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► **To cite this version:**

Eric Duchêne, Arthur Dumas, Nacim Oijid, Aline Parreau, Eric Rémila. The Maker-Maker domination game in forests. *Discrete Applied Mathematics*, 2023, 254, pp.34:1–34:19. 10.1016/j.dam.2024.01.023 . hal-04123172

HAL Id: hal-04123172

<https://hal.science/hal-04123172v1>

Submitted on 9 Jun 2023

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The Maker-Maker domination game in forests*

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June 9, 2023

Abstract

We study the Maker-Maker version of the domination game introduced in 2018 by Duchêne *et al.* Given a graph, two players alternately claim vertices. The first player to claim a dominating set of the graph wins. As the Maker-Breaker version, this game is PSPACE-complete on split and bipartite graphs. Our main result is a linear time algorithm to solve this game in forests. We also give a characterization of the cycles where the first player has a winning strategy.

Keywords: Positional Games, Maker-Breaker Games, Maker-Maker Games, PSPACE-completeness, domination game, Dominating set, Graphs

1 Introduction

Positional games have been introduced successively by Hales and Jewett in [10] and by Erdős and Selfridge in [7], and then widely studied in the literature (see the two books [1, 11] for an overview). They are played on an hypergraph of vertex set X , with a finite set $\mathcal{F} \subseteq 2^X$ of hyperedges. The set X is often called the *board* of the game, and an element of \mathcal{F} a *winning set*. The game involves two players that alternately claim a previously unclaimed vertex of X . The winner of the game is defined according to a convention. The original one is called *Maker-Maker* (or strong convention), where both players have the same objective, i.e. filling a whole winning set with their own claimed vertices. Such games may end in a draw if each winning set contains one vertex claimed by each player. When considering positional games, the main issue consists in determining which player has a winning strategy. In particular, it is well-known that in the Maker-Maker convention, the second player has no winning strategy. Thus, resolving a Maker-Maker game consists in determining whether the first player has a winning strategy or whether it ends in a draw. As mentioned in [11], despite this result, this convention has not been widely studied in the literature. The main reason is due to the hardness to tackle it, as the first player tries at the same time to fill a winning set while considering all the threats of his opponent. As a consequence, Maker-Maker instances often satisfy the so-called *extra set paradox*, which claims that adding new winning sets in the hypergraph is not necessarily better for the first player.

For all these reasons, the convention that has been mainly studied in the literature is the *Maker-Breaker convention*. In this convention, both players, called Maker and Breaker, have opposite objectives: Maker aims at filling a winning set while Breaker prevents her to do so. Constructive results are generally more affordable in this convention. In addition, the extra set paradox does not exist when playing in the Maker-Breaker convention: adding new winning sets is always better for Maker.

In the literature, there are many graph optimization problems that have been turned into positional games. One can cite for example the clique game, the connectivity game, the H -game [1], or the domination game [6]. If a large part of the studies is devoted to the case where they are played on complete or random graphs, there is a more recent approach that consists in playing such games on a general graph. It generally yields to algorithmic results,

*This work has been supported by the National Research Agency (ANR) projects P-GASE (ANR-21-CE48-0001-01).

both in terms of hardness proofs or the construction of polynomial time algorithms to compute the winner of the game [3, 5]. One can also find results about the parameterized complexity of such games [2]. In addition, general algorithmic results about Maker-Breaker games played on k -uniforms hypergraphs (i.e., all the winning sets are of size k) have been given by Rahman and Watson [12] and Galliot [8]. Indeed, they respectively proved that determining the winner of a Maker-Breaker positional game is PSPACE-complete when $k = 6$ and polynomial when $k = 3$.

When switching to the Maker-Maker convention, by putting together the above result of Rahman and Watson with an argument of Byskov [4], it has been derived that Maker-Maker games are PSPACE-complete on 7-uniforms hypergraphs. Beyond this result, connections between the two types of conventions are not well established. Indeed, there are very few algorithmic results, even for particular positional games derived from optimization problems. The objective of this paper is to investigate the Maker-Maker domination game, by highlighting the similarities, the implications, and the differences with the results known in the Maker-Breaker convention. It is known that the Maker-Breaker domination game is PSPACE-complete even for bipartite graphs and split graphs, and polynomial for cographs and forests [6]. More precisely, it is shown that Maker has a winning strategy playing second on a forest if and only if it admits a perfect matching. When switching to the Maker-Maker convention, we will see that some of these complexity results still hold. Yet, in the case of forests, the polynomial characterization of the winning positions is far more complex than simply finding a perfect matching. The major result of the current paper consists in determining this characterization. To the best of our knowledge, this is the first time in the literature that a non-trivial algorithm is given to determine the winner of a game played according to the Maker-Maker convention.

The paper is organized as follows: in Section 2, we present the main definitions that will be useful in the Maker-Maker domination game. In Section 3, we give results about the Maker-Maker convention that are derived from the Maker-Breaker one, including the PSPACE-hardness result. Then, we fully solve the case of paths and cycles in Section 4. Sections 5 and 6 are devoted to the resolution of the Maker-Maker domination game on forests in linear time. As the proof is rather complex, we decided to split it into two parts, where Section 5 corresponds to the overview of the proof with the presentation of the algorithm, and Section 6 to the proof of the most technical elements.

2 Preliminaries

2.1 Standard definitions of graph theory

In this paper, we will only consider finite, undirected and simple graphs. A graph G is defined by a couple (V, E) where V denotes the set of vertices and E the edges of the graph. The (*closed*) *neighborhood* of a vertex $x \in V$, denoted by $N[x]$, is the subset of vertices containing x and all the vertices that are adjacent to x . A vertex x is *universal* if $N[x] = V$. The *degree* of a vertex is the number of vertices adjacent to it. A *leaf* is a vertex of degree 1. If x is a vertex, $G \setminus \{x\}$ denotes the graph on the vertex set $V \setminus \{x\}$ with all the edges of E that are not incident to x .

Let X be a subset of vertices. X is a *independent set* if there are no adjacent vertices in X . X is a *cutset* if $G \setminus X$ is disconnected. The subgraph of G *induced by* X , denoted by $G[X]$, is the graph with vertex set X and edge set all the edges of E whose both extremities are in X . A *matching* M is a subset of edges that are two by two not incident. If $|M| = |V|/2$ (i.e. if all the vertices appear in some edge of M) then M is *perfect*.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs on disjoint vertex sets. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of G_1 and G_2 is the graph on vertex set $V_1 \cup V_2$ with edge set $E_1 \cup E_2 \cup E_\times$ where $E_\times = V_1 \times V_2$.

A *cograph* is a graph that is either a single vertex or the union of two cographs or the join of two cographs. A *bipartite* graph is a graph whose vertex set can be partitioned into two independent sets. A *split* graph is a graph whose vertex set can be partitioned into two sets K and I where K induces a complete graph and I an independent set. A *path* is a graph whose vertex set is $\{v_1, \dots, v_n\}$ and $v_i v_j$ is an edge if $|i - j| = 1$. A *cycle* is a path with the additional edge $v_1 v_n$. A *forest* is a graph without any cycle. A *tree* is a connected forest.

A vertex x *dominates* a vertex y if $y \in N[x]$. A subset of vertices S *dominates* a vertex y if there exists $x \in S$ that dominates y . A *dominating set* S of G is a subset of vertices that dominates all the vertices of the graph. The smallest size of a dominating set of G is denoted by $\gamma(G)$.

2.2 The Maker-Maker domination game

The Maker-Maker domination game is played on a graph $G = (V, E)$. Two players, Alice and Bob, alternately claim an unclaimed vertex of the graph, with Alice playing first. The game ends when the vertices claimed by one of players form a dominating set (in which case the corresponding player wins) or when all the vertices have been claimed and none of the players managed to claim a dominating set (in which case the game is a draw).

As a 2-player finite perfect information game, if both players play optimally, one of the players has a winning strategy or the game ends a draw. Furthermore, since this game is a Maker-Maker positional game, it is well known that the second player, Bob, does not have a winning strategy (see for example [11]). As a consequence, there are only two possible *outcomes* for the Maker-Maker domination game played on G : either Alice has a winning strategy, which will be denoted by $o(G) = \mathcal{A}$, or both players can ensure a draw, which will be denoted by $o(G) = \mathcal{D}$. The problem of deciding, given a graph G , if $o(G) = \mathcal{A}$ or $o(G) = \mathcal{D}$ is named the MAKER-MAKER DOMINATION GAME problem.

Positions. A *position* P of the game is a triple (G, V_A, V_B) where $G = (V, E)$ is a graph and $V_A, V_B \subseteq V$ are two subsets of vertices such that $V_A \cap V_B = \emptyset$. The vertices in V_A (respectively V_B) correspond to vertices claimed by Alice (resp. Bob). A vertex not in V_A nor V_B is *unclaimed*. If X is a set of vertices, the *subposition induced by X* is the position $(G[X], V_A \cap X, V_B \cap X)$. If $\{x, y\}$ is a pair of unclaimed vertices of P . The position $P_{x,y}$ is defined by $P_{x,y} = (G, V_A \cup \{x\}, V_B \cup \{y\})$. A set of unclaimed vertices S is a *winning set for Player t* if $S \cup V_t$ is a dominating set of G .

A *pointed position* is a position where the next player is specified, it will be denoted by a couple (P, t) where P is a position and $t \in \{A, B\}$ is the next player (A stands for Alice and B for Bob).

The *outcome* $o(P, t)$ of a pointed position (P, t) is defined for positions P satisfying the property that both V_A and V_B do not dominate G simultaneously. We have $o(P, t) = \mathcal{A}$, if, the next player being t , considering that Alice has already claimed all the vertices V_A and Bob has claimed vertices V_B , there is a winning strategy for Alice, i.e. a strategy which ensures Alice to dominate the graph G before Bob. We will say in this case that the position is \mathcal{A} -win. Otherwise, we say that the position is *Draw* and we denote it by $o(P, t) = \mathcal{D}$. In this latter case, note that it also covers the case where Bob first dominates the graph, as the pointed position P may not be balanced depending on the sets V_A and V_B . Such cases will be considered in the upcoming proofs when considering the different possible sequences of moves (but their outcome will still be considered as a draw, as Alice will avoid them in her optimal sequence of moves).

Note that the starting position on the graph G is the pointed position $((G, \emptyset, \emptyset), A)$ and we have $o(G) = o((G, \emptyset, \emptyset), A)$.

Ordering positions. Two positions P and P' are said *equivalent* if for any $t \in \{A, B\}$, $o(P, t) = o(P', t)$. Two pointed positions (P, t) and (P', t) with the same first player are said *equivalent* if $o(P, t) = o(P', t)$.

In addition, and by analogy with combinatorial game theory, it is standard to order outcomes, stating that $\mathcal{A} > \mathcal{D}$. With this convention, a pointed position (P, t) is *better* for Alice (respectively Bob) than a pointed position (P', t') if $o(P, t) \geq o(P', t')$ (respectively $o(P, t) \leq o(P', t')$).

As an illustration of these definitions, the two following observations ensure that:

- it is always better for any player to start.
- if two positions have exactly the same winning sets, there are equivalent.

Observation 2.1. *For any position P , we have $o(P, A) \geq o(P, B)$.*

The proof of the above result derives from a standard stealing strategy.

Observation 2.2. *Let $P = (G, V_A, V_B)$ and $P' = (G', V'_A, V'_B)$ be two positions such that there exists a bijection $f : V(G) \setminus V_A \cup V_B \rightarrow V(G') \setminus V'_A \cup V'_B$ between the sets of unclaimed vertices that satisfies that S is a winning set for Alice (respectively Bob) if and only if $f(S)$ is a winning set for Alice (respectively Bob). Then P and P' are equivalent.*

We say that a vertex x is *forced* for Alice (resp. Bob) in a position (P, A) (resp. (P, B)), if whenever Alice (resp. Bob) claims a vertex $y \neq x$ in P , the resulting position is *Draw* (resp. \mathcal{A} -win). Sometimes, a move is always better than another. This is for example the case when the neighborhood of one vertex contained another neighborhood.

The following lemma gives a general formal framework which allows us to eliminate uninteresting moves. Note that this lemma or its derivatives are often used when studying positional games, sometimes not explicitly. For the sake of completeness, we give a proof here.

Lemma 2.3. *Let $P = (G, V_A, V_B)$ be a position. Let x, y be two unclaimed vertices. Assume that for both $t \in \{A, B\}$, $N[x] \setminus N[V_t] \subseteq N[y] \setminus N[V_t]$, where $N[V_t] = \bigcup_{v \in V_t} N[v]$. Then there exists an optimal strategy in which y is claimed before x .*

Roughly speaking, the condition says that y dominates a superset of the vertices dominated by x for both players. The conclusion says that if there exists an optimal strategy (for any player) starting by claiming x , one can assume that there also exists an optimal strategy starting by claiming y .

Proof. Let $P = (G, V_A, V_B)$ be a position. Suppose that Alice has a winning strategy \mathcal{S} on P in which x is claimed before y . Up to consider more moves, consider a position P' where the next vertex that will be claimed is x or y . If it is y , there is nothing to do, so suppose it is x .

If x is claimed by Bob, by definition any strategy is losing for Bob, thus they are all equivalent, and he can claim y first instead without changing the outcome of the game. Therefore, we can suppose that Alice is going to claim x . Consider the following strategy \mathcal{S}' for Alice.

- Instead of claiming x , claim y .
- While Bob claims a vertex $v \neq x$, she claims the vertex w she would have claimed according to \mathcal{S} if Bob has claimed v when she had claimed x .
- If Bob claims x , she claims the vertex w she would have claimed according to \mathcal{S} if Bob had claimed y .

As \mathcal{S} was a winning strategy for Alice, at a certain moment of the game, she would obtain a set of vertices $S_A \supset V_A$ that is a dominating set of G , while $S_B \supset V_B$, the vertices claimed by Bob, is not. If both x and y are in S_A , then \mathcal{S}' will make both Alice and Bob claim exactly the same vertices, so Alice will win. Otherwise, by denoting by S'_A and S'_B the set of vertices claimed by Alice and Bob respectively after \mathcal{S}' , we have $S'_A = S_A \setminus \{x\} \cup \{y\}$ and $S'_B = S_B \setminus \{y\} \cup \{x\}$. We prove that S'_A is still a dominating set and S'_B is not, which proves that \mathcal{S}' is a winning strategy for Alice. Indeed, for S'_A , if u is dominated only by x in S_A , then it means that $u \notin N[V_A]$ and thus $u \in N[x] \setminus N[V_A]$. Then by definition, $u \in N[y] \setminus N[V_A]$ and thus is dominated by y . For S'_B , assume that it is a dominating set. Since S_B is not dominating, it means that there exists $u \in N[x]$ not dominated in S_B . Thus, $u \notin N[y]$ but we also have $u \notin N[V_B]$ as $V_B \subset S_B$. Hence, u is in $N[x] \setminus N[V_B]$ but not in $N[y] \setminus N[V_B]$, a contradiction.

The proof that if Bob has a drawing strategy is similar, by adding the fact that if Bob does not dominate the graph, then S_A is not a dominating set, and thus neither is $S_A \setminus \{y\} \cup \{x\}$. \square

Union and decomposition of positions. Let $P = (G, V_A, V_B)$ and $P' = (G', V'_A, V'_B)$ be two positions on disjoint sets of vertices¹. The *union* of P and P' , denoted by $P \cup P'$, is the position $(G \cup G', V_A \cup V'_A, V_B \cup V'_B)$. Note that one can remove a position where both players have a dominating set. It is a simple consequence of Observation 2.2 since the winning sets are the same.

Observation 2.4. *Let $P = (G, V_A, V_B)$ and $P' = (G', V'_A, V'_B)$ be two positions on disjoint graphs. Assume that V'_A and V'_B are dominating sets of G' . Then $P \cup P'$ and P are equivalent.*

One cannot in general determine the outcome of a position $P \cup P'$ knowing the outcome of P and P' . However, when both positions are \mathcal{A} -win when Bob starts, the union is still \mathcal{A} -win:

Observation 2.5. *Let P and P' be two positions such that $o(P, B) = o(P', B) = \mathcal{A}$. Then we have $o(P \cup P', B) = \mathcal{A}$.*

Proof. Alice follows her strategies as second player in both P and P' until she dominates one of the component, say P . Then she just claims in P' following her strategy. She might claim several times in a row in P' (if Bob goes on claiming on P), but by Observation 2.1, it is always better for Alice to play first than second. Thus, if she has to claim twice in a row in P' , she will be able to dominate P' , before Bob does. \square

¹In this paper, the union of two positions will always be done on disjoint sets of vertices. It could happen, for simplicity reasons, that two vertices in different components have the same name. They must be considered as distinct vertices.

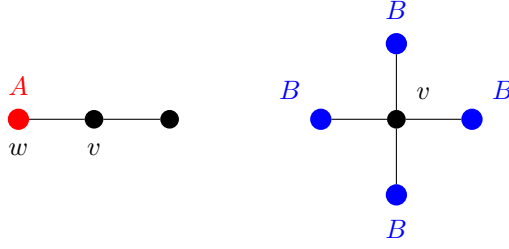


Figure 1: On the left, v is a B -trap and w may be isolated; on the right v is a A -trap

When considering a position, it could be useful to decompose it into several disjoint games. To do such a decomposition, the winning sets for both players should be the same. This is the case when a cut set is completely claimed and dominated by both players.

Lemma 2.6. *Let $P = (G, V_A, V_B)$ be a position. Assume $V(G)$ can be partitioned into three sets V_1, V_2, X such that:*

- *There are no edges between V_1 and V_2 ;*
- *All the vertices of X have been claimed: $X \subseteq V_A \cup V_B$.*
- *The vertices in X are already dominated by $V_A \cap X$ and $V_B \cap X$.*

Let P_1 and P_2 be the subpositions of P induced by $V_1 \cup X$ and $V_2 \cup X$ respectively (vertices of P_1 and P_2 are disjoint). Then the position P and the position $P_1 \cup P_2$ are equivalent.

