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# ON THE DEGREE OF REGULARITY OF A PARTICULAR LINEAR EQUATION

S. D. ADHIKARI, R. BALASUBRAMANIAN, S. ELIAHOU, AND D. J. GRYNKIEWICZ

ABSTRACT. There are many interesting results and open questions regarding the degree of regularity of Diophantine equations. The Fox and Kleitman conjecture [5] concerning the maximum degree of regularity of the equation  $x_1 + \cdots + x_k - y_1 - \cdots - y_k = b_k$ , as  $b_k$  runs over the positive integers, has recently been confirmed [7]. We here consider the problem of finding the degree of regularity of some specializations of the above equation.

## 1. INTRODUCTION

For given  $a_1, \dots, a_k$  and  $b$  in the set  $\mathbb{Z}$  of integers, we consider the linear Diophantine equation  $L$ :

$$\sum_{i=1}^k a_i x_i = b.$$

Following [6], given  $n \in \mathbb{N}_+$ , the set of positive integers, equation  $L$  is said to be  $n$ -regular if, for every  $n$ -coloring of  $\mathbb{N}_+$ , there exists a *monochromatic* solution  $x = (x_1, \dots, x_k) \in \mathbb{N}_+^k$  to  $L$ .

The *degree of regularity* of  $L$  is the largest integer  $n \geq 0$ , if any, such that  $L$  is  $n$ -regular. This (possibly infinite) number is denoted by  $\text{dor}(L)$ . If  $\text{dor}(L) = \infty$ , then  $L$  is said to be *regular*.

A well-known and challenging conjecture (known as *Rado's Boundedness Conjecture*) due to Rado [6] states that there is a function  $r: \mathbb{N}_+ \rightarrow \mathbb{N}_+$  such that, given any  $n \in \mathbb{N}_+$  and any equation  $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$  with integer coefficients, if this equation is not regular over  $\mathbb{N}_+$ , then it fails to be  $r(n)$ -regular. Even though there is a more general version, we state it here for a single homogeneous equation, as it has been proved by Rado [6] that if the conjecture is true for a single equation, then it is true for a system of finitely many linear equations, and as Fox and Kleitman [5] have shown, if the conjecture is true for a linear homogeneous equation, then it is true for any linear equation.

The first nontrivial case of the conjecture has been proved by Fox and Kleitman [5] by establishing the bound  $r(3) \leq 24$ . In the same paper [5], the authors made the following conjecture for a very specific linear Diophantine equation.

**Conjecture 1.1.** *Let  $k \geq 1$ . There exists an integer  $b_k \geq 1$  such that the degree of regularity of the  $2k$ -variable equation  $L_k(b_k)$ ,*

$$x_1 + \cdots + x_k - y_1 - \cdots - y_k = b_k,$$

is exactly  $2k - 1$ .

Fox and Kleitman [5] had proved the following.

**Proposition 1.2.** *For any  $b \in \mathbb{N}_+$ , the equation  $L_k(b)$  is not  $2k$ -regular.*

After some initial results ([1], [2]) for small values of  $k$ , the full conjecture of Fox and Kleitman has been very recently established by Schoen and Taczala in [7] by generalizing a theorem of Eberhard, Green and Manners [4].

In [3], Bialostocki et al. considered equation  $L$ , that is  $\sum_{i=1}^k a_i x_i = b$ , where  $\sum_{i=1}^k a_i = 0$  and  $b \neq 0$ . Among other things, the paper [3] established  $\text{dor}(x_1 + x_2 - 2y_1 = b)$  under the condition  $x_1 < y_1 < x_2$ . Here in Section 4, following some line of arguments in [1], we furnish a somewhat different proof for the result on  $\text{dor}(x_1 + x_2 - 2y_1 = b)$ ; because of Proposition 1.2, the result here is unconditional.

## 2. THE EQUATION $x_1 + x_2 - 2y_1 = b$

As mentioned in the introduction, Bialostocki et al. [3] established  $\text{dor}(x_1 + x_2 - 2y_1 = b)$ , under the condition  $x_1 < y_1 < x_2$ . Here, following the line of arguments in [1], we give a proof of the following.

**Theorem 2.1.** *Consider the equation  $L'(b)$ :*

$$x_1 + x_2 - 2y_1 = b.$$

*For all positive integers  $b$ , we have*

$$\text{dor}(L'(b)) = \begin{cases} 1 & \text{if } b \equiv 1 \pmod{2}, \\ 2 & \text{if } b \equiv 2, 4 \pmod{6}, \\ 3 & \text{if } b \equiv 0 \pmod{6}. \end{cases}$$

*Proof.* Because of Proposition 1.2,  $\text{dor}(L'(b)) \leq \text{dor}(L_2(b)) \leq 3$ . Again, since  $L'(b)$  is solvable in  $\mathbb{N}_+$ , we have  $1 \leq \text{dor}(L'(b))$ .

Thus,

$$1 \leq \text{dor}(L'(b)) \leq 3.$$

The proof will be complete with the following observations.

**Observation 1.** Consider the 2-coloring of  $\mathbb{N}_+$  given by coloring each integer according to its residue class modulo 2. Let  $(\lambda_1, \lambda_2, \lambda_3)$  be a monochromatic solution to  $L'(b)$  under this coloring.

This will imply

$$\lambda_1 + \lambda_2 - 2\lambda_3 \equiv 0 \pmod{2}.$$

Therefore, if  $b$  is odd, there cannot be a monochromatic solution in  $\mathbb{N}_+^3$  and hence

$$\text{dor}(L'(b)) = 1$$

in this case.

