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# Grading of affine Weyl semi-groups of Kac-Moody type

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## Abstract

For any Kac-Moody root data  $\mathcal{D}$ , D. Muthiah and D. Orr have defined a partial order on the semi-direct product  $W^+$  of the integral Tits cone with the vectorial Weyl group of  $\mathcal{D}$ , and a compatible length function. We classify covers for this order and show that this length function defines a  $\mathbb{Z}$ -grading of  $W^+$ , generalizing the case of affine ADE root systems and giving a positive answer to a conjecture of Muthiah and Orr.

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# Introduction

## Motivations

**Reductive groups over p-adic fields** Let  $\mathbf{G}$  be a split reductive group scheme with the data of a Borel subgroup  $\mathbf{B}$  containing a maximal torus  $\mathbf{T}$ . Let  $W^v = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$  be its vectorial Weyl group and  $Y$  be its coweight lattice:  $Y = \text{Hom}(\mathbb{G}_m, \mathbf{T})$ . The action of  $W^v$  on  $\mathbf{T}$  induces an action of  $W^v$  on  $Y$  and allows to form the semi-direct product  $W^a = Y \rtimes W^v$ . This group called the extended affine Weyl group of  $\mathbf{G}$ , appears naturally in the geometry and the representation theory of  $\mathbf{G}$  over discretely valued fields. A foundational work in this regard was done by N. Iwahori and H. Matsumoto in 1965 ([IM65]), when they exhibited a Bruhat decomposition of  $\mathbf{G}(\mathbb{Q}_p)$  indexed by  $W^a$ .

Let  $\mathcal{K}$  be a non-archimedean local field with ring of integer  $\mathcal{O}_{\mathcal{K}} \subset \mathcal{K}$ , uniformizer  $\pi \in \mathcal{O}_{\mathcal{K}}$  and residue field  $\mathbb{k}_{\mathcal{K}} = \mathcal{O}_{\mathcal{K}}/\pi$ . Let  $G = \mathbf{G}(\mathcal{K})$ , let  $K = \mathbf{G}(\mathcal{O}_{\mathcal{K}})$  its integral points and let  $I$  be its Iwahori subgroup, defined as  $I = \{g \in K \mid g \in \mathbf{B}(\mathbb{k}_{\mathcal{K}}) \pmod{\pi}\}$ . The extended affine Weyl group can be understood as  $N_G(\mathbf{T}(\mathcal{K}))/\mathbf{T}(\mathcal{O}_{\mathcal{K}})$ , so admits a lift in  $G$ . Then,  $G$  admits a decomposition in  $I$  double cosets indexed by  $W^a$ , the Iwahori-Matsumoto Bruhat decomposition:

$$G = \bigsqcup_{\pi^\lambda w \in W^a} I\pi^\lambda w I.$$

$W^a$  is a finite extension of a Coxeter group and thus admits a Bruhat order and a Bruhat length. The Bruhat order comes up in the geometry of the homogeneous space  $G/I$ : for any  $\pi^\lambda w \in W^a$ ,  $I\pi^\lambda w I$  is a subvariety of pure dimension  $\ell(\pi^\lambda w)$  in  $G/I$ , and its adherence admits a disjoint decomposition in  $I$  orbits:

$$\overline{I\pi^\lambda w I} = \bigsqcup_{\pi^\mu v \leq \pi^\lambda w} I\pi^\mu v I \tag{0.1}$$

which extends Iwahori-Matsumoto decomposition. The connection between the geometry of  $G/I$  and the combinatorial structure of  $W^a$  is deeper. In particular,  $R$ -Kazhdan Lusztig polynomials, defined as the number of points of certain intersections in  $G/I$ , are also given by a recursive formula based on the Bruhat order and the Bruhat length of  $W^a$ .

These polynomials appear in many topics around reductive groups over local fields, we aim to develop analogous polynomials when  $\mathbf{G}$  is replaced by a general Kac-Moody group.

**Extension to Kac-Moody groups** Replace  $\mathbf{G}$  by a general split Kac-Moody group. Kac-Moody group schemes are entirely defined by the underlying Kac-Moody root data  $\mathcal{D}$ , as defined in [Rém02, §2], and reductive groups correspond to root datum of finite type. Then the Iwahori-Matsumoto decomposition no longer holds on  $G = \mathbf{G}(\mathcal{K})$ . However there is a partial Iwahori-Matsumoto decomposition: there exists a sub-semigroup  $G^+$  of  $G$  such that:

$$G^+ = \bigsqcup_{\pi^\lambda w \in W^+} I\pi^\lambda w I.$$

The indexing set for this decomposition  $W^+$  is a sub-semigroup of  $Y \rtimes W^v$ , and it appears naturally in other related contexts, for example when trying to construct a Iwahori-Hecke algebra for  $G$  [BKP16, BPGR16]. Let us briefly explain how  $W^+$  is defined:

Let  $\Phi$  be the real root system of the root data  $\mathcal{D}$ . It is an infinite set (unless  $\mathcal{D}$  is reductive) of linear forms on  $Y$  coming with a subset of positive roots  $\Phi_+ \subset \Phi$  such that  $\Phi = \Phi_+ \sqcup -\Phi_+$ . Let  $Y^{++} = \{\lambda \in Y \mid \forall \alpha \in \Phi_+, \alpha(\lambda) \geq 0\}$  and  $Y^+ = W^v \cdot Y^{++}$ . Then  $W^+$  is defined as  $Y^+ \rtimes W^v$ . In the reductive case,  $Y^+$  coincides with  $Y$  and thus  $W^+ = W^a$ . However,  $Y \rtimes W^v$  can no longer be conceived as a finite extension of a Coxeter system, hence there is a priori no Bruhat order on  $W^+$ , let alone on  $Y \rtimes W^v$ . A well-behaved topology on  $G^+/I$  would allow to define an order on  $W^+$  through the analog of decomposition (0.1), but  $G^+/I$  does not seem to have a natural variety, nor even an ind-variety structure.

**An order and two lengths on  $W^+$**  In appendix of their article on the construction of a Iwahori Hecke algebra for  $G$  an affine Kac-Moody group over a p-adic field [BKP16, Appendix B2], A. Braverman, D. Kazhdan and M. Patnaik propose the definition of a preorder on  $W^+$  which would replace the Bruhat order of  $W^a$  and they conjecture that it is a partial order. In [Mut18], D. Muthiah extends the definition of this preorder to any Kac-Moody group  $G$ , defines a  $\mathbb{Z} \oplus \varepsilon\mathbb{Z}$ -valued length compatible with this preorder and hence shows that it is an order. In a joint work [MO19], Muthiah and D. Orr then show that this length can be evaluated at  $\varepsilon = 1$  to obtain a  $\mathbb{Z}$ -valued length strictly compatible with the order on  $W^+$ .

In order to build a Kazhdan-Lusztig theory of p-adic Kac Moody groups, we want to understand how close this order is to the Bruhat order of an affine Coxeter group, which properties still hold and which do not. The definition of a  $\mathbb{Z}$ -length is already a significant step, but many important properties, which are known to hold for Bruhat orders, remain unknown in this context. Several were proved only for Kac-Moody root systems of affine simply laced type using the specific structure of affine Weyl group of  $W^v$  in this context.

**Choice of vocabulary** The order on  $W^+$  is often mentioned in the literature as "double affine Bruhat order" and the associated length as "double affine Bruhat length" because it is most studied in the case of  $G$  a Kac-Moody group of affine type (in which case  $W^v$  is already an affine Weyl group). In this paper we refer to it as "affine Bruhat order" and "affine Bruhat length", denoted  $\ell^a$ , because we do not suppose that  $W^v$  is an affine Weyl group. Note that, if  $W^v$  is finite, then the affine Bruhat length and order on  $W^+$  are just the ones induced by its Coxeter group structure.

**Main result** Our main result is a positive answer to [MO19, Conjecture 1.5] in full generality:

For any partial order  $\leq$  on a set  $X$ , we say that  $\mathbf{y}$  covers  $\mathbf{x}$  if  $\mathbf{x} \neq \mathbf{y}$  and  $\{\mathbf{z} \in X \mid \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\} = \{\mathbf{x}, \mathbf{y}\}$ . A grading of  $X$  is a length function on  $X$  strictly compatible with  $\leq$  and such that  $\mathbf{y}$  covers  $\mathbf{x}$  if and only if  $\mathbf{x} \leq \mathbf{y}$  and  $\ell(\mathbf{y}) - \ell(\mathbf{x}) = 1$ . Gradings thus give an easy

classification of covers and more generally of maximal chains in  $X$ . The Bruhat length for a Coxeter group equipped with the Bruhat order is the prototypical example of a grading.

In [MO19], Muthiah and Orr prove that if  $\Phi$  is of affine ADE type, the affine Bruhat length gives a  $\mathbb{Z}$ -grading of  $W^+$  for the affine Bruhat order and conjecture this to be true in general. The main result of this paper is a positive answer to this conjecture:

**Theorem.** *Let  $\mathcal{D}$  be any Kac-Moody root data. Then the affine length  $\ell^a$  on  $W^+$  defines a  $\mathbb{Z}$ -grading of  $W^+$  strictly compatible with the affine Bruhat order. Otherwise said, let  $\mathbf{x} \leq \mathbf{y} \in W^+$ . Then:*

$$\mathbf{y} \text{ covers } \mathbf{x} \text{ if and only if } \ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) = 1.$$

Along the way, we obtain several geometric properties of covers for the affine Bruhat order which we expect to be insightful even if the root data is reductive (so  $W^v$  is finite and  $W^+$  is an affine Weyl group) as they only rely on the Coxeter structure of  $W^v$ .

**Further directions** In an upcoming joint work with A. Hebert, we prove that segments for the affine Bruhat order are finite in full generality, which generalizes a result proved by A. Welch [Wel22] in the affine ADE case. We use this finiteness in the context of measures to define  $R$ -Kazhdan Lusztig polynomials, following Muthiah's strategy exposed in [Mut19]. We hope that our understanding of covers would be helpful to compute these  $R$ -polynomials and to use them to define  $P$ -Kazhdan Lusztig polynomials.

Another interesting (but quite long reach) question is the following:  $W^+$  appears as the affinization of  $W^v$ , which may be taken as an affinized version of a finite Coxeter group. Can we iterate the affinization process, e.g to obtain a valid theory for reductive or Kac-Moody groups on valued fields of higher dimensions?

Finally, little is known on the preorder defined on the whole semi-direct product  $Y \rtimes W^v$ , it could be insightful to study it and to connect it to the failure of the full Iwahori-Matsumoto decomposition of  $G$ .

## Organization of the paper

**Scheme of proof** The proof relies on elementary methods, and is quite straightforward. The aim is to construct a non-trivial chain from  $\mathbf{x}$  to  $\mathbf{y}$  every time  $\mathbf{y} \geq \mathbf{x}$  verifies  $\ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) > 1$ . Let  $\text{proj}^{Y^+}$  denote the projection  $W^+ = Y^+ \rtimes W^v \rightarrow Y^+$ . We distinguish two cases which depend on the form of  $\mathbf{x}$  and  $\mathbf{y}$ : The first case is when  $\text{proj}^{Y^+}(\mathbf{y})$  lies in the orbit of  $\text{proj}^{Y^+}(\mathbf{x})$ , and the other case is when it does not.

In the first case, we show that the affine Bruhat order on  $\{\mathbf{z} \in W^+ \mid \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\}$  is, in some sense, a lift of several Bruhat-like orders on  $W^v$ , and we are able to construct chains between  $\mathbf{x}$  and  $\mathbf{y}$  from chains in  $W^v$ . The second case is more involved. Through a careful study of the relation between the vectorial chambers of  $\text{proj}^{Y^+}(\mathbf{x})$  and  $\text{proj}^{Y^+}(\mathbf{y})$ , we show that the length difference  $\ell^a(\mathbf{y}) - \ell^a(\mathbf{x})$  can be rewritten in a more workable form, making the conditions for which it is equal to one clear. Then the game is to build, explicitly, a non-trivial chain every time one of these conditions is not satisfied.

**Organization** Section 1 consists of preliminaries. In Subsection 1.1 we formally define everything we mentioned in this introduction. In particular we give the definition of the affine Bruhat order and the two affine Bruhat lengths as they are given in [MO19]. To be more flexible, we chose to define the affine Bruhat order on the whole semi-group  $Y \rtimes W^v$ , on which it may not be a preorder.

We show that we indeed recover the affine Bruhat order on  $W^+$  from this preorder in Subsection 1.3 at the end of this section, and we give other preliminary results, more classical, which are quite useful for our study of the affine Bruhat length and for the construction of chains.

We also give, in Subsection 1.2, a geometric interpretation of  $W^+$  and its affine Bruhat order, which is to be compared with the Coxeter complex of a Coxeter group, and its interpretation of the Bruhat order. Even though it is not clearly mentioned in the rest of the paper, this geometric interpretation was very useful to construct chains and understand  $W^+$ .

In Section 2, we show the result for covers such that  $\text{proj}^{Y^+}(\mathbf{y}) \in W^v \cdot \text{proj}^{Y^+}(\mathbf{x})$ . Firstly we define a relative version of the Bruhat length on  $W^v$  in Subsection 2.1 and we show that it appears naturally in the expression of the affine Bruhat length of  $W^+$  in Subsection 2.2. In Subsection 2.3, we then use this to lift chains from  $W^v$  to  $W^+$  to show the result.

In Section 3, we deal with covers such that  $\text{proj}^{Y^+}(\mathbf{y}) \notin W^v \cdot \text{proj}^{Y^+}(\mathbf{x})$ . We first show in Subsection 3.1 that these covers are of a very specific form. Namely, if  $\mathbf{x} = \pi^{v(\lambda)}w$  with  $v, w \in W^v$  and  $\lambda \in Y^{++}$ , then  $\mathbf{y}$  needs to be of the form  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  or  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  for some  $\beta \in \Phi_+$ .

The strategy is then to get enough necessary conditions on  $v, w, \lambda, \beta$  for  $\mathbf{y}$  to cover  $\mathbf{x}$ , in order to obtain a simplified expression for  $\ell^a(\mathbf{y}) - \ell^a(\mathbf{x})$ . Proposition 3.2 gives a first result in this direction. In Subsection 3.2 we fully exploit this strategy to obtain Expression (3.2) for the length difference.

Finally, in Subsection 3.3, we construct various chains from  $\mathbf{y}$  to  $\mathbf{x}$  to prove that the quantities appearing in Expression (3.2) need to be minimal when  $\mathbf{y}$  covers  $\mathbf{x}$ , which allows us to conclude.

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# 1 Preliminaries

## 1.1 Definition and notations

Let  $\mathcal{D} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  be a Kac-Moody root data as defined in [R02, §8]. It is a quintuplet such that:

- $I$  is a finite indexing set and  $A = (a_{ij})_{(i,j) \in I \times I}$  is a generalized Cartan matrix
- $X$  and  $Y$  are two dual free  $\mathbb{Z}$ -modules of finite rank, we write  $\langle \cdot, \cdot \rangle$  the duality bracket.
- $(\alpha_i)_{i \in I}$  (resp.  $(\alpha_i^\vee)_{i \in I}$ ) is a family of linearly independent elements of  $X$  (resp.  $Y$ ), the simple roots (resp. simple coroots)
- For all  $(i, j) \in I^2$  we have  $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$

**Vectorial Weyl group** For every  $i \in I$  set  $s_i \in \text{Aut}_{\mathbb{Z}}(X) : x \mapsto x - \langle \alpha_i^\vee, x \rangle \alpha_i$ . The generated group  $W^v = \langle s_i \mid i \in I \rangle$  is the **vectorial Weyl group** of the Kac-Moody root data. The duality bracket  $\langle Y, X \rangle$  induces a contragredient action of  $W^v$  on  $Y$ , explicitly  $s_i(y) = y - \langle y, \alpha_i \rangle \alpha_i^\vee$ . The bracket  $\langle \cdot, \cdot \rangle$  is then  $W^v$ -invariant.

$W^v$  is a Coxeter group, in particular it has a Bruhat order  $<$  and a length function  $\ell$  compatible with the Bruhat order.

**Vectorial distance on a Coxeter group** On any Coxeter group  $W$  we define a map  $d^W : W \times W \rightarrow W$  by  $d^W(v, w) = v^{-1}w$ , called the **vectorial distance** of  $W$ , it is  $W$ -invariant:  $d^W(rv, rw) = d^W(v, w)$  for any  $r, v, w \in W$ . We also define  $d^{\mathbb{N}} = \ell_W \circ d^W$  where  $\ell_W$  is the Bruhat length on  $W$ . These maps have properties analogous to the standard distance axioms, which justify the name. This is made precise in the context of buildings in [Ron89, Chapter 3 §1].

**Real roots** Let  $\Phi = W^v \cdot \{\alpha_i \mid i \in I\}$  be the set of real roots of  $\mathcal{D}$ , it is a root system in the classical sense, but possibly infinite. In particular let  $\Phi_+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i \cap \Phi$  be the set of positive real roots, then  $\Phi = \Phi_+ \sqcup -\Phi_+$ , we write  $\Phi_- = -\Phi_+$  the set of negative roots.

