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On a conjecture of Fox–Kleitman and additive combinatorics

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Abstract. Let D_k denote the maximum degree of regularity of the equation $x_1 + \dots + x_k - y_1 - \dots - y_k = b_k$ as b_k runs over the positive integers. The Fox and Kleitman conjecture, stating that D_k should equal $2k - 1$, has been confirmed by Schoen and Taczala (*Moscow J. Combin. Number Theory* **7** (2017) 79–93). Their proof is achieved by generalizing a theorem of Eberhard *et al.* (*Ann. Math.* **180** (2014) 621–652) on sets with doubling constant less than 4. Using much simpler methods and a result of Lev in additive combinatorics, our main result here is that the degree of regularity of the same equation for the specific value $b_k = c_{k-1} = \text{lcm}\{i : i = 1, \dots, k - 1\}$ is at least $k - 1$. This shows in a simple and explicit way that D_k behaves linearly in k .

Keywords. Degree of regularity; Fox–Kleitman conjecture.

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1. Introduction

For a given a_1, \dots, a_k and b in the set \mathbb{Z} of integers, we consider the linear diophantine equation L :

$$\sum_{i=1}^k a_i x_i = b.$$

Following [9], given $n \in \mathbb{N}_+$, the set of positive integers, equation L is said to be *n-regular* if, for every n -coloring of \mathbb{N}_+ , there exists a *monochromatic* solution $x = (x_1, \dots, x_k) \in \mathbb{N}_+^k$ to L .

The *degree of regularity* of L is the largest integer $n \geq 0$, if any, such that L is n -regular. This (possibly infinite) number is denoted by $\text{dor}(L)$. If $\text{dor}(L) = \infty$, then L is said to be *regular*.

A well-known and challenging conjecture (known as *Rado's boundedness conjecture*) due to Rado [9] states that there is a function $r: \mathbb{N}_+ \rightarrow \mathbb{N}_+$ such that, given any $n \in \mathbb{N}_+$ and any equation $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$ with integer coefficients, if this equation is not regular over \mathbb{N}_+ , then it fails to be $r(n)$ -regular. Even though there is a more general version, we state it here for a single homogeneous equation, as it has been proved by Rado [9] that if the conjecture is true for a single equation, then it is true for a system of finitely many linear equations, and as Fox and Kleitman [4] have shown, if the conjecture is true for linear homogeneous equations, then it is true for all linear equations.

The first nontrivial case of the conjecture has been proved by Fox and Kleitman [4] by establishing the bound $r(3) \leq 24$. In the same paper [4], the authors made the following conjecture for a very specific linear diophantine equation.

Conjecture 1.1. Let $k \geq 1$. There exists an integer $b_k \geq 1$ such that the degree of regularity of the $2k$ -variable equation $L_k(b_k)$,

$$x_1 + \dots + x_k - y_1 - \dots - y_k = b_k$$

is exactly $2k - 1$.

Fox and Kleitman [4] proved the following result, showing that $2k - 1$ is best possible in this context.

PROPOSITION 1.2

For any $b \in \mathbb{N}_+$, the equation $L_k(b)$ is not $2k$ -regular.

When $k = 2$, Adhikari and Eliahou [1] proved the Fox–Kleitman conjecture by establishing the following more general result,

Theorem 1.3 [1]. For all positive integers b , we have

$$\text{dor}(L_2(b)) = \begin{cases} 1 & \text{if } b \equiv 1 \pmod{2}, \\ 2 & \text{if } b \equiv 2, 4 \pmod{6}, \\ 3 & \text{if } b \equiv 0 \pmod{6}. \end{cases}$$

A shorter proof of the above has been given in [2].

