j\in\mathbb{C}$ $(1\leq j\leq 3)$. The equation $Bv=\lambda v$ yields

$$\sum_{j=1}^{3}\big([a_j(r_j^3+r_j)+b_j\lambda e^{-r_j L}]e^{r_j x}+[a_j\lambda e^{r_j L}-b_j(r_j^3+r_j)]e^{-r_j x}\big)=0.$$

This holds if and only if $b_j=-ie^{r_j L}a_j$ for $1\leq j\leq 3$. Thus v takes the form

$$v(x)=\sum_{j=1}^{3}a_j[e^{r_j x}-ie^{r_j(L-x)}]. \tag{4.27}$$

Then the conditions $v(L)=0$, $v(0)=0$ and $v'(L)=0$ are equivalent to

$$\sum_{j=1}^{3}a_j(e^{r_j L}-i) = 0, \tag{4.28}$$

$$\sum_{j=1}^{3}a_j(1-ie^{r_j L}) = 0, \tag{4.29}$$

$$\sum_{j=1}^{3}r_j a_j(e^{r_j L}+i) = 0. \tag{4.30}$$

The equations (4.28)-(4.29) yield $\sum_{j=1}^{3}a_j=\sum_{j=1}^{3}a_j e^{r_j L}=0$, i.e.

$$\begin{aligned} a_3 &= -a_1-a_2,\\ a_1(e^{r_1 L}-e^{r_3 L})+a_2(e^{r_2 L}-e^{r_3 L}) &= 0. \end{aligned} \tag{4.31}$$

Substituting the values of a_3 and a_2 in (4.30) results in

$$a_1\big[r_1(e^{r_1 L}+i)(e^{r_2 L}-e^{r_3 L})+r_2(e^{r_2 L}+i)(e^{r_3 L}-e^{r_1 L})+r_3(e^{r_3 L}+i)(e^{r_1 L}-e^{r_2 L})\big]=0.$$

Thus if $e^{r_2 L} \neq e^{r_3 L}$, then a_2 can be expressed in terms of a_1, and the system has a solution $(a_1, a_2, a_3) \neq (0,0,0)$ if and only if

$$r_1(e^{r_1 L}+i)(e^{r_2 L}-e^{r_3 L}) + r_2(e^{r_2 L}+i)(e^{r_3 L}-e^{r_1 L}) + r_3(e^{r_3 L}+i)(e^{r_1 L}-e^{r_2 L}) = 0. \quad (4.32)$$

The case $e^{r_2 L} = e^{r_3 L}$ is actually impossible for $|\lambda|$ large enough, see below (4.33) and (4.37). We shall use several times the following classical result.

Theorem 4.3. *(Rouché's theorem, see e.g.* [35, 3.42]*) Let f and g be analytic inside and on a closed contour Γ, and such that $|f(z)-g(z)| < |f(z)|$ on Γ. Then f and g have the same number of zeros inside Γ.*

• Assume that $\lambda \to +\infty$. Then $|r| \to +\infty$ and from the equation $r^3(1+\frac{1}{r^2}) = i\lambda$, we infer that

$$r^3 \sim i\lambda.$$

Then we can choose r_1, r_2, r_3 so that

$$r_1 \sim -i\lambda^{\frac{1}{3}},\ r_2 \sim -ij\lambda^{\frac{1}{3}},\ r_3 \sim -ij^2\lambda^{\frac{1}{3}}$$

where $j = e^{i\frac{2\pi}{3}}$. More precisely, for any $C > 1/3$ and for λ large enough, there is only one solution r of $r^3(1+\frac{1}{r^2}) = i\lambda$ in the disk $\{r \in \mathbb{C};\ |r-(-i\lambda^{\frac{1}{3}})| < C\lambda^{-\frac{1}{3}}\}$. Indeed, letting $f(r) = r^3 - i\lambda$, $g(r) = r^3 + r - i\lambda$, and considering $r = -i\lambda^{\frac{1}{3}} + \rho$ with $|\rho| = C\lambda^{-\frac{1}{3}}$, we have that $|f(r)-g(r)| = |r| = \lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}})$, while

$$|f(r)| = \left|(-i\lambda^{\frac{1}{3}}+\rho)^3 - i\lambda\right| = \left|3(-i\lambda^{\frac{1}{3}})^2\rho + 3(-i\lambda^{\frac{1}{3}})\rho^2 + \rho^3\right| = 3C\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}).$$

The conclusion for $r_1 \sim r_1^* := -i\lambda^{\frac{1}{3}}$ follows then from Rouché's theorem. We can do exactly the same for the two other roots $r_2 \sim r_2^* := jr_1^*$ and $r_3 \sim r_3^* := j^2 r_1^*$.