The set  $\Phi^\vee = W^v \cdot \{\alpha_i^\vee \mid i \in I\}$  is the set of **coroots**, and its subset  $\Phi_+^\vee = \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee \cap \Phi^\vee$  is the set of **positive coroots**.

To each root  $\beta$  corresponds a unique coroot  $\beta^\vee$ : if  $\beta = w(\alpha_i)$  then  $\beta^\vee = w(\alpha_i^\vee)$ . This map  $\beta \mapsto \beta^\vee$  is well defined, bijective between  $\Phi$  and  $\Phi^\vee$  and sends positive roots to positive coroots. Note that  $\langle \beta^\vee, \beta \rangle = 2$  for all  $\beta \in \Phi$ .

Moreover to each root  $\beta$  we associate a reflection  $s_\beta \in W^v$ : If  $\beta = w(\pm\alpha_i)$  then  $s_\beta := ws_iw^{-1}$ . Explicitly it is the map  $x \mapsto x - \langle \beta^\vee, x \rangle \beta$ .  $s_\beta = s_{-\beta}$  and the map forms a bijection between the the set of positive roots and the set  $\{r \in W^v \mid r^2 = 1\}$  of reflections of  $W^v$ .

**Inversion sets** For any  $w \in W^v$ , let  $\text{Inv}(w) = \Phi_+ \cap w^{-1}.\Phi_- = \{\alpha \in \Phi_+ \mid w(\alpha) \in \Phi_-\}$ . These sets are strongly connected to the Bruhat order, as by [Kum02, 1.3.13], for all  $\alpha \in \Phi_+$ :

$$\alpha \in \text{Inv}(w) \iff ws_\alpha < w \iff s_\alpha w^{-1} < w^{-1}.$$

Moreover, they are related to the Bruhat length :  $\ell(w) = |\text{Inv}(w)|$  ([Kum02, 1.3.14]).

They also have a geometric interpretation in the Coxeter complex of  $W^v$  as described below.

**Integral fundamental chamber and Tits cone** We define the (integral) fundamental chamber by  $Y^{++} = \{\lambda \in Y \mid \langle \lambda, \alpha_i \rangle \geq 0 \forall i \in I\}$ . Then, the integral Tits cone is  $Y^+ = W^v.Y^{++}$ . It is a convex cone of  $Y$ , in particular it is a semi-group for the group operation of  $Y$ , and it is equal to  $Y$  if and only if  $W^v$  is finite, if and only if  $\Phi$  is finite, if and only if  $A$  is of finite type (see [Kum02, 1.4.2])

$Y^{++}$  is a fundamental domain for the action of  $W^v$  on  $Y^+$ , and for any  $\lambda \in Y^+$  we define  $\lambda^{++}$  to be the unique element of  $Y^{++}$  in its  $W^v$ -orbit.

There is a height function on  $Y^+$ , defined as follows:

**Definition 1.1.** Let  $(\Lambda_i)_{i \in I}$  be a set of fundamental weights, that is to say  $\langle \alpha_i^\vee, \Lambda_i \rangle = \delta_{ij}$  for any  $i, j \in I$ . We fix it once and for all. Let  $\rho = \sum_{i \in I} \Lambda_i$ . Then for any  $\lambda \in Y$  define the **height** of  $\lambda$  as:

$$\text{ht}(\lambda) = \langle \lambda, \rho \rangle.$$

The height depends on the choice of fundamental weights, but its restriction to  $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$  does not:  $\text{ht}(\sum_{i \in I} n_i \alpha_i^\vee) = \sum_{i \in I} n_i$ .

**Affine Weyl group** The action of  $W^v$  on  $Y$  allows to form the semi-direct product  $Y \rtimes W^v$  and we denote its elements by  $\pi^\lambda w$  with  $\lambda \in Y, w \in W^v$ .

By definition,  $Y^+ \subset Y$  is stable by the action of  $W^v$  on  $Y$ , therefore we can form  $W^+ = Y^+ \rtimes W^v$  which is a sub semi-group of  $Y \rtimes W^v$ . This semi-group is called the affine Weyl group (although it is not a group if  $W^v$  is infinite). In [MO19] D. Muthiah and D. Orr define a Bruhat order and an associated length function on  $W^+$  which we aim to study in this article.

Denote by  $\text{proj}^{Y^+} : W^+ \rightarrow Y^+$  the canonical projection, which sends  $\pi^\lambda w$  onto  $\lambda$ . Moreover denote by  $\text{proj}^{Y^{++}} : W^+ \rightarrow Y^{++}$  the projection to  $Y^{++}$ :  $\text{proj}^{Y^{++}}(\mathbf{x}) = (\text{proj}^{Y^+}(\mathbf{x}))^{++}$ . Let us call  $\text{proj}^{Y^+}(\mathbf{x})$  the coweight of  $\mathbf{x}$ , and  $\text{proj}^{Y^{++}}(\mathbf{x})$  its **dominance class**.

**Affine roots** Let  $\Phi^a = \Phi \times \mathbb{Z}$  be **the set of affine roots** and denote by  $\beta + n\pi$  the affine root  $(\beta, n)$ .  $\beta + n\pi$  is said to be positive if  $n > 0$  or  $(n = 0$  and  $\beta \in \Phi_+)$  and we write  $\Phi_+^a$  for the set of positive affine roots. We have  $\Phi^a = \Phi_+^a \sqcup -\Phi_+^a$ .

$Y \rtimes W^v$  acts on  $\Phi^a$  by:

$$\pi^\lambda w(\beta + n\pi) = w(\beta) + (n + \langle \lambda, w(\beta) \rangle)\pi. \quad (1.1)$$



For any  $n \in \mathbb{Z}$ , its sign is denoted  $\text{sgn}(n) \in \{-1, +1\}$ , with the convention that  $\text{sgn}(0) = +1$ .

For  $n \in \mathbb{Z}$  and  $\beta \in \Phi_+$ , set:

$$\beta[n] = \text{sgn}(n)\beta + |n|\pi \in \Phi_+^a \quad (1.2)$$

$$s_{\beta[n]} = \pi^{n\beta^\vee} s_\beta. \quad (1.3)$$

If  $n \neq 0$  we also define  $\beta[n] \in \Phi_+^a$  for  $\beta \in \Phi_-$ , by  $\beta[n] = (-\beta)[-n]$ .

**Bruhat order on  $W^+$**  Recall Braverman, Kazhdan and Patnaik's definition of the Bruhat order  $<$  introduced in [BKP16, Section B. 2]: Let  $\mathbf{x} \in W^+$  and let  $\beta[n] \in \Phi_+^a$  be such that  $\mathbf{x}s_{\beta[n]} \in W^+$ . Then:

$$\mathbf{x} < \mathbf{x}s_{\beta[n]} \iff \mathbf{x}(\beta[n]) \in \Phi_+^a. \quad (1.4)$$

Explicitly, if  $\mathbf{x} = \pi^\lambda w \in W^+$ , this is written:

$$\begin{aligned} |n| + \text{sgn}(n)\langle \lambda, w(\beta) \rangle &> 0 \\ \text{or } |n| + \text{sgn}(n)\langle \lambda, w(\beta) \rangle &= 0 \text{ and } \text{sgn}(n)w(\beta) > 0. \end{aligned}$$

Then we extend this relation by transitivity, which makes it a preorder. Originally, they defined it only for affine vectorial Weyl groups, but the definition extends to any vectorial Weyl group and Muthiah showed in [Mut18] that it is an order on  $W^+$  in general.

**Extension to  $Y \rtimes W^v$**  As the whole semi-direct product  $Y \rtimes W^v$  acts on  $\Phi_+^a$ , Formula (1.4) makes sense for any  $\mathbf{x} \in Y \rtimes W^v$ , and in this paper we define  $<$  on the whole semi-direct product  $Y \rtimes W^v$ , as the closure by transitivity of the relation defined through (1.4) for  $\mathbf{x} \in Y \rtimes W^v$ . We show in the next section that if  $\mathbf{x} < \mathbf{y}$  and  $\mathbf{y} \in W^+$ , then  $\mathbf{x} \in W^+$ , this ensures that the restriction of the  $Y \rtimes W^v$ -preorder to  $W^+$  coincides with Braverman, Kazhdan, Patnaik's order on  $W^+$ . However  $<$  may not be an order on  $Y \rtimes W^v$ .

**Bruhat order through a right action** The Bruhat order can also be recovered using a right action of  $Y \rtimes W^v$  on  $\Phi_+^a$ , namely:

$$s_{\beta[n]}\pi^\lambda w > \pi^\lambda w \iff (\pi^\lambda w)^{-1}(\beta[n]) = \text{sgn}(n)w^{-1}(\beta) + (|n| - \text{sgn}(n)\langle \lambda, \beta \rangle)\pi \in \Phi_+^a. \quad (1.5)$$

Note that this formula holds if  $\beta \in \Phi_+$  and  $n \in \mathbb{Z}$ , but also if  $\beta \in \Phi_-$  and  $n \in \mathbb{Z} \setminus \{0\}$ . In case  $\beta \in \Phi_-$  and  $n = 0$  however, it is no longer correct and Formula (1.5) needs to be applied with the root  $-\beta[0]$ .

Applying reflections on the left is better suited for the geometric interpretation we will give in the next paragraph.

**Length functions on  $W^+$**  Muthiah and Orr associated a length function strictly compatible with the Bruhat order on  $W^+$ , generalizing the classical Bruhat length on Coxeter groups. They use it to show that the Bruhat order on  $W^+$  is anti-symmetric (so it is an order and not only a preorder). However, this length function takes values in  $\mathbb{Z} \oplus \varepsilon\mathbb{Z}$  and not in  $\mathbb{N}$ , setting  $\varepsilon = 1$  gives a well-behaved length with integral values, but which can take negative values. Let us now give its definition, it heavily relies on the inversion sets we introduced above.

**Definition 1.2.** *The **affine length function** is the map  $W^+ \rightarrow \mathbb{Z} \oplus \varepsilon\mathbb{Z}$  defined by:*

$$\ell_\varepsilon^\alpha(\pi^\lambda w) = 2\text{ht}(\lambda^{++}) + \varepsilon(|\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}| - |\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}|).$$

The **affine length with integral values** is the affine length function on which we set  $\varepsilon = 1$ :

$$\ell^\alpha(\pi^\lambda w) = 2\text{ht}(\lambda^{++}) + (|\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}| - |\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}|).$$

In [MO19] the authors show that both lengths functions are strictly compatible with the Bruhat order on  $W^+$ , in the sense that:

$$\mathbf{x}s_{\beta[n]} > \mathbf{x} \iff \ell_\varepsilon^\alpha(\mathbf{x}s_{\beta[n]}) > \ell_\varepsilon^\alpha(\mathbf{x}) \iff \ell^\alpha(\mathbf{x}s_{\beta[n]}) > \ell^\alpha(\mathbf{x}).$$

In what follows we will mostly use  $\ell^\alpha$  and rarely mention  $\ell_\varepsilon^\alpha$ .

## 1.2 Geometric interpretation

We introduced everything in a very algebraic way, but there is a strong geometric intuition behind root systems, vectorial Weyl groups and the vectorial Bruhat order, developed for instance in the context of buildings in [Ron89]. There is also a geometrical interpretation of the Bruhat order on  $W^+$  which we develop in this paragraph, it takes place in the standard apartment of the measure associated to a Kac-Moody group with underlying Kac-Moody data  $\mathcal{D}$ .

Let  $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$ ,  $X$  embeds in its dual  $V^\vee$  and the vectorial Weyl group  $W^v$  acts naturally on it. Inside  $V$  we have the fundamental chamber  $C_0^v = \{v \in V \mid \langle v, \alpha_i \rangle \geq 0\}$  and the Tits cone  $\mathcal{T} = W^v.C_f^v$ . A **vectorial chamber** is a set of the form  $w.C_f^v$  for  $w \in W^v$ . Since the interior of  $C_f^v$  has trivial stabiliser in  $W^v$ , the set of chambers is in natural bijection with  $W^v$  by  $w \mapsto wC_0^v = C_w^v$ .

To each root  $\beta \in \Phi_+$  let  $M_\beta = \{x \in V \mid \langle x, \beta \rangle = 0\}$ , it is an hyperplane of  $V$  and, if  $\beta = w(\alpha_i)$  with  $\alpha_i$  a simple root, then  $M_\beta \cap \mathcal{T} = C_w^v \cap C_{ws_i}^v$ .

We can put a structure of simplicial complex on  $\mathcal{T}$ , for which chambers are the cells of maximal rank and the walls are the cells of maximal rank within non-chambers. The walls split the Tits cone in two parts, and separate the set of vectorial chambers in two:

say that  $C_w^v$  is on the positive side of  $M_\beta$  if  $w^{-1}(\beta) > 0$ . In particular since  $\beta$  is a positive root, the positive side is always the one which contain the fundamental chamber.

Then the vectorial Bruhat order can be interpreted by:  $s_\beta w > w$  if and only if, when we split  $\mathcal{T}$  along  $M_\beta$  the chambers  $C_w^v$  and  $C_0^v$  are in the same connected component of  $\mathcal{T}$ , that is to say  $C_w^v$  is on the positive side of  $M_\beta$ .

Moreover  $\text{Inv}(w^{-1})$  the inversion set of  $w^{-1}$  can be interpreted as the set of walls separating the chamber  $C_w^v = w.C_0$  from the fundamental chamber  $C_0^v$ .

In Figure 1.1 we represent the Tits cone and its structure for a root system of rank 2 with Cartan matrix  $\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$ , which is of indefinite type. The Tits cone is colored in blue, and the vectorial chamber  $C_w^v$  is labelled by  $w$ . It is an approximation since  $W^v$  is infinite.

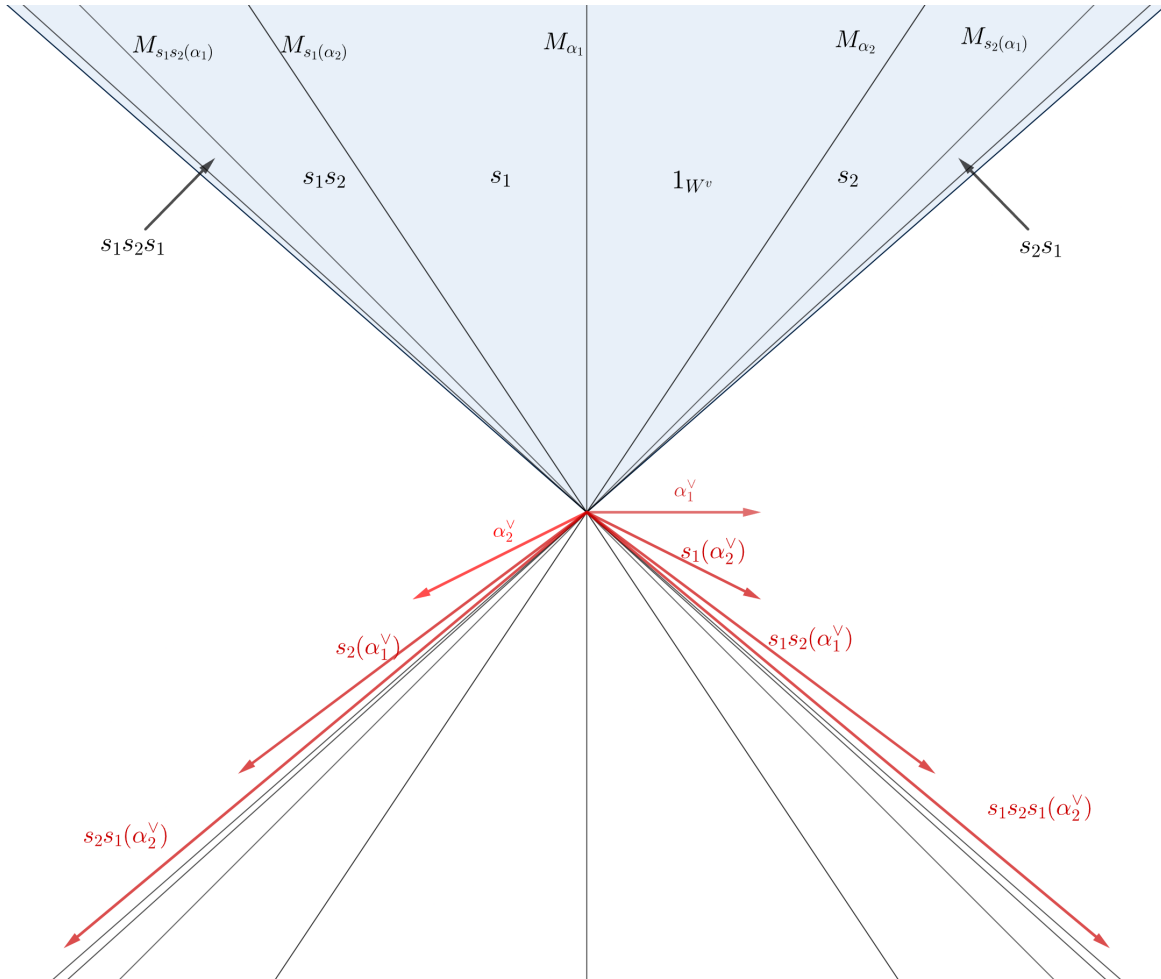


Figure 1.1: The Tits cone for a root system of Cartan Matrix  $\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$ .