Proof. We consider the trivial bijective map f between the unclaimed vertices of P and $P_1 \cup P_2$. Let S be a set of unclaimed vertices of P and $t \in \{A, B\}$. Assume first that S is a winning set of P for Player t . Let $S_1 = f(S) \cap V_1$. We prove that S_1 is a winning set of P_1 for Player t . Let u be a vertex of P_1 and $u' = f^{-1}(u)$. Either $u' \in X$ and thus is dominated by a vertex of $V_t \cap X$ in P and thus u is still dominated in P_1 . Or $u' \notin X$, and then u' is dominated by some vertex s of $S \cup V_t$ in P . Since $u' \notin X$ and X is a cutset, s should be in $V_1 \cup X$ and $f(s)$ is still in $S_1 \cup V_t$. Similarly, we can prove that $S_2 = f(S) \cap V_2$ is a winning set of P_2 for Player t , and thus $f(S)$ is a winning set of $P_1 \cup P_2$ for Player t . The reverse is easier: the union of two winning sets in P_1 and P_2 is clearly a winning set of P . Therefore, Observation 2.2 applies and the two positions are equivalent. \square

2.3 Traps

We will frequently use the notion of *trap* that is defined in this section. Roughly speaking, a A -trap (respectively B -trap) is a vertex of a game position such that, if it is not claimed by Alice (resp. Bob) by the end of the game, means that Alice (resp. Bob) will never build a dominating set. Formally, traps can be defined as follows:

Definition 2.7. *Let $P = (G, V_A, V_B)$ be a position of the game. A A -trap (respectively B -trap) is an unclaimed vertex v such that there exists a vertex w with $N[w] \cap V \setminus V_B = \{v\}$ (resp. $N[w] \setminus V_A = \{v\}$)*

In this definition, v corresponds to the vertex that must be claimed, and w to the vertex that will not be dominated if v is not claimed. Figure 1 illustrates the notion of traps.

The next lemma shows that if there is a A -trap in a position, one can consider that the next player will claim it immediately.

Lemma 2.8. *Let P be a position of the game and v be a A -trap of P . Claiming v is an optimal move for both players. Moreover, $o(P, B) = \mathcal{D}$.*

Proof. Let w such that $N[w] \cap V \setminus V_B = \{v\}$. Suppose it is Bob's turn. By claiming v , Bob isolates the vertex w as now $N[w] \subseteq V_B$. Therefore, Alice cannot dominate w any longer and the outcome is \mathcal{D} .

If it is Alice's turn, if she does not claim v , Bob claims it and once again isolates w . Therefore, v is forced for Alice. Thus, v is an optimal move. \square

Corollary 2.9. *Let P be a position of the game. If there exist two A -traps v_1 and v_2 of P such that $v_1 \neq v_2$, then $o(P, A) = o(P, B) = \mathcal{D}$.*

Proof. Since there are two distinct A -traps, the two vertices w_1 and w_2 that might be isolated are necessarily distinct (otherwise v_1 and v_2 would both be in their neighborhood, contradicting the definition of a trap). Hence, even if Bob is not the first player, he will be able to claim in one of the two traps and hence isolate either w_1 or w_2 . \square

The next lemma proposes another example of a forced move for Alice when there exists an unclaimed P_5 in a position as a subgraph.

Lemma 2.10. *Let (P, B) be a pointed position of the game with $P = (G, V_A, V_B)$. If there exists a path $G' = (v_1, v_2, v_3, v_4, v_5)$ such that G' is a subgraph of G with $V(G') \cap (V_A \cup V_B) = \emptyset$ and v_2, v_3, v_4 of degree 2, then if Bob claims v_3 , Alice is then forced to answer on v_2 or v_4 .*

Proof. If Bob claims v_3 , then if Alice answers elsewhere than on v_1, v_2 or v_4 , Bob claims his second move on v_2 and creates two A -traps in v_1 and v_4 . Indeed, since the vertices v_2 and v_3 are of degree 2, there remains only one way for Alice to dominate v_2 (i.e. by claiming v_1) and v_3 (i.e. by claiming v_4). By corollary 2.9, the resulting position is \mathcal{D} raw. If Alice claims v_1 , then Bob claims v_4 and by symmetry creates two A -traps equivalent to the previous case. \square

3 General results derived from the Maker-Breaker convention

Whereas the Maker-Maker domination game has not been studied yet, the Maker-Breaker version has been defined in [6]. In this section, we recall some results in this convention that have consequences for the Maker-Maker convention.

3.1 Basics

In Maker-Breaker games, only the winning condition differs from the Maker-Maker games. For more convenience and according to the terminology of [6], we will call the players Dominator and Staller. Dominator wins if she claims a dominating set, otherwise Staller wins. In particular, there is no draw. In this convention, since the two players have different roles, one need to precise who the first player is. Note that both players have interest to start since one can prove that if a player has a winning strategy playing second, he also has a strategy playing first.

When playing on the same graph, a winning strategy for Staller going second is a drawing strategy for Bob. Indeed, Bob will prevent Alice to claim a dominating set and thus Alice has no winning strategy. The contrary is not true since Bob can threaten Alice to create a dominating set in the Maker-Maker convention whereas Staller cannot. As a counter-example, we will prove that Bob has a drawing strategy on the cycle on ten vertices (see Theorem 4.7) whereas Dominator has a winning strategy for this graph [6]. However, strategies for Dominator can sometimes be used by Alice (see for example Lemma 3.4). Furthermore, if Alice manages to prevent Bob from claiming a dominating set, then she can play as Dominator in the Maker-Breaker convention in the rest of the game. Using this fact, one can prove that deciding the outcome in the Maker-Maker convention is as difficult as deciding the outcome in the Maker-Breaker convention.

Theorem 3.1. *MAKER-MAKER DOMINATION GAME is PSPACE-complete even if G is bipartite or split.*

Proof. First note MAKER-MAKER DOMINATION GAME is in PSPACE, by application of Lemma 2.2 from Schaefer [13], since there are most $|V|$ turns and during each turn, at most $|V|$ moves are available.

We prove that it is PSPACE-hard by a reduction from MAKER-BREAKER DOMINATION GAME: given a graph G and a first player (Dominator or Staller), who has a winning strategy ? MAKER-BREAKER DOMINATION GAME has been proven PSPACE-complete in [6], even if Staller starts, and even if the graph is split or bipartite.

We do the reduction as follows. Let $G = (V, E)$ be a graph. Consider $G' = (V \cup \{v_0\}, E)$ with v_0 a new isolated vertex. Note that the property of being split or bipartite is maintained by this operation. We prove that Dominator wins the MAKER-BREAKER DOMINATION GAME on G going second, if and only if Alice wins the MAKER-MAKER DOMINATION GAME on G' going first.

Suppose first that Dominator has a winning strategy \mathcal{S} on G going second. We define the following strategy for Alice on G' : first claim v_0 , then apply \mathcal{S} . By hypothesis, this strategy is a winning strategy for Dominator, thus, the set of vertices claimed by Alice at the end of the game will dominate the graph. As Bob cannot dominate v_0 , he cannot dominate before her, thus Alice wins.

Reciprocally, suppose that Staller has a winning strategy \mathcal{S} on G going first. We define the following strategy for Bob on G' . If Alice does not claim first v_0 , claims it. Alice cannot dominate v_0 any longer, so the outcome is at least a Draw. Otherwise, apply \mathcal{S} . By hypothesis, this strategy is a winning strategy for Staller, thus, the set of vertices claimed by Alice at the end of the game will not dominate G , and the outcome is Draw. \square

Deciding the outcome of the union of graphs when the outcome of each graph is known is trivial in Maker-Breaker convention (see [6]). Thus, to study a class of graphs, one just need to consider the connected case. In Maker-Maker, the situation is slightly different, one cannot in general say anything about the union.

3.2 Pairing strategies

A standard strategy in positional games is the *pairing strategy*. Let \mathcal{S} be a set of disjoint pair of vertices. A pairing strategy using \mathcal{S} consist in, whenever the opponent claims a vertex of a pair, answering by claiming the other one. This strategy ensures that the player who follows it takes at least one vertex in each pair, i.e. takes a *transversal* of \mathcal{S} . For the Maker-Breaker domination game, this strategy can be used by Dominator if any transversal of \mathcal{S} is a dominating set. This corresponds to the definition of a *pairing dominating set*.

Definition 3.2 ([6]). *Let $G = (V, E)$ be a graph. A subset of pair of vertices $\mathcal{S} = \{(u_1, v_1), \dots, (u_k, v_k)\}$ of V is a pairing dominating set if all the vertices are distinct and if the intersections of the closed neighborhoods of each pair cover all the vertices of the graph:*

$$V = \cup_{i=1}^k N[u_i] \cap N[v_i].$$

In other words, any transversal of \mathcal{S} is a dominating set.

Lemma 3.3 ([6]). *If G has a pairing dominating set, then in the Maker-Breaker convention, Dominator wins playing first or second.*

A particular case of pairing dominating sets are perfect matchings. If G is a forest, both notions are equivalent. In general, the reverse of Lemma 3.3 is not true. However, for some classes of graphs like forests or cographs, the reverse holds: if Dominator has a winning strategy playing second (and thus first), then G admits a pairing dominating set [6]. In the Maker-Maker convention, Alice can also use pairing strategies. A first possibility for Alice is when there exists a pairing dominating set of size $\gamma(G)$, the minimum size of a dominating set.

Lemma 3.4. *Let G be a graph. If G has a pairing dominating set of size $\gamma(G)$, then Alice has a winning strategy in G .*

Proof. Let G be a graph. Suppose G has a pairing dominating set of size $\gamma(G)$. By claiming her γ first moves in it, and claiming in the same pair as Bob if Bob claims a first vertex in a pair, Alice can dominate in γ moves. Bob cannot dominate before by definition of $\gamma(G)$. Therefore, Alice has a winning strategy in G . \square

This lemma can be applied in particular for connected cographs. Indeed, if G is a connected cograph with a universal vertex, then Alice wins at her first move. Otherwise, G has a pairing dominating set of size 2 and $\gamma(G) = 2$, thus by Lemma 3.4, Alice has a winning strategy. As mentioned previously, dealing with disconnected graphs is not easy in the Maker-Maker convention. Actually, we did not manage to determine the outcome of a general cograph and let it as an open problem. Note that determining the outcome for cographs in Maker-Breaker convention is polynomial (see [6]), but finding the minimum number of moves needed by Dominator to win is surprisingly open [9].

Another application of pairing strategies will be when there is some B -trap in a position. If it is Alice's turn, she can claim the trap and then play any strategy of Dominator, like a pairing strategy. For that, we will use a pairing dominating set that are taking into consideration the vertices already claimed. We call such a pairing dominating set a A -pairing.

Definition 3.5. *Let $G = (V, E)$ be a graph and $P = (G, V_A, V_B)$ a position. A set of disjoint pairs of unclaimed vertices $\mathcal{S} = \{(u_1, v_1), \dots, (u_k, v_k)\}$ of V is a A -pairing of position P if for each transversal T of \mathcal{S} , the set $V_A \cup T$ dominates V .*

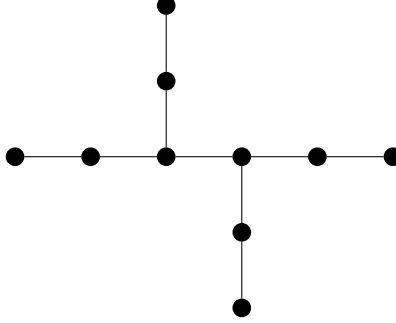


Figure 2: Example of a graph where Bob wins in Maker-Maker, but if we remove any leaf with its neighbor, Alice wins.

If Alice can prevent Bob to dominate the graph, A -pairings are then enough for Alice. This can be used in particular where there are some B -traps in the game.

Lemma 3.6. *Let P be a position with a B -trap and a A -pairing. Then, we have $o(P, A) = A$.*

Proof. Let P be a position containing a B -trap v and a A -pairing \mathcal{S} . Alice playing first can claim v . Now, Bob cannot dominate G anymore. Therefore, by following a pairing strategy using \mathcal{S} , Alice will claim a transversal of \mathcal{S} . Thus, she will dominate the whole graph and win. \square

Lemma 3.7. *Let P be a position with two B -traps v and v' and a A -pairing that contains nor v nor v' in P . Then, we have $o(P, B) = A$.*

Proof. Alice follows a pairing strategy using $\mathcal{S} \cup \{v, v'\}$. Bob cannot dominate since she will claim either v or v' and thus, there will be a vertex not dominated by Bob. She will dominate the graph since she will claim a transversal of \mathcal{S} . \square

3.3 Removing leaves

The key ingredient to solve trees in the Maker-Breaker convention is to remove leaves using the following lemma:

Lemma 3.8 ([6]). *Let G be a graph, ℓ be a vertex of degree 1 and u be its unique neighbor. If u has degree 2, then G and $G \setminus \{\ell, u\}$ have the same outcome in Maker-Breaker convention: Dominator has a winning strategy in G if and only if she has a winning strategy in $G \setminus \{\ell, u\}$ (whoever is first).*

Let T be a tree. One can apply successively Lemma 3.8 until obtaining a *reduced* tree T' where the unique neighbor of any leaf has degree at least 3. Then there are only three possibilities for T' . Either T' is empty, which means that T has a perfect matching. In this case, Dominator wins playing first or second. The second possibility is that T' is a single vertex or a star. Then the first player has a winning strategy. The last possibility is that T' contains two disjoint *cherries*. A cherry is a vertex u connected to two leaves. In this last case, Staller wins playing first or second. This operation of removing leaves of a tree until obtaining a reduced tree correspond to the algorithm given in [6] to solve trees, and thus forests using unions of trees.

The situation is a quite different in the Maker-Maker convention. Indeed, Lemma 3.8 is not true anymore and one cannot reduce trees as in Maker-Breaker convention. Indeed, leaves are still playing an important role as shown by Figure 2. Actually, one can prove that it is always optimal for Bob to claim the unique neighbor of a leaf:

Lemma 3.9. *Let $P = (G, V_A, V_B)$ be a position with an unclaimed leaf ℓ for which its unique neighbor u is also unclaimed. Then $o(P, B) = o(P_{\ell, u}, B)$ ².*

²In this equality, it is assumed that $P_{\ell, u}$ is really a position, i.e. $V_B \cup \{u\}$ and $V_A \cup \{\ell\}$ do not both dominate the graph. In this latter particular case, we obviously have $o(P, B) = \mathcal{D}$.

Proof. Suppose first that $(P_{\ell,u}, B)$ is \mathcal{D} raw. Bob can, in (P, B) , claim u first. Then ℓ is a A -trap and by Lemma 2.8, Alice must claim it. Thus, the game is now $(P_{\ell,u}, B)$ that is \mathcal{D} raw by hypothesis. Thus, (P, B) is \mathcal{D} raw.

We prove the other implication by contraposition. Suppose $(P_{\ell,u}, B)$ is \mathcal{A} -win. Let \mathcal{S} be a winning strategy for Alice in $(P_{\ell,u}, B)$. We consider a strategy \mathcal{S}' for Alice in (P, B) defined as follows:

- If Bob claims a vertex in $\{\ell, u\}$, Alice claims the other one.
- Otherwise, Alice claims according to \mathcal{S} (ignoring the moves on ℓ and u) until she dominates all the vertices of $V \setminus \{\ell, u\}$. If u, ℓ have been claimed at this moment, either Bob has claimed u and Alice ℓ , and she wins since the strategy played is the same as \mathcal{S}' , or Bob has claimed ℓ and Alice u , which dominates more vertices than ℓ . Therefore, in both cases, her winning strategy in \mathcal{S}' ensures her that she dominates first. Otherwise, when she dominates all the vertices of $V \setminus \{\ell, u\}$, Bob does not dominate in the game played on $(P_{\ell,u}, B)$. Thus, claiming u does not make him dominate G . Therefore, by claiming at her next move a vertex in $\{\ell, u\}$, Alice wins.

Thus, Alice has a winning strategy in (P, B) , finishing the proof. \square

In other words, after the first move of Alice, one can always assume that Bob has claimed all the vertices that are adjacent to leaves of G , and that Alice has answered by claiming all the leaves. This will be particularly important when dealing with trees.

4 Paths and cycles

In this section, we consider paths and cycles. The structure of the positions obtained after some moves will be basically union of paths where the extremities are claimed. Thus, we first deal with these paths and derive some general results on them that will help us to solve paths and cycles (and also forests in the next sections).

4.1 Bounded Paths

A *bounded path* is a path on at least four vertices where the four vertices at its extremities (the two leftmost and the two rightmost) are already claimed by Alice and Bob in such a way that the four vertices are already dominated by both players.

Definition 4.1. A bounded path of length n is a position (G, V_A, V_B) such that:

- G is a path $(v_{-1}, v_0, v_1, \dots, v_n, v_{n+1}, v_{n+2})$;
- the unclaimed vertices are exactly vertices v_1 to v_n ;
- exactly one vertex among $\{v_{-1}, v_0\}$ (respectively $\{v_{n+1}, v_{n+2}\}$) is in V_A , the other being in V_B .

According to this definition, the knowledge of the label of v_0 and v_{n+1} is sufficient to deduce the label of v_{-1} and v_{n+2} . Therefore, for t, t' in $\{A, B\}$, we will denote by $[to^n t']$ the bounded path of size n such that $v_0 \in V_t$ and $v_{n+1} \in V_{t'}$. See Figure 3 for an illustration of $[Ao^5A]$ and $[Bo^3A]$.

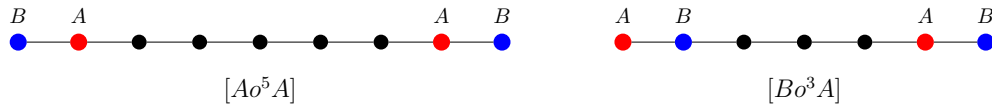


Figure 3: The bounded paths $[Ao^5A]$ and $[Bo^3A]$.

In some situations, bounded paths can be considered as a neutral structure that preserves the outcome when adjoined to another position. The next lemma illustrates a first case in which this may occur. It will also lead to a natural resolution of paths.

Lemma 4.2. For any position P and any integer n , $o(P, B) = o(P \cup [Ao^n B], B)$.

Proof. Let $P = (G, V_A, V_B)$ be a position. We do the proof by induction on the number of unclaimed vertices in $P \cup [Ao^n B]$. First note that if $n \leq 1$, the result is true since $[Ao^n B]$ is already dominated by both players. Hence, by Observation 2.4, the result is true when the number of unclaimed vertices is at most 1.

Now assume there are at least two unclaimed vertices and $n \geq 2$. Suppose first that $(P, B) = \mathcal{D}$. In this case, Bob starts by claiming v_{n-1} in $P \cup [Ao^n B]$ creating a A -trap in v_n . By Lemma 2.8, Alice has to answer v_n . By induction hypothesis $o(P \cup [Ao^{n-2} B], B) = \mathcal{D}$, which ensures that $o(P \cup [Ao^n B], B) = \mathcal{D}$.