Though the full conjecture of Fox and Kleitman [4] has been very recently established by Schoen and Taczala in [10] by generalizing a theorem of Eberhard *et al.* [3] in Theorem 3.3 of section 3, we give a very short proof of the fact that, writing $c_{k-1} = \text{lcm}\{i : i = 1, 2, \dots, k-1\}$, the equation $L_k(c_{k-1})$ is $(k-1)$ -regular. Apart from giving a lower bound for the degree of regularity of $L_k(b_k)$ for the particular value $b_k = c_{k-1}$, our much simpler proof (which uses a result of Lev [7]), nonetheless achieves the correct order of magnitude, with a linear constant of 1 rather than the precise value 2, which is much improved as compared to earlier knowledge (as has been mentioned in [4], from a result of Strauss [11], it followed that, for an appropriate b_k , the equation $L_k(b_k)$ was $\Omega(\log k)$ -regular). Note that when $k = p$ is a prime, then $k - 1$ is the exact degree of regularity of

$L(c_{k-1})$, since there is no monochromatic solution to $L(c_{p-1})$ for the p -coloring given by congruence mod p . We also show that, apart from the first few values $k \leq 5$, it suffices to color the first $c_{k-1} + 1$ positive integers to find a monochromatic solution to $L(c_{k-1})$, with the solution occurring in the densest color class.

We now state the following result which was established in [2].

Theorem 1.4 [2]. *We have $\text{dor}(L_3(24)) = 4$.*

Thinking that it is worth recording, a very simple proof of Theorem 1.4 needing only Kneser’s theorem will be given in the next section.

We observe that the proof of the Fox–Kleitman conjecture by Schoen and Taczala and our proofs of Theorem 1.4 and Theorem 3.3 are by applications of results from additive combinatorics.

In what follows, for integers a, b with $a \leq b$, the set of integers x with $a \leq x \leq b$ will be denoted by the integer interval $[a, b]$. For a finite set $A \subseteq \mathbb{Z}$, we shall write $\text{diam } A = \max A - \min A$ to denote the diameter of A . Given two subsets A and B from an additive abelian group, we let $A + B = \{a + b : a \in A, b \in B\}$ denote their sumset and $A - B = \{a - b : a \in A, b \in B\}$ denote their difference set. If $n \geq 0$ is an integer, then $nA = \underbrace{A + \dots + A}_n$ denotes the n -fold iterated sumset, where $0A := \{0\}$, while $n \cdot A = \{na : a \in A\}$ denotes the dilation of A .

Let G be an abelian group and let $A, B \subseteq G$ be nonempty subsets. We let $H(A) = \{h \in G : h + A = A\}$ denote the stabilizer of A , which is a subgroup of G . Note that $H = H(A)$ is equivalent to H being the maximal subgroup for which A is a union of H -cosets. The set A is called *periodic* if $H(A)$ is nontrivial, and otherwise is called *aperiodic*. We will make use of Kneser’s theorem (see [5, Chapter 6]), which states that $|A + B| \geq |A + H| + |B + H| - |H|$ for $H = H(A + B)$. Equivalently, $|A + B| \geq |A| + |B| - 1$ when $A + B$ is aperiodic. Iterating Kneser’s theorem gives $|\sum_{i=1}^n A_i| \geq \sum_{i=1}^n |A_i + H| - (n - 1)|H|$ for $H = H(\sum_{i=1}^n A_i)$.

2. $\text{dor}(L_3(24)) = 4$

For the sake of completeness, we now give an expanded version of the proof of Proposition 1.2 due to Fox and Kleitman [4] which says that for any $b \in \mathbb{N}_+$, the equation $L_k(b)$ is not $2k$ -regular.

Proof. If b is not a multiple of k , then considering the coloring given by the residue class modulo k , there is no monochromatic solution to the equation $L_k(b)$ and the equation not even being k -regular, we are through.

So, we assume that b is a multiple of k and consider the following $2k$ -coloring of \mathbb{N}_+ : for $1 \leq i \leq 2k$, the set of integers colored i is defined to be

$$X_i = \bigcup_{j \geq 0} ([(i - 1)b/k + 1, ib/k] + 2bj).$$

Now, the set $X_i - X_i$ is independent of i . Since the set $k(X_1 - X_1) = \bigcup_{j \in \mathbb{Z}} ([-b + k, b - k] + 2jb)$ is a union of translates of $[-b + k, b - k]$ by integer multiples of $2b$, it

cannot contain b . Therefore, for any i , $1 \leq i \leq 2k$, $k(X_i - X_i)$ does not contain b . This shows that $L_k(b)$ is not $2k$ -regular. \square

We proceed to prove that $\text{dor}(L_3(24)) = 4$ using Kneser's theorem. Since 5 does not divide 24, considering the mod 5 coloring shows that $L_3(24)$ is not 5-regular, and hence we only have to show that $L_3(24)$ is 4-regular, which in turn will follow from the result below and the pigeonhole principle. That result was first stated and proved in [2].