From $r^3 = i\lambda(1+r^{-2})^{-1} = i\lambda(1-r^{-2}+r^{-4}+\cdots)$, we obtain that

$$\begin{aligned}
r_1 &= -i\lambda^{\frac{1}{3}}(1+\frac{1}{3}\lambda^{-\frac{2}{3}}+\cdots) = -i\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}) \sim -i\lambda^{\frac{1}{3}},\\
r_2 &= -ij\lambda^{\frac{1}{3}}(1+\frac{1}{3}j^{-2}\lambda^{-\frac{2}{3}}+\cdots) = -ij\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}) \sim (\frac{\sqrt{3}}{2}+\frac{i}{2})\lambda^{\frac{1}{3}},\\
\text{and}\quad r_3 &= -ij^2\lambda^{\frac{1}{3}}(1+\frac{1}{3}j^{-4}\lambda^{-\frac{2}{3}}+\cdots) = -ij^2\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}) \sim (-\frac{\sqrt{3}}{2}+\frac{i}{2})\lambda^{\frac{1}{3}}.
\end{aligned}$$

Clearly,

$$|e^{r_1 L}| \to 1,\quad |e^{r_2 L}| \to +\infty \quad \text{and} \quad e^{r_3 L} \to 0. \quad (4.33)$$

Plugging these expressions in (4.32), we arrive to

$$e^{r_1 L}(r_1 - r_2) + i(r_1 - r_3) = O(\lambda^{\frac{1}{3}} e^{-\frac{\sqrt{3}}{2}\lambda^{\frac{1}{3}}}). \quad (4.34)$$

On the other, we have that

$$r_1 - r_2 = -i\lambda^{\frac{1}{3}}(1-j) + O(\lambda^{-\frac{1}{3}}),\quad r_1 - r_3 = -i\lambda^{\frac{1}{3}}(1-j^2) + O(\lambda^{-\frac{1}{3}}),$$

and hence

$$e^{r_1 L} = e^{-i\lambda^{\frac{1}{3}}L + O(\lambda^{-\frac{1}{3}})} = ij^2 + O(\lambda^{-\frac{2}{3}}) = e^{-i\frac{\pi}{6}} + O(\lambda^{-\frac{2}{3}}). \quad (4.35)$$

We infer that for λ large enough,

$$\lambda^{\frac{1}{3}}L = \frac{\pi}{6} + 2n\pi + O(\lambda^{-\frac{1}{3}})$$

for some $n \in \mathbb{Z}$, so that $\lambda_n \sim L^{-3}\left(\frac{\pi}{6} + 2n\pi\right)^3$. On the other hand, we claim that there is a simple eigenvalue $\lambda_n \sim L^{-3}\left(\frac{\pi}{6} + 2n\pi\right)^3$ for all $n \in \mathbb{N}$ large enough. Indeed, we infer from $r^3 + r - i\lambda = 0$ that $r_2 + r_3 = -r_1$ and $r_2 r_3 = i\lambda/r_1$, so that

$$r_{2,3} = \frac{-r_1 \pm (r_1^2 - \frac{4i\lambda}{r_1})^{\frac{1}{2}}}{2} = \frac{-r_1 \pm (-3r_1^2 - 4)^{\frac{1}{2}}}{2}.$$