Let us now turn to the interpretation of the  $W^+$ -Bruhat order. Let  $\mathbb{A}$  be a real affine space with underlying vectorial space  $V$ , we call  $\mathbb{A}$  the (standard) affine apartment associated to  $\mathcal{D}$ . The tangent space of  $\mathbb{A}$  is canonically isomorphic to  $T\mathbb{A} = \mathbb{A} \times V$ , with, for any  $x \in \mathbb{A}$ ,  $T_x\mathbb{A} = \{x\} \times V$ .

The semi-group  $W^+$  has an affine action on  $\mathbb{A}$ , given by  $\pi^\lambda w(x) = -\lambda + w(x)$ , it induces an action on the tangent space  $T\mathbb{A}$  given by  $\pi^\lambda w((x, v)) = (-\lambda + w(x), w(v))$ . To any positive affine root  $\beta[n] \in \Phi_+^a$  corresponds an affine hyperplane  $M_{\beta[n]} = \{x \in \mathbb{A} \mid \langle x, \beta \rangle + n = 0\}$ , the **affine wall** associated to the affine root  $\beta[n]$ . For any  $x \in M_{\beta[n]}$  we have  $T_x M_{\beta[n]} = \{x\} \times M_\beta \subset T_x \mathbb{A}$ .

For any  $\pi^\lambda w \in W^+$  let  $C_{\pi^\lambda w} = \{-\lambda\} \times C_w^v \subset T_\lambda \mathbb{A} \subset T\mathbb{A}$ , we call it the **alcove of type  $\pi^\lambda w$** . Mirroring the classical situation,  $C_0 = \{0\} \times C_f^v$  is a fundamental domain for the action of  $W^+$  on  $Y^+ \times \mathcal{T} \subset T\mathbb{A}$  and  $W^+$  acts on  $\{C_{\mathbf{x}} \mid \mathbf{x} \in W^+\}$  simply transitively. Affine walls separate naturally the set of alcoves in two and we call the side containing  $C_0$  the positive side.

Then the  $W^+$ -Bruhat order can be interpreted by:  $s_{\beta[n]}\pi^\lambda w > \pi^\lambda w$  if and only if  $C_{\pi^\lambda w}$  is on the positive side of  $M_{\beta[n]}$ .

We give an illustration of the affine apartment in Figure 1.2 below.

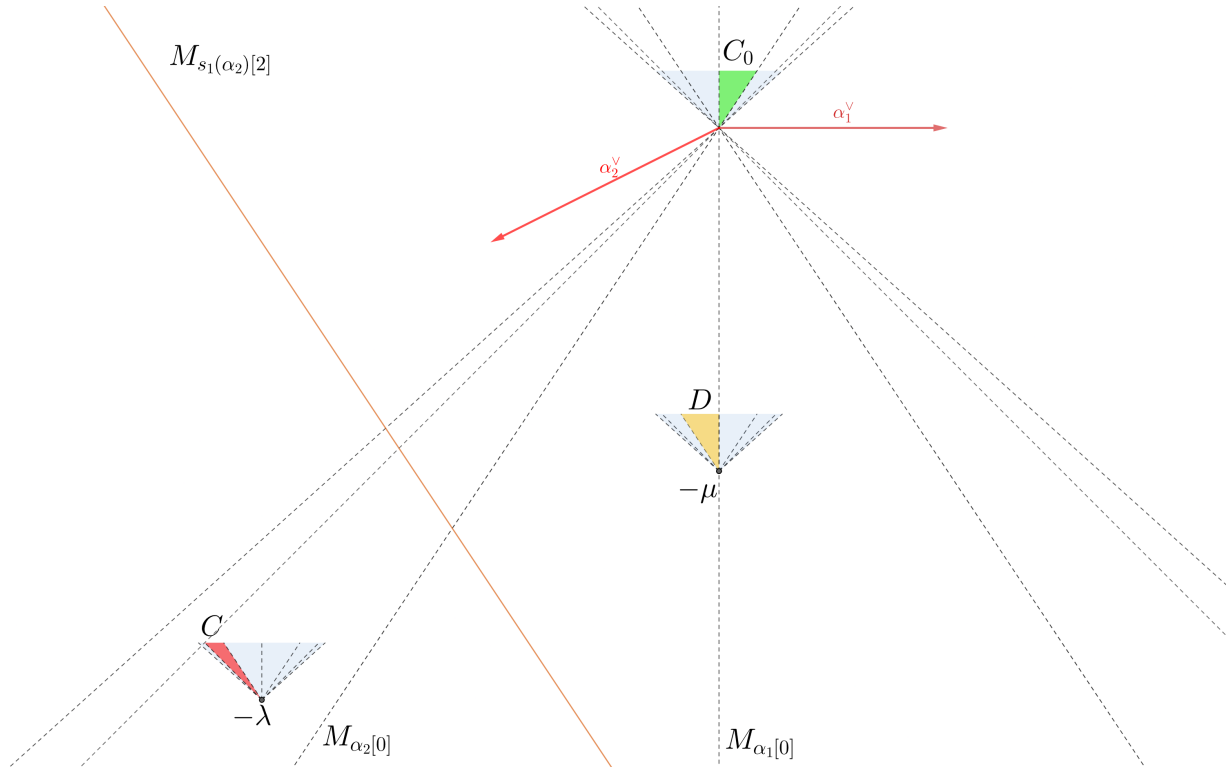


Figure 1.2: The affine apartment for a root system of Cartan Matrix  $\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$ .

In Figure 1.2 we represent the affine apartment for the same root data as in Figure 1.1. The blue polygons represent the local Tits cones at three different points: the origin,  $-\lambda \in -Y^+$  and  $-\mu$ , which is the image of  $-\lambda$  by the reflection along the wall  $M_{s_1(\alpha_2)[2]}$  (represented in green).

We have highlighted three alcoves: In orange the alcove  $C_0$ ; in red the alcove  $C = C_{\pi^\lambda s_1 s_2}$  and in green  $D = C_{\pi^\mu s_1}$  which is the image of  $C$  by  $s_{s_1(\alpha_2)[2]}$ . We see that  $D$  is on the same side of  $M_{s_1(\alpha_2)[2]}$  as the fundamental alcove  $C_0$ , thus:  $\pi^\lambda s_1 s_2 = s_{s_1(\alpha_2)[2]}(\pi^\mu s_1) > \pi^\mu s_1$ .

Note that  $-\lambda$  lies in the negative vectorial chamber  $-s_2 C_f^v$ , that is to say that  $s_2 \lambda$  is dominant. Therefore  $\pi^\lambda s_2$  is the minimal length element of  $\pi^\lambda W^v$ . We will make this more explicit in Section 2.2.

### 1.3 Preliminary results

In this paper, we consider the action of reflections on  $W^+$  on the left. To switch between the right and left actions note that:

$$\pi^\lambda w s_{\beta[n]} = \pi^{\lambda + n w(\beta^\vee)} s_{w(\beta)} w = \pi^{s_{w(\beta)} \lambda + (n + \langle \lambda, w(\beta) \rangle) w(\beta^\vee)} s_{w(\beta)} w = s_{w(\beta)[n + \langle \lambda, w(\beta) \rangle]} \pi^\lambda w.$$

Since the affine Bruhat order is generated on  $W^+$  by the relations  $s_{\beta[n]} \mathbf{x} > \mathbf{x} \iff \ell^a(s_{\beta[n]} \mathbf{x}) > \ell^a(\mathbf{x})$  for affine roots  $\beta[n] \in \Phi_+^a$ , covers are always of this form.

**Lemma 1.3.** *Let  $\pi^\lambda w \in Y \rtimes W^v$  and  $\beta[n] \in \Phi_+^a$ . Write  $\pi^\mu w'$  for  $s_{\beta[n]} \pi^\lambda w$  and suppose that  $(\pi^\lambda w)^{-1}(\beta[n]) \in \Phi_+^a$ . Then  $\lambda \in [\mu, s_\beta \mu]$ . In particular,  $\mu \in Y^+ \implies \lambda \in Y^+$ .*

*Proof.* Explicitely:

$$\pi^\mu w' = \pi^{n \beta^\vee} s_\beta \pi^\lambda w = \pi^{s_\beta \lambda + n \beta^\vee} s_\beta w.$$

Thus  $\mu = s_\beta \lambda + n \beta^\vee = \lambda + (n - \langle \lambda, \beta \rangle) \beta^\vee$  and  $s_\beta \mu = \lambda - n \beta^\vee$ .

Moreover, since  $(\pi^\lambda w)^{-1}(\beta[n]) \in \Phi_+^a$ , by Formula (1.5),  $|n| - \text{sgn}(n) \langle \lambda, \beta \rangle = \text{sgn}(n)(n - \langle \lambda, \beta \rangle) \geq 0$ .

Therefore, unless  $n - \langle \lambda, \beta \rangle = 0$ ,  $n$  and  $n - \langle \lambda, \beta \rangle$  have same sign and thus,  $\lambda = s_\beta \mu + n \beta^\vee = \mu - (n - \langle \lambda, \beta \rangle) \beta^\vee$  lies in  $[s_\beta \mu, \mu]$ . If  $n - \langle \lambda, \beta \rangle = 0$  then  $\mu = \lambda$  and the result remains true.

In particular, since  $Y^+$  is convex and  $W^v$ -stable, if  $\mu$  lies in  $Y^+$  then so does  $\lambda$ .  $\square$

We directly obtain from the last point that:

**Corollary 1.4.** *The affine Bruhat order defined on  $W^+$  coincides with the restriction of the preorder defined through (1.4) on the whole semi-direct product  $Y \rtimes W^v$*

We give here a few elementary results on the height function, which will be useful in our study of the affine Bruhat length. They are also used in [MO19, Section 3].

**Lemma 1.5.** *For any  $w \in W^v$ ,*

$$\rho - w^{-1}(\rho) = \sum_{\gamma \in \text{Inv}(w)} \gamma.$$

*Proof.* This is [Kum02, 1.3.22 Corollary 3], we prove it by induction on the length of  $w$ :

- If  $w$  is a simple reflection  $s_\alpha$  then  $\text{Inv}(s_\alpha) = \{\alpha\}$  and  $\rho - s_\alpha(\rho) = \langle \alpha^\vee, \rho \rangle \alpha = \alpha$  since  $\langle \alpha^\vee, \rho \rangle = 1$  by definition of  $\rho$ .
- Suppose the result true for elements of length  $n$ , and suppose that  $\ell(w) = n + 1$  then write  $w = w_1 s_\alpha$  for  $\alpha$  a simple root and  $w_1$  an element of length  $n - 1$ . Then :  
 $\rho - w(\rho) = \rho - w_1(\rho) + w_1(\rho - s_\alpha(\rho)) = \sum_{\gamma \in \text{Inv}(w_1^{-1})} \gamma + w_1(\alpha)$  and since  $\text{Inv}(w^{-1}) = \text{Inv}(w_1^{-1}) \sqcup \{w_1(\alpha)\}$  we get the result for  $w$ .

□

**Corollary 1.6.** *For any positive root  $\beta \in \Phi_+$  we have:*

$$2\text{ht}(\beta^\vee) = \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle. \quad (1.6)$$

*Moreover all the terms of the summand are positive.*

*Proof.* This is a direct application of Lemma 1.5. Moreover for any  $\gamma \in \text{Inv}(s_\beta)$ , by definition  $\gamma \in \Phi_+$  and  $s_\beta(\gamma) = \gamma - \langle \beta^\vee, \gamma \rangle \beta^\vee \in \Phi_-$  so, since  $\beta$  is a positive root the coefficient  $\langle \beta^\vee, \gamma \rangle$  is necessarily positive. □

**Corollary 1.7.** *Let  $\mu \in Y^+$  and  $u \in W^v$  be such that  $\mu = u\mu^{++}$ . Then:*

$$\text{ht}(\mu^{++}) = \text{ht}(\mu) - \sum_{\tau \in \text{Inv}(u^{-1})} \langle \mu, \tau \rangle. \quad (1.7)$$

*Moreover the terms in this summand are non-positive.*

*Proof.* By definition  $\text{ht}(\mu^{++}) = \langle u^{-1}(\mu), \rho \rangle = \langle \mu, u(\rho) \rangle$ , and by Lemma 1.5:

$$\text{ht}(\mu^{++}) = \langle \mu, u(\rho) \rangle = \langle \mu, \rho - \sum_{\tau \in \text{Inv}(u^{-1})} \tau \rangle = \text{ht}(\mu) - \sum_{\tau \in \text{Inv}(u^{-1})} \langle \mu, \tau \rangle.$$

Moreover, for any  $\tau \in \Phi$ , we have  $\langle \mu, \tau \rangle = \langle \mu^{++}, u^{-1}(\tau) \rangle$ , so  $\tau \in \text{Inv}(u^{-1}) \implies \langle \mu, \tau \rangle \leq 0$  and the terms of the above summand are all non-positive. □

We end this section with a classical result on Coxeter complexes which will be useful to construct many chains for the affine Bruhat order. Recall that, for any Coxeter group  $W$  with simple reflection set  $S$ , a minimal gallery in  $W$  from  $w$  to  $v$  is a sequence  $w = w_1, \dots, w_n = v$  such that  $w_{i+1} = w_i s_i$  for some simple reflection  $s_i \in S$ . A gallery is said to be minimal if its length  $n$  is equal to  $d^{\mathbb{N}}(w_1, w_n)$ . We refer to [Ron89, Chapter 2] for the elementary properties of minimal galleries in Coxeter complexes. The next Lemma is another formulation of [Ron89, Proposition 2.8].

**Lemma 1.8.** *Let  $W$  be a Coxeter group and let  $v_1, v_2, w \in W$  be such that  $v_2$  is not on a minimal gallery from  $v_1$  to  $w$ . Then there is a reflection  $r \in W$  such that  $d^W(v_1, rw) > d^W(v_1, w)$  and  $d^W(v_2, rw) < d^W(v_2, w)$ .*

*Proof.* If  $v_2$  is not on a minimal gallery from  $v_1$  to  $w$ , there exists a root (seen as a half-apartment)  $\alpha$  such that  $v_1, w \in \alpha$  and  $v_2 \notin \alpha$ , by [Ron89, Proposition 2.8]. Then consider the folding along  $\alpha$ , defined by:

$$\forall u \in W, \rho_\alpha(u) = \begin{cases} r_\alpha u & \text{if } u \notin \alpha \\ u & \text{otherwise.} \end{cases}$$

It reduces the vectorial distance, hence:

$$d^W(v_1, w) = d^W(\rho_\alpha(v_1), \rho_\alpha(r_\alpha w)) < d^W(v_1, r_\alpha w).$$

$$d^W(v_2, r_\alpha w) = d^W(r_\alpha v_2, w) = d^W(\rho_\alpha(v_2), \rho_\alpha(w)) < d^W(v_2, w).$$

□

## 2 The case of constant dominance class

In this section, we study the affine Bruhat order restricted to a dominance class, that is to say, for a given  $\lambda^{++} \in Y^{++}$ , the restriction of the affine Bruhat order to the subset  $(\text{proj}^{Y^{++}})^{-1}(\lambda^{++}) = \{\pi^\mu w \in W^+ \mid \mu \in W^v \cdot \lambda^{++}\}$ .

We start this section by defining a relative length and a relative Bruhat order on  $W^v$ , which naturally appear in the affine length  $\ell^a$  on  $W^+$ . This connection was already observed by Muthiah and Orr in [MO18].

### 2.1 Relative length on $W^v$

**Definition 2.1.** *For any  $v, w \in W^v$  let:*

$$\ell_v(w) = |\text{Inv}(w^{-1}) \setminus \text{Inv}(v^{-1})| - |\text{Inv}(w^{-1}) \cap \text{Inv}(v^{-1})|.$$

*This is a signed version of the Bruhat length, in particular  $\ell_1 = \ell$ .*

*We associate an order to  $\ell_v$  by setting, for any element  $w \in W^v$  and any reflection  $r \in W^v$ :  $w <_v wr$  if and only if  $\ell_v(w) < \ell_v(wr)$  and then let  $<_v$  be the order generated by these relations, it is strictly compatible with  $\ell_v$ . In particular  $<_1$  is the classical Bruhat order.*

As does the Bruhat length, the lengths  $\ell_v$  have a geometric interpretation on the Coxeter complex associated to  $W^v$ : it is the summand  $\ell_v(w) = \sum \epsilon_v(M)$  over every wall separating the chamber 1 and the chamber  $w$ , where  $\epsilon_v(M)$  is  $-1$  if  $M$  separates 1 and  $v^{-1}$  and  $+1$  otherwise.

We will use this relative length to give an alternative definition of the affine length. Let us first give an explicit formula for  $\ell_v$  depending only on the classical length  $\ell = \ell_1$ .