Now assume that (P, B) is \mathcal{A} -win and let \mathcal{S} be a winning strategy for Alice in this pointed position. We give a strategy for Alice in $(P \cup [Ao^n B], B)$. Let v_B be the vertex claimed by Bob in $(P \cup [Ao^n B], B)$. Alice answers as follows:

- If $v_B \in V(G)$ and if Alice does not dominate P , she claims the same vertex v_A she would have answered following \mathcal{S} if Bob had claimed v_B in (P, B) . The resulting position has two vertices less, and is \mathcal{A} -win by induction hypothesis.
- If $v_B \in V(G)$ and Alice already dominates P , we necessarily have $n \geq 2$, otherwise, she would already have won. She claims v_2 . The resulting position is better than $(P \cup [AoA] \cup [Ao^{n-2} B], B)$, which itself is better than $([AoA] \cup [Ao^{n-2} B], B)$, as Alice already dominates P . Thus, by induction hypothesis applied with $P' = [AoA]$, which is not dominated by Bob, the position is \mathcal{A} -win.
- If v_B is a vertex of $[Ao^n B]$, then v_B corresponds to some vertex v_k , with $k \in \{1, \dots, n\}$. If $k = n$, Alice answers v_{n-1} . Otherwise, she claims v_{k+1} . Note that if $k = n$, the position obtained is better for Alice than the one obtained by $k = n - 1$ (since she dominates a superset of vertices). Therefore, and without loss of generality, we will assume that $k < n$. By Lemma 2.6, the resulting position is then equivalent to $(P \cup [Ao^{k-1} B] \cup [Ao^{n-1-k} B], B)$ which has less unclaimed vertices than the original one. By induction hypothesis applied twice, it has the same outcome as $(P \cup [Ao^{k-1} B], B)$, which also has the same outcome as (P, B) , which is \mathcal{A} -win by hypothesis.

This analysis ensures that $o(P \cup [Ao^n B], B) = \mathcal{A}$, since for each claim of Bob, there exists an answer of Alice leading to an \mathcal{A} -win position. \square

The next lemma presents another situation where the adjunction of some bounded paths with particular constraints does not change a winning outcome for Alice.

Lemma 4.3. *Let $P = (G, V_A, V_B)$ be a position, let n be an integer such that $n \not\equiv 0 \pmod{3}$ and let n' be an integer such that $n' \equiv 0 \pmod{2}$. Then if $o(P, B) = \mathcal{A}$, then $o(P \cup [Ao^n A] \cup [Bo^{n'} B], B) = \mathcal{A}$.*

Proof. We prove this result by induction on the number m of unclaimed vertices of $P \cup [Ao^n A] \cup [Bo^{n'} B]$. For initialization, if $m = 1$, then P has no unclaimed vertices, $n = 1$ and $n' = 0$. Thus, Alice dominates $P \cup [Ao^n A] \cup [Bo^{n'} B]$ and the result is true, by definition.

Assume now that $m \geq 2$. If Alice dominates $(P \cup [Ao^n A] \cup [Bo^{n'} B])$ (which implies that $n \leq 2$ and $n' = 0$) then the result is true, by definition. Otherwise, we have to prove that, for each vertex x claimed by Bob, there exists an answer y for Alice such that $o(P \cup [Ao^n A] \cup [Bo^{n'} B], B) = \mathcal{A}$, the pointed position obtained after the two claim is \mathcal{A} -win.

Denote by (u_1, \dots, u_n) the unclaimed vertices of $[Ao^n A]$ and by $(v_1, \dots, v_{n'})$ the unclaimed vertices of $[Bo^{n'} B]$. We consider all the possible moves for Bob.

- If Bob claims u_1 or u_2 in $[Ao^n A]$ and $n \leq 2$, we consider two subcases. If $n' = 0$, then Alice dominates $[Ao^n A] \cup [Bo^{n'} B]$, which obviously gives the result. If $n' \geq 2$, Alice claims v_1 . By Observation 2.4, the resulting game is equivalent to $P \cup [Ao^{n-1} B]$, which is equivalent to P by Lemma 4.2.
- If Bob claims u_1 or u_2 in $[Ao^n A]$ and $n \geq 4$, by Lemma 2.3, we can suppose he claims u_2 . then Alice claims u_3 . Thus, the resulting position is equivalent to $P \cup [Ao^{n-3} A] \cup [Bo^{n'} B]$, which gives the result using the induction hypothesis. The case where Bob claims u_n or u_{n-1} is symmetric.
- If Bob claims a vertex u_k in $[Ao^n A]$ with $k \notin \{1, 2, n-1, n\}$, Alice answers by claiming a vertex u_i with $i \in \{k-1, k+1\}$ such that the resulting component is $[Ao^j A] \cup [Ao^{j'} B]$ with $j \not\equiv 0[3]$. Note that, since $n \not\equiv 0[3]$, this vertex always exists. By Lemmas 2.6 and 4.2, the resulting position is equivalent to $P \cup [Ao^j A] \cup [Bo^{n'} B]$, which gives the result using the induction hypothesis.

- If Bob claims some vertex v_k in $[Bo^{n'}B]$, Alice answers by claiming a vertex v_i with $i \in \{k-1, k+1\}$ such that the resulting component is $[Bo^jB] \cup [Ao^{j'}B]$ with $j \equiv 0[2]$. Note that as $n' \equiv 0[2]$, this vertex always exists. By Lemma 4.2 and Lemma 2.6, the resulting position is equivalent to $P \cup [Ao^nA] \cup [Bo^jB]$, which gives the result using the induction hypothesis.
- If Bob claims a vertex x in P , we consider two subcases. If Alice does not dominate G yet, she claims the vertex she would have claimed as an answer to x in her winning strategy in P . The resulting position has two vertices less, which gives the result using the induction hypothesis.

If Alice already dominates G , and $n' \geq 2$, then Alice claims v_1 in $[Bo^{n'}B]$, turning it into $[Ao^{n'-1}B]$. By Lemma 4.2, the game is now equivalent to $P \cup [Ao^nA]$, and Alice already dominates G , so by induction hypothesis, it is a winning position for her. For $n' = 0$ and $n \geq 4$ (cases where $n \leq 2$ are trivial), then Alice claims u_2 . The resulting position is better for Alice than the position $P \cup [AoA] \cup [Ao^{n-2}B]$, which, by Lemma 4.2, is equivalent to $P \cup [AoA]$ and therefore is a winning position for Alice. \square

We finish this subsection by proving that bounded paths $[Bo^nB]$, where n is odd, and $[Ao^nA]$, when $n \equiv 0 \pmod{3}$, are not good for Alice when Bob starts. The first one is natural and give a condition for a \mathcal{D} raw whatever the rest of the position is. The second one is more surprising. Here Bob obtains a \mathcal{D} raw mostly by threatening Alice to dominate before her. Thus, one cannot add any position and maintain a \mathcal{D} raw.

Lemma 4.4. *For any position P and any odd integer n , $o(P \cup [Bo^nB], B) = \mathcal{D}$.*

Proof. We prove the result by induction on n . If $n = 1$, there is a A -trap: by Lemma 2.8, $o(P \cup [BoB], B) = \mathcal{D}$. Now, let $n \geq 3$. Bob claims v_2 which forces Alice to claim v_1 . The position is then equivalent to $(P \cup [Bo^{n-2}B], B)$ which is \mathcal{D} raw by induction. \square

Lemma 4.5. *For any positive integer k , $o([Ao^{3k}A], B) = \mathcal{D}$.*

Proof. We prove the result by induction on k .

If $k = 1$, Bob claims v_2 and directly dominates $[Ao^3A]$. Thus, $o([Ao^3A], B) = \mathcal{D}$.

If $k = 2$, Bob claims v_2 . If Alice does not answer in $\{v_1, v_3, v_4\}$, Bob claims v_3 at his second turn and create two A -traps in v_1 and v_4 which ensures a \mathcal{D} raw. Thus, Alice should claim a vertex among $\{v_1, v_3, v_4\}$ and does not dominate the graph at her first claim. Then Bob can claim v_5 and dominates the graph. Hence $o([Ao^6A], B) = \mathcal{D}$.

Assume now that $k \geq 3$ and that the result is true for any positive $k' < k$. Consider the position $([Ao^{3k}A], B)$. Bob claims v_5 at his first turn. By Lemma 2.10 applied on $(v_3, v_4, v_5, v_6, v_7)$, Alice should answer on v_4 or v_6 . If she claims v_4 , then by Lemma 5.5, the position is equivalent to the position $([Ao^3A] \cup [Bo^{3k-5}A], B)$, which is equivalent, by Lemma 4.2 to $([Ao^3A], B)$ which is \mathcal{D} raw.

If she claims v_6 , in the same way, the position is equivalent to $([Ao^4B] \cup [Ao^{3(k-2)}A], B)$, which is equivalent to $([Ao^{3(k-2)}A], B)$ which is \mathcal{D} raw by induction. \square

4.2 Paths

Lemma 4.2 can be used to prove that Alice always wins on paths.

Theorem 4.6. *Let P_n be the path of length n . Then $o(P_n) = \mathcal{A}$.*

Proof. Let v_1, \dots, v_n be the vertices of the path. In $n \leq 3$, then Alice wins at her first turn, so we can assume that $n \geq 4$. Alice starts by claiming v_2 . By Lemma 3.9, we can assume that Bob claims v_{n-1} and Alice answers by claiming v_n . Let (P, B) be the actual pointed position of the game. Using Observation 2.2, and since v_1 should be in any winning set of Bob, this position is equivalent to the position $([AoA] \cup [Ao^{n-4}B], B)$. By Lemma 4.2, $o([AoA] \cup [Ao^{n-4}B], B) = o([AoA], B) = \mathcal{A}$, which ensures that $o(G) = \mathcal{A}$. \square

When playing in the Maker-Breaker convention, note that this result implies that Dominator also has a winning strategy on any path playing first. This result was already known since [6], but the strategy developed here is different from the other one.

4.3 Cycles

The case of cycles in the Maker-Maker convention is more subtle than for the Maker-Breaker convention, where Dominator always wins. More precisely, we will show that there are infinitely many \mathcal{A} -win and \mathcal{D} outcomes that depend on the size of the cycle modulo 3.

From now, we will denote by C_n the cycle of order n . The vertices of C_n will be denoted by v_0 to v_{n-1} .

Theorem 4.7. *Let n be an integer. We have $o(C_n) = \mathcal{D}$ if and only if $n \geq 10$ and $n \equiv 1 \pmod{3}$.*

Proof. We first prove the “if” part. Let $n = 3k + 1$, with $k \geq 3$, and let C_n be a cycle of order n . By symmetry, we can assume that Alice first claims v_0 and thus $o(C_n) = o(C_n, \{v_0\}, \emptyset, B)$. We give a strategy for Bob to obtain at least a draw. Bob first claims v_5 . By Lemma 2.10 with $G' = (v_3, v_4, v_5, v_6, v_7)$, Alice has to answer v_4 or v_6 .

- If Alice claims v_6 . Then Bob can claim v_3 which forces Alice to claim the A -trap in v_4 and then Bob can claim v_1 which forces Alice to claim the A -trap in v_2 . At this point, the position is equivalent to $([Ao^{3(k-2)}A], B)$ which is \mathcal{D} by Lemma 4.5 (remember that $k - 2 \geq 1$).
- If Alice claims v_4 . Bob can claim successively all the v_{2i+1} starting from v_7 , creating a A -trap in v_{2i} that Alice is forced to claim. If n is even, Bob follows this strategy until claiming v_{n-1} . Then Alice has to claim v_{n-2} and does not dominate the cycle (she does not dominate v_2). Then Bob can dominate the cycle by claiming v_2 .

If n is odd, Bob follows this strategy until claiming v_{n-4} . After Alice has claimed v_{n-5} , Bob claims v_2 . Alice is forced to claim v_{n-1} , otherwise Bob wins by claiming it. Then Bob claims v_{n-2} , creating a trap in v_{n-3} , forcing Alice to claim it. At this point, Alice still does not dominate v_2 . Then Bob can dominate the cycle by claiming v_1 .

In all cases, the game either ends in a draw or Bob dominates the graph, thus $o(C_n) = \mathcal{D}$.

We now prove the “only if” part. First consider that $n \not\equiv 1 \pmod{3}$. Without loss of generality, one can assume that Alice will claim v_0 at her first turn, and thus we consider the pointed position $(P, B) = ((C_n, \{v_0\}), \emptyset, B)$. Consider now the position P' obtained from P by adding a vertex claimed by Bob adjacent only to v_0 . This position is better for Bob since he dominates more vertices than in P . Thus, it is enough to prove that (P', B) is \mathcal{A} -win to ensure that (P, B) is \mathcal{A} -win. Using Observation 2.2, (P', B) is equivalent to $([Ao^{n-1}A], B)$. By Lemma 4.3, since $n - 1 \not\equiv 0 \pmod{3}$, $o([Ao^{n-1}A], B) = \mathcal{A}$, and thus $o(C_n) = \mathcal{A}$.

It remains to prove that Alice wins on C_4 and C_7 . On C_4 , Alice wins by claiming any two vertices and as it is not possible to dominate in one move, she will dominate first. On C_7 , Alice can claim v_0 , and then use a pairing strategy with pairs (v_2, v_3) and (v_4, v_5) . This way, she dominates the cycle in three moves, and Bob cannot dominate before since at least three vertices are required to dominate C_7 . \square

5 Forests

In this section, we give the essential elements to solve the case of forests, given by Theorem 5.1 below. In particular, we will first reduce the problem to any *standard* forest, meaning that non-standard forests correspond to cases that can be solved easily. Then we will give necessary conditions about the first move of Alice, yielding to the introduction of the skeleton of a forest. From that definition, we will be able to present the general algorithm that computes the outcome of any forest, as depicted by the decision tree of Figure 8. In the next section, we will give the full proof of the validity of the algorithm.

Theorem 5.1. *Deciding the outcome of a forest can be done in linear time.*

5.1 Removing small components

If there is an isolated vertex in the forest, this is like playing in the Maker-Breaker convention.

Lemma 5.2. *Let F be a forest with an isolated vertex v_0 . Then $o(F) = \mathcal{A}$ if and only if $F \setminus \{v_0\}$ contains a perfect matching.*

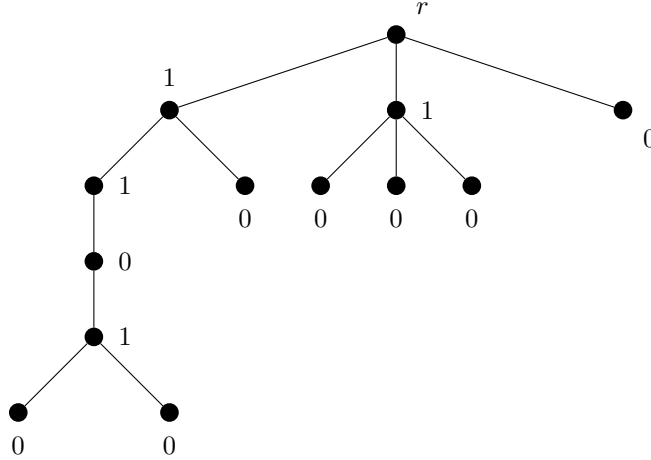


Figure 4: Example of labelling

Proof. Alice has to claim first v_0 . Then Bob is playing first in $F \setminus \{v_0\}$. Since he cannot dominate anymore, he has the same role as Staller in the Maker-Breaker Domination Game. In [6], it is proved that Dominator playing second in a forest in the Maker-Breaker Domination Game wins if and only if there is a perfect matching, which gives the result. \square

Isolated edges can be removed without changing the outcome.

Lemma 5.3. *Let F be a forest with an isolated edge $e = uv$. Then $o(F) = o(F \setminus \{u, v\})$.*

Proof. If Alice (respectively Bob) has a winning (resp. draw) strategy in $F \setminus \{u, v\}$, then she can apply her strategy in F by pairing u with v . The resulting strategy will still be winning (resp. leading to a draw) in F . \square

By applying Lemma 5.2 and Lemma 5.3, one can consider in what follows that all the connected components have at least three vertices.

5.2 Bottom-to-top strategies for Bob

In this subsection, we describe a strategy for Bob that will often be considered to obtain draws or to reduce trees. Let T be a tree rooted on a vertex r , vertices of T except r are labeled inductively with values 0 and 1, starting from the leaves as follows:

- If all children of v are labeled 1, then v is labeled 0 (hence, all the leaves are labeled 0);
- If at least a child of v is labeled by 0, then v is labeled by 1.

Figure 4 gives an example of such a labeling.

Let T be a tree rooted in r and consider the pointed position (P, B) with $P = (T, V_A, \emptyset)$ and $V_A \subseteq \{r\}$. A *bottom-to-top strategy* for Bob on (P, B) consists, at each step, in claiming a vertex v labeled by 1 such that all the successors of v labeled by 1 are already claimed by Bob. The following property is maintained during this process: Alice is forced to claim only vertices labeled by 0, and any vertex claimed by Alice (except r) has his parent claimed by Bob. Indeed, it is true before the first claim of Bob. Assume it is true before Bob's turn. Let v be a vertex labeled by 1 such that all the successors of v labeled by 1 are already claimed by Bob. By definition of the labeling and the assumption, v has a child u labeled by 0 that is unclaimed. All the children of u are by definition labeled by 1 and thus already claimed by Bob. Thus, u is a A -trap and Alice is forced to claim it, maintaining the property true. Such a strategy can also be applied when all the leaves of T are adjacent to vertices already claimed by Bob. In this strategy, Bob can claim all the vertices labeled by 1. A particular interesting case for Bob is when there exists a vertex v labeled by 1 with two children labeled by 0:

Lemma 5.4. *Let T be a tree and consider the labeling of T rooted in v_0 . If there exists a vertex labeled by 1 with two children labeled by 0, then the position (P, B) with $P = (T, \{v_0\}, \emptyset)$ is Draw.*

Proof. Bob uses a bottom-to-top strategy on T . When claiming v , he will create two A -traps on the two children of v labeled by 0, which concludes the proof by Corollary 2.9. \square

Bob can also use bottom-to-top strategies to reduce the forest to a smaller one where he has a \mathcal{D} raw strategy.

Lemma 5.5. *Let F be a forest and v_0 a vertex. Consider the labeling of F rooted in v_0 (for the components not containing v_0 , root on any vertex). Let $v \neq v_0$ be a vertex labeled by 1 and S_v be the set of successors of v in the rooted tree. Let $F_{\bar{v}}$ be the tree obtained by removing all vertices of S_v and adding a leaf v' connected to v . Then, if $o(F_{\bar{v}}, \{v_0, v'\}, \{v\}, B) = \mathcal{D}$, then $o(F, \{v_0\}, \emptyset, B) = \mathcal{D}$.*

Proof. Bob follows by a bottom-to-top strategy on S_v and can claim all the vertices of S_v labeled by 1 until claiming v . Alice is always forced to claim a child of the claimed vertex. Afterward, Bob plays his strategy to obtain a \mathcal{D} raw in $(F_{\bar{v}}, \{v_0, v'\}, \{v\}, B)$, leading to a \mathcal{D} raw for $(F, \{v_0\}, \emptyset, B)$. \square

5.3 Cherries

A particular case where the bottom-to-top strategy will be useful is when there are some cherries in the forest. Recall that a cherry is a vertex c connected to two leaves ℓ_1 and ℓ_2 . It will be denoted by the triple $C = (c, \ell_1, \ell_2)$. If F contains some cherries, the outcome of F can be easily computed.