Since

$$r_2 \sim r_2^* = jr_1^* = \frac{-r_1^* + \sqrt{3}ir_1^*}{2} = \frac{-r_1^* + (-3(r_1^*)^2)^{\frac{1}{2}}}{2},$$

we have that

$$r_2 = \frac{-r_1 + (-3r_1^2 - 4)^{\frac{1}{2}}}{2}, \quad r_3 = \frac{-r_1 - (-3r_1^2 - 4)^{\frac{1}{2}}}{2}. \tag{4.36}$$

Let

$$g(r_1) = \big[r_1(e^{r_1L} + i)(e^{r_2L} - e^{r_3L}) + r_2(e^{r_2L} + i)(e^{r_3L} - e^{r_1L}) + r_3(e^{r_3L} + i)(e^{r_1L} - e^{r_2L})\big]e^{-r_2L}$$

where r_2 and r_3 are as in (4.36), and let $f(r_1) = e^{r_1L}r_1(1 - j) + r_1 i(1 - j^2)$. Then for λ large enough, $f(r_1) = 0$ has a solution $r_1^* = -i\lambda_n^{\frac{1}{3}}$ where $\lambda_n = [\frac{1}{L}(\frac{\pi}{6} + 2\pi n)]^3$, $n \in \mathbb{Z}$. On the other hand, $|f(r_1) - g(r_1)| = O(\lambda^{-\frac{1}{3}}) < 1 < |f(r_1)|$ if $|r_1 - r_1^*| = 1/L$ and λ is large enough. As $f(r_1^*) = 0$, we infer from Rouché's theorem that $g(r_1) = 0$ has only one root in the disk $\{r_1 \in \mathbb{C};\ |r_1 - r_1^*| < \frac{1}{L}\}$. Combined with (4.35), this yields for all $n \in \mathbb{N}$ large enough an eigenvalue λ_n such that

$$\lambda_n^{\frac{1}{3}}L = \frac{\pi}{6} + 2n\pi + O(n^{-1}),$$

and hence

$$\lambda_n = L^{-3}\left(\frac{\pi}{6} + 2n\pi\right)^3 + O(n),$$

and we see from (4.27)-(4.30) that the associated eigenspace is onedimensional.

• Assume that $\lambda \to -\infty$. We still choose r_1, r_2, r_3 so that

$$r_1 \sim -i\lambda^{\frac{1}{3}},\ r_2 \sim -ij\lambda^{\frac{1}{3}},\ r_3 \sim -ij^2\lambda^{\frac{1}{3}}.$$

Then we obtain that

$$\begin{aligned} r_1 &= -i\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}) \sim i|\lambda|^{\frac{1}{3}}, \\ r_2 &= -ij\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}) \sim -(\frac{\sqrt{3}}{2} + \frac{i}{2})|\lambda|^{\frac{1}{3}}, \\ \text{and} \quad r_3 &= -ij^2\lambda^{\frac{1}{3}} + O(\lambda^{-\frac{1}{3}}) \sim (\frac{\sqrt{3}}{2} - \frac{i}{2})|\lambda|^{\frac{1}{3}}. \end{aligned}$$

Clearly,

$$|e^{r_1L}| \to 1, \quad |e^{r_2L}| \to 0 \quad \text{and} \quad |e^{r_3L}| \to +\infty. \tag{4.37}$$

Plugging these expressions in (4.32), we arrive to

$$e^{r_1 L}(r_3 - r_1) + i(r_2 - r_1) = O(|\lambda|^{\frac{1}{3}} e^{-\frac{\sqrt{3}}{2}|\lambda|^{\frac{1}{3}}}).$$

On the other, we have that

$$r_3 - r_1 = -i\lambda^{\frac{1}{3}}(j^2 - 1) + O(\lambda^{-\frac{1}{3}}), \quad r_2 - r_1 = -i\lambda^{\frac{1}{3}}(j - 1) + O(\lambda^{-\frac{1}{3}}),$$

and hence

$$e^{r_1 L} = e^{-i\lambda^{\frac{1}{3}} L + O(\lambda^{-\frac{1}{3}})} = ij + O(\lambda^{-\frac{1}{3}}) = e^{i\frac{7\pi}{6}} + O(\lambda^{-\frac{2}{3}}).$$

We infer that

$$-\lambda^{\frac{1}{3}} L = \frac{7\pi}{6} + 2n\pi + O(\lambda^{-\frac{1}{3}}),$$

for some $n \in \mathbb{Z}$, so that

$$\lambda = -L^{-3}\left(\frac{7\pi}{6} + 2n\pi\right)^3 + O(n). \tag{4.38}$$

On the other hand, we can prove as above that for all $n \in \mathbb{Z}$ with $n < 0$ and $|n|$ large enough, there is indeed an eigenvalue

$$\lambda_n = -L^{-3}\left(\frac{7\pi}{6} + 2|n|\pi\right)^3 + O(n),$$

and that it is simple.