**Lemma 2.2.** *If  $sv > v$  with  $v \in W^v$  and  $s$  a simple reflection then for any  $w \in W^v$ ,  $\ell_{sv}(w) = \ell_v(sw) - 1$ .*

*Proof.* For any  $w \in W^v$ , the map  $\gamma \mapsto s\gamma$  defines a bijection:

$$\text{Inv}(w^{-1}) \setminus \{\alpha_s\} \cong \text{Inv}(w^{-1}s) \setminus \{\alpha_s\}.$$

Moreover because  $sv > v$ ,  $\alpha_s \in \text{Inv}(v^{-1}s)$  and  $\alpha_s \notin \text{Inv}(v^{-1})$ .

Therefore:

$$|\text{Inv}(w^{-1}) \cap \text{Inv}(v^{-1}s) \setminus \{\alpha_s\}| = |\text{Inv}(w^{-1}s) \cap \text{Inv}(v^{-1})|$$

and

$$|\text{Inv}(w^{-1}) \setminus \text{Inv}(v^{-1}s)| = |\text{Inv}(w^{-1}s) \setminus (\text{Inv}(v^{-1}) \cup \{\alpha_s\})|.$$

- If  $\alpha_s \in \text{Inv}(w^{-1})$  then  $\alpha_s \notin \text{Inv}(w^{-1}s)$  and  $\ell_{sv}(w) = |\text{Inv}(w^{-1}s) \setminus \text{Inv}(v^{-1})| - (|\text{Inv}(w^{-1}s) \cap \text{Inv}(v^{-1})| + 1) = \ell_v(sw) - 1$ .
- If  $\alpha_s \notin \text{Inv}(w^{-1})$  then  $\alpha_s \in \text{Inv}(w^{-1}s)$  and  $\ell_{sv}(w) = (|\text{Inv}(w^{-1}s) \setminus \text{Inv}(v^{-1})| - 1) - |\text{Inv}(w^{-1}s) \cap \text{Inv}(v^{-1})| = \ell_v(sw) - 1$ .

□

**Proposition 2.3.** *For all  $v, w \in W^v$  we have:*

$$\ell_v(w) = \ell(v^{-1}w) - \ell(v).$$

*Proof.* Since  $\ell = \ell_1$ , we take a reduced expression for  $v$  and apply Lemma 2.2 recursively to get the result. □

The order  $<_v$  also has a geometric interpretation which will be important later on, and given by the following corollary:

**Corollary 2.4.** *For any root  $\alpha \in \Phi$  and element  $w \in W^v$ , we have that  $w <_v s_\alpha w$  if and only if, in the Coxeter complex of  $W^v$ ,  $C_w^v$  and  $C_v^v$  are on the same side of the wall  $M_\alpha$ .*

*Proof.* We have  $\ell_v(sw) - \ell_v(w) = \ell(v^{-1}sw) - \ell(v^{-1}w)$  and by the definition of the Coxeter complex this is positive if and only if  $C_v^v$  and  $C_w^v$  are on the same side of the wall  $M_s$ . □

Therefore  $<_v$  can be interpreted as a shift of the classical Bruhat order, corresponding geometrically to taking  $C_v$  as fundamental chamber in the Coxeter complex.



## 2.2 Relation with the affine Bruhat length

For  $\lambda \in Y^+$ , let  $\Phi_\lambda$  denote the set  $\{\alpha \in \Phi \mid \langle \lambda, \alpha \rangle = 0\}$  and  $W_\lambda = \text{Stab}_{W^v}(\lambda)$ . Let  $v \in W^v$  be such that  $\lambda = v\lambda^{++}$ . Then  $W_\lambda v = vW_{\lambda^{++}}$  and, since  $\lambda^{++}$  is dominant,  $W_{\lambda^{++}}$  is a standard parabolic subgroup, that is a group of the form  $W_J = \langle s \mid s \in J \rangle$  where  $J \subset S$  is a set of simple reflections. More precisely,  $J = \{s \in S \mid s.\lambda^{++} = \lambda^{++}\}$ .

By standard Coxeter group theory (see for instance [BB05, Section 2.2]), for any  $u \in W^v$ , the left coset  $uW_{\lambda^{++}} = uW_J$  has a unique representative of minimal length which we denote  $u^J$ , and one has a decomposition  $u = u^J u_J$  with  $u_J \in W_J$  such that:

$$\ell(u) = \ell(u^J) + \ell(u_J). \quad (2.1)$$

**Definition 2.5.** • For any  $\lambda \in Y^{++}$ , we denote by  $W^\lambda$  the set of minimal length representatives for  $W_\lambda$  cosets:  $w \in W^\lambda \iff \forall \tilde{w} \in W_\lambda, \ell(w\tilde{w}) > \ell(w)$

- For any  $\lambda \in Y^+$ , we denote by  $v^\lambda$  the minimal length element in  $W^v$  which verifies  $\lambda = v^\lambda \lambda^{++}$ . In other words, for any  $u \in W^v$  such that  $\lambda = u\lambda^{++}$ , we have  $v^\lambda = u^J$ , where  $J$  is the set of simple reflections such that  $W_J = W_{\lambda^{++}}$ .

**Proposition 2.6.** For any coweight  $\lambda = v\lambda^{++} \in Y^+$ , for any  $w \in W^v$ , we have:

$$|\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}| - |\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}| = \ell_{v^\lambda}(w).$$

Thus:

$$\ell_\varepsilon^a(\pi^\lambda w) = 2\text{ht}(\lambda^{++}) + \varepsilon \ell_{v^\lambda}(w) \text{ and } \ell^a(\pi^\lambda w) = 2\text{ht}(\lambda^{++}) + \ell_{v^\lambda}(w)$$

*Proof.* For  $\lambda \in Y^+$  and  $v \in W^v$  such that  $\lambda = v\lambda^{++}$ , then,  $\alpha \in \Phi_+$  verifies  $\langle \lambda, \alpha \rangle \geq 0$  if and only if  $\alpha \in \Phi_\lambda \cup (\Phi_+ \setminus \text{Inv}(v^{-1}))$ , so:

$$|\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}| = (\text{Inv}(w^{-1}) \setminus \text{Inv}(v^{-1})) \bigsqcup (\text{Inv}(w^{-1}) \cap \text{Inv}(v^{-1}) \cap \Phi_\lambda)$$

$$\text{Inv}(w^{-1}) \cap \text{Inv}(v^{-1}) = \{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\} \bigsqcup (\text{Inv}(w^{-1}) \cap \text{Inv}(v^{-1}) \cap \Phi_\lambda).$$

Therefore we have:

$$|\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}| - |\{\alpha \in \text{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}| = \ell_v(w) + 2|\text{Inv}(w^{-1}) \cap \text{Inv}(v^{-1}) \cap \Phi_\lambda|$$

We show that  $\text{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda = \emptyset$  which is enough to prove the result.

Let  $\alpha \in \text{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda$ , then since  $\alpha \in \Phi_\lambda$ ,  $s_\alpha$  fixes  $\lambda$ , that is  $s_\alpha \in W_\lambda$ . Moreover  $(v^\lambda)^{-1}(\alpha) < 0$  so  $s_\alpha v^\lambda < v^\lambda$ , this contradicts the minimality of  $v^\lambda$  (note that, as  $W_\lambda v^\lambda = v^\lambda W_{\lambda^{++}}$ ,  $v^\lambda$  is also the minimal representative for the right coset  $W_\lambda v^\lambda$ ). Hence  $\text{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda = \emptyset$ .  $\square$

**Corollary 2.7.** Let  $\lambda \in Y^+$  and  $w \in W^v$ . Suppose that  $\pi^\mu w' = s_{\beta[n]} \pi^\lambda w$  for some affine root  $\beta[n] \in \Phi^a$  is such that  $\mu^{++} = \lambda^{++}$ . Then:

$$\pi^\mu w' > \pi^\lambda w \iff \ell_{v^\mu}(w') > \ell_{v^\lambda}(w).$$

In particular:

- If  $w, w' \in W^v$ , then:

$$\pi^\lambda w < \pi^\lambda w' \iff w <_{v^\lambda} w'.$$

In particular  $\pi^\lambda v^\lambda$  is the minimal element of  $\pi^\lambda W^v$ .

- If  $r \in W^v$ , then:

$$\pi^{r^\lambda} r w > \pi^\lambda w \iff r v^\lambda < v^\lambda.$$

**Remark 2.8.** The fact that  $\text{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda = \emptyset$  is visible geometrically in the Coxeter complex of  $W$ , in which  $\Phi_\lambda$  is the set of walls containing  $\lambda$ ,  $\text{Inv}(v^{-1})$  is the set of walls separating  $C_f^v$  and  $vC_f^v$ , and  $v^\lambda C_f^v$  is the projection of the fundamental chamber on the face containing  $\lambda$ , that is to say the closest chamber from the fundamental one amongst the chambers containing  $\lambda$  in their closure.

**Remark 2.9.** By combining Formula (1.7) with Proposition 2.6, we obtain the formulas already given by Muthiah and Orr in [MO19, Proposition 3.10].

## 2.3 Covers with constant dominance class

We now use the relative vectorial Bruhat order and length to prove that, for any  $\lambda^{++} \in Y^{++}$ , covers in  $\text{proj}^{Y^{++}}(\lambda^{++})$  increase the affine Bruhat length by exactly one. This is a first step towards a characterization of general covers.

**Theorem 2.10.** If  $\mathbf{x} \triangleleft \mathbf{y}$  is a cover in  $W^+$  with  $\text{proj}^{Y^+}(\mathbf{y}) \in W^v \cdot \text{proj}^{Y^+}(\mathbf{x})$ , then  $\ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1$ .

*Proof.* Let  $\lambda = \text{proj}^{Y^+}(\mathbf{x})$  and let  $v$  of minimal length be such that  $\lambda = v\lambda^{++}$  (so  $v = v^\lambda$  with the above notation). We can write  $\mathbf{y} = s_{\beta[n]}\mathbf{x}$  with  $\beta \in \Phi_+$  and  $n \in \mathbb{Z}$ . We suppose that  $\ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) = N > 1$  and we show that there exists a non-trivial chain from  $\mathbf{x}$  to  $\mathbf{y}$ .

Suppose first that  $\text{proj}^{Y^+}(\mathbf{y}) = \text{proj}^{Y^+}(\mathbf{x}) = \lambda$ , then  $n = \langle \lambda, \beta^\vee \rangle$  and we can write  $\mathbf{y} = \pi^\lambda s_\beta w$  with  $\mathbf{x} = \pi^\lambda w$ .

Then,  $N = \ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) = \ell_v(s_\beta w) - \ell_v(w)$  and by Corollary 2.7, it is enough to exhibit  $w' \in W^v$  such that  $w <_v w' <_v w s_\beta$ .

By Lemma 2.3,  $\ell(v^{-1}s_\beta w) - \ell(v^{-1}w) = N > 1$  and there is a chain  $v^{-1}w < v^{-1}wr_1 < \dots < v^{-1}wr_1 \dots r_N = v^{-1}w s_{w(\beta)}$ . So we have a  $<_v$ -chain  $w <_v wr_1 <_v \dots <_v wr_1 \dots r_N = s_\beta w$ , in particular  $\mathbf{y}$  does not cover  $\mathbf{x}$ .

Suppose now that  $\mathbf{y} = s_{\beta[n]}\mathbf{x} > \mathbf{x} = \pi^\lambda w$  with  $\text{proj}^{Y^+}(\mathbf{y}) \in W^v \cdot \lambda \setminus \{\lambda\}$ . To simplify notations, write  $s$  for  $s_\beta$ . We know from Corollary 2.7 that  $\mathbf{y} = \pi^{s^\lambda} s w$  with  $sv < v$ , suppose for now that  $\ell(sv) = \ell(v) - 1$ .

Let  $u \in W_J$  be such that  $svu$  is the minimal representative  $(sv)^J = v^{s^\lambda}$  of the coset  $svW_J$ . Then by equality 2.1,  $\ell(sv) = \ell(sv u) + \ell(u)$  and, since  $\ell(sv) = \ell(v) - 1$ , we obtain:

$$\begin{aligned} \ell_{v^{s^\lambda}}(sw) - \ell_v(w) &= \ell((vu)^{-1}w) - \ell(v^{-1}w) + \ell(v) - \ell(sv u) \\ &= \ell((vu)^{-1}w) - \ell(v^{-1}w) + \ell(u) + 1 \\ &= \ell(u) + 1 - (d^{\mathbb{N}}(v, w) - d^{\mathbb{N}}(vu, w)). \end{aligned} \tag{2.2}$$

Since  $d^{\mathbb{N}}(vu, v) = \ell(u)$ , by triangular inequality  $d^{\mathbb{N}}(v, w) - d^{\mathbb{N}}(vu, w) \leq \ell(u)$  hence  $\ell_{v^{s\lambda}}(sw) - \ell_v(w)$  is greater or equal to 1, with equality if and only if there is a minimal gallery from  $w$  to  $v$  going through  $vu$ . Equivalently, if and only if, in  $W_{\lambda^{++}} = W_J$ ,  $u$  is on a minimal gallery from 1 to  $v^{-1}\text{proj}_{vW_J}(w)$ , where  $\text{proj}_{vW_J}(w)$  denotes the closest element of  $vW_J$  from  $w$ .

Indeed, by standard Coxeter group theory (e.g. [Ron89, Theorem 2.9]), for any element  $\tilde{v}$  of  $vW_J$ , there is a minimal gallery from  $\tilde{v}$  to  $w$  which goes through  $\text{proj}_{vW_J}(w)$ , so if  $vu \in vW_J$  is on a minimal gallery from  $v$  to  $w$ , then one can suppose that this gallery also goes through  $\text{proj}_{vW_J}(w)$ .

We thus want to produce, when  $u$  is not on a minimal gallery from 1 to  $v^{-1}\text{proj}_{vW_J}(w)$  in  $W_J$ , an element of  $W^+$  greater than  $\pi^\lambda w$  and lesser than  $\pi^{s\lambda}sw$ . By Lemma 1.8, there is  $r \in W_J$  such that  $d^{W_J}(1, rv^{-1}\text{proj}_{vW_J}(w)) > d^{W_J}(1, v^{-1}\text{proj}_{vW_J}(w))$  and  $d^{W_J}(u, rv^{-1}\text{proj}_{vW_J}(w)) < d^{W_J}(u, v^{-1}\text{proj}_{vW_J}(w))$ . In  $W^v$ , this implies  $d^{W^v}(vr, w) > d^{W^v}(v, w)$  and  $d^{W^v}(vru, w) < d^{W^v}(vu, w)$ .

Let  $\tilde{w} = vrv^{-1}w$ . We compute:

$$\begin{aligned} \ell_v(\tilde{w}) - \ell_v(w) &= \ell(rv^{-1}w) - \ell(v^{-1}w) > 0 \\ \ell_{v^{s\lambda}}(sw) - \ell_{v^{s\lambda}}(s\tilde{w}) &= \ell((vu)^{-1}w) - \ell((vru)^{-1}w) > 0 \\ \ell_{v^{s\lambda}}(s\tilde{w}) - \ell_v(\tilde{w}) &= \ell(u) + 1 - (d^{\mathbb{N}}(v, \tilde{w}) - d^{\mathbb{N}}(vu, \tilde{w})) > 0. \end{aligned}$$

Hence by Corollary 2.7,  $\pi^\lambda w < \pi^\lambda \tilde{w} < \pi^{s\lambda} s\tilde{w} < \pi^{s\lambda} sw$ .

Finally, if  $\ell(sv) < \ell(v) - 1$ , let  $r_1 \dots r_m$  be reflections of  $W^v$  such that  $\ell(r_k \dots r_1 v) = \ell(v) - k$  for all  $k \in \llbracket 1, m \rrbracket$  and  $sv = r_m \dots r_1 v$ . Geometrically, the  $r_k$  are the reflections corresponding to the walls crossed by a minimal gallery from  $v$  to  $sv$ . By computation (2.2) we get that  $\pi^{r_k \dots r_1 \lambda} r_k \dots r_1 w < \pi^{r_{k+1} \dots r_1 \lambda} r_{k+1} \dots r_1 w$  for all  $k$  and in particular  $\pi^{s\lambda} sw$  does not cover  $\pi^\lambda w$ .  $\square$

## 3 Covers with varying dominance class

### 3.1 A few properties of covers with varying dominant coweight

We now turn to the case of covers  $\pi^\lambda w < \pi^\mu w'$  in  $W^+$  with  $\mu^{++} \neq \lambda^{++}$ .

By the formula (1.5), if  $\pi^\mu s_\beta w = s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$  then  $n \in \mathbb{Z} \setminus ]0, \langle \lambda, \beta \rangle[$  and conversely if  $n \in \mathbb{Z} \setminus [0, \langle \lambda, \beta \rangle]$  then  $s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$ , however  $s_{\beta[n]} \pi^\lambda w$  may not be in  $W^+$  as  $\lambda + \mathbb{Z}\beta^\vee \not\subset Y^+$ . The limit cases  $n \in \{0, \langle \lambda, \beta \rangle\}$  correspond to  $\lambda^{++} = \mu^{++}$  dealt with in the previous section.