Lemma 5.6. *Let F be a forest. If F has two cherries $C = (c, \ell_1, \ell_2)$ and $C' = (c', \ell'_1, \ell'_2)$, with $c \neq c'$, then $o(F) = \mathcal{D}$.*

Proof. Let F be such a forest. After Alice has claimed her first vertex, she cannot have claimed both c and c' . Suppose without loss of generality that Alice has claimed c' . Then Bob claims c . The resulting pointed position contains two A -traps and thus is \mathcal{D} raw by Corollary 2.9. \square

Lemma 5.7. *Let F be a forest with exactly one cherry $C = (c, \ell_1, \ell_2)$. Then $o(F) = \mathcal{A}$ if and only if there is a matching in $F \setminus \{c\}$ that covers $V(F) \setminus N[c]$.*

Proof. Suppose first that F has a matching M in $F \setminus \{c\}$ that covers $V(F) \setminus N[c]$. Then Alice claims c , which creates a double B -trap in ℓ_1 and ℓ_2 . The matching M is actually a A -pairing, and, we can suppose it contains neither ℓ_1 nor ℓ_2 since these two vertices have their neighborhood included in $N[c]$. Then by Lemma 3.7, Thus $o(F) = \mathcal{A}$.

Suppose now that F has no such matching M . We define a strategy for Bob as follows. If Alice's first claim is not element of the cherry, then Bob claims c and creates two A -traps, leading to a \mathcal{D} raw position. Thus, we can assume that Alice's first claim r is an element of $\{c, \ell_1, \ell_2\}$. Let T be the connected component of F containing r . If there exists another connected component T' of F that has no perfect matching, then Bob can apply the strategy of Staller playing first in T' that prevents Dominator to dominate T' (see [6]). Thus, one can assume that there is a perfect matching in all the components of F distinct from T . Now label the vertices of T rooted in r as defined in Section 5.2. We want to prove that there exists a vertex labeled 1 with two children labeled 0, which will ensure a \mathcal{D} raw strategy for Bob by applying the bottom to top strategy. If it is not the case, then consider the matching M' where all the vertices labeled 1 (except c) are paired with their unique child labeled 0. We claim that M' covers $V(T) \setminus N[c]$. Indeed, assume $x \in V(T) \setminus N[c]$ is not covered by M' . Then x must be labeled 0 and its parent should be r since it is the only vertex not labeled. But then $x \in N[c]$. Thus, there exists a matching in $F \setminus \{c\}$ that covers $V(F) \setminus N[c]$: take the union of M' and the perfect matchings of all the other components. \square

5.4 Definition of the skeleton and easy cases

Considering Lemmas 5.6 and 5.7, we will assume now that there is no cherry in F , no isolated vertex and no isolated P_2 . From Lemma 3.9, one can assume that after the first turn of Alice, Bob will claim all the unclaimed vertices of F with a leaf as a neighbor. Alice is then forced to answer on each leaf. This motivates us to define the *skeleton* of F , denoted by S_F , as the vertices that are not a leaf nor a parent of a leaf.

More formally, we denote by L_F the leaves of F and by M_F their parents. Remark that, with the hypothesis that there is neither cherry nor isolated vertex, we have $L_F \cap M_F = \emptyset$ and the mapping $\rho : L_F \rightarrow M_F$, which associates to each vertex v of L_F its parent, is bijective. Then, let S_F be defined by $S_F = V(F) \setminus (L_F \cup M_F)$. Figure 5 is an illustration of a forest with its skeleton.

In some simple cases, we can directly give the outcome of F .

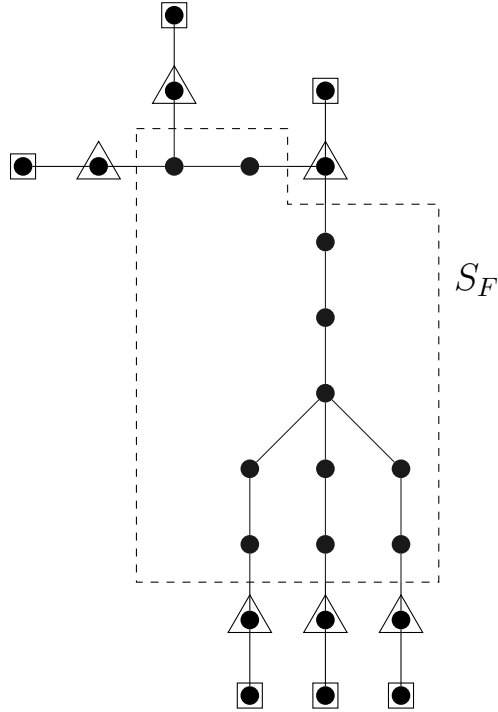


Figure 5: A forest F and its skeleton S_F . Note that F is connected but S_F is not. M_F is the set of vertices in triangles, and L_F is the set of vertices in squares.

Lemma 5.8. *If S_F is empty, then $o(F) = \mathcal{A}$.*

Proof. Set $L_F = \{\ell_1, \dots, \ell_k\}$. By hypothesis, F has no cherry, therefore, no vertex can be adjacent to two of these leaves. Thus, we can denote $M_F = \{m_1, \dots, m_k\}$ with m_i adjacent to ℓ_i for $1 \leq i \leq k$. Note that k vertices (one on each set $\{\ell_k, m_k\}$) are necessary and sufficient to dominate F . Thus $\gamma(F) = k$ and there is a pairing dominating set of size k (the set of pairs $\{(\ell_i, m_i), i \in \{1, \dots, k\}\}$). By Lemma 3.4, Alice has a winning strategy in F . \square

Lemma 5.9. *Let T be a connected component of F such that S_T is empty. Then $o(F) = o(F \setminus T)$.*

Proof. Assume first that $o(F \setminus T) = \mathcal{A}$ and let x be the first claim of Alice in a winning strategy. Then Alice claims x in F . By Lemma 3.9, Bob will claim all the vertices of M_T and Alice will answer all the vertices of L_T . After these moves, T is completely claimed and dominated by both players. By Observation 2.4, we can remove this component. The game is then equivalent to $(F \setminus T, \{x\}, \emptyset)$ which is \mathcal{A} -win when Bob starts, leading to $o(F) = \mathcal{A}$.

Assume now that $o(F \setminus T) = \mathcal{D}$. Consider the game played in F and a first claim x of Alice. If $x \in V(F \setminus T)$, then as before, the position $(F, \{x\}, \emptyset)$ is equivalent to the position $(F \setminus T, \{x\}, \emptyset)$ which is \mathcal{D} when Bob starts. If $x \in V(T)$, let x' be the unique neighbor of x if x is a leaf or the leaf connected to x if $x \in M_T$. Then we can assume by Observation 2.4 that Bob claims all the vertices in M_T (except x or x') and that Alice answers by claiming all the vertices in L_T (except x or x'). At this point, all the vertices of T have been claimed except x' . Then Bob claims x' . Then both players dominate T and the position is equivalent to $(F \setminus T, \mathcal{A})$ which is a \mathcal{D} position. In conclusion, whatever Alice claims, Bob can ensure a draw in F . Thus $o(F) = \mathcal{D}$. \square

Lemma 5.10. *If S_F induces a star of center c such that c has no neighbor in M_F , then $o(F) = \mathcal{A}$.*

Proof. Alice claims the center c as a first move. Then as explained above, Bob claims all the vertices of M_F and Alice answers all the leaves of L_F . After that, Alice dominates the whole graph and Bob does not, since he does not dominate c . \square

We say that that F is *standard* if all its connected components have at least four vertices and a non-empty skeleton, if F has no cherries, and if S_F does not induce a star of center c where c has no neighbor in M_F . We now focus on the standard forests.

5.5 First move of Alice

If F is standard, next lemmas say that it can be assumed that Alice must claim outside S_F and must connect it. The main idea behind this result is that, if Alice claims in S_F , she claims too far from the leaves and Bob can win with a bottom-to-top strategy.

Lemma 5.11. *Let F be a standard forest and $v_0 \in S_F$. We have $o((F, \{v_0\}, \emptyset), B) = \mathcal{D}$.*

Proof. Assume first that the graph induced by S_F is either a unique vertex, a unique edge, or a star (whose center has necessarily a neighbor in M_F). Let $L_F = \{\ell_1, \dots, \ell_k\}$ and $M_F = \{m_1, \dots, m_k\}$ such that for $1 \leq i \leq k$, ℓ_i and m_i are neighbors. Starting from $(F, \{v_0\}, \emptyset)$, Bob successively claims m_1, \dots, m_k , which forces Alice to reply ℓ_1, \dots, ℓ_k . When Bob has just claimed m_k , he dominates the whole graph while Alice does not yet dominate ℓ_k . This ensures that $o((F, \{v_0\}, \emptyset), B) = \mathcal{D}$.

It can now be assumed that the graph induced by S_F is neither a unique vertex, a unique edge, nor a star. Thus, we can consider the position $P' = (F, \{v_0\} \cup L_F, M_F)$, and from successive applications of Lemma 3.9, we just need to prove that $o(P', B) = \mathcal{D}$.

First focus on components of S_F which do not contain v_0 . Let C such a component, if each vertex of C is dominated by a vertex of M_F , then Bob dominates C while Alice does not. Otherwise, let v_1 be a leaf of the subtree T_C of F induced by C . Bob plays the bottom-to-top strategy in T_C rooted in v_1 . It can be done since the leaves of T_C are only connected to vertices of M_F already claimed by Bob. If two vertices labeled by 0 have the same parent, then the strategy creates a double trap, which ensures a \mathcal{D} draw. If v_1 has all its children labeled by 1, when Bob claims the last vertex labeled by 1, a double trap is created, since v_1 will be a trap. This also ensures the \mathcal{D} draw. The only non directly conclusive case is when v_1 has one child labeled by 0, the reached position after Bob has followed the bottom-to-top strategy is such that Bob dominates C while Alice does not. Indeed, Bob dominates v_1 since it is a leaf of C and thus should be connected to M_F but Alice does not dominate v_1 .

Bob follows this strategy on each component of S_F not containing v_0 . All the answers from Alice are forced. If a double trap appears, we are done. Otherwise, each component is dominated by Bob and not by Alice. Moreover, it is Bob's turn. In such a case, we now need to focus on the component C_0 of S_F which contains v_0 .

- If the diameter of C_0 at most 1, (which implies that the graph induced by S_F is not connected), Bob already dominates C_0 , and therefore F , but Alice does not.
- If the diameter of C_0 is 2, the graph induced by C_0 is a star centered in a vertex c . Since it is assumed that the graph induced by S_F is not a star, C_0 is not the only component of the graph induced by S_F . Thus, if Bob does not dominate C_0 yet, he claims c or any of its neighbor and dominates the whole graph while Alice does not.
- If the diameter of C_0 is at least 3, there exists a vertex v_2 at distance exactly 2 from v_0 in the subgraph induced by C_0 . Let (v_0, v_1, v_2) be the path from v_0 to v_2 . Bob can then use a bottom-to-top strategy in the tree induced by C_0 rooted in v_0 .
 - If v_1 is labeled by 0, then v_2 is labeled by 1. Bob can play a bottom-to-top strategy claiming all the vertices labeled by 1 except v_2 . Then he claims v_1 . At this moment, Bob dominates F but Alice does not dominate v_2 .
 - If v_1 is labeled by 1, Bob plays a bottom-to-top strategy in all branches, finishing by claiming v_1 . At this moment, he dominates F but, by construction, Alice does not dominate v_2 .

Thus, in any case, Bob has a strategy which leads to a \mathcal{D} draw position, and thus $o((F, \{v_0\}, \emptyset), B) = \mathcal{D}$. □

Lemma 5.12. *If F is standard and $o(F) = \mathcal{A}$, then there exists a vertex $v_0 \in M_F$ such that $o((F, \{v_0\}, \emptyset), B) = \mathcal{A}$. Moreover, v_0 satisfies:*

1. *the subgraph of F induced by $S_F \cup \{v_0\}$ is a tree ;*
2. *in the labeling of F rooted in v_0 , each vertex v labeled by 1 has a unique child.*

Proof. As it is supposed that $o(F, \mathcal{A}) = \mathcal{A}$, there exists a vertex $v_0 \in V(F)$ such that $o((F, \{v_0\}, \emptyset), B) = \mathcal{A}$. From Lemma 5.11, we have $v_0 \in M_F \cup L_F$. From Lemma 2.3, it can be assumed that $v_0 \notin L_F$. Indeed, if $l \in L_F$ and $m \in M_F$ is its private neighbor, $N[l] \subset N[m]$.

Thus, we can assume that $v_0 \in M_F$. We now prove the two other properties.

1. Since F has no cherry, there exists a unique vertex $v_{-1} \in L_F$ which is a neighbor of v_0 . Assume that the graph induced by $S_F \cup \{v_0\}$ is not connected. We prove that, under this hypothesis, $o((F, \{v_0\}, \emptyset), B) = \mathcal{D}$, which gives the result by contraposition. First, using Lemma 3.9, we have $o((F, \{v_0\}, \emptyset), B) = o((F, \{v_0\} \cup L_F \setminus \{v_{-1}\}, M_F \setminus \{v_0\}), B)$.

With the same arguments as in the previous lemma, the implementation of a bottom-to-top strategy on each component of the graph induced by $S_F \cup \{v_0\}$ not containing v_0 leads to either a double trap (which gives the result), or a position where each of these components is dominated by Bob but not by Alice. Now, focus on the component C containing v_0 . Bob then plays a bottom-to-top strategy on the tree induced by C rooted in v_0 . Then Bob claims v_{-1} after all the vertices labeled by 1 have been claimed. By this way, if no trap has appeared before, Bob dominates the whole forest F , while Alice does not totally dominate F . This gives the result.

2. For the second item, consider the labeling of F (that is actually a tree) rooted in v_0 . First note that all the leaves of S_F are labeled by 0. Thus, each vertex labeled by 1 has at least one child. Assume that there exists a vertex v labeled by 1 with has at least two children.
 - If two children v' and v'' of v are labeled by 0, Bob can then use a bottom-to-top strategy until v is claimed. When he claims v , he creates a double A -trap in v' and v'' and thus obtains a \mathcal{D} raw.
 - Otherwise, there exists one child v' of v labeled by 0 and the other one, v'' labeled by 1. Then Bob uses a bottom-to-top strategy but without claiming v'' , until all the vertices labeled by 1 (except v'' are claimed). All replies of Alice remain forced. After this is done, Bob claims v_{-1} , the neighbor of v_0 which belongs to L_F , and then dominates the whole graph while Alice does not dominate v'' , which ensures that $o((T, \{v_0\}, \emptyset), B) = \mathcal{D}$. \square

As a consequence of this result, if F is standard and winning for Alice, then F is necessarily a tree.

5.6 Splitting the graph

By Lemma 5.12, it can be assumed that Alice first claims a vertex $v_0 \in M_F$ that is connected to all the components of S_F (note that if S_F is not connected, there is at most one such vertex). It can also be assumed that each vertex labeled by 1 in F rooted in v_0 has degree exactly 2. In all the remaining, v_0 will denote this first claim of Alice. Let v_{-1} be the leaf connected to v_0 . After this first move, it can be assumed, using Lemma 3.9, that Bob will claim all the other vertices of M_F one by one. At each time, Alice must answer to the corresponding leaf in L_F . After this step, the free vertices are the vertices in S_F with the vertex v_{-1} . Formally, the obtained position is $P = (F, L_F \setminus \{v_{-1}\} \cup \{v_0\}, M_F \setminus \{v_0\})$ and we have $o(F) = o(P, B)$. In that follows, we will split the graph into several components defined from the connected components of the skeleton.

Definition 5.13. *For a connected component C of S_F , let T be the connected component of $F \setminus \{v_0\}$ that contains C . The position P_C is defined as the position induced by $T \cup \{v_0, v'_0\}$ in the position $P = (F, L_F \setminus \{v_{-1}\} \cup \{v_0\}, M_F \cup \{v'_0\} \setminus \{v_0\})$, where v'_0 is an additional leaf connected to v_0 and claimed by Bob.*

Figure 6 illustrates the two positions P_C and $P_{C'}$ derived from the forest F of Figure 5 when played on v_0 . Lemma 5.14 shows that this splitting (with an additional $[AoA]$ position) yields to an equivalent position.

Lemma 5.14. *The position $P = (F, L_F \setminus \{v_{-1}\} \cup \{v_0\}, M_F \setminus \{v_0\})$ is equivalent to the position*

$$\left(\bigcup_{C \in CC(S_F)} P_C \right) \cup [AoA]$$

where $CC(S_F)$ denotes the set of connected components of S_F .

Proof. Let P' be the position $(\bigcup_{C \in CC(S_F)} P_C) \cup [AoA]$. Note that the unclaimed vertices are in a one-to-one correspondence in the two positions (v_{-1} is corresponding to the unclaimed vertex of $[AoA]$). By Observation 2.2, we just need to prove that P and P' have the same winning sets for both players.