• Finally, we can relabel the λ_n's so that the eigenvalues of B form a sequence $(\lambda_n)_{n\in\mathbb{Z}}$ as in Claim A.4.

Step 3. Diagonalization of the operator A

We denote by $(v_n)_{n\in\mathbb{Z}}$ an orthonormal basis in $L^2(0, L)$ such that $Bv_n = \lambda_n v_n$ for all $n \in \mathbb{Z}$. Let for $n \in \mathbb{Z}$

$$\theta_n^+(x) := -\frac{i}{\sqrt{2}} v_n(L - x),\ \theta_n^-(x) := \frac{i}{\sqrt{2}} v_n(L - x), \quad \text{and } u_n^+(x) = u_n^-(x) := \frac{1}{\sqrt{2}} v_n(x).$$

Then $(\theta_n^+, u_n^+)_{n\in\mathbb{Z}} \cup (\theta_n^-, u_n^-)_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2_{\mathbb{C}}(0, L) \times L^2_{\mathbb{C}}(0, L)$ (endowed with the natural scalar product $((\theta, u), (\varphi, w)) = \int_0^L [\theta(x)\overline{\varphi(x)} + u(x)\overline{w(x)}]dx$) composed of eigenvectors of A:

$$A(\theta_n^+, u_n^+) = i\lambda_n(\theta_n^+, u_n^+), \quad A(\theta_n^-, u_n^-) = (-i\lambda_n)(\theta_n^-, u_n^-).$$

Let $(\theta^0, u^0) \in [L^2_{\mathbb{C}}(0, L)]^2$ be decomposed as

$$\theta^0(x) = \sum_{n\in\mathbb{Z}} a_n v_n(L - x), \quad u^0(x) = \sum_{n\in\mathbb{Z}} b_n v_n(x).$$

(Note that $(v_n(L - \cdot))_{n\in\mathbb{Z}}$ is also an orthonormal basis in $L^2(0, L)$.) Then we can write

$$(\theta^0, u^0) = \sum_{n\in\mathbb{Z}} [c_n^+(\theta_n^+, u_n^+) + c_n^-(\theta_n^-, u_n^-)]$$

where

$$c_n^+ = \frac{1}{\sqrt{2}}(ia_n + b_n), \quad c_n^- = \frac{1}{\sqrt{2}}(b_n - ia_n).$$

It follows that the solution (θ, u) of (2.18) can be written as

$$(\theta, u) = \sum_{n\in\mathbb{Z}} [c_n^+ e^{i\lambda_n t}(\theta_n^+, u_n^+) + c_n^- e^{-i\lambda_n t}(\theta_n^-, u_n^-)]. \tag{4.39}$$

Let us show that there is an asymptotic spectral gap for the spectrum.
It follows from (4.22) and (4.23) that

$$\lambda_{n+1} - \lambda_n \sim \frac{24\pi^3}{L^3} n^2 \quad \text{as } n \to \infty, \tag{4.40}$$

$$\lambda_n - \lambda_{n-1} \sim \frac{24\pi^3}{L^3} n^2 \quad \text{as } n \to -\infty. \tag{4.41}$$