Theorem 2.1. *For any subset $X \subset [0, 32]$ of cardinality $|X| = 9$, we have*

$$24 \in 3(X - X).$$

Proof. Suppose the result is not true and let $X \subset [0, 32]$ be a counterexample.

Thus, writing $S = X - X$, we have

$$24 \notin S + S + S.$$

Since $0 \in S$, this implies that none of the numbers 8, 12, 24 are in S .

If $4 \in S$, then none of the numbers 16, 20, 28, 32 are in S . Therefore, $4 \in S$ would imply $S \cap [0, 32] \cap 4\mathbb{Z} = \{0, 4\}$. Hence, $4 \in S = X - X$ implies that, for all $i = 0, 1, 2, 3$,

$$|X \cap (4\mathbb{N} + i)| \leq 2$$

and hence $|X| \leq 8$, a contradiction to our assumption.

Therefore, none of the numbers 4, 8, 12 nor 24 are in S .

From the above observation, the difference between consecutive elements of $X_i := X \cap (4\mathbb{Z} + i)$, for any $i \in [0, 3]$ is at least 16. Thus, if $|X_i| \geq 3$, then this is only possible if $i = 0$ and $X_0 = \{0, 16, 32\}$. Since $|X| \geq 9$ ensures by the pigeonhole principle that $|X_i| \geq 3$ for some i , we must have $|X_1| = |X_2| = |X_3| = 2$ and $X_0 = \{0, 16, 32\}$.

Now, $X_0 \subset X$, and therefore it follows that $\{16, 32\} \subset S$.

Since $24 = 20 + 20 - 16 = 28 + 28 - 32$, it follows that $20, 28 \notin S$ and hence

$$S \cap 4\mathbb{N} \cap [0, 32] = \{0, 16, 32\}.$$

Therefore,

$$X = \{0, 16, 32\} \cup \{a, a + 16\} \cup \{b, b + 16\} \cup \{c, c + 16\},$$

where $a \equiv 1 \pmod{4}$, $b \equiv 2 \pmod{4}$, $c \equiv 3 \pmod{4}$ and $1 \leq a, b, c \leq 15$.

Writing $Y = X \cap [0, 15]$, we have $Y = \{0, a, b, c\}$. Let $A = (Y - Y) + (Y - Y) + (Y - Y)$. Since $Y - Y \subset [-15, 15]$, we have

$$A \subset [-45, 45].$$

Suppose there exists $\alpha \in A$ with $\alpha \equiv 8 \pmod{16}$. Since $A = -A$, we may assume $\alpha \in \{8, 24, 40\}$.

If $\alpha = 24$, then $24 \in A \subset S + S + S$, and we are through.

If $\alpha = (y_1 - y'_1) + (y_2 - y'_2) + (y_3 - y'_3) = 8$ with $y_i, y'_i \in Y$, then $y_1 + 16 \in X$ and

$$\alpha + 16 = (y_1 + 16 - y'_1) + (y_2 - y'_2) + (y_3 - y'_3) = 24 \in S + S + S,$$

and once again we are through.

Finally, if $\alpha = 40$, then observing that $y'_1 + 16 \in X$, we have

$$\alpha - 16 = (y_1 - (y'_1 + 16)) + (y_2 - y'_2) + (y_3 - y'_3) = 24 \in S + S + S.$$

Therefore, if we can show that A contains an element $\equiv 8 \pmod{16}$, the theorem will be proved.

For a subset $Z \subseteq \mathbb{Z}$, let $\bar{Z} \subseteq \mathbb{Z}/16\mathbb{Z}$ denote its image modulo 16. Now, considering Y modulo 16 as a subset of $\mathbb{Z}/16\mathbb{Z}$, \bar{Y} has 4 elements and $0 \in \bar{A}$. If \bar{A} is periodic, it must contain 8 as all nontrivial subgroups of $\mathbb{Z}/16\mathbb{Z}$ contain 8.