A set S of unclaimed vertices is winning for Alice in P and in P' if and only if it is dominating all the vertices of S_F except the ones connected to (a copy of) v_0 , which corresponds to the same condition in both positions. A

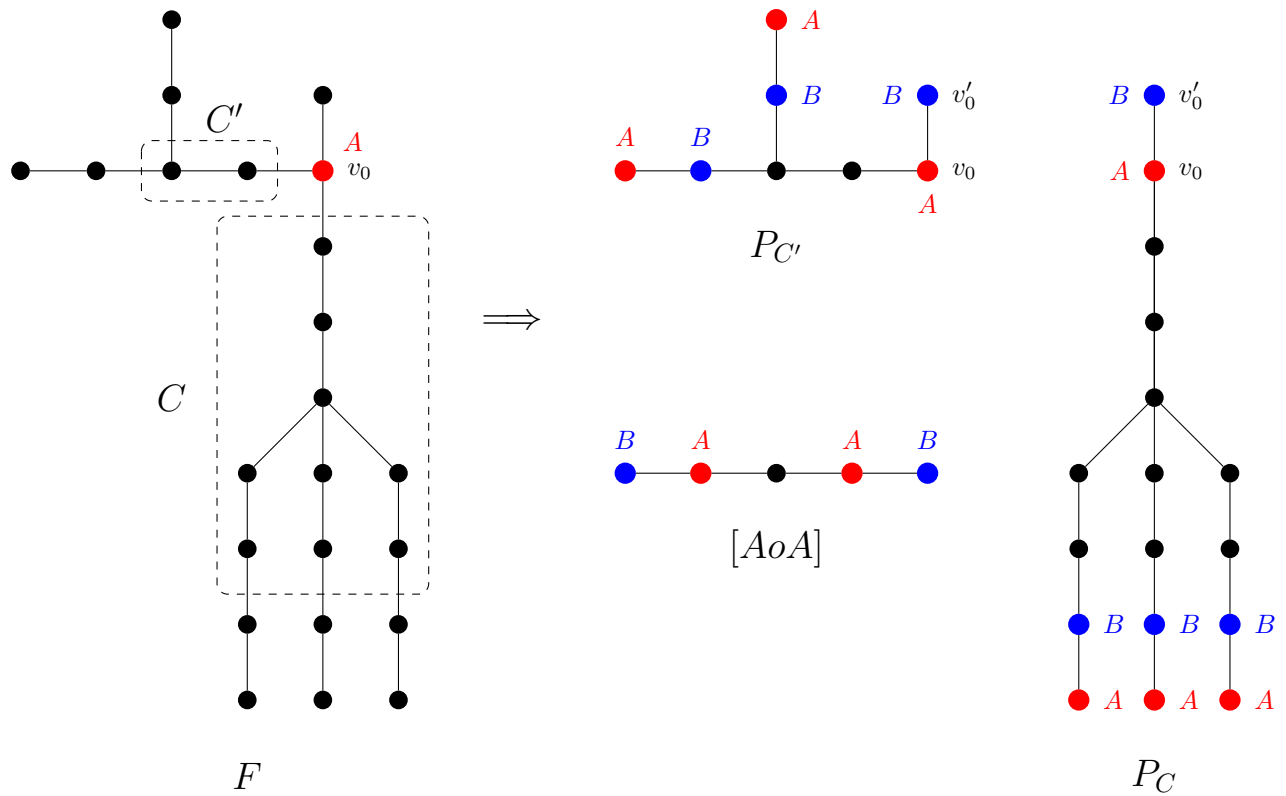


Figure 6: On the left a forest F . On the right, the equivalent position obtained by splitting F in Lemma 5.14. The component C is a fork whereas C' is a P_2 .

set S of unclaimed vertices is winning for Bob in P if and only if $v_{-1} \in S$ and S is dominating all the vertices of S_F that are not connected to a vertex of $M_F \setminus \{v_0\}$. In P' , S is winning for Bob if it contains the unclaimed vertex of $[AoA]$ and if it is dominating all the unclaimed vertices not already dominated by Bob, that are exactly all the vertices of S_F not connected to $M_F \setminus \{v_0\}$. Thus, the winning sets are in bijection and by Observation 2.2, the positions are equivalent. \square

Using this decomposition, we now prove that Bob can just focus on a subset of components where he has a Draw strategy.

Lemma 5.15. *Assume there exists a set \mathcal{S} of connected components of S_F such that $o((\bigcup_{C \in \mathcal{S}} P_C) \cup [AoA], B) = \mathcal{D}$. Then $o(F) = \mathcal{D}$.*

Proof. Using Lemma 5.14, it suffices to prove that $o((\bigcup_{C \in CC(S_F)} P_C) \cup [AoA], B) = \mathcal{D}$. For each $C \notin \mathcal{S}$, Bob plays a bottom-to-top strategy on P_C rooted on v_0 . At each time Alice is forced to answer in the same component C , and, at the end, Bob dominates the component C . This is successively done for all such components.

Afterward, the remaining position is equivalent to $((\bigcup_{C \in \mathcal{S}} P_C) \cup [AoA], B)$ which is Draw by hypothesis. \square

5.7 Favorable skeletons for Alice

In this subsection, we give some necessary and sufficient conditions for a component C to be winning for Alice.

Definition 5.16. *A fork is a star with at least three branches where each is subdivided exactly once.*

On Figure 6, the component C is an example of a fork with four branches.

Lemma 5.17. *If $((F, \{v_0\}, \emptyset), B)$ is \mathcal{A} -win, then all the connected components of S_F must induce a path or a fork that is connected to v_0 by a leaf.*

Proof. Proceeding by contraposition, assume that there exists a component C of S_F , which is neither a fork nor a path connected to v_0 by a leaf. We will prove $o(P_C \cup [AoA], B) = \mathcal{D}$, which gives the result using Lemma 5.15.

Since C is not a path connected by a leaf to v_0 , there exists a vertex $c \in C$ of degree at least 3 in the tree induced by $C \cup \{v_0\}$. Let $P_a = (c, a_1, a_2, \dots, a_p)$ be the path linking c to v_0 (a_p is adjacent to v_0), $P_b = (c, b_1, b_2, \dots, b_q)$ be a path of C of maximal length such that $b_1 \neq a_1$ and $P_c = (c, c_1, c_2, \dots, c_r)$ be another maximal path of S_F starting in c . Note that possibly $p = 0$, but that $q \geq r \geq 1$. In the labeling of F rooted in v_0 , the vertex c is necessarily labeled by 0 since it has degree 3. This implies that both q and r are even since b_q and c_r , as leaves of C , are labeled by 0. Moreover, all the vertices labeled by 1 have degree 2. This implies that all the vertices of the three paths P_a , P_b and P_c that are connected to other vertices of F must be labeled by 0. In particular, using Lemma 5.5, it is enough to prove that there is a Draw strategy when C is reduced to these three paths.

Assume first that $q \geq 4$. By Lemma 5.5, it is enough to give a Draw strategy for Bob for $q = 4$ and $r = 2$ with C reduced to the union of the three paths. The first claim of Bob is b_2 . By Lemma 2.10, Alice should claim either b_1 or b_3 .

- If Alice replies by claiming b_3 , then Bob claims c , which forces Alice to claim b_1 . Then, he successively claims a_2, a_4 , and so on until a_{p-1} is dominated by B . Successive replies of Alice are forced: when Bob claims a_{2i} , Alice necessarily replies in a_{2i-1} . Finally, Bob claims in $[AoA]$ and gets a Draw by dominating before Alice (Alice does not dominate c_1 and c_2).
- If Alice replies by claiming b_1 , then Bob claims c_1 , which forces Alice to claim c_2 . Then Bob claims the unclaimed vertex of $[AoA]$. Then the position is equivalent to the position $([Ao^{p-1}B] \cup [Bo^2B], A)$. By Lemma 4.2, this position is equivalent to $([Bo^2B], A)$ which is Draw since Bob already dominates.

Assume now that $r = q = 2$. As before, it is enough to give a strategy for C restricted to the three paths.

- If $p \geq 5$, then Bob claims a_3 , which enforces Alice to reply either a_2 or a_4 by Lemma 2.10.
 - If Alice claims a_4 , then Bob successively claims b_1, c_1 (with forced Alice to claim b_2 and c_2), and then a_1 , which creates two A -traps in a_2 and c .

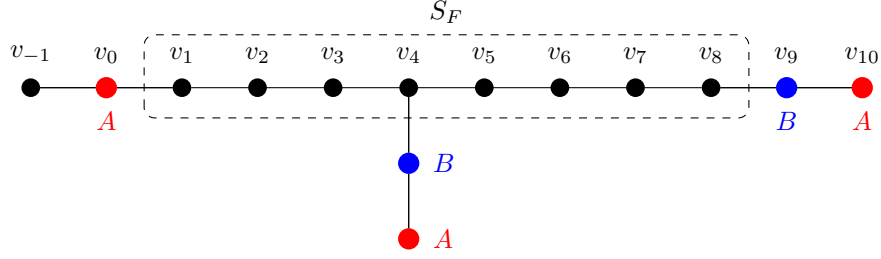


Figure 7: Example of a tree F such that $o(F) = \mathcal{D}$ but S_F is a path. Alice should claim first v_0 or v_9 . If Alice claims v_0 , Bob claims the other vertices of M_F . Then Bob can win by claiming v_2 (Alice should answer v_1, v_3 or v_4), v_6 (Alice should answer v_5, v_7 or v_8) and v_{-1} . The case where Alice claims first v_9 is similar.

- If Alice replies by a_2 , then, first, Bob claims, a_5, a_7, \dots , and so on until a_p is dominated by Bob, replies of Alice being forced on a_4, a_6, \dots . Second, Bob claims c . If Alice does not answer in the set $\{a_1, b_1, b_2, c_1, c_2\}$, Bob successively claim b_1 creating a A -trap in b_2 , c_1 , creating a A -trap in c_2 and a_1 isolating c . Thus, Alice must claim a vertex in the set $\{a_1, b_1, b_2, c_1, c_2\}$. Then Bob claims the unclaimed vertex of $[AoA]$ and dominates the whole position before Alice.
- If $p = 4$, the position can be treated as for $p = 5$ if Alice replies in a_2 or a_4 . But she can also claim a_1 . In this case, Bob claims the unclaimed vertex of $[AoA]$ with the threat to claim c and dominate before Alice. Even if Alice replies in c , Bob succeeds in dominating before Alice by successively claiming b_1, c_1 and a_2 (Alice will not dominate a_3 during this time).
- If $p = 1$ or $p = 3$, then Bob claims c . If Alice claims the unclaimed vertex of $[AoA]$, a_2 or a_3 , then Bob can claim b_1 and c_1 , forcing Alice to reply by claiming b_2 and c_2 . Then Bob can isolate c by claiming a_1 . Thus, Alice should answer by claiming a vertex in $\{a_1, b_1, b_2, c_1, c_2\}$. Then Bob can claim the unclaimed vertex of $[AoA]$. If $p = 1$, he wins. If $p = 3$, he can dominate in one move by claiming either a_2 or a_3 whereas Alice need at least two moves to dominate.
- If $p = 2$, then we have a fork (since all the other branches of C starting from c must have length 2 and the vertices adjacent to C are labeled by 1, and thus of degree 2), which is not possible, by hypothesis.
- If $p = 0$, then Bob claims the unclaimed vertex of $[AoA]$. Alice needs at least two moves to dominate. If Alice does not claim c , Bob wins at his second turn by claiming it. If Alice claims c she still need two moves to dominate. Then Bob can dominate before by claiming b_1 and c_1 . \square

Lemma 5.17 gives us the possible structures of the connected components of S_F to have a position \mathcal{A} -win. But actually, this condition is not sufficient: there are for example trees where S_F is a path, but Bob can obtain a \mathcal{D} raw (see for example Figure 7). We need to consider which vertices of S_F are already dominated by Bob.

Definition 5.18. Let $X, Y \in \{A, B\}$, n be a positive integer, and a subset $U \subseteq \{1, 2, \dots, n\}$. We denote by $[Xo^nY]^U$ the position obtained from the bounded path $[Xo^nY]$ where for each $i \in U$ a pendant edge x_iy_i is added to the vertex v_i , with x_i is linked to v_i and claimed by Bob, and y_i claimed by Alice.

Informally, an $[Xo^nY]^U$ is a bounded path, where some vertices are already dominated by Bob. As an illustration, if we set that v_{-1} is claimed by Bob on Figure 7, then we obtain the position $[Ao^8B]^{\{4\}}$. When U is empty, $[Xo^nY]^U = [Xo^nY]$. If $Y = B$ (respectively $X = B$), one can assume that $n \notin U$ (resp. $1 \notin U$). Indeed, v_n (resp. v_1) is already dominated by Bob.

Finally, if C is a connected component of S_F that is a path connected to v_0 by a leaf, then P_C is equivalent to a position $[Ao^nB]^U$, for a fixed n and a fixed U . Indeed, set for U all the integers i such that v_i is connected to a vertex of $M_F \setminus \{v_0\}$.

The following three observations give natural properties about bounded paths. The first one is about the existence of a pairing.

Observation 5.19. A position $[Xo^nY]^U$ contains a A -pairing, except if $X = Y = B$ and n is odd.

Roughly speaking, the next two observations say that it is always better for Alice to play on a bounded path with the extremities claimed by her, and with fewer vertices dominated by Bob in U .

Observation 5.20. For any integer n and any set $U \in \{1, \dots, n\}$, and any $X, \in \{A, B\}$, we have

$$o([Ao^n A]^U, X) \geq o([Ao^n B]^U, X) \geq o([Bo^n B]^U, X).$$

Observation 5.21. For any integer n and any sets $U \subseteq U' \subseteq \{1, \dots, n\}$, and each $X, Y, Z \in \{A, B\}$, we have

$$o([Xo^n Y]^U, Z) \geq o([Xo^n Y]^{U'}, Z)$$

In the next definition, we define the components C that are *favorable* for Alice, and among them, the ones that are *strongly* and *weakly* favorable. Theorem 5.23 will justify this terminology.

Definition 5.22. Let v_0 be the first move of Alice, satisfying Lemma 5.12. Let C be a connected component of S_F . We say that C is favorable for Alice if it satisfies one of the following case:

1. C is a path connected to v_0 by a leaf, i.e, $P_C = [Ao^n B]^U$ with $U \subseteq \{1, 2, \dots, n-1\}$, and at least one of the following cases holds:

- (a) $n \in \{1, 2\}$ and $U = \emptyset$;
- (b) $n = 3$ and $U \subseteq \{1\}$ or $U \subseteq \{2\}$;
- (c) $n \geq 4$ and $U \subseteq \{2, 3, n-2\}$;
- (d) $n \geq 9$, n is odd and $U \subseteq \{2, 5, n-2\}$;
- (e) $n \in \{9, 11\}$ and $\{3, 5\} \subseteq U \subseteq \{2, 3, 5, n-2\}$;

2. C induces a fork and the only vertices of C that can be connected to M_F are the center c , the leaves except the one connected to v_0 , and eventually the neighbor of c between c and v_0 .

Moreover, if C belongs to the cases (1.a), (1.b) or (1.c), we say that it is *strongly favorable*. On the opposite, if C belongs to the cases (1.e) or (2), we say that it is *weakly favorable*.³

We can now state the final theorem that ends the characterization of trees. This theorem is actually technical to prove, and thus the proof is postponed to Section 6.

Theorem 5.23. Let F be a tree and $v_0 \in M_F$ be a first move of Alice that satisfies the condition of Lemma 5.12. The position $((F, \{v_0\}, \emptyset), B)$ is \mathcal{A} -win if and only if all the components of S_F are favorable to Alice and at most one of them is weakly favorable.

5.8 Proof of Theorem 5.1

We now have all the ingredients to complete the study of forests and prove Theorem 5.1. We summarize the algorithm into the diagram of Figure 8.

Proof of Theorem 5.1. Let F be a forest. If F has an isolated vertex v_0 , then by Lemma 5.2, $o(F) = \mathcal{A}$ if and only if $F \setminus \{v_0\}$ has a perfect matching. If F has an isolated edge $e = uv$, then by Lemma 5.3, F has the same outcome as $F \setminus \{u, v\}$. Thus, we can assume that all the connected components of F have at least three vertices. If F has two cherries, then $o(F) = \mathcal{D}$ by Lemma 5.6. If it has one cherry (c, l, l') , then by Lemma 5.7, $o(F) = \mathcal{A}$ if and only if there is a matching in $F \setminus \{c\}$ that covers $V(F) \setminus N[c]$. Thus, we can assume that F has no cherry. If $S_F = \emptyset$, then $o(F) = \mathcal{A}$ (Lemma 5.8). Otherwise, one can remove components of F that have an empty skeleton (Lemma 5.9). If S_F induces a star of center c such that c is not adjacent to M_F , then $o(F) = \mathcal{A}$ (Lemma 5.10). Otherwise, F is standard. If F is not connected, then $o(F) = \mathcal{D}$. If S_F is not connected, Alice should claim the vertex v_0 of M_F that connects S_F (Lemma 5.12) if it exists (otherwise $o(F) = \mathcal{D}$). Then Alice wins if and only if all the components of S_F are favorable to her and at most one is weakly favorable (Theorem 5.23). Assume now

³If C corresponds to the case (1.d), it is neither strongly nor weakly favorable.

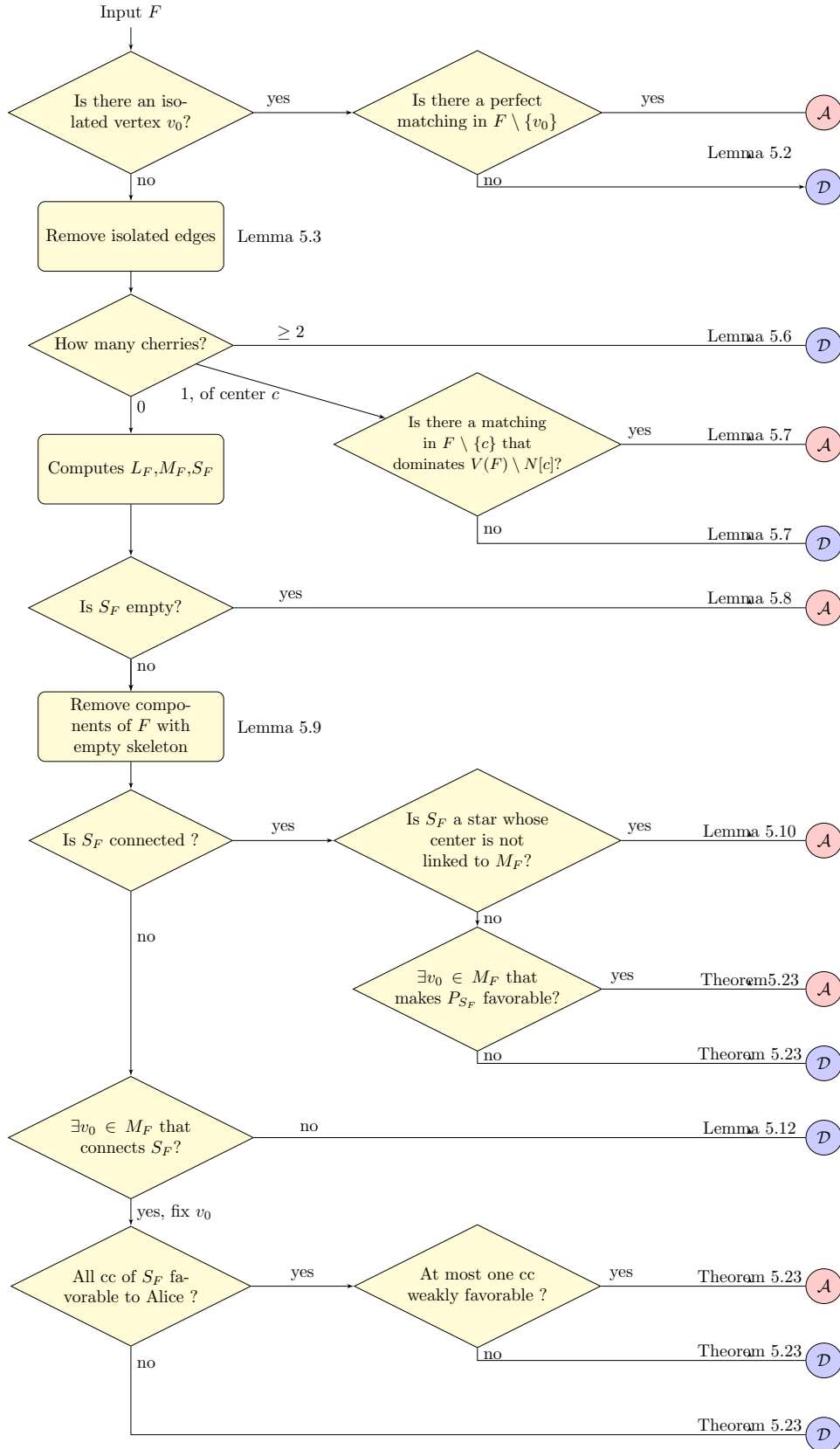


Figure 8: The decision tree to compute the outcome of any forest.

that S_F is connected. Let C be the tree induced by S_F . Alice should claim in M_F and connected to a leaf of C (Lemma 5.12). If there exists $v_0 \in M_F$ that is adjacent to a leaf of C such that P_C is favorable to Alice, then $o(F) = \mathcal{A}$, otherwise $o(F) = \mathcal{D}$ (Theorem 5.23).