We next show that covers only occur for minimal  $n$ :

**Proposition 3.1.** *Let  $\lambda \in Y^+$  and  $w \in W^v$ , let  $\beta \in \Phi$  and  $n \in \mathbb{Z}$ . Let us denote  $\sigma = \text{sgn}(\langle \lambda, \beta \rangle) \in \{1, -1\}$ . If  $\pi^\mu w' = s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$  is a cover with  $\lambda^{++} \neq \mu^{++}$ , then  $n \in \{-\sigma, \langle \lambda, \beta \rangle + \sigma\}$ .*

*Proof.* For any  $\nu \in Y^+$  if we identify the Coxeter complex of  $W^v$  with the positive Tits cone  $\mathcal{T} \subset \mathbb{A}$ ,  $C_{\nu^v}^v$  is the closest vectorial chamber, from the fundamental chamber, containing  $\nu$  in

its closure. All the elements of  $\lambda + \sigma\mathbb{Z}_{>0}\beta^\vee$  are on the same side of  $M_\beta$ , hence by Corollary 2.4, for any two such  $\nu, \nu' \in \lambda + \sigma\mathbb{Z}_{>0}\beta^\vee$  and any  $w \in W^v$ ,  $w <_{\nu} s_\beta w \iff w <_{\nu'} s_\beta w$ .

Suppose first that  $n \in \langle \lambda, \beta \rangle + \sigma\mathbb{Z}_{>1}$  and let  $\mu = \lambda + (n - \langle \lambda, \beta \rangle)\beta^\vee$ , then:

- If  $w <_{v^\mu} s_\beta w$ , we have a chain:

$$\pi^\lambda w < s_{\beta[\langle \lambda, \beta \rangle + \sigma]} \pi^\lambda w = \pi^{\lambda + \sigma\beta^\vee} s_\beta w < \pi^\mu w < \pi^\mu s_\beta w.$$

The second inequality comes from  $\pi^\mu w = s_{\beta[n+\sigma]} \pi^{\lambda + \sigma\beta^\vee} s_\beta w$  with the positivity condition being easily verified, the third comes from the hypothesis and Corollary 2.7.

- If  $s_\beta w <_{v^\mu} w$  then we have a chain:

$$\pi^\lambda w < s_{\beta[\langle \lambda, \beta \rangle + \sigma]} \pi^\lambda w = \pi^{\lambda + \sigma\beta^\vee} s_\beta w < \pi^{\lambda + \sigma\beta^\vee} w < \pi^\mu s_\beta w.$$

Here the second inequality comes from the argument above:  $s_\beta w <_{v^\mu} w \iff s_\beta w <_{v^{\lambda + \sigma\beta^\vee}} s_\beta w$ , and Corollary 2.7. The third comes from  $\pi^\mu s_\beta w = s_{\beta[n+\sigma]} \pi^{\lambda + \sigma\beta^\vee} w$  with the positivity condition (1.5) being verified.

Either way, for  $n \in \langle \lambda, \beta \rangle + \sigma\mathbb{Z}_{>1}$ ,  $s_{\beta[n]} \pi^\lambda w$  does not cover  $\pi^\lambda w$ . For  $n \in -\sigma\mathbb{Z}_{>1}$  the argument is similar, because all the elements of  $s_\beta \lambda - \sigma\mathbb{Z}_{>0}\beta^\vee$  are on the same side of  $M_\beta$ . Hence the only possible covers (with varying coweights) are for  $n \in \{-\sigma, \langle \lambda, \beta \rangle + \sigma\}$ .  $\square$

This is still far from a sufficient condition and many cases of potential covers can still be eliminated. For example, if  $s_\beta v^\lambda < v^\lambda$  (i.e.  $\sigma = -1$ ) then:

- If  $w <_{v^\lambda} s_\beta w$ :  $\pi^\lambda w < \pi^\lambda s_\beta w < s_\beta \pi^\lambda s_\beta w = \pi^{s_\beta \lambda} w < s_{\beta[-\langle \lambda, \beta \rangle - \sigma]} \pi^{s_\beta \lambda} w = s_{\beta[-\sigma]} \pi^\lambda w$ .
- If  $s_\beta w <_{v^\lambda} w$ , then  $w <_{v^{s_\beta \lambda}} s_\beta w$  and:  $\pi^\lambda w < s_\beta \pi^\lambda w = \pi^{s_\beta \lambda} s_\beta w < \pi^{s_\beta \lambda} w < \pi^{s_\beta \lambda + \beta^\vee} s_\beta w = s_{\beta[-\sigma]} \pi^\lambda w$ .

We give another necessary condition for  $\pi^\mu s_\beta w = s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$  to be a cover, this is a generalisation of the chains produced in the proof of Theorem 2.10:

**Proposition 3.2.** *Let  $\pi^\mu s_\beta w = s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$  with  $\mu^{++} \neq \lambda^{++}$ . Suppose that  $s_\beta v^\mu$  is not on a minimal gallery from  $w$  to  $v^\lambda$ . Then  $\pi^\mu s_\beta w > \pi^\lambda w$  is not a cover.*

*Proof.* We express the difference of  $\varepsilon$ -length using Proposition 2.6:

$$\ell_\varepsilon^a(\pi^\mu s_\beta w) - \ell_\varepsilon^a(\pi^\lambda w) = 2\text{ht}(\mu^{++} - \lambda^{++}) + \varepsilon(\ell_{v^\mu}(s_\beta w) - \ell_{v^\lambda}(w)).$$

If there exists a reflection  $r \in W^v$  such that  $\ell_{v^\lambda}(rw) > \ell_{v^\lambda}(w)$  and  $\ell_{v^\mu}(s_\beta rw) < \ell_{v^\mu}(s_\beta w)$  then since the  $\varepsilon$ -length is increasing along this sequence, we have a chain:

$$\pi^\lambda w < \pi^\lambda rw < \pi^\mu s_\beta rw < \pi^\mu s_\beta w.$$

Since  $\ell_v(rw) - \ell_v(w) = \ell(v^{-1}rw) - \ell(v^{-1}w)$  for  $v, r, w \in W^v$ , Lemma 1.8 guaranties the existence of  $r$ , which proves the proposition.

In Figure 3.1 below, we give an example of a chain constructed this way in the  $A_1$ -affine case (with Cartan matrix  $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ ).  $\square$

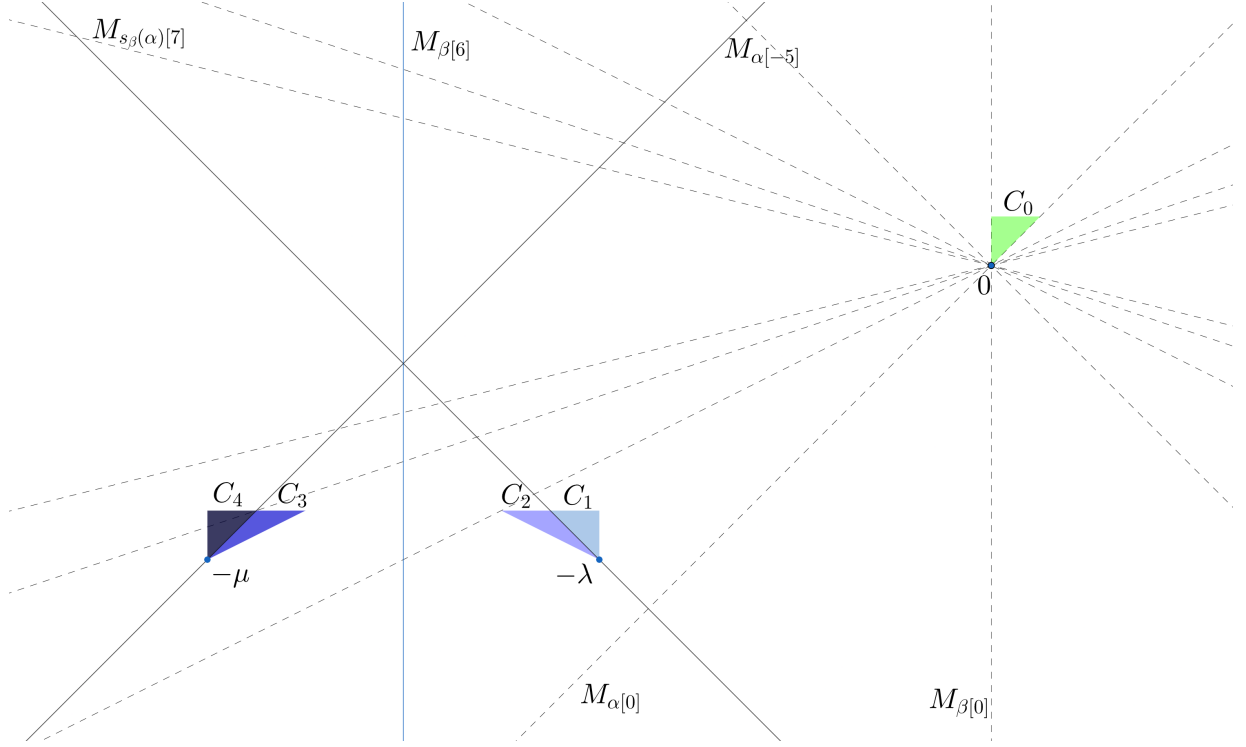


Figure 3.1: Example of a chain constructed as in Proposition 3.2.

In this example,  $\alpha$  and  $\beta$  are the simple roots of a  $A_1$ -affine root system, and we have chosen  $\lambda, w$  and  $\beta[n]$  such that  $v^\lambda = s_\alpha$ ,  $v^\mu = s_\alpha s_\beta$  and  $w = s_\beta$ .  $\pi^\lambda w$  corresponds to the alcove  $C_1$  in light blue, and its image  $\pi^\mu s_\beta w$  by  $s_{\beta[6]}$  corresponds to  $C_4$ . Since  $r = s_\beta s_\alpha s_\beta$  verifies  $d^{W^v}(v^\lambda, rw) = s_\alpha s_\beta s_\alpha > d^{W^v}(v^\lambda, w) = s_\alpha s_\beta$ , and  $d^{W^v}(s_\beta v^\mu, rw) = s_\beta < s_\beta s_\alpha = d^{W^v}(s_\beta v^\mu, w)$ , there is a chain  $\pi^\lambda w < \pi^\lambda r w < \pi^\mu s_\beta r w < \pi^\mu s_\beta w$  which corresponds to the sequence of alcoves  $(C_1, C_2, C_3, C_4)$  on the figure.

Outside of the case of constant dominant coweight dealt with in Theorem 2.10, if we write  $\mathbf{x} = \pi^{v^\lambda} w$  with  $\lambda \in Y^{++}$ ,  $v, w \in W^v$  with  $v$  of minimal length in  $vW_\lambda$ , by Proposition 3.1 the only covers are of the form  $\mathbf{y} \in \{\pi^{v(\lambda+\beta^v)} s_{v(\beta)} w, \pi^{vs_\beta(\lambda+\beta^v)} s_{v(\beta)} w\}$  for some  $\beta \in \Phi_+$ , so the rest of this paper is dedicated to covers of this sort.

**Remark 3.3.** Let  $v, w \in W^v$  and  $\alpha_0 \in \Phi$ . To produce chains, note that Formula (1.5) applied with the affine reflections  $s_{v(\alpha_0)[m+\langle \mu_0, \alpha_0 \rangle]}$  and  $s_{v(\alpha_0)[-m]}$  give for any  $\mu_0 \in Y$  that:

$$\forall m \in \mathbb{Z} \setminus \llbracket -\langle \mu_0, \alpha_0 \rangle, 0 \rrbracket \implies \begin{cases} \pi^{v(\mu_0+m\alpha_0^\vee)} s_{v(\alpha_0)} w & > \pi^{vs_{\alpha_0}(\mu_0)} w \\ \pi^{v(\mu_0+m\alpha_0^\vee)} s_{v(\alpha_0)} w & > \pi^{v(\mu_0)} w. \end{cases} \quad (3.1)$$

The limit cases  $m \in \{-\langle \mu_0, \alpha_0 \rangle, 0\}$  need to be treated more carefully, they depend on the sign of the root  $v(\alpha_0)$  (because Formula (1.5) holds for the affine reflection  $s_{v(\alpha_0)[0]}$  only if  $v(\alpha_0) \in \Phi_+$ ), on the sign of  $\langle \mu_0, \alpha_0 \rangle$  and on the vectorial element  $w$ .

### 3.2 Another expression for the affine length difference

In this subsection, we give another expression for the length difference  $\ell^a(\mathbf{y}) - \ell^a(\mathbf{x})$  when  $\mathbf{x}$  is of the form  $\pi^{v(\lambda)}w$  and  $\mathbf{y} \in \{\pi^{v(\lambda+\beta^V)}s_{v(\beta)}w, \pi^{vs_\beta(\lambda+\beta^V)}s_{v(\beta)}w\}$ , with  $\lambda \in Y^{++}$ ,  $v \in W^\lambda$ ,  $w \in W^v$  and  $\beta \in \Phi_+$  such that  $\lambda + \beta^V \in Y^+$ .

The two next lemmas give information on the vectorial chamber of  $v(\lambda + \beta^V)$ .

**Lemma 3.4.** *Let  $\lambda \in Y^{++}$  and  $\beta^V \in \Phi_+^V$ , let  $u \in W^v$  be such that  $\lambda + \beta^V$  belongs to the vectorial chamber  $C_u^v$ , that is to say  $u^{-1}(\lambda + \beta^V) \in Y^{++}$ . Then  $\ell(s_\beta u) = \ell(s_\beta) + \ell(u)$*

*Proof.* Let  $s_{\tau_1} \dots s_{\tau_n}$  be a reduced expression of  $u$ , so that  $\ell(u) = n$  and:

$$\text{Inv}(u^{-1}) = \{\tau_1, s_{\tau_1}(\tau_2), \dots, s_{\tau_1} \dots s_{\tau_{n-1}}(\tau_n)\}.$$

We show that  $s_{\tau_{k+1}} \dots s_{\tau_1} s_\beta > s_{\tau_k} \dots s_{\tau_1} s_\beta$  for all  $k \leq n-1$ .

Since  $\lambda + \beta^V \in C_u^v$ , for any  $\alpha \in \text{Inv}(u^{-1})$  we have  $\langle \lambda + \beta^V, \alpha \rangle \leq 0$  and, in particular since  $\lambda$  is dominant,  $\langle \beta^V, \alpha \rangle \leq 0$ .

Let  $k \in \llbracket 0, n-1 \rrbracket$ . Since  $s_{\tau_1} \dots s_{\tau_k}(\tau_{k+1}) \in \text{Inv}(u^{-1})$ ,  $\langle \beta^V, s_{\tau_1} \dots s_{\tau_k}(\tau_{k+1}) \rangle \leq 0$ , so:  $s_\beta(s_{\tau_1} \dots s_{\tau_k}(\tau_{k+1})) = s_{\tau_1} \dots s_{\tau_k}(\tau_{k+1}) - \langle \beta^V, s_{\tau_1} \dots s_{\tau_k}(\tau_{k+1}) \rangle \beta$  is a positive root as a sum of positive roots. Thus  $s_{\tau_{k+1}} \dots s_{\tau_1} s_\beta > s_{\tau_k} \dots s_{\tau_1} s_\beta$  for any  $k \leq n-1$  and therefore  $\ell(s_\beta u) = \ell(u^{-1} s_\beta) = n + \ell(s_\beta) = \ell(s_\beta) + \ell(u)$ .  $\square$

**Lemma 3.5.** *Let  $\lambda \in Y^{++}$  and  $\beta \in \Phi_+$  be such that  $\lambda + \beta^V \in Y^+$ . Let  $v \in W^\lambda$  and let  $u$  denote the element  $v^{\lambda+\beta^V}$ .*

*Then, if  $\pi^{v(\lambda+\beta^V)}s_{v(\beta)}w$  (resp.  $\pi^{vs_\beta(\lambda+\beta^V)}s_{v(\beta)}w$ ) covers  $\mathbf{x} = \pi^{v(\lambda)}w$ , we have:*

$$\ell(vu) = \ell(v) + \ell(u) \quad (\text{resp. } \ell(vs_\beta u) = \ell(v) + \ell(s_\beta u)).$$

*Proof.* Let us write to simplify notation,  $W_J$  for  $W_{(\lambda+\beta^V)^{++}}$ . Note that  $vu = \text{proj}_{vuW_J}(v)$  since  $u$  is the element  $u^J$  of minimal length in  $uW_J$ .

Suppose by contradiction that  $\ell(vu) < \ell(v) + \ell(u)$ . Then  $d^{\mathbb{N}}(1, vu) = \ell(vu) < d^{\mathbb{N}}(1, v) + d^{\mathbb{N}}(v, vu) = \ell(v) + \ell(u)$ , so  $v$  is not on a minimal gallery from 1 to  $vu$ . Therefore by Lemma 1.8, there is a reflection  $r \in W^v$  such that  $d^{W^v}(1, rvu) > d^{W^v}(1, vu)$  and  $d^{W^v}(1, rv) < d^{W^v}(1, v)$ , that is to say  $rv < v$  and  $rvu > vu$ .

