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Controllability and ergodicity of 3D primitive equations driven by a finite-dimensional force

Pierre-Marie Boulvard* Peng Gao† Vahagn Nersesyan‡ §

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Abstract

We study the problems of controllability and ergodicity of the system of 3D primitive equations modeling large-scale oceanic and atmospheric motions. The system is driven by an additive force acting only on a finite number of Fourier modes in the temperature equation. We first show that the velocity and temperature components of the equations can be simultaneously approximately controlled to arbitrary position in the phase space. The proof is based on Agrachev–Sarychev type geometric control approach.

Next, we study the controllability of the linearization of primitive equations around a non-stationary trajectory of the randomly forced system. Assuming that the probability law of the forcing is decomposable and observable, we prove almost sure approximate controllability by using the same Fourier modes as in the nonlinear setting. Finally, combining the controllability of the linearized system with a criterion from [KNS20a], we establish exponential mixing for the nonlinear primitive equations with a random force.

AMS subject classifications: 35Q35, 35Q86, 35R60, 37A25, 37N10, 60H15, 86A10, 93B05

Keywords: 3D primitive equations, degenerate control/noise, approximate controllability, exponential mixing, geometric control theory.

*Institut de Mathémathiques de Jussieu–Paris Rive Gauche, Université Paris Diderot, UMR 7586, Sorbonne Paris Cité, F-75013, Paris, France; e-mail: Pierre-Marie.Boulvard@imj-prg.fr

†School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, China; e-mail: gaop428@nenu.edu.cn

‡NYU-ECNU Institute of Mathematical Sciences, NYU Shanghai, 3663 Zhongshan Road North, Shanghai, 200062, China, e-mail: Vahagn.Nersesyan@nyu.edu

§Université Paris-Saclay, UVSQ, CNRS, Laboratoire de Mathématiques de Versailles, 78000, Versailles, France

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0 Introduction

The system of *3D primitive equations* (PEs) of meteorology and oceanology is an important model of geophysical fluid dynamics. Today, most numerical weather prediction and climate simulation models are based on them. This system is derived, using hydrostatic approximation, from the 3D Navier–Stokes equations with Coriolis force coupled with the thermodynamic equation (see the book by Zeitlin [Zei18]). The mathematical study of these equations has attracted a lot of attention in the last two decades. Following the framework introduced by Lions, Temam, and Wang [LTW92a, LTW92b], we consider in this paper the

PEs written in the form:

$$\partial_t v + L_1 v + \langle v, \nabla \rangle v + w\, \partial_z v + f v^\perp + \nabla p = h_1, \tag{0.1}$$

$$\partial_z p + \theta = 0, \tag{0.2}$$

$$\operatorname{div} v + \partial_z w = 0, \tag{0.3}$$

$$\partial_t \theta + L_2 \theta + \langle v, \nabla \rangle \theta + w\, \partial_z \theta = h_2 + \eta. \tag{0.4}$$

The unknowns are the 3D velocity field of the fluid (v_1, v_2, w), where[1] $v = (v_1, v_2)$ and w are the horizontal and vertical velocity components, the temperature θ, and the pressure p. The number f is the Coriolis rotation frequency, the functions h_1 and h_2 are given source terms, and η is an external perturbation—a *control* or a *random noise*. The operators

$$L_1 = -\nu_1 \Delta - \mu_1 \partial_{zz},$$
$$L_2 = -\nu_2 \Delta - \mu_2 \partial_{zz}$$

are the viscosity and heat diffusions, where the numbers $\nu_1, \mu_1 > 0$ are the horizontal and vertical viscosities, while $\nu_2, \mu_2 > 0$ are the horizontal and vertical heat diffusivity coefficients. We denote by Δ, ∇, div the 2D (horizontal) Laplacian, gradient, divergence operators:

$$\Delta = \partial_{xx} + \partial_{yy}, \quad \nabla = (\partial_x, \partial_y), \quad \operatorname{div} = \langle \nabla, \cdot \rangle,$$

and $\langle v, \nabla \rangle = v_1 \partial_x + v_2 \partial_y$.

The space variable (x, y, z) is assumed to belong to the torus $\mathbb{T}^3 = \mathbb{R}^3 / 2\pi \mathbb{Z}^3$, i.e., all the above functions are 2π-periodic in x, y, and z. Furthermore, we assume that the functions v, p, h_1 are even and the functions w, θ, h_2, η are odd in z. As a consequence, w, θ, h_2, η vanish at $z = 0$.

The unknown functions in system (0.1)-(0.4) can be divided into two types: the prognostic unknowns v and θ, which are determined through an initial boundary value problem, and diagnostic ones w and p, which can be expressed as functions of v and θ. Indeed, from the conservation of mass equation (0.3) and the boundary condition $w|_{z=0} = 0$ it follows that

$$w(t, x, y, z) = -\int_0^z \operatorname{div} v(t, x, y, \mathfrak{z})\, \mathrm{d}\mathfrak{z}, \tag{0.5}$$

and from the hydrostatic balance (0.2) that

$$p(t, x, y, z) = p_s(t, x, y) - \int_0^z \theta(t, x, y, \mathfrak{z})\, \mathrm{d}\mathfrak{z}. \tag{0.6}$$

Using equalities (0.5) and (0.6), the following equivalent formulation is obtained