In terms of complexity, almost all the operations described in the algorithm are elementary (finding a matching in a forest, identifying the isolated vertices, edges and cherries, computing L_F , M_F and S_F , deciding if S_F is connected or a star) and can be done in linear time by examining the tree from the leaves. When S_F is not connected, there exists at most one v_0 that can connect them and this is easy to check. If S_F is connected and thus reduce to a single component, one have to check all the possible v_0 that could make P_{S_F} favorable. If S_F induces a path, there are at most two possibilities for v_0 since it should be connected to an extremity of the path. If S_F induces a fork (easy to check) of center c , to be favorable, there must be at most one neighbor of c that is adjacent to M_F . If there is exactly one neighbor a_1 of c adjacent to M_F , then the leaf of S_F adjacent to a_1 must have only one neighbor in M_F and this neighbor will be the vertex v_0 . If all the neighbors at distance 1 of c in S_F are not adjacent to vertices of M_F , then there must be at least one leaf of S_F adjacent to exactly one vertex of M_F . This vertex of M_F would be v_0 . In all the other cases, P_{S_F} cannot be favorable. Thus, finding v_0 can be done in linear time. Finally, once v_0 is fixed, deciding whether a component is favorable or not can also be done in linear time (because it is a fork or a path with pending edges). \square

6 Proof of Theorem 5.23

We here prove Theorem 5.23 by considering successively both directions of the equivalence. The next subsection proves that the favorable components are necessary conditions. The sufficient condition is then proved in Subsection 6.2.

6.1 The direct part

In this subsection, we assume that $o((F, \{v_0\}, \emptyset), B) = \mathcal{A}$ and we prove that all the components are favorable and at most one is weakly favorable. We already know by Lemma 5.17 that all the connected components of S_F are paths or forks. In Lemma 6.1, we prove that if a component is a fork, it should satisfy the conditions of Point 2 in Definition 5.22. In Lemma 6.2, we prove that if a component is a path, it should satisfy the conditions of Point 1 in Definition 5.22. Finally, in Lemma 6.6, we prove that if two components are weakly favorable, then the outcome is \mathcal{D} .

Lemma 6.1. *Assume that $o((F, \{v_0\}, \emptyset), B) = \mathcal{A}$. Let C be a connected component of S_F which induces a fork. Then C is favorable.*

Proof. Proceeding by contraposition, we assume that C is not favorable. We will prove that $o([AoA] \cup P_C, B) = \mathcal{D}$ which gives the result using Lemma 5.15.

Let c denotes the center of the fork. Let a_1 and a_2 be the two vertices on the path between c and v_0 with a_2 adjacent to v_0 , and v_{-1} be the unclaimed vertex of $[AoA]$. Since C is not favorable, it means that either a_2 or a neighbor of c but not a_1 is dominated by Bob in P_C . We consider the two cases.

- If a_2 is already dominated by Bob, then Bob claims c .
 - If Alice replies in v_{-1} then Bob can claim all the neighbors of c distinct from a_1 and Alice is forced to answer the leaf of C adjacent to the vertex claimed by Bob. Afterward, Bob claims a_1 , isolating the vertex c , which ensures that Alice will not dominate the graph.
 - otherwise, Bob claims v_{-1} and, therefore, dominates the graph before Alice.
- If Bob already dominates a neighbor b_1 of c (with $b_1 \neq a_1$), then Bob claims all the other neighbors of c distinct from a_1 and b_1 , forcing Alice to claim the leaves adjacent to them. Then Bob claims the unclaimed vertex of $[AoA]$. At this point, he needs only one move to dominate the graph that can be either a_1 or a_2 , whereas Alice still need to dominate a_1 and b_2 , which cannot be dominated in a single move. Thus, Bob will dominate before Alice. \square

Lemma 6.2. *Assume that $o((F, \{v_0\}, \emptyset), B) = \mathcal{A}$, and let C be a connected component of S_F which induces a path. Then C is favorable.*

Proof. Let $P_C = [Ao^n B]^U$. We proceed by contraposition. Assume that C is not favorable. We will prove that $o([AoA] \cup P_C, B) = \mathcal{D}$, which gives the result by using Lemma 5.15. We split the proof into a few claims. We denote by v_{-1} the unclaimed vertex of $[AoA]$ and by v_1, \dots, v_n the unclaimed vertices of P_C . We consider for the labeling that P_C is rooted in v_0 .

Claim 6.3. *If n is even and $5 \in U$, then $o([AoA] \cup P_C, B) = \mathcal{D}$.*

Proof of the claim. When $n = 6$, Bob claims v_3 , which enforces Alice to reply in v_4 (if Alice replies in v_2 , then Bob claims v_5 and create a double trap). Then Bob successively claims v_1 and v_{-1} and, therefore dominates before Alice.

If $n \geq 8$, v_7 is labeled by 1. By Lemma 5.5, the position can be reduced to the case $n = 6$, and thus is also Draw. \diamond

Claim 6.4. *If $n \geq 4$ and if there exists $i \in U$ such that $i \notin \{2, 3, 5, n - 2\}$, then $o([AoA] \cup P_C, B) = \mathcal{D}$.*

Proof of the claim. Assume first that $i = 1$.

- If $n \in \{4, 5\}$, then Bob first claims v_3 , (which forces Alice to answer v_4 or v_2). Then Bob claims v_{-1} and dominates before Alice (who does not dominate v_2 or v_4).
- If $n \geq 6$, either v_5 or v_6 is labelled by 1. Then as before, we can apply Lemma 5.5 to reduce to the case $n \in \{4, 5\}$.

We can now assume that $i \geq 4$. There are two cases, according to the parity of the value $n - i$. In each case, Lemma 5.5 is used.

- If $n - i$ is odd,
 - If $n - i = 1$, that is $i = n - 1$, then Bob claims v_{n-3} (this is possible since $4 \leq i < n$),
 - * For $n \geq 6$, Alice necessarily replies in v_{n-2} . Indeed, by Lemma 2.10, Alice should answer in v_{n-4} or v_{n-2} , but if she claims v_{n-4} then a component $[Bo^{2k+1}B]$ will appear which is Draw by Lemma 4.4. After Alice has claimed v_{n-2} , the resulting position P is then equivalent to a position $[AoA] \cup [Ao^{n-4}B]^{U'} \cup [Ao^2B]^{\{1\}}$, for some set U' . We thus have
$$o(P, B) \leq o([AoA] \cup ([Ao^{n-4}B] \cup [Ao^2B]^{\{1\}}), B) = o([AoA] \cup [Ao^2B]^{\{1\}}, B) = \mathcal{D}.$$
The first equality comes from Lemma 4.2 and the second one is obvious since Bob can dominate in one move. This gives the desired result in this case.
 - * For $n = 5$, Alice can also reply in v_{n-1} . In such a situation, Bob claims v_{-1} and directly dominates before Alice. If Alice claims v_{n-2} we are in the same situation as the previous case.
 - If $n - i \geq 3$, then v_{i+2} is labeled by 1 and Lemma 5.5 applies to reduce the instance to the case where $i = n - 1$.
- Assume now that $n - i$ is even. Since $i \notin \{n - 2, n\}$, we have $n - i \geq 4$. If $n - i \geq 6$, then v_{i+5} is labeled by 1, and Lemma 5.5 applies to reduce the instance to the case where $n - i = 4$. Thus, we can assume that $i = n - 4$. Furthermore, since $i \notin \{1, 2, 3, 5\}$, we can assume that $n \geq 8$ and $n \neq 9$.
 - If $n \geq 12$, Bob claims v_{n-9} , which, as before, enforces Alice to reply in v_{n-8} to avoid a component $[Bo^{2k+1}B]$. Afterward, Bob successively claims $v_{n-11}, v_{n-13}, \dots$ and so on until v_1 is dominated by Bob, the answers of Alice being forced.

Bob now claims v_{n-6} , which enforces Alice to reply v_{n-7}, v_{n-5} or v_{n-4} . Then Bob claims v_{n-2} , which enforces Alice to reply v_{n-1} , or v_{n-3} . In the resulting position, Bob dominates each vertex i , $1 \leq i \leq n$, while Alice does not. Indeed, there exists a vertex $i \geq n - 6$, which is not dominated by Alice, as she has claimed only two moves that can dominate vertices from v_i for $n - 6 \leq i \leq n$, and each of them dominates at most three vertices. Finally, Bob claims v_{-1} and dominates before Alice.

- if $n = 11$, Bob claims v_9 . By Lemma 2.10, Alice should answer in $\{v_8, v_{10}\}$. If Alice claims v_8 , Bob claims v_5 . Alice has to claim either v_4 or v_6 . Then, Bob claims v_{-1} . Alice now needs to claim at least two more vertices to dominate: one to dominate $\{v_4, v_6\}$ and one to dominate v_{10} , while Bob will dominate with his next claim in $\{v_1, v_2\}$.

If Alice claims v_{10} , Bob also claims v_5 . Alice has to claim v_6 , otherwise v_7 creates a double trap. Now Bob claims v_{-1} and will dominate with his next claim in $\{v_1, v_2\}$, while Alice needs two vertices to dominate.

- if $n = 10$, then Bob successively claims v_8, v_4 . As, by Lemma 2.10, Alice has to claim a vertex adjacent to the vertex that Bob has claimed, her moves are almost forced. If she has claimed v_9 and v_3 , Bob claims v_6 and creates two traps in v_5 and v_7 . Otherwise, he claims v_{-1} and dominates before Alice by a final claim in $\{v_1, v_2\}$.
- if $n = 8$, then Bob successively claims v_6, v_2, v_{-1} and dominates before Alice. Indeed, she has only claimed two moves to dominate vertices between v_2 and v_7 , and the only dominating set of size two on this path contains v_6 .

◇

Claim 6.5. *Assume that $n \geq 13$, n is odd and $\{3, 5\} \subseteq U$, then $o([AoA] \cup P_C, B) = \mathcal{D}$.*

Proof of the claim. It suffices to prove it for $n = 13$, since, for $n > 13$, Lemma 5.5 applies. Bob claims v_{11} , which enforces Alice to reply in v_{10} (if Alice replies in v_{12} , then Claim 6.3 applies). Then, Bob claims v_8 .

- If Alice replies in v_7 , then Bob successively claims v_5 and v_{-1} . Afterward, Bob finally succeeds in dominating before Alice by claiming either v_1 or v_2 .
- If Alice replies in v_6 , then Bob claims v_2 , which enforces Alice to claim v_1, v_3 , or v_4 . Then Bob claims v_{-1} . At this step, Bob threatens to claim v_5 and dominate before Alice. Thus, the reply of Alice is necessarily v_5 . Then, Bob claims v_7 , which enforces the reply v_9 from Alice. Finally, Bob claims either v_3 or v_4 (one of these vertices is free) and dominates before Alice.

◇

We can now finish the proof of the lemma. Since C is not favorable, we can split the cases according to n as follows:

- If $n = 1$, then the result is obvious, since there is no unfavorable component.
- If $n = 2$ and $1 \in U$, then Bob claims v_{-1} and then dominates the graph before Alice, which is a contradiction.
- If $n = 3$ and $1, 2 \in U$, then Bob claims v_{-1} and then dominates the graph before Alice, which is a contradiction.
- If $n \in \{4, 5, 6, 7\}$ then Claim 6.3 and Claim 6.4 give the result.
- If $n \geq 8$ and n is even, the combination of Claims 6.3 and 6.4 gives that $U \subseteq \{2, 3, n - 2\}$.
- If n is odd and $n \geq 13$, then the combination of Claims 6.4 and 6.5 gives that either $U \subseteq \{2, 3, n - 2\}$ or $U \subseteq \{2, 5, n - 2\}$, which gives the result.
- If $n \in \{9, 11\}$, then Claim 6.4 allows to conclude. □

Lemma 6.6. *Assume there are two connected components C, C' of S_F that are weakly favorable.*

Then $o(F, \{v_0\}, \emptyset, B) = \mathcal{D}$.

Proof. Assume first that both C and C' both induce forks. We prove the result for forks with exactly three branches. Indeed, if Bob has a strategy in this case, he will have a strategy for forks with more branches using Lemma 5.5 since all the neighbors of c are labeled by 1. Let c denote the center of the fork induced by C , a_1, b_1, c_1 denote the neighbors of c , in such a way that such that a_1 is between c and v_0 , and a_2, b_2, c_2 respectively denote the neighbors of a_1, b_1, c_1 different from c . We define in the same way $c', a'_1, b'_1, c'_1, a'_2, b'_2$ and c'_2 for C' . Bob start by claiming v_{-1} . By symmetry, we can suppose that Alice replies one vertex among $\{c, a_1, a_2, b_1, b_2\}$.

- If Alice replies in c , then Bob successively claims b_1 and c_1 (the replies of Alice are forced). Now Bob claims c' , then Bob needs two more moves to dominate (one in (a_1, a_2) and one in (a'_1, a'_2)) whereas Alice needs three moves.
- If Alice replies in a_1 , then Bob claims a_2 .
 - If Alice replies in a'_1 , then Bob claims a'_2 . At this step, Alice needs at least four moves to dominate while Bob can dominate in three moves in a lot of manners, with a center of a fork, and two vertices of the other fork. Alice cannot avoid Bob to dominate using three moves, and therefore, before Alice.
 - If Alice replies in c , then Bob successively claims b_1 and c_1 . After the forced replies of Alice, Bob claims in c' and dominates before Alice with one last move in (a'_1, a'_2) while Alice cannot dominate in one move. The case where Alice replies in c' can be treated symmetrically.
 - If Alice replies in b_1 , then Bob claims successively c_1 and c' and dominates before Alice by claiming one vertex in (c, b_2) .
 - If Alice replies in b'_1 , then Bob claims c' . At this step, Alice needs at least four moves to dominate while Bob can dominate in three moves by a pairing strategy with the pairs (a'_1, a'_2) , (b_1, b_2) and (c_1, c_2) .
- If Alice replies in a_2 , then Bob claims a_1 , and afterward, all is similar to the previous case.
- If Alice replies in b_1 or b_2 , then Bob claims c' . Bob can now dominate in three moves by pairing (a'_1, a'_2) , (a_1, a_2) and (c, c_1) while Alice needs at least four moves.

Assume now that both C and C' induce paths. Since they satisfy the conditions 1.e of Definition 5.22, their length is 9 or 11. Bob first reduce the paths to length 9 if needed so that both paths have length 9. Let (v_1, v_2, \dots, v_9) (respectively $(v'_1, v'_2, \dots, v'_9)$) the path induced by C (resp. C'), with v_1 and v'_1 connected to v_0 . Since the paths are weakly favorable, Bob already dominates v_3, v_5, v'_3 and v'_5 .

First, Bob successively claims v_7 and v'_7 . By Lemma 2.10, Alice should answer first v_6 or v_8 and then v'_6 or v'_8 . We have three cases according to the replies of Alice.

- If the replies are v_8 and v'_8 , then Bob continues by claiming v_3 and v'_3 which enforces Alice to reply in v_4 and v'_4 (if, for instance, Alice does not reply v_4 , then Bob claims v_5 and forbids Alice to dominate at the following claim). Afterward, Bob claims v_{-1} , and achieves to dominate before Alice as he only needs one move in (v_1, v_2) and one in (v'_1, v'_2) while Alice needs to dominate v_2, v'_2, v_6 and v'_6 , each of them requiring a different move.
- If the replies are v_6 and v'_8 (note that v_8 and v'_6 is symmetric), then Bob continues by claiming v'_3 which enforces Alice to reply in v'_4 . Afterward, Bob claims v_{-1} . Now Bob has a paired dominating set of size 3 $\{(v_1, v_2), (v'_1, v'_2), (v_3, v_4)\}$, but Alice needs to dominate v_2, v_8, v'_2 and v'_6 , each of them requiring a different move. Thus, Bob can dominate first.
- If the replies are v_6 and v'_6 , Bob claims v_{-1} . At this time Bob and Alice need four moves to dominate, but each set of four moves for Alice contains the pair $\{v_3, v'_3\}$. Thus, after the reply of Alice, Bob can claim one element of the pair $\{v_3, v'_3\}$ and, by this way, dominate before Alice.

Assume now that C induces a path and C' induces a fork. As before, C' can be assumed to have exactly three branches and C nine vertices. We denote the vertices of the fork and the path as in the previous cases.

First, Bob claim v_7 . Once again, Alice has to claim either v_6 or v_8 , otherwise Bob ensure a draw by Lemma 2.10 There are two cases according to the reply of Alice.