By minimality of  $u$ ,  $r$  is not in  $vuW_J(vu)^{-1}$ : otherwise  $rvu \in vuW_J$  verifies  $d^{W^v}(v, rvu) = d^{W^v}(rv, vu) < d^{W^v}(v, vu)$ , because foldings reduce the vectorial distance and  $v, vu$  are on different sides of the wall associated to  $r$ .

Since  $vu$  is the projection of  $v$  on  $vuW_J$  which is convex (see [Ron89, Lemma 2.10]), and since the wall  $M_r$  associated to the reflection  $r$  separates  $v$  and  $vu$ , any element of  $vuW_J$  is on the same side of the wall  $M_r$  as  $vu$ , so  $rvu\tilde{u} > vu\tilde{u}$  for any  $\tilde{u} \in W_J$ . In particular, let  $\tilde{u} \in W_J$  be such that  $rvu\tilde{u}$  is the minimal coset representative of  $rvuW_J$ . Then by Corollary 2.7 (second point), since  $rvu\tilde{u} > vu\tilde{u}$ , we have:

$$\pi^{rv(\lambda+\beta^V)}r s_{v(\beta)}w = \pi^{rvu\tilde{u}((\lambda+\beta^V)^{++})}r s_{v(\beta)}w < \pi^{vu\tilde{u}((\lambda+\beta^V)^{++})}s_{v(\beta)}w = \pi^{v(\lambda+\beta^V)}s_{v(\beta)}w.$$

Therefore by Corollary 2.7 for the left and right hand side inequalities and Formula (3.1) applied with  $(\mu_0, \alpha_0, m) = (\lambda, \beta, 1)$  for the middle one, we have a chain:

$$\pi^{v(\lambda)}w < \pi^{rv(\lambda)}rw < \pi^{rv(\lambda+\beta^\vee)}s_{rv(\beta)}rw = \pi^{rv(\lambda+\beta^\vee)}rs_{v(\beta)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

The case of  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  is similar with the chain:

$$\pi^{v(\lambda)}w < \pi^{rv(\lambda)}rw < \pi^{rvs_\beta(\lambda+\beta^\vee)}s_{rv(\beta)}rw = \pi^{rvs_\beta(\lambda+\beta^\vee)}rs_{v(\beta)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

□

**Proposition 3.6.** *Let  $\lambda \in Y^{++}$ ,  $v \in W^\lambda$ ,  $w \in W^v$ . Let  $\beta \in \Phi_+$  be a positive root such that  $\lambda + \beta^\vee \in Y^+$  and let  $u$  denote  $v^{\lambda+\beta^\vee} \in W^{(\lambda+\beta^\vee)^{++}}$ .*

*Suppose that  $\mathbf{y} \in \{\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w, \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w\}$  covers  $\mathbf{x} = \pi^{v(\lambda)}w$ . Then:*

$$\ell^\alpha(\mathbf{y}) - \ell^\alpha(\mathbf{x}) = \left( \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle - \ell(s_\beta) \right) - 2 \left( \ell(u) + \sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle \right). \quad (3.2)$$

*Proof.* Let  $W_J$  denote  $W_{(\lambda+\beta^\vee)^{++}}$ . Recall that  $u = v^{\lambda+\beta^\vee}$  is the minimal element of  $W^v$  such that  $u((\lambda + \beta^\vee)^{++}) = \lambda + \beta^\vee$ , so it is the minimal representative of the coset  $uW_J$ . By Proposition 2.6 we have:

$$\begin{aligned} \ell^\alpha(\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w) - \ell^\alpha(\pi^{v(\lambda)}w) &= 2\text{ht}((\lambda + \beta^\vee)^{++}) - 2\text{ht}(\lambda) + \ell_{v^v(\lambda+\beta^\vee)}(s_{v(\beta)}w) - \ell_v(w) \\ \ell^\alpha(\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w) - \ell^\alpha(\pi^{v(\lambda)}w) &= 2\text{ht}((\lambda + \beta^\vee)^{++}) - 2\text{ht}(\lambda) + \ell_{v^{vs_\beta(\lambda+\beta^\vee)}}(s_{v(\beta)}w) - \ell_v(w). \end{aligned}$$

We unwrap these formulas with the help of previous results.

- In the case  $\mathbf{y} = \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ , let  $\tilde{u} \in W_J$  be such that  $v\tilde{u} = (vu)^J = v^{v(\lambda+\beta^\vee)}$ . The term  $\ell_{v^v(\lambda+\beta^\vee)}(s_{v(\beta)}w) - \ell_v(w)$  rewrites as  $\ell((u\tilde{u})^{-1}s_\beta v^{-1}w) - \ell(vu\tilde{u}) - \ell(v^{-1}w) + \ell(v)$ . Since  $\mathbf{y} > \mathbf{x}$  is a covering, by Proposition 3.2,  $vs_\beta u\tilde{u} = s_{v(\beta)}(vu)^J$  is on a minimal gallery from  $v$  to  $w$ , so  $\ell(v^{-1}w) = \ell((vs_\beta u\tilde{u})^{-1}w) + \ell(s_\beta u\tilde{u})$ . Moreover by Lemma 3.4,  $\ell(s_\beta u\tilde{u}) = \ell(s_\beta) + \ell(u\tilde{u})$  and, by Lemma 3.5,  $\ell(vu) = \ell(v) + \ell(u)$ . Finally, by Equation (2.1), since  $u = u^J = v^{\lambda+\beta^\vee}$  and  $v\tilde{u} = (vu)^J = v^{v(\lambda+\beta^\vee)}$ , we have  $\ell(u\tilde{u}) = \ell(u) + \ell(\tilde{u})$  and  $\ell(vu) = \ell(vu\tilde{u}) + \ell(\tilde{u})$ . Thus:

$$\begin{aligned} \ell_{v^v(\lambda+\beta^\vee)}(s_{v(\beta)}w) - \ell_v(w) &= \ell((u\tilde{u})^{-1}s_\beta v^{-1}w) - \ell(v^{-1}w) - \ell(vu\tilde{u}) + \ell(v) \\ &= -\ell(s_\beta u\tilde{u}) - \ell(vu) + \ell(\tilde{u}) + \ell(v) \\ &= -\ell(s_\beta) - \ell(u\tilde{u}) - \ell(u) + \ell(\tilde{u}) \\ &= -\ell(s_\beta) - 2\ell(u). \end{aligned}$$

- In the second case, let  $\tilde{u} \in W_J$  be such that  $vs_\beta u\tilde{u} = (vs_\beta u)^J = v^{vs_\beta(\lambda+\beta^\vee)}$ , then  $\ell_{v^{vs_\beta(\lambda+\beta^\vee)}}(s_{v(\beta)}w) - \ell_v(w)$  rewrites as  $\ell((u\tilde{u})^{-1}v^{-1}w) - \ell(vs_\beta u\tilde{u}) - \ell(v^{-1}w) + \ell(v)$ . By Proposition 3.2,  $\ell((u\tilde{u})^{-1}v^{-1}w) = \ell(v^{-1}w) - \ell(u\tilde{u})$ . By Equation (2.1),  $\ell(u\tilde{u}) =$

$\ell(u) + \ell(\tilde{u})$  and  $\ell(vs_\beta u \tilde{u}) = \ell(vs_\beta u) - \ell(\tilde{u})$ . By Lemma 3.5 and 3.4,  $\ell(vs_\beta u) = \ell(v) + \ell(s_\beta u) = \ell(v) + \ell(s_\beta) + \ell(u)$ .

Thus in this case:

$$\begin{aligned} \ell_{v, vs_\beta(\lambda+\beta^\vee)}(s_{v(\beta)}w) - \ell_v(w) &= \ell((u\tilde{u})^{-1}v^{-1}w) - \ell(vs_\beta u \tilde{u}) - \ell(v^{-1}w) + \ell(v) \\ &= \ell(v^{-1}w) - \ell(u\tilde{u}) - (\ell(vs_\beta u) - \ell(\tilde{u})) - \ell(v^{-1}w) + \ell(v) \\ &= -\ell(s_\beta) - 2\ell(u). \end{aligned}$$

- By Formula (1.7) we have  $2\text{ht}((\lambda + \beta^\vee)^{++}) = 2\text{ht}(\lambda + \beta^\vee) - 2 \sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle$ , and by Formula (1.6),  $2\text{ht}(\beta^\vee) = \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle$

Combining these points, we get:

$$\begin{aligned} \ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) &= 2\text{ht}(\lambda) + 2\text{ht}(\beta^\vee) - 2 \sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle - 2\text{ht}(\lambda) - \ell(s_\beta) - 2\ell(u) \\ &= \left( \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle - \ell(s_\beta) \right) - 2 \left( \ell(u) + \sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle \right). \end{aligned}$$

□

It is easy to see that  $\sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle - \ell(s_\beta)$  is always positive and that, on the contrary,  $\ell(u) + \sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle$  is always non-positive. Therefore, the length difference is equal to 1 if and only if the first term is equal to 1 and the second term cancels out. In the next subsection we construct chains to show that this is the case under a certain assumption on  $\beta$ .

### 3.3 End of the proof

The result we prove in this section is the following:

**Theorem.** *Suppose that  $\mathbf{y}$  covers  $\mathbf{x}$ . Then:*

$$\ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1.$$

We first deal with the second term of Expression (3.2), through the following Proposition:

**Proposition 3.7.** *Let  $\lambda \in Y^{++}$ ,  $\beta \in \Phi_+$ ,  $v, w \in W^v$  be such that  $v$  is a minimal coset representative in  $vW_\lambda$  and such that  $\lambda + \beta^\vee \in Y^+$ . Suppose that  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  or  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  covers  $\pi^{v(\lambda)}w$ . Then:*

$$\forall \tau \in \Phi_+, \langle \lambda + \beta^\vee, \tau \rangle \geq -1.$$



It is deduced from the two following technical lemma, we give their proofs after showing how they imply the proposition.

**Lemma 3.8.** *Let  $\lambda \in Y^{++}$ ,  $v, w \in W^v$ ,  $\beta \in \Phi_+$  be such that  $v$  is a minimal coset representative of  $vW_\lambda$ . Suppose that there exists a pair  $(\tau, n) \in \Phi_+ \times \mathbb{Z}$  such that :*

- (i)  $n > 0$
- (ii)  $\langle \lambda + n\tau^\vee, \beta \rangle \geq -1$
- (iii)  $n < -\langle \lambda + \beta^\vee, \tau \rangle$ .

*Then,  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  and  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  do not cover  $\pi^{v(\lambda)}w$ .*

**Lemma 3.9.** *Let  $\lambda \in Y^{++}$  and  $\beta \in \Phi_+$  be such that  $\lambda + \beta^\vee$  lies in  $Y^+$ . Let  $\tau \in \Phi_+$  be such that  $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$  and suppose that  $\langle \tau^\vee, \beta \rangle \leq -2$ . Then  $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1$ .*

*Proof of Proposition 3.7.* We prove the contrapositive: Let  $\tau \in \Phi_+$  be a positive root such that  $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$  then  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  and  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  do not cover  $\pi^{v(\lambda)}w$ .

Since  $\langle \tau^\vee, \beta \rangle$  and  $\langle \beta^\vee, \tau \rangle$  have the same sign ([Bar96, Lemma 1.1.10]), we have that  $\langle \tau^\vee, \beta \rangle \leq -1$ .

Suppose first that  $\langle \tau^\vee, \beta \rangle \leq -2$ , then we show that  $(\tau, -(\langle \lambda + \beta^\vee, \tau \rangle + 1))$  is a pair which satisfy the conditions of Lemma 3.8:

- (i) Since  $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$  and  $\lambda$  is dominant,  $\langle \beta^\vee, \tau \rangle \leq -2$  and  $-(\langle \beta^\vee, \tau \rangle + 1) > 0$ .
- (ii) By Lemma 3.9,  $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1$ , thus:

$$\begin{aligned} \langle \lambda - (\langle \lambda + \beta^\vee, \tau \rangle + 1)\tau^\vee, \beta \rangle &= \langle s_\tau(\lambda + \beta^\vee) - \beta^\vee - \tau^\vee, \beta \rangle \\ &= \langle \lambda + \beta^\vee, s_\tau(\beta) \rangle - 2 - \langle \tau^\vee, \beta \rangle \\ &\geq \langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1. \end{aligned}$$

- (iii) Clearly  $-(\langle \beta^\vee, \tau \rangle + 1) < -\langle \beta^\vee, \tau \rangle$ .

Suppose now that  $\langle \tau^\vee, \beta \rangle = -1$ , we show that  $(\tau, 1)$  is a pair satisfying the conditions of Lemma 3.8:

- (i) The first point is trivially verified.
- (ii) Since  $\langle \tau^\vee, \beta \rangle = -1$  and  $\lambda$  is dominant,  $\langle \lambda + \tau^\vee, \beta \rangle \geq -1$ .
- (iii) Since  $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$ ,  $1 < -\langle \lambda + \beta^\vee, \tau \rangle$ .

Hence, either way we can apply Lemma 3.8 to prove that if such a  $\tau \in \Phi_+$  exists,  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  and  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  do not cover  $\pi^{v(\lambda)}w$ .  $\square$

*Proof of Lemma 3.8.* We use conditions (i),(ii), (iii) to produce chains from  $\pi^{v(\lambda)}w$  to  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  and  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ .

- By Condition (i), since  $\lambda$  is dominant and  $\tau$  is a positive root, using Formula (3.1) with  $(\mu_0, \alpha_0, m) = (\lambda, \tau, n)$ , we have the inequalities :

$$\begin{aligned}\pi^{v(\lambda)}w &< \pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w \\ \pi^{v(\lambda)}w &< \pi^{vs_\tau(\lambda+n\tau^\vee)}s_{v(\tau)}w.\end{aligned}$$

- On the other hand, since  $\langle \tau^\vee, \tau \rangle = 2$ , Condition (iii) is equivalent to  $-n < -\langle \lambda + \beta^\vee + n\tau, \tau^\vee \rangle$ , so, using Formula (3.1) for  $(\mu_0, \alpha_0, m) = (\lambda + \beta^\vee + n\tau^\vee, \tau, -n)$ , we get the inequalities:

$$\begin{aligned}\pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w &< \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w \\ \pi^{vs_\beta(\lambda+\beta^\vee+n\tau)}s_{v(\beta)}s_{v(\tau)}w &< \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.\end{aligned}$$

- Let us first suppose that Inequality (ii) is strict, so  $\langle \lambda + n\tau^\vee, \beta \rangle \geq 0$ . Then by Formula (3.1) applied with  $(\mu_0, \alpha_0, m) = (\lambda + n\tau^\vee, \beta, 1)$ , we get the inequalities:

$$\begin{aligned}\pi^{vs_\tau(\lambda+n\tau^\vee)}s_{v(\tau)}w &< \pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w \\ \pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w &< \pi^{vs_\beta(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w.\end{aligned}$$

Thus, if Inequality (ii) is strict, we have chains :

$$\pi^{v(\lambda)}w < \pi^{vs_\tau(\lambda+n\tau^\vee)}s_{v(\tau)}w < \pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w \quad (3.3)$$

$$\pi^{v(\lambda)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee+n\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w \quad (3.4)$$

Now suppose that  $\langle \lambda + n\tau^\vee, \beta \rangle = -1$ . Then note that  $\lambda + n\tau^\vee + \beta^\vee = s_\beta(\lambda + n\tau^\vee)$ , and Formula (3.1) can not be used for the middle inequality anymore.

- If  $vs_\tau(\beta) \in \Phi_+$  then we can apply Formula (1.5) to the element  $\pi^{vs_\tau(\lambda+\tau^\vee)}s_{v(\tau)}w$  and the positive affine root  $vs_\tau(\beta)[0]$ , and since  $\langle \lambda + n\tau^\vee, \beta \rangle = -1 < 0$ , we still have:

$$\pi^{vs_\tau(\lambda+n\tau^\vee)}s_{v(\tau)}w < s_{vs_\tau(\beta)[0]}\pi^{vs_\tau(\lambda+\tau^\vee)}s_{v(\tau)}w = \pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w$$

and the chain (3.3) still holds.

- If  $vs_\tau(\beta) \in \Phi_-$ , that is to say  $s_\tau(\beta) \in \text{Inv}(v)$ , then by the second point of Corollary 2.7:

$$\pi^{v(\lambda)}w < \pi^{vs_{s_\tau(\beta)}(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w$$

and by Formula (3.1) applied with  $(\mu_0, \alpha_0, m) = (\lambda, \tau, 1)$  we get:

$$\pi^{vs_\tau s_\beta s_\tau(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\tau s_\beta(\lambda+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w = \pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w.$$

We thus obtain a chain:

$$\pi^{v(\lambda)}w < \pi^{vs_\tau s_\beta s_\tau(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{v(s_\tau(\lambda+\beta^\vee+n\tau^\vee))}s_{v(\tau)}s_{v(\beta)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

For the element  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  in the case  $\langle \lambda + n\tau^\vee, \beta \rangle = -1$ , we have the following distinction of cases:

- If  $w^{-1}vs_\tau(\beta) \in \Phi_-$ , by Formula (1.5) applied to  $\pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w$  and the affine root  $v(\beta)[\langle \lambda + n\tau^\vee, \beta \rangle]$ , since  $\langle \lambda + n\tau^\vee, \beta \rangle < 0$ , we get:

$$\pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w = \pi^{vs_\beta(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w$$

and the chain (3.4) still holds.