On the other hand, for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have that

$$\lambda_n - (-\lambda_{k-n}) = (a-b)(a^2 + ab + b^2) + O(n)$$

where $a = (\frac{\pi}{6} + 2\pi(k_1 + n))/L$, $b = (\frac{7\pi}{6} + 2\pi(k_2 - k + n))/L$. Then

$$|a - b| = \left| \frac{2\pi(k_1 - k_2 + k) - \pi}{L} \right| \geq \frac{\pi}{L}$$

and

$$a^2 + ab + b^2 \geq \frac{1}{2}(a^2 + b^2).$$

It follows that there is some $n_0 \in \mathbb{N}^*$ such that for any $k \in \mathbb{Z}$ and some $C \in \mathbb{R}_+$,

$$|\lambda_n - (-\lambda_{k-n})| = \frac{\pi}{2L^3} \left| \frac{\pi}{6} + 2\pi(k_1 + n) \right|^2 + O(n) \geq Cn^2 \quad \text{for } n \geq n_0.$$

Combined with (4.40)-(4.41), we infer that for all $A > 0$ we can find some $n_1 \geq n_0$ such that we have the spectral gap relation

$$\begin{aligned} &|(i\lambda_n) - (i\lambda_m)| \geq A, \quad \forall m \neq n \ \text{ with } \ |m| \geq n_1, \ |n| \geq n_1, \\ &|(i\lambda_n) - (-i\lambda_m)| \geq A, \quad \forall m, n \ \text{ with } \ |m| \geq n_1, \ |n| \geq n_1. \end{aligned}$$

Pick any $T > 0$. It follows then from Ingham's lemma that there exist an integer $N \in \mathbb{N}$ and a constant $K > 0$ such that for all $(c_n^+)_{n\in\mathbb{Z}}, (c_n^-)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, we have

$$K^{-1} \sum_{|n|\geq N} \left(|c_n^+|^2 + |c_n^-|^2\right) \leq \int_0^T \Big| \sum_{|n|\geq N} \left(c_n^+ e^{i\lambda_n t} + c_n^- e^{-i\lambda_n t}\right)\Big|^2 dt \leq K \sum_{|n|\geq N} \left(|c_n^+|^2 + |c_n^-|^2\right).$$

Next, we compare $|v_n''(0)|$ to $|v_n''(L)|$ as $|n| \to +\infty$.

Claim A.5 There are two numbers $K_+, K_- \in \mathbb{C}^*$ such that $v_n''(0) \sim K_+ v_n''(L)$ as $n \to +\infty$ and $v_n''(0) \sim K_- v_n''(L)$ as $n \to -\infty$.

It follows from (4.27) that

$$v_n''(0)=\sum_{j=1}^{3}a_jr_j^2(1-ie^{r_jL}),\quad v_n''(L)=\sum_{j=1}^{3}a_jr_j^2(e^{r_jL}-i)$$

where $r_j=r_j(n)$ and $a_j=a_j(n)$. Assume first that $n\to+\infty$. Then

$$a_3=-a_1-a_2,\quad a_1(e^{r_1L}-e^{r_3L})+a_2(e^{r_2L}-e^{r_3L})=0.$$

It follows that

$$\begin{aligned}v_n''(0)&=a_1r_1^2(1-ie^{r_1L})+a_2r_2^2(1-ie^{r_2L})+a_3r_3^2(1-ie^{r_3L})\\&=a_1\Big(r_1^2(1-ie^{r_1L})+r_2^2ie^{r_1L}-r_3^2+O(\lambda_n^{\frac{2}{3}}e^{-\frac{\sqrt{3}}{2}\lambda_n^{\frac{1}{3}}})\Big)\\&=a_1\big((1-j^4)(-\lambda_n^{\frac{2}{3}})+ie^{r_1L}(j^2-1)(-\lambda_n^{\frac{2}{3}})+O(1)\big)\\&=a_1(1-j)(-\lambda_n^{\frac{2}{3}})(1+j^2ie^{r_1L}+O(\lambda_n^{-\frac{2}{3}})).\end{aligned}$$