- If Alice replies v_8 , then Bob successively claims b_1, c_1 and a_1 (replies of Alice being forced in b_2, c_2 and c). Afterward, Bob claims v_3 . Alice has to claim either v_2 or v_5 by Lemma 2.10. If she claims v_2 , Bob claims v_5 and creates two traps. If she claims v_4 , Bob claims v_{-1} . Now Bob only needs one move in either v_1 or v_2 to dominate while Alice needs at least two, as she does not dominate v_2 nor v_6 .
- If Alice replies v_6 , then Bob claims v_{-1}

- If Alice replies c , then Bob successively claims b_1, c_1 (replies of Alice are again forced in b_2 and c_2) and then v_3 . At this time, Bob needs two moves to dominate (one in v_1, v_2 and one in (a_1, a_2)), while Alice needs to dominate v_2, v_4 and v_8 , and each of them requires a different move. Therefore, Bob dominates before Alice by this way.
- If Alice replies v_3 , then Bob claims c . Now $(a_1, a_2), (v_1, v_2)$ and (v_4, v_5) is a paired dominating set for Bob of size three, while Alice needs to dominated a_2, b_1, c_1 and v_8 , each of these vertices requiring a different move.
- If Alice replies v_1 or v_2 , by Lemma 2.3, as $N[v_1] \setminus N[V_t] \subset N[v_2] \setminus N[V_t]$ for $t \in \{A, B\}$, we can suppose, she plays v_2 . Then Bob successively claims b_1, c_1, a_1 (replies are forced in b_2, c_2, c respectively) and then v_1 . At this time Bob needs one move to dominate in (v_3, v_4) , while Alice needs to dominate v_4 and v_8 which she cannot dominate in a single move.
- If Alice replies a_1 or a_2 , by Lemma 2.3, as $N[a_2] \setminus N[V_t] \subset N[a_1] \setminus N[V_t]$ for $t \in \{A, B\}$, we can suppose she claims a_1 . Bob claims a_2 .
 - * If Alice replies c , then Bob successively claims b_1, c_1 , (replies are forced) and v_3 . At this time Bob can dominate in two moves, one in (v_1, v_2) and one in (v_4, v_5) while Alice cannot, as she still has to dominate v_2, v_4 and v_8 .
 - * If Alice replies v_1 or v_2 , by Lemma 2.3, as $N[v_1] \setminus N[V_t] \subset N[v_2] \setminus N[V_t]$ for $t \in \{A, B\}$, we can suppose she replies v_2 . Then Bob successively claims b_1, c_1 , (replies are forced for Alice) and v_1 . At this time Bob needs one move in (v_3, v_4) to dominate, while Alice needs two moves to dominate v_4 and v_8 .
 - * If Alice replies elsewhere, then Bob claims c . At this time Bob needs two moves to dominate, one in (v_1, v_2) and one in (v_3, v_4, v_5) . By hypothesis, at least one will be available. Alice needs at least three moves to dominate, as she does not dominate at least three of the four vertices v_3, v_8, b_1 and c_1 , each of them requiring a different move.
- If Alice replies elsewhere, then Bob claims c . At this time Bob needs three moves to dominate, one in (v_1, v_2) , one in (a_1, a_2) and one in (v_3, v_4, v_5) . By hypothesis, at least one will be available in each of these sets at any moment. Alice needs at least four moves to dominate, as she does not dominate at least four of the five vertices a_1, b_1, c_1, v_3 and v_8 , each of them requiring a different move. \square

6.2 The converse part

In this final subsection, we prove the reverse part of Theorem 5.23: if all the connected components of S_F are favorable and at most one is weakly favorable, then Alice has a winning strategy. This part is naturally harder than the other one, as Alice cannot force Bob to answer where she would like to. Hence, all the possible answers of Bob must be considered, which was not the case previously, as Alice was often forced to answer locally to a move of Bob.

We first prove in Subsection 6.2.1, that we can remove the strongly favorable components (i.e. corresponding to the cases 1.a to 1.c of Definition 5.22). In Subsection 6.2.2, we consider only one weakly favorable component and give a strategy for Alice in this case. Finally, in Subsection 6.2.3, we define a class of positions \mathcal{C}_1 that contains the starting positions (without strongly favorable components) and for which Alice can ensure either to win or to stay in this class after a move of Bob. By induction, this will imply that Alice has a winning strategy in this class. Note that one can always consider U to be maximal in Definition 5.22. Indeed, if Alice have a strategy for U maximal, she will have a strategy with any subset of U .

6.2.1 Removing strongly favorable components

Lemma 6.7. *Let Q be a position where Bob is not dominating. Consider a strongly favorable component $[Ao^nB]^U$. We have*

$$o(Q, B) \leq o(Q \cup [Ao^nB]^U, B).$$

In other words, adding a strongly favorable component can only be favorable to Alice.

Proof. First note that since $[Ao^nB]^U$ is strongly favorable, we have $U \subseteq \{2, 3, n-2\}$.

We prove by induction of the number p of unclaimed vertices of $Q \cup [Ao^n B]^U$ that if $o(Q, B) = \mathcal{A}$, then $o(Q \cup [Ao^n B]^U, B) = \mathcal{A}$. Note that $p \geq n$. Let v_1, v_2, \dots, v_n denotes the sequence of unclaimed vertices of $[Ao^n B]^U$. First we can assume that $n > 1$. Indeed, if $n = 1$, then $U = \emptyset$ and by Observation 2.4, $o(Q \cup [AoB]^U, B) = o(Q, B)$.

Thus, we can assume that $p \geq 2$ and $n \geq 2$. If $p = n = 2$, Q contains no unclaimed vertices. Since $o(Q, B) = \mathcal{A}$, Alice dominates Q while Bob does not. Thus, in $Q \cup [Ao^2 B]^U$ Bob claims v_1 or v_2 , and Alice answers by claiming the other one. By this way, Alice dominates $Q \cup [Ao^n B]^U$ while Bob does not. Thus, the resulting position is \mathcal{A} -win.

Consider now that $p \geq 3$. Let y be the vertex claimed by Bob. Assume first that y is an unclaimed vertex of Q . Note that Bob cannot dominate Q in one move unless Alice already does, otherwise we would have $o(Q, B) = \mathcal{D}$.

- If Q is not dominated by Alice yet, then Alice claims x according to a winning strategy in Q . Thus, by definition, $o(Q_{x,y}, B) = \mathcal{A}$ and by induction hypothesis $o(Q_{x,y} \cup [Ao^n B]^U, B) = \mathcal{A}$.
- if G is already dominated by Alice:
 - if $2 \leq n \leq 4$, then Alice claims v_3 and dominates the whole graph before Bob;
 - if $n = 5$, the Alice claims v_3 . We have:

$$o((Q \cup [Ao^n B]^U)_{v_3,y}, B) \geq o([Ao^n B]^U_{v_3,y}, B) \geq o([Ao^2 A]^{\{2\}} \cup [Ao^2 B]), B) = \mathcal{A},$$

. where the first inequality comes from the fact that Q is dominated by Alice, the second inequality comes from Lemma 2.6 and from the fact that $U \subseteq \{2, 3\}$, which is strongly favorable according to Definition 5.22.1.a, and the final equality is obtained by induction hypothesis, using $Q' = [Ao^2 A]^{\{2\}}$.

- if $n = 6$, then Alice claims v_3 . We have:

$$o((Q \cup [Ao^n B]^U)_{v_3,y}, B) \geq o([Ao^n B]^U_{v_3,y}, B) \geq o([Ao^2 A]^{\{2\}} \cup [Ao^3 B]^{\{1\}}, B) = \mathcal{A},$$

where the first inequality comes from the fact that Q is dominated by Alice, the second inequality comes from Lemma 2.6 and from the fact that $U \subseteq \{2, 3, n-2\}$, which is strongly favorable according to Definition 5.22.1.b, and the final equality is obtained by induction hypothesis, using $Q' = [Ao^2 A]^{\{2\}}$.

- if $n \geq 7$, then Alice claims v_3 . We have:

$$o((Q \cup [Ao^n B]^U)_{v_3,y}, B) \geq o([Ao^n B]^U_{v_3,y}, B) \geq o([Ao^2 A]^{\{2\}} \cup [Ao^{n-3} B]^{\{n-5\}}, B) = \mathcal{A},$$

where the first inequality comes from the fact that Q is dominated by Alice, the second inequality comes from Lemma 2.6 and from the fact that $U \subseteq \{2, 3, n-2\}$, which is strongly favorable according to Definition 5.22.1.c, and the final equality is obtained by induction hypothesis, using $Q' = [Ao^2 A]^{\{2\}}$.

Assume now that Bob claimed a vertex v_i of $[Ao^n B]^U$.

- If $2 \leq i \leq n-1$, then Alice claims v_{i+1} . We have

$$o((Q \cup [Ao^n B]^U)_{v_i, v_{i+1}}, B) = o(Q \cup [Ao^{i-1} B]^{\{2,3\}} \cup [Ao^{n-i-1} B]^{\{n-i-3\}}, B) = \mathcal{A},$$

where the first equality comes from Lemma 2.6 and the inequality comes from two applications of the induction hypothesis.

- If $i = 1$, and $n \geq 3$ then Alice claims v_3 . We have

$$o((Q \cup [Ao^n B]^U)_{v_3, v_1}, B) \geq o(Q \cup [Ao^{n-3} B]^{\{n-5\}}, B) = \mathcal{A},$$

by induction hypothesis.

If $n = 2$ then Alice claims v_2 . We have $o((Q \cup [Ao^2 B]^U)_{v_2, v_1}, B) = o(Q, B) = \mathcal{A}$, from Observation 2.4

- If $i = n$, then Alice claims v_{n-1} . We have

$$o((Q \cup [Ao^n B]^U)_{v_{n-1}, v_n}, B) \geq o(Q \cup [Ao^{n-2} B]^{\{2,3\}}, B) = \mathcal{A},$$

by induction hypothesis.

Thus, for each claim y of Bob on $Q \cup [Ao^n B]^U$, there exists an answer x of Alice such that $o((Q \cup [Ao^n B]^U)_{x,y}, B) = \mathcal{A}$. This ensures that $o(Q \cup [Ao^n B]^U, B) = \mathcal{A}$. \square

6.2.2 Dealing with the weakly favorable component

Lemma 6.8. *Let C be a connected component of S_F that is weakly favorable, then $o(P_C \cup [AoA], B) = \mathcal{A}$. Moreover, Alice can play to ensure that after each of her move, there is a A -pairing.*

Proof. Recall that by Observation 5.19, there is A -pairing in any bounded path $[Xo^nY]$ except if $X = Y = B$ and n is odd.

Assume first that C is a fork. Let c be the center of C , a_1 the neighbor of c on the path to v_0 and a_2 the vertex between a_1 and v_0 . Let b_1 be another neighbor of c and b_2 the other neighbor of y . We prove the result by induction on the number of branches in the fork. Note that, by Lemma 2.3, as $N[a_2] \setminus N[V_t] \subset N[a_1] \setminus N[V_t]$ for $t \in \{A, B\}$, a_1 is always a better move than a_2 , thus we can suppose that it will not be played as next move.

- if Bob claims c , then Alice replies in a_1 . Since there is a double B -trap and a A -pairing, disjoint from the traps, Alice wins by Lemma 3.7. Moreover, by following the pairing strategy, she will keep the fact there is always a A -pairing.
- If Bob claims b_1 , then Alice replies in b_2 . If there are strictly more than three branches, the position is still a fork where the center is dominated by Bob (which is still weakly favorable). The result is true by induction. If there were exactly three branches, then the actual position is $((P_C \cup [AoA])_{b_2, b_1}, B) = ([Ao^5B]^{\{2,3\}} \cup [AoA], B)$. By Lemma 6.7, $o((P_C \cup [AoA])_{b_2, b_1}, B) \geq o([AoA], B) = \mathcal{A}$. Furthermore, there is A -pairing by Observation 5.19.
- If Bob claims b_2 , then Alice replies in b_1 . If there are strictly more than three branches, then Alice follows the strategy with one less branch. Since she dominates one more vertex, it can only be better for her. If there are exactly three branches, we have as before:

$$o((P_C \cup [AoA])_{b_1, b_2}, B) \geq o([Ao^5B]^{\{2,3\}} \cup [AoA], B) \geq o([AoA], B) = \mathcal{A}.$$

- If Bob claims a_1 , then Alice replies by claiming c . We have

$$o((P_C \cup [AoA])_{c, a_1}, B) = o([Ao^2B] \cup \dots \cup [Ao^2B] \cup [AoA], B) \geq o([AoA], B) = \mathcal{A}.$$

There is a A -pairing by pairing together the two vertices belonging to the same paths.

- If Bob claims v_{-1} , then Alice replies by claiming c . We have

$$o((P_C \cup [AoA])_{c, v_{-1}}, B) \geq o([Ao^2A]^{\{2\}} \cup [Ao^2B] \cup \dots \cup [Ao^2B], B) \geq o([Ao^2A]^{\{2\}}, B) = \mathcal{A}.$$

Again, there is a A -pairing by pairing together the two vertices belonging to the same paths.

Assume now that P_C is a bounded path. Alice will never let components of the form $[Bo^{2\ell}B]$ to Bob and thus there will always be a A -pairing. Assume first that P_C has length 9. We just need to prove that Alice has a strategy in the worst case, that is for $U = \{2, 3, 5, 7\}$. Thus, let $P_C = [Ao^9B]^{\{2,3,5,7\}}$. Note that, by Lemma 2.3, as $N[v_1] \setminus N[V_t] \subset N[v_2] \setminus N[V_t]$ for $t \in \{A, B\}$, v_2 is always a better move than v_1 , thus we can suppose that it will not be played as next move.

- If Bob claims v_1 , then Alice replies in v_3 . We have

$$o((([Ao^9B]^{\{2,3,5,7\}})_{v_3, v_1} \cup [AoA]), B) \geq o([Ao^6B]^{\{2,4\}} \cup [AoA], B) \geq o([AoA], B) = \mathcal{A},$$

where the last inequality comes from Lemma 6.7.

- If Bob claims v_3 , then Alice replies in v_2 . Then $o((P_C \cup [AoA])_{w_2, w_3}, B) = \mathcal{A}$ since there are two B -traps and a A -pairing.
- If Bob claims v_i , $4 \leq i \leq 8$, then Alice replies in v_{i+1} . We have

$$o((([Ao^9B]^{\{2,3,5,7\}})_{v_{i+1}, v_i} \cup [AoA]), B) = o([Ao^{i-1}B]^{\{2,3,5,7\} \cap \{1, \dots, i-2\}} \cup [Ao^{9-i-1}B]^{7-i-1} \cup [AoA], B) \geq o([AoA], B) = \mathcal{A},$$

where the last inequality comes from two applications of Lemma 6.7 (since both created paths satisfy hypotheses of Lemma 6.7).

- If Bob claims v_9 , then Alice replies in v_8 . We have

$$o(([Ao^9B]^{\{2,3,5,7\}})_{v_8,v_9} \cup [AoA], B) \geq o([Ao^7A]^{\{2,3,5,7\}} \cup [AoA], B) \geq o([Ao^7B]^{\{2,3,5\}} \cup [AoA], B) \geq o([AoA], B) = \mathcal{A},$$

where the last inequality comes from two applications of Lemma 6.7.

- If Bob claims v_{-1} , then Alice replies in v_3 . We have

$$o(([Ao^9B]^{\{2,3,5,7\}} \cup [AoA])_{v_3,v_{-1}}, B) \geq o([Ao^6B]^{\{2,4\}} \cup [Ao^2A], B) \geq o([Ao^2A], B) = \mathcal{A},$$

where the last inequality comes from Lemma 6.7.

Assume now that P_C has length 11. As before, we can assume that $P_C = [Ao^{11}B]^{\{2,3,5,9\}}$.

- If Bob claims v_i , $1 \leq i \leq 8$, then it can be done like for a path of length 9.

- If Bob claims v_9 , then Alice replies in v_8 ,

- If now Bob claims v_3 , then Alice replies in v_2 , creating two B -traps. Thus $o(Q_{v_2,v_3}, B) = \mathcal{A}$, by Lemma 3.7,

- If now Bob claims v_5 , then Alice replies in v_6 , creating two B -traps. Thus $o(Q_{v_6,v_5}, B) = \mathcal{A}$, by Lemma 3.7

- If now Bob claims v_{10} (or v_{11}), then Alice replies in v_{11} (or v_{11}). The resulting position is equivalent to $[Ao^7A]^{\{2,3,5\}} \cup [AoA]$, and, from Lemma 6.7,

$$o([Ao^7A]^{\{2,3,5\}} \cup [AoA], B) \geq o([AoA], B) = \mathcal{A}.$$

- If now Bob claims v_i , with $i \in \{-1, 1, 2, 4\}$, then Alice replies in v_3 . Afterward Alice can dominate with two more claims, one in $\{v_5, v_6\}$, one in $\{v_{10}, v_{11}\}$, whatever the strategy of Bob; therefore Alice dominates before Bob

- If now Bob claims v_i , with $i \in \{6, 7\}$, then Alice replies in v_5 . Afterward Alice can dominate with two more claims, one in $\{v_2, v_3\}$, one in $\{v_{10}, v_{11}\}$, whatever the strategy of Bob; therefore Alice dominates before Bob.

- if Bob claims in v_{10} , then Alice replies in v_{11} , which reduces the problem to the previous case with 9 vertices.

- If Bob claims v_{11} , then Alice replies in v_{10} . The resulting position is equivalent to $[Ao^9A]^{\{2,3,5,9\}} \cup [AoA]$, and by Observation 5.20, we have

$$o([Ao^9A]^{\{2,3,5,9\}} \cup [AoA], B) \geq o([Ao^9B]^{\{2,3,5\}} \cup [AoA], B) = \mathcal{A},$$

as seen before.

- If Bob claims v_{-1} , then Alice replies in v_3 . We have

$$o(([Ao^{11}B]^{\{2,3,5,9\}} \cup [AoA])_{v_3,v_{-1}}, B) \geq o([Ao^8B]^{\{2,6\}} \cup [Ao^2A], B) \geq o([Ao^2A], B) = \mathcal{A},$$

where the last inequality comes from Lemma 6.7. □

6.2.3 A stable class of positions

In order to define our stable class \mathcal{C}_1 , we first define the class \mathcal{C}_0 , that informally corresponds to the positions derived from a weakly favorable position when Alice follows a winning strategy.

Definition 6.9 (\mathcal{C}_0). *The class \mathcal{C}_0 is defined recursively as follows. A position P is an element of \mathcal{C}_0 if:*

- $P = P_C$ where C is a weakly favorable component;
- there exists $P' \in \mathcal{C}_0$, with two unclaimed vertices x and y , such that $P = P'_{x,y}$ and $o(P \cup [AoA], B) = \mathcal{A}$.

Next corollary is a direct application of Lemma 6.8.

Corollary 6.10. *For each $P \in \mathcal{C}_0$, we have $o(P \cup [AoA], B) = \mathcal{A}$.*

We can now define the stable class of positions \mathcal{C}_1

Definition 6.11 (\mathcal{C}_1). *A position P belongs to the class \mathcal{C}_1 if P is the union of the following positions:*

1. *at most one position in \mathcal{C}_0 .*
2. *a union of positions of the type $[Ao^n B]^{\{2,5,n-2\}}$ $n \geq 9$, n odd.*
3. *a union of k positions of the type $[Bo^{2t} B]^{\{2t-2\}}$, $t \geq 1$,*
4. *a union of k' positions of the type $[Ao^n A]^{\{2,5\}}$, with $k' \geq k$ and $n \geq 1$.*

Remark 6.12. *By definition, each position formed by at most one weakly favorable position and some positions of the alternative 1.d of Definition 5.22 is an element of \mathcal{C}_1 .*

Next corollary is a direct application of Lemma 6.8 and Observation 5.19.