**Observation 2.** Let  $b$  be even and write  $h = b/2$  with  $h \in \mathbb{N}_+$ .

The following three vectors in  $\mathbb{N}_+^3$  are solutions to  $L'(b)$ :

$$\begin{aligned} (b+1, 1, 1), \\ (h+1, h+1, 1), \\ (b+1, b+1, h+1). \end{aligned}$$

Since, for any 2-coloring of  $\mathbb{N}_+$ , at least two elements in the set  $\{b+1, h+1, 1\}$  must be of the same color, at least one of the above three solutions must be monochromatic, and hence  $\text{dor}(L'(b)) \geq 2$  when  $b$  is even.

**Observation 3.** If  $b \not\equiv 0 \pmod{3}$ , then coloring each integer according to its residue class modulo 3 gives a coloring of  $\mathbb{N}_+$  for which there cannot be any monochromatic solution to  $L'(b)$ , and hence  $\text{dor}(L'(b)) \leq 2$  in this case.

**Observation 4.** Here we consider the case  $b \equiv 0 \pmod{6}$ . Since the sum of coefficients is zero, it is easy to see that if  $L'(6)$  is proved to be 3-regular, then so is  $L'(b)$ .

Let  $c: \mathbb{N}_+ \rightarrow \{0, 1, 2\}$  be an arbitrary 3-coloring of  $\mathbb{N}_+$ .

Consider the following families of special solutions to  $L'(6)$  parametrized by  $a \in \mathbb{N}_+$ :

$$\begin{aligned} (a+6, a, a), \\ (a+5, a+1, a), \\ (a+4, a+2, a), \\ (a+3, a+3, a), \\ (a+8, a, a+1), \\ (a+1, a+9, a+2). \end{aligned}$$

The underlying sets for each of these solutions can be assumed to be multi-chromatic, and thus all sets from

$$\mathcal{E} = \{\{a, a+3\}, \{a, a+6\}, \{a, a+2, a+4\}, \{a, a+1, a+5\}, \{a, a+1, a+8\}, \{a+1, a+9, a+2\}\},$$

where  $a$  ranges through  $\mathbb{N}_+$ , are multi-chromatic sets under  $c$ .

As just observed, the integer  $a$  must be colored distinctly from both  $a+3$  and  $a+6$ . Moreover, if  $c(a+6) = c(a+3)$ , then we would obtain the monochromatic solution  $(a+6, a+6, a+3)$ . It follows that

$$\{c(a), c(a+3), c(a+6)\} = \{0, 1, 2\} = \{c(a+3), c(a+6), c(a+9)\},$$

with the second equality following by the same argument used for the first, only replacing  $a$  by  $a+3$ . Hence

$$c(a) = c(a+9).$$

Thus the color of an integer only depends on its residue class modulo 9. So, denoting the elements of  $\mathbb{Z}/9\mathbb{Z}$  by  $0, 1, \dots, 8$  and their respective colors under  $c$  by  $c_0, c_1, \dots, c_8$  (with indices modulo 9), we may depict the distribution of colors by the following table:

TABLE 1. The color table  $C$ 

$c_0$	$c_1$	$c_2$
$c_3$	$c_4$	$c_5$
$c_6$	$c_7$	$c_8$

Since the sets  $\{a, a+2, a+4\}$ ,  $\{a, a+1, a+5\}$  and  $\{a+1, a+2, a+9\}$  belong to  $\mathcal{E}$  for all  $a \in \mathbb{N}_+$ , and are assumed to be multichromatic under  $c$ , for all  $i \in \mathbb{Z}/9\mathbb{Z}$ , we have

$$\begin{aligned}
(1) \quad & |\{c_i, c_{i+2}, c_{i+4}\}| \geq 2, \\
(2) \quad & |\{c_i, c_{i+1}, c_{i+5}\}| \geq 2, \\
(3) \quad & |\{c_i, c_{i+1}, c_{i+2}\}| \geq 2.
\end{aligned}$$

We may assume that the first column  $(c_0, c_3, c_6)$  of  $C$  is equal to  $(0, 1, 2)$  and the table is as follows:

TABLE 2

0	$c_1$	$c_2$
1	$c_4$	$c_5$
2	$c_7$	$c_8$

The second and third columns of  $C$  being permutations of its first column, there are nine possible pairs holding the remaining two 0's in  $C$ :

$$\begin{aligned}
& (c_1, c_2), (c_1, c_5), (c_1, c_8); \\
(4) \quad & (c_4, c_2), (c_4, c_5), (c_4, c_8); \\
& (c_7, c_2), (c_7, c_5), (c_7, c_8).
\end{aligned}$$

However, recalling that  $c_0 = 0$ , we have

$$\begin{aligned}
& |\{c_0, c_1, c_2\}| \geq 2 \text{ by (3), } |\{c_0, c_1, c_5\}| \geq 2 \text{ by (2), } |\{c_8, c_0, c_1\}| \geq 2 \text{ by (3);} \\
& |\{c_0, c_2, c_4\}| \geq 2 \text{ by (1), } |\{c_4, c_5, c_0\}| \geq 2 \text{ by (2), } |\{c_8, c_0, c_4\}| \geq 2 \text{ by (2);} \\
& |\{c_7, c_0, c_2\}| \geq 2 \text{ by (1), } |\{c_5, c_7, c_0\}| \geq 2 \text{ by (1), } |\{c_7, c_8, c_0\}| \geq 2 \text{ by (3).}
\end{aligned}$$

Hence none of the pairs from (4) can equal  $(0, 0)$ , contradicting that the two remaining 0's in  $C$  must lie in one of the pairs from (4).  $\square$

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