But $e^{r_1L}\sim e^{-i\lambda_n^{\frac{1}{3}}L}\sim e^{-i\frac{\pi}{6}}=ij^2$, and hence

$$v_n''(0)\sim a_1(1-j)^2(-\lambda_n^{\frac{2}{3}}).$$

Similarly, we obtain

$$\begin{aligned}v_n''(L)&=a_1r_1^2(e^{r_1L}-i)+a_2r_2^2(e^{r_2L}-i)+a_3r_3^2(e^{r_3L}-i)\\&=a_1\Big(r_1^2(e^{r_1L}-i)-r_2^2e^{r_1L}+ir_3^2+O(\lambda_n^{\frac{2}{3}}e^{-\frac{\sqrt{3}}{2}\lambda_n^{\frac{1}{3}}})\Big)\\&=a_1\big(e^{r_1L}(1-j^2)(-\lambda_n^{\frac{2}{3}})+i(j-1)(-\lambda_n^{\frac{2}{3}})+O(1)\big)\\&=a_1(1-j)(-\lambda_n^{\frac{2}{3}})(-j^2e^{r_1L}-i+O(\lambda_n^{-\frac{2}{3}})),\end{aligned}$$

and hence

$$v_n''(L)\sim a_1(1-j)\,ij^2(-\lambda_n^{\frac{2}{3}}).$$

We conclude that $v_n''(0)\sim\frac{1-j}{ij^2}v_n''(L)$ as $n\to+\infty$. We can prove in the same way that $v_n''(0)\sim K_-v_n''(L)$ as $n\to-\infty$ for some constant $K_-\neq0$.

We are in a position to complete the proof of Theorem 3.11. Pick any $(\theta^0, u^0) \in X_2$. From (4.39), we have that

$$\begin{aligned}
\theta_{xx}(t,L) &= \sum_{n\in\mathbb{Z}}[c_n^+ e^{i\lambda_n t}\theta_{n,xx}^+(L) + c_n^- e^{-i\lambda_n t}\theta_{n,xx}^-(L)] \\
&= \frac{i}{\sqrt{2}}\sum_{n\in\mathbb{Z}}[-c_n^+ e^{i\lambda_n t}v_n''(0) + c_n^- e^{-i\lambda_n t}v_n''(0)], \\
\theta_{xx}(t,0) &= \frac{i}{\sqrt{2}}\sum_{b\in\mathbb{Z}}[-c_n^+ e^{i\lambda_n t}v_n''(L) + c_n^- e^{-i\lambda_n t}v_n''(L)], \\
u_{xx}(t,L) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}[c_n^+ e^{i\lambda_n t}v_n''(L) + c_n^- e^{-i\lambda_n t}v_n''(L)], \\
\text{and} \quad u_{xx}(t,0) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}[c_n^+ e^{i\lambda_n t}v_n''(0) + c_n^- e^{-i\lambda_n t}v_n''(0)].
\end{aligned}$$

Then by (3.30), Claim A.4 and Claim A.5, we have that (with a constant C that may change from line to line)

$$\begin{aligned}
\|(\theta^0,u^0)\|_{X_2}^2 &\le C\int_0^T [|\theta_{xx}(t,L)|^2 + |u_{xx}(t,L)|^2]\,dt \\
&\le C\left(\int_0^T \left|\sum_{n\in\mathbb{Z}}[-c_n^+ e^{i\lambda_n t}v_n''(0) + c_n^- e^{-i\lambda_n t}v_n''(0)]\right|^2 dt\right. \\
&\qquad \left. + \int_0^T \left|\sum_{n\in\mathbb{Z}}[c_n^+ e^{i\lambda_n t}v_n''(L) + c_n^- e^{-i\lambda_n t}v_n''(L)]\right|^2 dt\right) \\
&\le C\left(\|(\theta^0,u^0)\|_{X_0}^2 + \sum_{|n|\ge N}(|c_n^+|^2 + |c_n^-|^2)(|v_n''(0)|^2 + |v_n''(L)|^2)\right) \\
&\le C\left(\|(\theta^0,u^0)\|_{X_0}^2 + \sum_{|n|\ge N}(|c_n^+|^2 + |c_n^-|^2)|v_n''(0)|^2\right) \\
&\le C\left(\|(\theta^0,u^0)\|_{X_0}^2 + \int_0^T |\theta_{xx}(t,L)|^2\,dt\right).
\end{aligned}$$

Thus (3.50) is proved. The proof of (3.51) is similar, and therefore it is omitted. The proof of Theorem 3.11 is complete. □

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