Otherwise, \bar{A} is aperiodic and hence Kneser's theorem (see remarks after the statement of Kneser's theorem in Sect. 6.1 in [5]) implies

$$|\bar{A}| \geq 6|\bar{Y}| - 6 + 1 = 24 - 6 + 1 = 19,$$

which is not possible. □

3. The equation $L_k(c_{k-1})$

Here we improve upon the result of Strauss, mentioned in the Introduction, by establishing that, for some integer b_k , the degree of regularity of the equation $L_k(b_k): (x_1 - y_1) + \dots + (x_k - y_k) = b_k$ is at least $k - 1$. Specifically, we show that this holds with $b_k = c_{k-1} = \text{lcm}\{i : i = 1, 2, \dots, k - 1\}$.

The following is a result of Lev (Corollary, [7]). Here, the case $h = 1$ is trivial.

Theorem A. *Let $A \subseteq \mathbb{Z}$ be a finite set of integers with $|A| \geq 2$ and $\gcd(A - A) = 1$. Let $s = \lfloor \frac{\text{diam } A - 1}{|A| - 2} \rfloor$ (for $|A| \geq 3$), and set $s = 1$ for $|A| = 2$. Let $h_1, h_2 \geq 0$ be integers with $h := h_1 + h_2 \geq 1$.*

- (1) *If $h \leq s$, then $|h_1 A - h_2 A| \geq \frac{h(h+1)}{2}|A| - h^2 + 1$.*
- (2) *If $h \geq s$, then $|h_1 A - h_2 A| \geq \frac{s(s+1)}{2}|A| - s^2 + 1 + (h - s) \text{diam } A$.*

The following is a basic consequence of the pigeonhole principle [8, Lemma 1].

Lemma 3.1. *Let $A \subseteq \mathbb{Z}$ be a finite, nonempty set of integers with $\text{diam } A \leq 2|A| - 2$. Let $e = 2|A| - 2 - \text{diam } A$. Then*

$$[-e, e] \subseteq A - A.$$

Using the above, we can prove the following lemma.

Lemma 3.2. *Let $n > r \geq 1$ be integers. Suppose $X \subseteq \mathbb{Z}$ is a subset of integers with $|X| \geq n + 1$, $\text{diam } X \leq rn$ and $d = \gcd(X - X)$. Then*

$$d\mathbb{Z} \cap [-rdn, rdn] \subseteq (r + 1)X - (r + 1)X.$$

Proof. Observing that the lemma is translation invariant, we may w.l.o.g. assume $0 = \min X$. If $r = 1$, then $X = [0, rn] = [0, n]$, in which case $(r + 1)X - (r + 1)X = 2X - 2X = [-2n, 2n]$, and the lemma holds. Therefore, we may assume $r \geq 2$, and thus $|X| \geq n + 1 \geq r + 2 \geq 4$. Let $N = \max X = \text{diam } X \leq rn$.

Suppose $d \geq 2$. Then all elements of X will be divisible by d (in view of $0 \in X$). Let $X' = \frac{1}{d} \cdot X = \{x/d : x \in X\}$ and observe that $\gcd(X' - X') = 1$ with $X' \subseteq [0, \lfloor \frac{rn}{d} \rfloor] \subseteq [0, rn]$ and $|X'| = |X| \geq n + 1$. Consequently, if we knew the lemma held whenever $d = 1$, then we could apply this case to X' to conclude that $[-rn, rn] \subseteq (r + 1)X' - (r + 1)X'$, implying (by multiplying everything by d) that $d\mathbb{Z} \cap [-rdn, rdn] \subseteq (r + 1)X - (r + 1)X$, as desired. So we see that it suffices to consider the case when $d = 1$, i.e., when $\gcd(X - X) = 1$, which we now assume.