Corollary 6.13. *For each $P \in \mathcal{C}_1$, P admits an A -pairing.*

We will now prove that the class \mathcal{C}_1 (with a trap $[AoA]$ adjoined) is either stable after Alice's answer to Bob's claim, or directly winning for Alice. We will consider the two cases according to whether Bob claims the unclaimed vertex v_{-1} of $[AoA]$ or not. The first case where Bob does not claim v_{-1} is proved by Lemma 6.15 that requires Claim 6.14. The second case is when Bob claims v_{-1} and is solved by Lemma 6.17 that requires Claim 6.16 as a particular case.

Claim 6.14. *Let Q be any position, and t be a positive integer. Let $(w_1, w_2, \dots, w_{2t})$ be the unclaimed vertices of $[Bo^{2t} B]^{\{2t-2\}}$. For each w_i , there exists $w_j \neq w_i$ and $0 \leq t' < t$ such that*

$$o(Q \cup [Bo^{2t'} B]^{\{2t'-2\}}, B) \leq o(Q \cup [Bo^{2t} B]_{w_j, w_i}^{\{2t-2\}}, B)$$

(with the convention that $[Bo^0 B]^{\{-2\}}$ is empty).

The idea behind this claim is that if a position contains a bounded path $[Bo^{2t} B]^{\{2t-2\}}$ and Bob claims a vertex in it, then Alice can reply in the same bounded path, preserving the global structure of the position.

Proof. We have three cases.

- If $i = 2t$ then Alice claims w_{2t-1} , using Observations 5.20 and 5.21, we have

$$o(Q \cup ([Bo^{2t} B]^{\{2t-2\}})_{w_{2t-1}, w_{2t}}, B) = o(Q \cup [Bo^{2t-2} A]^{\{2t-2\}}, B) \geq o(Q \cup [Bo^{2t-2} B], B) \geq o(Q \cup [Bo^{2t-2} B]^{\{2t-4\}}, B)$$

- If $i = 2k$, $1 \leq k < t$, then Alice claims w_{2k-1} . We have

$$o(Q \cup [Bo^{2t} B]_{w_{2k-1}, w_{2k}}^{\{2t-2\}}, B) = o(Q \cup [Bo^{2k-2} A] \cup [Bo^{2(t-k)} B]^{\{2(t-k)-2\}}, B).$$

Lemma 6.7 applies, and we get

$$o(Q \cup [Bo^{2t} B]_{w_j, w_i}^{\{2t-2\}}, B) \geq o(Q \cup [Bo^{2(t-k)} B]^{\{2(t-k)-2\}}, B).$$

- If $i = 2k - 1$, $1 \leq k \leq t$, then Alice claims w_{2k} . We have

$$o(Q \cup [Bo^{2t} B]_{w_j, w_i}^{\{2t-2\}}, B) = o((Q \cup [Bo^{2k-2} B] \cup [Ao^{2(t-k)} B]^{\{2(t-k)-2\}}, B),$$

Lemma 6.7 applies, and we get

$$o(Q \cup (Bo^{2t} B]_{w_j, w_i}^{\{2t-2\}}, B) \geq o(Q \cup [Bo^{2k-2} B], B) \geq o(Q \cup [Bo^{2k-2} B]^{\{2k-4\}}, B).$$

□

Lemma 6.15 (Bob does not claim v_{-1}). *Let $P \in \mathcal{C}_1$, such that P is not dominated by Alice. Let $Q = P \cup [AoA]$ and y be an unclaimed vertex of P . There exists an unclaimed vertex $x \neq y$ such that at least one of the following alternatives holds:*

- $o(Q_{x,y}, B) = \mathcal{A}$;
- or there exists $P' \in \mathcal{C}_1$ with at least two unclaimed vertices less than P , such that $o(P' \cup [AoA], B) \leq o(Q_{x,y}, B)$.

Proof. We denote by v_{-1} the unclaimed element of $[AoA]$ in Q . We assume that Bob claims y . We have several cases, according to the component that contains y .

- Assume first that y is an element of the component P' of P that belongs to \mathcal{C}_0 . If there exists a free vertex x of P' such that $P'_{x,y} \in \mathcal{C}_0$, then we are done, since $P_{x,y} \in \mathcal{C}_1$.

Otherwise, by definition of \mathcal{C}_0 , $(P' \cup [AoA], B)$ is \mathcal{A} -win. Since there is no winning answer in P' for Alice to the claim y of Bob, we necessarily have $o((P' \cup [AoA])_{v_{-1},y}, B) = \mathcal{A}$. From the position $(P' \cup [AoA])_{v_{-1},y}$, Bob cannot dominate the whole graph. Alice can follow her strategy on $(P' \cup [AoA])_{v_{-1},y}$ and a pairing strategy on the rest of the graph (that exists by Corollary 6.13). This ensures that $o(Q_{v_{-1},y}, B) = \mathcal{A}$

- Assume now that y is an unclaimed vertex of $[Ao^n B]^{\{2,5,n-2\}}$, with n odd, $n \geq 9$. Let (w_1, w_2, \dots, w_n) be the sequence of free vertices of $[Ao^n B]^{\{2,5,n-2\}}$. By Lemma 2.3, as $N[w_1] \setminus N[V_t] \subset N[w_2] \setminus N[V_t]$ for $t \in \{A, B\}$, one can assume that $y \neq w_1$. Let P' be the position such that $P = P' \cup [Ao^n B]^{\{2,5,n-2\}}$ and let $Q' = P' \cup [AoA]$. Note that $P' \in \mathcal{C}_1$.

– If $y = w_2$ then take $x = w_3$. By Lemma 2.6, Observation 2.4 and Lemma 6.7,

$$o(Q_{x,y}, B) \geq o(Q' \cup [Ao^{n-3} B]^{\{2,n-5\}}, B) \geq o(Q', B).$$

– If $y = w_3$, then take $x = w_2$. There is double B -trap in Q_{w_2,w_3} . Thus, by Lemmas 3.7 and 6.13, we have $o(Q_{w_2,w_3}, B) = \mathcal{A}$.

– If $y = w_i$ $i \geq 4$ with i even, then take $x = w_{i+1}$. We have

$$o(Q_{x,y}, B) = o(Q' \cup [Ao^{i-1} B]^{\{2,5\}} \cup [Ao^{n-i} B]^{\{n-i-2\}}, B) \geq o(Q' \cup [Ao^{i-1} B]^{\{2,5\}}, B)$$

by application of Lemma 6.7. Note that $P' \cup [Ao^{i-1} B]^{\{2,5\}}$ is an element of \mathcal{C}_1 .

– if $y = w_i$ with i odd and $i \geq 5$ then take $x = w_{i-1}$. We have

$$o(Q_{x,y}, B) = o(Q' \cup [Ao^{i-2} A]^{\{2,5\}} \cup [Bo^{n-i} B]^{\{n-i-2\}}, B).$$

Note that $P' \cup [Ao^{i-2} A]^{\{2,5\}} \cup [Bo^{n-i} B]^{\{n-i-2\}}$ is an element of \mathcal{C}_1 , since $n-i$ is even and since we add a position $[Ao^{i-2} A]^{\{2,5\}}$.

- If y is an unclaimed vertex of a component of the form $[Bo^{2t} B]^{\{2t-2\}}$, then the position can be reduced by Claim 6.14.

- Assume now that y is an unclaimed vertex of $[Ao^n A]^{\{2,5\}}$. As before, we denote by (w_1, w_2, \dots, w_n) the vertices of $[Ao^n A]^{\{2,5\}}$, by P' the position P without the component $[Ao^n A]^{\{2,5\}}$ and by Q' the position $P' \cup [AoA]$. If $n \leq 3$, then take $x = v_{-1}$. We have $o(Q_{v_{-1},y}, B) = \mathcal{A}$ since the position $Q_{v_{-1},y}$ admits a A -pairing by Corollary 6.13 and Bob. Thus, it can be assumed that $n \geq 4$.

– If $y \in \{w_1, w_2\}$, then take $x = w_3$. We have

$$o(Q_{w_3,y}, B) \geq o(Q' \cup [Ao^{n-3} A]^{\{2\}}, B).$$

Note that $P' \cup [Ao^{n-3} A]^{\{2\}}$ is in \mathcal{C}_1 .

– If $y = w_3$, then take $x = w_2$. The position Q_{w_2,w_3} admits an A -pairing, by Corollary 6.13. Thus, by lemma 3.7, we have $o(Q_{w_2,w_3}) = \mathcal{A}$.

– if $y = w_i$, with $i \geq 4$, then take $x = w_{i-1}$. We have

$$o(Q_{x,y}, B) = o(Q' \cup [Ao^{i-2}A]^{\{2,5\}} \cup [Ao^{n-i}B], B) \geq o(Q' \cup [Ao^{i-2}A]^{\{2,5\}}, B)$$

according to Lemma 6.7. We are done since $P' \cup [Ao^{i-2}A]^{\{2,5\}}$ is in \mathcal{C}_1 . \square

Claim 6.16. *Let P be any position, and t be a nonnegative integer. We have*

$$o(P \cup [AoA], B) \leq o(P \cup [AooAooA]^{\{2,5\}} \cup [Bo^{2t}B]^{\{2t-2\}}, B)$$

where $[AooAooA]^{\{2,5\}}$ is the position obtained from $[Ao^5A]^{\{2,5\}}$ by adding the central vertex v_3 in the set V_A .

Proof. Let $(w_1, w_2, \dots, w_{2t})$ be the sequence of unclaimed vertices of $[Bo^{2t}B]^{\{2t-2\}}$, (v_1, v_2, v_4, v_5) the sequence of unclaimed vertices of $[AooAooA]^{\{2,5\}}$, from left to right, and v_{-1} denote the unclaimed vertex of $[AoA]$. To lighten notation, we also state $R = P \cup [AooAooA]^{\{2,5\}} \cup [Bo^{2t}B]^{\{2t-2\}}$

Assume that $o(P \cup [AoA], B) = \mathcal{A}$. We prove that $o(R, B) = \mathcal{A}$ by induction on the number p of unclaimed vertices of R . For initialization, if $p \in \{4, 5\}$, then $t = 0$ and all the vertices of P are claimed except possibly one. Thus, since $o(P \cup [AoA], B) = \mathcal{A}$, Alice dominates P , and therefore Alice dominates $R = P \cup [AooAooA]^{\{2,5\}}$.

Now assume that $p \geq 6$. We have several alternatives according to the claim y of Bob in R .

- If y is an unclaimed vertex of $[Bo^{2t}B]^{\{2t-2\}}$, then Claim 6.14 applies, and the induction gives the result.
- Assume now that y is an unclaimed vertex of P . If there exists an unclaimed vertex x of P such that $o(P_{x,y} \cup [AoA], B) = \mathcal{A}$. We conclude by the induction hypothesis. If it is not possible, it means that once Bob has claimed y in the position $P \cup [AoA]$, Alice is forced to claim v_{-1} . Then, in R , Alice claims v_2 , creating a double B -trap in v_1 and v_4 .

In the position $((P \cup [AoA])_{v_{-1},y}, B)$, Alice has a strategy which allows her to dominate P . Thus, Alice can win in $R_{v_2,y}$ playing component by component, until, the game ends as follows:

- If Bob claims in P , Alice also claims in P according to the strategy in $((P \cup [AoA])_{v_{-1},y}, B)$.
- If Bob claims in $[AooAooA]^{\{2,5\}}$, then Alice isolates one vertex using one of the B -trap.
- If Bob claims in $[Bo^{2t}B]^{\{2t-2\}}$, Alice follows a pairing strategy with the A -pairing of this component.

This way, Alice will prevent Bob to dominate since Bob has to claim at some point in $[AooAooA]$. The two other items ensure that Alice while dominate the whole graph.

- Assume finally that y is an unclaimed vertex of $[AooAooA]$. The position (P, A) is \mathcal{A} -win since it corresponds to the position $(P \cup [AoA], B)$ where Bob has claimed v_{-1} . Let x be a unclaimed vertex of P such that $o(P_x, B) = \mathcal{A}$, where P_x is the position obtained from P when Alice has claimed x . Alice answers x in the game R . We prove that $o(R_{x,y}, B) = \mathcal{A}$. Alice plays as follows:
 - If Bob claims a vertex in P_x , Alice answers in P_x according to her strategy in (P_x, B) .
 - If Bob claims a vertex in $[Bo^{2t}B]^{\{2t-2\}}$, then Claim 6.14] can be used.
 - If Bob claims again a vertex in $[AooAooA]$, Alice claims w_1 .

The first item ensures that Alice dominates P before Bob. The second item ensures that the fact that Bob claims in the component $[Bo^{2t}B]^{\{2t-2\}}$ is irrelevant. In the case of third item, one can consider using Lemma 6.7 that the component $[Bo^{2t}B]^{\{2t-2\}}$ disappears. The component $[AooAooA]$ becomes also irrelevant since each player dominates all vertices of this component. Thus, it remains only the component derived from P_x where Alice dominates before Bob. \square

Lemma 6.17 (Bob claims v_{-1}). *Let $P \in \mathcal{C}_1$ such that P is not dominated by Alice and $Q = P \cup [AoA]$. There exists an unclaimed vertex x of P such that at least one of the following alternatives holds:*

- $o(Q_{x,v_{-1}}, B) = \mathcal{A}$,
- there exists $P' \in \mathcal{C}_1$ with at least one unclaimed vertex less than P , such that $o(P' \cup [AoA], B) \leq o(Q_{x,v_{-1}}, B)$.

Proof. We have different cases according to the structure of P .

- If P contains a component $[Ao^n B]^{\{2,5,n-2\}}$, $n \geq 9$, n odd, let v_1, v_2, \dots, v_n denote the sequence of free vertices of this component. Take $x = v_2$. Let P' be the position such that $P = P' \cup [Ao^n B]^{\{2,5,n-2\}}$ and let $Q' = P' \cup [AoA]$. Note that $P' \in \mathcal{C}_1$.

We now have

$$o(Q_{v_2, v_{-1}}, B) \geq o(Q' \cup [Ao^{n-2} B]^{\{3,n-4\}}, B) \geq o(Q', B)$$

from Lemma 6.7.

- If P contains a component $[Ao^n A]^{\{2,5\}}$, let w_1, w_2, \dots, w_n denote the sequence of vertices of $[Ao^n A]^{\{2,5\}}$. Let P' be the position such that $P = P' \cup [Ao^n A]^{\{2,5\}}$ and let $Q' = P' \cup [AoA]$.

- If $n \geq 6$, then take $x = w_{n-1}$. We have

$$o(Q_{w_{n-1}, v_{-1}}, B) \geq o(Q' \cup [Ao^{n-2} A]^{\{2,5\}}, B)$$

and note that $P' \cup [Ao^{n-2} A]^{\{2,5\}}$ is an element of \mathcal{C}_1 .

- If $3 \leq n \leq 4$, then take $x = w_2$. We have

$$o(Q_{w_2, v_{-1}}, B) \geq o(Q' \cup [Ao^{n-2} A], B)$$

and we conclude as previously.

- If $n = 1$, then take $x = w_1$. The position $Q_{w_1, v_{-1}}$ admits an A -pairing from Corollary 6.13. Thus, from 3.6, we have $o(Q_{w_1, v_{-1}}, B) = \mathcal{A}$.
- If $p = 5$ and $k \geq 1$, take $x = w_3$, which creates an $[AooAooA]$. We define P'' such that

$$P' = P'' \cup ([Bo^{2t} B]^{\{2t-2\}}).$$

Note that $P'' \in \mathcal{C}_1$ as one component of each type 3 and 4 (in Definition 6.11) has been removed. Lemma 6.16 applies, thus

$$o(P'' \cup [AoA], B) \leq o(P'' \cup [AooAooA]^{\{2,5\}} \cup [Bo^{2t} B]^{\{2t-2\}}, B) = o(Q_{w_3, v_{11}}, B)$$

- If P cannot be treated in one of the previous cases, then P only contains some subpositions of the type $[Ao^5 A]^{\{2,5\}}$ and at most one position $P_0 \in \mathcal{C}_0$. In this case, we will prove that $o(Q, B) = \mathcal{A}$, which implies the result.

First we have $o([Ao^5 A]^{\{2,5\}}, B) = \mathcal{A}$. Indeed, Bob is forced to claim w_3 , since otherwise Alice claims w_3 and wins. After this claim, there exists an A -pairing of size 2, while Bob needs at least two more claims to dominate the graph. Thus $Q = P \cup [AoA]$ is formed by a union of winning positions (when Bob starts): one is $P_0 \cup [AoA]$ (or simply $[AoA]$ if P_0 does not exist) and the other ones are positions $[Ao^5 A]^{\{2,5\}}$. Thus, by Observation 2.5, we get $o(Q, B) = \mathcal{A}$. \square

Corollary 6.18. *For each position $P \in \mathcal{C}_1$, we have $o(P \cup [AoA], B) = \mathcal{A}$.*

Proof. We prove the result by induction on the number of unclaimed vertices in $Q = P \cup [AoA]$. If Alice dominates P , then we directly have $o(P \cup [AoA], B) = \mathcal{A}$. Otherwise, consider a move x of Bob. If $x \neq v_{-1}$ (respectively $x = v_{-1}$), then by Lemma 6.15 (resp. Lemma 6.17), there exists an unclaimed vertex y of P such that either Alice wins or there exists a position P' of \mathcal{C}_1 with less unclaimed vertices such that $o(P' \cup [AoA], B) \leq o(Q_{x,y}, B)$. By induction, $o(P' \cup [AoA], B) = \mathcal{A}$ and thus $o(Q_{x,y}, B) = \mathcal{A}$.

Thus, for any claim x of Bob, Alice can claim a vertex y such that $o(Q_{x,y}, B) = \mathcal{A}$. Therefore, $o(Q, B) = \mathcal{A}$. \square

6.2.4 Conclusion

Putting together all the previous results, we can prove the reverse part of Theorem 5.23:

Corollary 6.19. *If all the components of S_F are favorable to Alice and at most one of them is weakly favorable, then $((F, \{v_0\}, \emptyset), B)$ is \mathcal{A} -win.*

Proof. Let $Q = (F, \{v_0\}, \emptyset)$ be a position such that all the components of S_F are favorable to Alice and at most one of them is weakly favorable. By Lemma 6.7, one can assume that there is no strongly favorable component in S_F . Let P such that $Q = P \cup [A_0A]$. By Remark 6.12, P is an element of \mathcal{C}_1 . Corollary 6.18 leads to the desired result. \square

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