- If  $w^{-1}vs_\tau(\beta) \in \Phi_+$ , then using Formula (1.5) with  $\pi^{v(\lambda)}w$  and the affine reflection  $vs_\tau(\beta)[\langle \lambda, s_\tau(\beta) \rangle]$  (which is always possible because, if  $\langle \lambda, s_\tau(\beta) \rangle = 0$  then by minimality of  $v$ ,  $vs_\tau(\beta) \in \Phi_+$ ), we get:

$$\pi^{v(\lambda)}w < \pi^{v(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w$$

and then, by Formula (3.1) applied with  $(\mu_0, \alpha_0, m) = (\lambda, \tau, n)$ , we get:

$$\pi^{v(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w$$

hence we obtain a chain:

$$\pi^{v(\lambda)}w < \pi^{v(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

Therefore, in all cases, if such a pair  $(\tau, n)$  exists, then  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$  and  $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$  do not cover  $\pi^{v(\lambda)}w$ .  $\square$

*Proof of Lemma 3.9.* The proof relies on the assumption that  $\lambda + \beta^\vee$  lies in the Tits cone, which is equivalent to saying that there is only a finite number of positive roots  $\alpha$  such that  $\langle \lambda + \beta^\vee, \alpha \rangle < 0$ .

We will show that  $\langle \lambda + \beta^\vee, (s_\tau s_\beta)^n(\tau) \rangle \geq 0$  for  $n$  large enough implies  $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1$ , which implies the lemma. To shorten the computation, let us write  $a = -\langle \beta^\vee, \tau \rangle$  and  $a^\vee = -\langle \tau^\vee, \beta \rangle$ . So the assumptions  $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$  and  $\langle \tau^\vee, \beta \rangle \leq -2$  imply that  $a \geq 2 + \langle \lambda, \tau \rangle$  and  $a^\vee \geq 2$ . In the basis  $(\beta, \tau)$  of  $\mathbb{R}\beta \oplus \mathbb{R}\tau$ , the matrix of  $s_\tau s_\beta$  is  $M = \begin{pmatrix} -1 & a \\ -a^\vee & aa^\vee - 1 \end{pmatrix}$ . We have  $\chi_M = X^2 + (2 - aa^\vee)X + 1$  thus, since  $aa^\vee \geq 4$ ,  $M^2 = (aa^\vee - 2)M - I_2$ . Write  $M^n = \mu_n M + \nu_n I_2$  for  $n \in \mathbb{N}$ , then an easy computation shows that  $\nu_n = -\mu_{n-1}$  and  $\mu_{n+1} = (aa^\vee - 2)\mu_n - \mu_{n-1}$ . In particular since  $aa^\vee - 2 \geq 2$  and  $\mu_0 = 0 < \mu_1$ , by iteration  $(\mu_n)$  is strictly increasing.

Let  $x = \langle \lambda, \beta \rangle \geq 0$  and  $y = \langle \lambda, \tau \rangle \in [0, a - 2]$ . Then:

$$\begin{aligned} \langle \lambda + \beta^\vee, (s_\tau s_\beta)^n(\tau) \rangle &= \langle \lambda + \beta^\vee, a\mu_n\beta + ((aa^\vee - 1)\mu_n - \mu_{n-1})\tau \rangle \\ &= (x + 2)\mu_n a + ((aa^\vee - 1)\mu_n - \mu_{n-1})(y - a). \end{aligned}$$

Since this is non-negative for  $n$  large enough, we have for  $n$  large:  $(x+2)\mu_n a \geq (a-y)((aa^\vee - 1)\mu_n - \mu_{n-1}) > (a-y)\mu_n(aa^\vee - 2)$  since  $\mu_{n-1} < \mu_n$ . Hence:

$$(x+2) > (a-y)(a^\vee - \frac{2}{a}) = aa^\vee - a^\vee y - 2 + 2\frac{y}{a}.$$

Therefore  $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle = x + 2 + a^\vee y - aa^\vee > -2 + 2\frac{y}{a}$  and, since it is an integer, we deduce  $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1 \geq 1 - a^\vee$ , which proves the result.  $\square$

**Corollary 3.10.** *Let  $\lambda \in Y^{++}$ ,  $v \in W^\lambda$ ,  $w \in W^v$ . Let  $\beta \in \Phi_+$  be a positive root such that  $\lambda + \beta^\vee \in Y^+$ . Suppose that  $\mathbf{y} \in \{\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w, \pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w\}$  covers  $\mathbf{x} = \pi^{v(\lambda)} w$ .*

*Then:*

$$\ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) = \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle - \ell(s_\beta). \quad (3.5)$$

*Proof.* We first deduce from Proposition 3.7 that  $\sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle = -\ell(u)$ , where  $u = v^{\lambda+\beta^\vee} \in W^v$ .

By Proposition 3.7,  $\langle \lambda + \beta^\vee, \tau \rangle \geq -1$ , and we show that for any  $\tau \in \text{Inv}(u^{-1})$  this is an equality. By Corollary 1.7, the terms  $\langle \lambda + \beta^\vee, \tau \rangle$  are all non-positive.

Moreover, since  $\tau \in \text{Inv}(u^{-1})$  we have  $u^{-1}s_\tau < u^{-1}$  or, equivalently,  $s_\tau u < u$ , so by minimality of  $u$ ,  $s_\tau u((\lambda + \beta^\vee)^{++}) \neq \lambda + \beta^\vee$ , and thus  $\langle \lambda + \beta^\vee, \tau \rangle \neq 0$ . Hence, we obtain  $\langle \lambda + \beta^\vee, \tau \rangle = -1$  for all  $\tau \in \text{Inv}(u^{-1})$  and, since  $|\text{Inv}(u^{-1})| = \ell(u^{-1}) = \ell(u)$ , we get the equality:

$$\sum_{\tau \in \text{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle = -\ell(u).$$

We then directly obtain Expression (3.5) from Expression (3.2).  $\square$

We now prove in Lemma 3.11 that, under a certain assumption, for such covers,  $\sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle$  is actually equal to  $\ell(s_\beta) + 1$ , which is enough to conclude that the length difference is exactly 1.

**Lemma 3.11.** *Let  $\lambda \in Y^{++}$ ,  $v, w \in W^v$  be such that  $v$  is the minimal coset representative of  $vW_\lambda$ . Let  $\beta \in \Phi_+$  and let  $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}$  be such that  $\langle \beta^\vee, \gamma \rangle \geq 2$  and suppose that  $\beta \notin \text{Inv}(s_\gamma)$ . Then  $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$  and  $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$  do not cover  $\pi^{v(\lambda)} w$ .*

*Proof.* By Proposition 3.7, we can suppose that  $\langle \lambda + \beta^\vee, \tau \rangle \geq -1$  for any  $\tau \in \Phi_+$ . Let  $\gamma$  be as in the statement and write  $\alpha = s_\gamma(\beta) \in \Phi_+$ . We will construct non-trivial chains in the same fashion as in the proof of Lemma 3.8. Beforehand, we show by computation that  $\langle \lambda + \gamma^\vee, \alpha \rangle \geq -1$ . If  $\langle \gamma^\vee, \beta \rangle = 1 = -\langle \gamma^\vee, \alpha \rangle$  it is clear since  $\lambda$  is dominant, else if  $\langle \gamma^\vee, \beta \rangle \geq 2$ :

$$\begin{aligned} \langle \lambda + \gamma^\vee, \alpha \rangle &= \langle \lambda + \beta^\vee - \alpha^\vee + (1 - \langle \beta^\vee, \gamma \rangle)\gamma^\vee, \alpha \rangle \\ &= \langle \lambda + \beta^\vee, \alpha \rangle + (1 - \langle \beta^\vee, \gamma \rangle)\langle \gamma^\vee, \alpha \rangle - 2 \\ &= \langle \lambda + \beta^\vee, \alpha \rangle + (\langle \beta^\vee, \gamma \rangle - 1)\langle \gamma^\vee, \beta \rangle - 2. \end{aligned}$$

Since  $\langle \beta^\vee, \gamma \rangle \geq 2$  and  $\langle \gamma^\vee, \beta \rangle \geq 2$ ,  $(\langle \beta^\vee, \gamma \rangle - 1)\langle \gamma^\vee, \beta \rangle \geq 2$ , and by Proposition 3.7,  $\langle \lambda + \beta^\vee, \alpha \rangle \geq -1$ . Thus,  $\langle \lambda + \gamma^\vee, \alpha \rangle \geq -1$  either way.

We construct chains which are slight modifications of the ones constructed in the proof of Lemma 3.8, we give the detail for sake of completeness.

- Suppose first that  $\langle \lambda + \gamma^\vee, \alpha \rangle \geq 0$ . Then we show that we have the following chains:

$$\pi^{v(\lambda)}w < \pi^{v(\lambda+\gamma^\vee)}s_{v(\gamma)}w < \pi^{v(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\alpha)}s_{v(\gamma)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w \quad (3.6)$$

$$\pi^{v(\lambda)}w < \pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{v(\gamma)}w < \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w. \quad (3.7)$$

Indeed:

1.  $\lambda$  is dominant and  $\gamma \in \Phi_+$ , so  $\langle \lambda, \gamma \rangle \geq 0$  and Formula (3.1) applied with  $(\mu_0, \alpha_0, m) = (\lambda, \gamma, 1)$  gives the leftmost inequalities.
  2. Since  $\langle \lambda + \gamma^\vee, \alpha \rangle \geq 0$ , Formula (3.1) applied to  $(\mu_0, \alpha_0, m) = (\lambda + \gamma^\vee, \alpha, 1)$  gives the second inequalities.
  3. Since  $\lambda + \beta^\vee = (\lambda + \gamma^\vee + \alpha^\vee) + (\langle \beta^\vee, \gamma \rangle - 1)\gamma^\vee$ , by applying Formula (3.1) to  $(\mu_0, \alpha_0, m) = (\lambda + \gamma^\vee + \alpha^\vee, \gamma, \langle \beta^\vee, \gamma \rangle - 1)$ , as  $\langle \beta^\vee, \gamma \rangle - 1 > \max(0, -\langle \lambda + \gamma^\vee + \alpha^\vee, \gamma \rangle)$  (explicitly  $-\langle \lambda + \gamma^\vee + \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle - \langle \lambda, \gamma \rangle - 2$ ) we obtain the rightmost inequalities.
- We now suppose that  $\langle \lambda + \gamma^\vee, \alpha \rangle = -1$ . Then  $\lambda + \gamma^\vee + \alpha^\vee = s_\alpha(\lambda + \gamma^\vee)$  and the above chains do not always hold. We focus here on the case of  $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ .
    1. If  $v(\alpha) \in \Phi_+$ , since  $\langle \lambda + \gamma^\vee, \alpha \rangle < 0$ , the inequality  $\pi^{vs_\alpha(\lambda+\gamma^\vee)}s_{v(\alpha)}s_{v(\gamma)}w > \pi^{v(\lambda+\gamma^\vee)}s_{v(\gamma)}w$  still holds, by Formula (1.5) applied with  $s_{v(\alpha)[0]}$ . The rest of the chain (3.6) still holds and the whole chain remains correct.
    2. If  $v(\alpha) \in \Phi_-$ , then  $vs_\alpha < v$ , and we have the chain:

$$\pi^{v(\lambda)}w < \pi^{vs_\alpha(\lambda)}s_{v(\alpha)}w < \pi^{vs_\alpha(\lambda+\gamma^\vee)}s_{v(\alpha)}s_{v(\tau)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

The reflection used for the first inequality is  $s_{-v(\alpha)[0]}$ , and it holds by Formula (1.5) because  $\langle v(\lambda), -v(\alpha) \rangle = -\langle \lambda, \alpha \rangle < 0$ . Note that this is non-zero because  $v$  is the minimal representative of  $vW_\lambda$  and thus  $vs_\alpha < v$  implies  $s_\alpha \notin W_\lambda$  so  $\langle \lambda, \alpha \rangle \neq 0$ . For the second and the third inequalities we use Formula (3.1) with  $(\mu_0, \alpha_0, m)$  equal to  $(\lambda, \gamma, 1)$  and  $(\lambda + \alpha^\vee + \gamma^\vee, \gamma, \langle \beta^\vee, \gamma \rangle - 1)$  respectively.

- We suppose that  $\langle \lambda + \gamma^\vee, \alpha \rangle = -1$  and we deal with the case of  $\pi^{vs_{v(\beta)}(\lambda+\beta^\vee)}s_{v(\beta)}w$ . Then:

$$\pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w = \pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{vs_\gamma(\alpha)}s_{v(\gamma)}w = s_{vs_\gamma(\alpha)[\langle \lambda+\gamma^\vee, \alpha \rangle]} \pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{v(\gamma)}w.$$

Moreover  $(s_{v(\gamma)}w)^{-1}(vs_\gamma(\alpha)) = w^{-1}v(\alpha)$ . Thus, since  $\langle \lambda + \gamma^\vee, \alpha \rangle < 0$  we have:

1. If  $w^{-1}v(\alpha) \in \Phi_-$ , by Formula (1.5),  $\pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{v(\gamma)}w < \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w$  and the chain (3.7) still holds.
2. If  $w^{-1}v(\alpha) \in \Phi_+$ , then, since  $\langle \lambda, \alpha \rangle = \langle \gamma^\vee, \beta \rangle - 1 > 0$ , by Formula (1.5),  $\pi^{v(\lambda)}w < s_{v(\alpha)}[\langle \lambda, \alpha \rangle]\pi^{v(\lambda)}w = \pi^{v(\lambda)}s_{v(\alpha)}w$ . Then by Formula (3.1) applied to  $(\mu_0, \alpha_0, m) = (\lambda, \gamma, 1)$  we have  $\pi^{v(\lambda)}s_{v(\gamma)}s_{v(\beta)}s_{v(\gamma)}w < \pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{v(\beta)}s_{v(\gamma)}w = \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w$  and we have a chain:

$$\pi^{v(\lambda)}w < \pi^{v(\lambda)}s_{v(\alpha)}w < \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

□

**Lemma 3.12.** *Let  $\beta \in \Phi_+$  and suppose that there exists  $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}$  such that  $\langle \beta^\vee, \gamma \rangle \geq 2$  and  $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \neq 3$ . Then  $\gamma$  can be chosen such that  $\beta \notin \text{Inv}(s_\gamma)$ .*

*Proof.* Note that, by [Bar96, Lemma 1.1.10], for any  $\beta, \gamma \in \Phi$ ,  $\langle \beta^\vee, \gamma \rangle$  and  $\langle \gamma^\vee, \beta \rangle$  have the same sign, so if  $\langle \beta^\vee, \gamma \rangle \geq 2$  and  $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \neq 3$ , either  $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \geq 4$ , either  $\langle \beta^\vee, \gamma \rangle = 2$  and  $\langle \gamma^\vee, \beta \rangle = 1$ . We treat separately these cases:

- Let us first suppose that there exists  $\gamma \in \text{Inv}(s_\beta)$  such that  $\langle \beta^\vee, \gamma \rangle = 2$  and  $\langle \gamma^\vee, \beta \rangle = 1$ . Suppose that  $\beta \in \text{Inv}(s_\gamma)$ , so  $s_\gamma(\beta) = \beta - \gamma < 0$ , and  $s_\beta(\gamma) = \gamma - 2\beta < 0$ . Then we show that  $\beta \notin \text{Inv}(s_{\tilde{\gamma}})$  for  $\tilde{\gamma} = -s_\beta(\gamma)$ :

$$s_{\tilde{\gamma}}(\beta) = s_\beta s_\gamma s_\beta(\beta) = -s_\beta(\beta - \gamma) = \gamma - \beta = -s_\gamma(\beta) > 0.$$

Moreover  $s_\beta(\tilde{\gamma}) = -\gamma < 0$  and  $\langle \beta^\vee, \tilde{\gamma} \rangle = \langle \beta^\vee, \gamma \rangle = 2$  therefore,  $\gamma$  can be chosen such that  $\beta \notin \text{Inv}(s_\gamma)$ .

- Let us now suppose that there exists  $\gamma \in \text{Inv}(s_\beta)$  such that  $\langle \beta^\vee, \gamma \rangle \geq 2$  and  $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \geq 4$ . Write  $\beta = v_\beta(\beta_0) = s_{\alpha_1} \dots s_{\alpha_n}(\beta_0)$  where the  $\alpha_i$  and  $\beta_0$  are simple roots, and suppose that  $n$  is of minimal length amongst possible expressions of  $\beta$ . Therefore  $s_{\alpha_1} \dots s_{\alpha_n} s_{\beta_0} s_{\alpha_n} \dots s_{\alpha_1}$  is a reduced expression of  $s_\beta$  and:

$$\text{Inv}(s_\beta) = \{s_{\alpha_1} \dots s_{\alpha_{p-1}}(\alpha_p) \mid p \leq n\} \sqcup \{\beta\} \sqcup \{s_{\alpha_1} \dots s_{\alpha_n} s_{\beta_0} s_{\alpha_n} \dots s_{\alpha_{n+1-p}}(\alpha_{n-p}) \mid p \leq n\}.$$

Let  $k$  be the smallest such that  $\gamma_k = s_{\alpha_1} \dots s_{\alpha_{k-1}}(\alpha_k)$  verifies  $\langle \beta^\vee, \gamma_k \rangle \geq 2$  and  $\langle \beta^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \beta \rangle \geq 4$ .