Since $|X| \geq n + 1$, $N \leq rn$, and $n > r \geq 2$, we have

$$s := \left\lfloor \frac{N - 1}{|X| - 2} \right\rfloor \leq \frac{N - 1}{|X| - 2} \leq \frac{N - 1}{n - 1} \leq \frac{rn - 1}{n - 1} < r + 1. \tag{1}$$

Consequently, applying Theorem A to X (using $h = h_1 = r + 1$ and $h_2 = 0$), we find that

$$|(r + 1)X| \geq \frac{s(s + 1)}{2} |X| - s^2 + 1 + (r + 1 - s)N.$$

Note that $\text{diam}((r + 1)X) = (r + 1)N$, $(s + 1)(|X| - 2) \geq N$, $N \geq s(n - 1) + 1$ and $r \geq s$. Thus

$$\begin{aligned} M &:= 2|(r + 1)X| - 2 - \text{diam}((r + 1)X) = 2|(r + 1)X| - 2 - (r + 1)N \\ &\geq s(s + 1)(|X| - 2) + 2s + (r + 1 - 2s)N \\ &\geq sN + 2s + (r + 1 - 2s)N = 2s + (r + 1 - s)N \\ &\geq 2s + (r + 1 - s)(s(n - 1) + 1). \end{aligned}$$

The above bound is quadratic in s with the coefficient of s^2 negative (since $n > 1$). The bound for M is thus minimized at a boundary value for s . As a result, since $1 \leq s \leq r$ in view of (1), we conclude that $M \geq rn + 2 > 0$. Hence we can apply Lemma 3.1 using $A = (r + 1)X$ to conclude that $[-rn, rn] \subseteq [-M, M] \subseteq (r + 1)X - (r + 1)X$, completing the proof. \square

The least common multiple of the first r integers has been well studied. Bounds from Hong and Feng [6] give

$$c_r := \text{lcm}\{i : i = 1, 2, \dots, r\} \geq 2^{r-1},$$

for instance, while the first few values are easily computed to be $c_1 = 1$, $c_2 = 2$, $c_3 = 6$, $c_4 = 12$, $c_5 = 60$, $c_6 = 60$, and $c_7 = 420$.

Theorem 3.3. *Let $k \geq 2$ be an integer and let $c_{k-1} = \text{lcm}\{i : i = 1, 2, \dots, k - 1\}$. Then the equation*

$$(x_1 - y_1) + \dots + (x_k - y_k) = c_{k-1}$$

is $(k - 1)$ -regular.

Proof. Let $r = k - 1 \geq 1$, $c = c_r$ for $r \geq 5$, $c = 3c_r = 3c_2$ when $r = 2$ and let $c = 2c_r$ for $r \leq 4$ with $r \neq 2$. Thus c_r is divisible by every integer from $[1, r]$ and $n := \frac{c}{r} > r$ (in view of the basic lower bound mentioned above for c_r as well as the first few explicit values given above). Let $\chi : [1, c + 1] \rightarrow [1, r]$ be an arbitrary r -coloring. We will show that there is a monochromatic solution to the equation $(x_1 - y_1) + \dots + (x_k - y_k) = c_r$, which will show the equation to be r -regular, as desired.

Observe that $[1, c + 1] = [1, rn + 1]$ with $n = \frac{c}{r} > r$. Thus, by the pigeonhole principle, there is a monochromatic subset $X \subseteq [1, rn + 1]$ with $|X| \geq n + 1 \geq r + 2 \geq 3$ and $\text{diam } X \leq rn$. Let $d = \gcd(X - X)$. Then $X \subseteq [1, rn + 1]$ is contained in an arithmetic progression with difference d . However, since $|X| \geq n + 1$, this is only possible if $d \in [1, r]$. Thus $d \mid c_r$ by construction with $c_r \leq c = rn$, ensuring that $c_r \in d\mathbb{Z} \cap [1, rn]$. Applying Lemma 3.2 to X now yields $c_r \in (r + 1)X - (r + 1)X = kX - kX$. Thus there are $x_1, \dots, x_k, y_1, \dots, y_k \in X$ such that $(x_1 - y_1) + \dots + (x_k - y_k) = c_r = c_{k-1}$, and since all elements in X are monochromatic, this provides a monochromatic solution, completing the proof. \square

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