The expression  $s_{\alpha_1} \dots s_{\alpha_{k-1}} s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_1}$  is an expression of  $s_{\gamma_k}$ , thus:

$$\text{Inv}(s_{\gamma_k}) \subset \{s_{\alpha_1} \dots s_{\alpha_{p-1}}(\alpha_p) \mid p \leq k-1\} \sqcup \{\gamma_k\} \sqcup \{s_{\alpha_1} \dots s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_{k+1-p}}(\alpha_{k-p}) \mid p \leq k-1\}.$$

Suppose by contradiction that  $\beta \in \text{Inv}(s_{\gamma_k})$ . Since  $v_\beta$  is of minimal length,  $\beta$  is not in the first set, hence there is  $p \in \llbracket 1, k-1 \rrbracket$  such that  $\beta = s_{\alpha_1} \dots s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_{k+1-p}}(\alpha_{k-p})$ .

We show that  $\gamma_{k-p} = s_{\alpha_1} \dots s_{\alpha_{k-p-1}}(\alpha_{k-p}) \in \text{Inv}(s_\beta)$  verifies  $\langle \beta^\vee, \gamma_{k-p} \rangle \geq 2$ , which contradicts the minimality of  $k$ . Note that  $\beta = -s_{\gamma_k}(\gamma_{k-p})$ . We compute:

$$\begin{aligned} \langle \beta^\vee, \gamma_{k-p} \rangle &= \langle -s_{\gamma_k}(\gamma_{k-p}^\vee), \gamma_{k-p} \rangle \\ &= -(2 - \langle \gamma_{k-p}^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \gamma_{k-p} \rangle) \\ &= \langle \beta^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \beta \rangle - 2. \end{aligned}$$

So since  $\langle \beta^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \beta \rangle \geq 4$ , we get  $\langle \beta^\vee, \gamma_{k-p} \rangle \geq 2$ , and with a similar computation, we find that  $\langle \gamma_{k-p}^\vee, \beta \rangle = \langle \beta^\vee, \gamma_{k-p} \rangle \geq 2$  as well, so  $\langle \beta^\vee, \gamma_{k-p} \rangle \langle \gamma_{k-p}^\vee, \beta \rangle \geq 4$ . This contradicts the minimality of  $k$  and thus  $\beta \notin \text{Inv}(s_{\gamma_k})$ . □

**Proposition 3.13.** *Let  $\lambda \in Y^{++}$ ,  $v, w \in W^v$  be such that  $v$  is the minimal coset representative of  $vW_\lambda$ . Let  $\beta \in \Phi_+$  and let  $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}$  be such that  $\langle \beta^\vee, \gamma \rangle \geq 2$ . Then  $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$  and  $\pi^{v s_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w$  do not cover  $\pi^{v(\lambda)} w$ .*

*Proof.* If  $\beta \notin \text{Inv}(s_\gamma)$  this is Lemma 3.11, and by Lemma 3.12 it extends to every  $\gamma \in \text{Inv}(s_\beta)$  such that  $\langle \beta^\vee, \gamma \rangle \geq 2$  and  $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \neq 3$ . Therefore the only case left is if  $\beta \in \text{Inv}(s_\gamma)$  and:

$$\langle \beta^\vee, \gamma \rangle = 3, \quad \langle \gamma^\vee, \beta \rangle = 1.$$

We then show that, in this case,  $\beta, \gamma$  appear as positive roots of a root subsystem of  $\Phi$  isomorphic to  $G_2$ , and we use this system to construct chains replacing the ones in the proof of Lemma 3.11.

First, note that  $-s_\gamma(\beta)$  lies in  $\text{Inv}(s_\beta)$  (so  $s_\beta s_\gamma(\beta)$  is positive). Indeed, we can write, as in the proof of Lemma 3.12,  $\beta = s_{\alpha_1} \dots s_{\alpha_n}(\beta_0)$  for a minimal  $n$ , and  $\gamma = s_{\alpha_1} \dots s_{\alpha_{k-1}}(\alpha_k)$  for some  $k \leq n$ . Then, since  $\beta \in \text{Inv}(s_\gamma)$ ,  $\beta$  is also of the form  $s_{\alpha_1} \dots s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_{k-p+1}}(\alpha_{k-p})$  for some  $p \leq k-1$ , and thus  $-s_\gamma(\beta) = s_{\alpha_1} \dots s_{\alpha_{k-p-1}}(\alpha_{k-p}) \in \text{Inv}(s_\beta)$ . Therefore we have the following positive roots, and their associated coroots (the notation will become clear afterwards):

- $\theta_1 := -s_\gamma(\beta) = \gamma - \beta \in \Phi_+$ , with associated coroot  $\theta_1^\vee = -s_\gamma(\beta^\vee) = 3\gamma^\vee - \beta^\vee$ .
- $\tilde{\beta} := -s_\beta(\gamma) = 3\beta - \gamma \in \Phi_+$ , with associated coroot  $\tilde{\beta}^\vee = -s_\beta(\gamma^\vee) = \beta^\vee - \gamma^\vee$ .
- $\tilde{\gamma} := s_\beta s_\gamma(\beta) = 2\beta - \gamma \in \Phi_+$ , with associated coroot  $\tilde{\gamma}^\vee = s_\beta s_\gamma(\beta^\vee) = 2\beta^\vee - 3\gamma^\vee$ .

Let us also denote  $\theta_2 = s_{\theta_1}(\gamma) = 3\beta - 2\gamma$ , with associated coroot  $\theta_2^\vee = \beta^\vee - 2\gamma^\vee$ . Then one can check that  $\{\theta_1, \theta_2\}$  form the positive simple roots of a  $G_2$  root system (in the sense that  $\langle \theta_1^\vee, \theta_2 \rangle = -3$  and  $\langle \theta_2^\vee, \theta_1 \rangle = -1$ ), such that  $\gamma = s_{\theta_1}(\theta_2)$ ,  $\beta = s_{\theta_1} s_{\theta_2}(\theta_1)$ ,  $\tilde{\gamma} = s_{\theta_2}(\theta_1)$  and  $\tilde{\beta} = s_{\theta_2} s_{\theta_1}(\theta_2)$ . However,  $\theta_2$  may not be a positive root in  $\Phi$ , and we thus need to distinguish these two cases.

Let us first suppose that  $\theta_2$  lies in  $\Phi_+$ . Notice that  $\theta_1^\vee + \tilde{\beta}^\vee + \theta_2^\vee = (3\gamma^\vee - \beta^\vee) + (\beta^\vee - \gamma^\vee) + (\beta^\vee - 2\gamma^\vee) = \beta^\vee$ , and  $s_{\theta_1} s_{\tilde{\beta}} s_{\theta_2} = s_{\theta_1} (s_{\theta_2} s_{\theta_1} s_{\theta_2} s_{\theta_1} s_{\theta_2}) s_{\theta_2} = s_{\theta_1} s_{\theta_2} s_{\theta_1} s_{\theta_2} s_{\theta_1} = s_{\theta_2} s_{\tilde{\beta}} s_{\theta_1} = s_\beta$ .

Moreover, we have  $\langle \theta_2^\vee, \tilde{\beta} \rangle = \langle \beta^\vee - 2\gamma^\vee, 3\beta - \gamma \rangle = 1 > 0$ , and  $\langle \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle = \langle 2\beta^\vee - 3\gamma^\vee, \gamma - \beta \rangle = -1$ .

Hence:

- Suppose first that  $\langle \lambda, \theta_2 \rangle > 0$ . Then, since  $\lambda$  is dominant and by the previous computations, using Formula (3.1) with  $(\mu_0, \alpha_0, m) = (\lambda, \theta_2, 1)$  for the first inequality,  $(\lambda + \theta_2^\vee, \tilde{\beta}, 1)$  for the second and  $(\lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1, 1)$  for the third, we obtain the chains:

$$\pi^{v(\lambda)} w < \pi^{v(\lambda + \theta_2^\vee)} s_{v(\theta_2)} w < \pi^{v(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\tilde{\beta})} s_{v(\theta_2)} w < \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w \quad (3.8)$$

$$\pi^{v(\lambda)} w < \pi^{vs_{\theta_2}(\lambda + \theta_2^\vee)} s_{v(\theta_2)} w < \pi^{vs_{\theta_2} s_{\tilde{\beta}}(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\theta_2)} s_{v(\tilde{\beta})} w < \pi^{vs_{\beta}(\lambda + \beta^\vee)} s_{v(\beta)} w. \quad (3.9)$$

- If  $\langle \lambda, \theta_1 \rangle = 0$ , then  $\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle = -1$  so the last inequality in the chains (3.8) and (3.9) do not always hold, we have the following disjunction of cases, which we already encountered in Lemma 3.11 and Lemma 3.8:

- If  $v(\theta_1) \in \Phi_+$ , the chain (3.8) still holds, else  $vs_{\theta_1} < v$ ,  $\lambda + \beta^\vee = s_{\theta_1}(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)$  and we instead have the chain:

$$\pi^{v(\lambda)} w < \pi^{vs_{\theta_1}(\lambda)} s_{v(\theta_1)} w < \pi^{vs_{\theta_1}(\lambda + \theta_2^\vee)} s_{v(\theta_1)} s_{v(\theta_2)} w < \pi^{vs_{\theta_1}(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\theta_1)} s_{v(\tilde{\beta})} s_{v(\theta_2)} w$$

where the last term is actually equal to  $\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$ .

- If  $w^{-1}v(\theta_1) \in \Phi_-$ , then since  $\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle < 0$ , by Formula (1.5) applied with the affine root  $vs_{\theta_2} s_{\tilde{\beta}}(\theta_1)[\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle]$ , the third inequality of Chain (3.9) still holds, and thus the whole chain remains correct. Otherwise if  $w^{-1}v(\theta_1) \in \Phi_+$  we instead have the chain:

$$\pi^{v(\lambda)} w < \pi^{v(\lambda)} s_{v(\theta_1)} w < \pi^{vs_{\theta_2}(\lambda + \theta_2^\vee)} s_{v(\theta_2)} s_{v(\theta_1)} w < \pi^{vs_{\theta_2} s_{\tilde{\beta}}(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\theta_2)} s_{v(\tilde{\beta})} s_{v(\theta_1)} w$$

where the last term is actually equal to  $\pi^{vs_{\beta}(\lambda + \beta^\vee)} s_{v(\beta)} w$  since  $\lambda + \theta_2^\vee + \tilde{\beta}^\vee = s_{\theta_1}(\lambda + \beta^\vee)$ .

We now turn to the case of  $\theta_2 \in \Phi_-$ . Notice that  $\beta^\vee = -\theta_2^\vee + \tilde{\gamma}^\vee + \gamma^\vee$  and  $s_\beta = s_\gamma s_{\tilde{\gamma}} s_{\theta_2} = s_{\theta_2} s_{\tilde{\gamma}} s_\gamma$ . Moreover,  $\langle -\theta_2^\vee, \tilde{\gamma} \rangle = \langle 2\gamma^\vee - \beta^\vee, 2\beta - \gamma \rangle = -1$  and  $\langle -\theta_2^\vee + \tilde{\gamma}^\vee, \gamma \rangle = \langle \beta^\vee - \gamma^\vee, \gamma \rangle = 1$ . Therefore, since  $\lambda$  is dominant and  $-\theta_2$  is a positive root:

- If  $\langle \lambda, \tilde{\gamma} \rangle > 0$ , then using Formula (3.1) with  $(\mu_0, \alpha_0, m) = (\lambda, -\theta_2, 1)$  for the first inequality,  $(\lambda - \theta_2^\vee, \tilde{\gamma}, 1)$  for the second and  $(\lambda - \theta_2^\vee + \tilde{\gamma}^\vee, \gamma, 1)$  for the third, we obtain the chains:

$$\pi^{v(\lambda)} w < \pi^{v(\lambda - \theta_2^\vee)} s_{v(\theta_2)} w < \pi^{v(\lambda - \theta_2^\vee + \tilde{\gamma}^\vee)} s_{v(\tilde{\gamma})} s_{v(\theta_2)} w < \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w \quad (3.10)$$

$$\pi^{v(\lambda)} w < \pi^{vs_{\theta_2}(\lambda - \theta_2^\vee)} s_{v(\theta_2)} w < \pi^{vs_{\theta_2} s_{\tilde{\gamma}}(\lambda - \theta_2^\vee + \tilde{\gamma}^\vee)} s_{v(\theta_2)} s_{v(\tilde{\gamma})} w < \pi^{vs_{\beta}(\lambda + \beta^\vee)} s_{v(\beta)} w. \quad (3.11)$$



• Suppose now that  $\langle \lambda, \tilde{\gamma} \rangle = 0$ , so  $\lambda - \theta_2^\vee + \tilde{\gamma} = s_{\tilde{\gamma}}(\lambda - \theta_2^\vee)$ . Then:

– If  $v(\tilde{\gamma}) \in \Phi_+$ , the chain (3.10) still holds. Else,  $vs_{\tilde{\gamma}} < v$  and we instead have the chain:

$$\pi^{v(\lambda)}w < \pi^{vs_{\tilde{\gamma}}(\lambda)}s_{v(\tilde{\gamma})}w < \pi^{vs_{\tilde{\gamma}}(\lambda - \theta_2^\vee)}s_{v(\tilde{\gamma})}s_{v(\theta_2)}w < \pi^{v(\lambda + \beta^\vee)}s_{v(\beta)}w$$

where the first inequality comes from Corollary 2.7 and the two others from Formula (3.1).

– If  $w^{-1}v(\tilde{\gamma}) \in \Phi_-$ , then the chain (3.11) still holds. Else  $w^{-1}v(\tilde{\gamma}) \in \Phi_+$  and we instead have the chain:

$$\pi^{v(\lambda)}w < \pi^{v(\lambda)}s_{v(\tilde{\gamma})}w < \pi^{vs_{\theta_2}(\lambda - \theta_2^\vee)}s_{v(\theta_2)}s_{v(\tilde{\gamma})}w < \pi^{vs_{\beta}(\lambda + \beta^\vee)}s_{v(\beta)}w$$

where the first inequality is deduced from Formula (1.5) used with the affine root  $v(\tilde{\gamma})[\langle \lambda, \tilde{\gamma} \rangle]$ , and the two others from Formula (3.1) as for the chain (3.11).

We thus have proved that, for any  $\beta \in \Phi_+$ , if there exists  $\gamma \in \text{Inv}(s_\beta)$  such that  $\langle \beta^\vee, \gamma \rangle \geq 2$ , then  $\pi^{v(\lambda + \beta^\vee)}s_{v(\beta)}w$  and  $\pi^{vs_{\beta}(\lambda + \beta^\vee)}s_{v(\beta)}w$  do not cover  $\pi^{v(\lambda)}w$ .  $\square$

We now have everything to prove the Theorem:

**Theorem 3.14.** *Suppose that  $\mathbf{y}$  covers  $\mathbf{x}$ . Then:*

$$\ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1.$$

*Proof.* Suppose that  $\mathbf{y}$  covers  $\mathbf{x}$ . The case  $\text{proj}^{Y^+}(\mathbf{y}) \in W^v \cdot \text{proj}^{Y^+}(\mathbf{x})$  is Theorem 2.10. Else, if  $\text{proj}^{Y^+}(\mathbf{y}) \notin W^v \cdot \text{proj}^{Y^+}(\mathbf{x})$ , by Proposition 3.1,  $\mathbf{y}$  is of the form  $\pi^{v(\lambda + \beta)}s_{v(\beta)}w$  or  $\pi^{vs_{\beta}(\lambda + \beta^\vee)}s_{v(\beta)}w$ , for  $\mathbf{x} = \pi^{v(\lambda)}w$  with  $\lambda \in Y^{++}$ ,  $v \in W^\lambda$ ,  $w \in W^v$  and  $\beta \in \Phi_+$ . Then, by Corollary 3.10, we have:

$$\ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) = \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle - \ell(s_\beta).$$

Moreover, by Proposition 3.13,  $\langle \beta^\vee, \gamma \rangle = 1$  for any  $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}$ . Since  $|\text{Inv}(s_\beta)| = \ell(s_\beta)$  and  $\beta \in \text{Inv}(s_\beta)$  verifies  $\langle \beta^\vee, \beta \rangle = 2$ , we deduce that  $\sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle = \ell(s_\beta) + 1$ , and

therefore for any form of  $\mathbf{y}$  covering  $\mathbf{x}$ :

$$\ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) = 1.$$

$\square$

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