[1] We denote $v^\perp = (-v_2, v_1)$.

for the PEs:

$$\partial_t v + L_1 v + \langle v, \nabla \rangle v - \int_0^z \operatorname{div} v(t,x,y,\mathfrak{z})\,\mathrm{d}\mathfrak{z}\,\partial_z v + f v^\perp$$
$$+\nabla p_s(t,x,y) - \int_0^z \nabla\theta(t,x,y,\mathfrak{z})\,\mathrm{d}\mathfrak{z} = h_1, \tag{0.7}$$
$$\partial_t \theta + L_2 \theta + \langle v, \nabla \rangle \theta - \int_0^z \operatorname{div} v(t,x,y,\mathfrak{z})\,\mathrm{d}\mathfrak{z}\,\partial_z \theta = h_2 + \eta. \tag{0.8}$$

The well-posedness of these equations has been studied by many authors. The existence of *weak solutions* is known from the works of Lions, Temam, and Wang [LTW92a, LTW92b], but the uniqueness is still an open problem. In this paper, we deal with *strong solutions* whose global existence and uniqueness is established by Cao and Titi [CT07] in the case of Neumann boundary conditions; see also the paper by Kobelkov [Kob07] for a different proof. In the case of periodic boundary conditions, the global existence of strong solutions is considered by Petcu [Pet06] and in the case of Dirichlet boundary conditions, by Kukavica and Ziane [KZ07]. The existence of a global attractor is obtained by Ju [Ju07] and Chueshov [Chu14]. We refer the reader to the reviews [TZ04, PTZ09] for more details and references.

In the periodic setting, the PEs (0.7), (0.8) are considered in the function spaces H and V recalled in Section 1. To formulate the first main result of this paper, we assume that the couple of source terms (h_1, h_2) is a smooth element of H, and η is a control taking values in the space $\mathcal{H} = \operatorname{span}\{\phi_i : i = 1, \dots, 10\}$, where ϕ_i are the following eigenfunctions of the heat diffusion operator L_2:

$$\cos jx \sin z, \quad \sin jx \sin z, \quad \cos jy \sin z, \quad \sin jy \sin z, \quad \sin jz, \quad j = 1, 2.$$

Theorem A. *Problem (0.7), (0.8) is approximately controllable by $\mathcal{H}$-valued controls. More precisely, for any $\varepsilon > 0$, any time $T > 0$, any initial condition $(v_0, \theta_0) \in V$, and any target $(v_1, \theta_1) \in H$, there is a control $\eta \in L^\infty([0,T], \mathcal{H})$ such that the unique strong solution (v, θ) of problem (0.7), (0.8) satisfies*

$$(v(0), \theta(0)) = (v_0, \theta_0), \tag{0.9}$$
$$\|(v(T), \theta(T)) - (v_1, \theta_1)\|_{L^2(\mathbb{T}^3, \mathbb{R}^3)} < \varepsilon. \tag{0.10}$$

Note that the space $\mathcal{H}$ of admissible values for the control η is *independent* of the physical parameters $h_1, h_2, f, \nu_i, \mu_i$, $i = 1, 2$. A more general version of this result is formulated in Theorem 2.3, where a saturation property is specified that ensures the approximate controllability of the system. We also show that if some controlled Fourier modes are added in the velocity equation (0.7), then approximate controllability holds with respect to the stronger norm of the space $H^1(\mathbb{T}^3, \mathbb{R}^3)$.

Approximate controllability of PDEs by additive finite-dimensional forces has been studied by many authors in the recent years. The first results are obtained by Agrachev and Sarychev [AS05, AS06], who considered the Navier–Stokes (NS) and Euler systems on the 2D torus (see also the review [AS08]). Their approach

has been generalized by Shirikyan [Shi06, Shi07] to the case of the 3D NS system; see also the papers [Shi14, Shi18] by Shirikyan, where the Burgers equation is considered on the real line and on a bounded interval with Dirichlet boundary conditions. Rodrigues and Phan [Rod06, PR19] established approximate controllability of the NS system on 2D and 3D rectangles with Lions boundary conditions. In the periodic setting, Nersisyan [Ner10, Ner11] considered 3D Euler systems for perfect compressible and incompressible fluids, Sarychev [Sar12] studied the 2D cubic Schrödinger equation, and Nersesyan [Ner15] considered the Lagrangian trajectories of the 3D NS system.

The proof of Theorem A is based on a technique of applying large controls on short time intervals. Previously, such ideas have been used mainly in the study of finite-dimensional control systems; e.g., see the works of Jurdjevic and Kupka [JK85, Jur97] and the references therein. Infinite-dimensional extensions of this technique appear in the above-cited papers of Agrachev and Sarychev. More recently, this approach has been used in the paper of Glatt-Holtz, Herzog, and Mattingly [GHHM18], where, in particular, a 1D parabolic PDE is considered with polynomial nonlinearity of odd degree, and in the paper of Nersesyan [Ner20], where the nonlinearity is a smooth function that grows polynomially without any restriction on the degree and on the space dimension.

The main difficulty of the problem considered in this paper comes from the *highly degenerate* nature of the control system. The form of the saturation property and the argument for its verification are more complicated than in the previously studied situations. When the control acts directly only on the temperature equation, we are able to check the saturation with respect to the L^2-norm. The latter is known to be poorly adapted for the stability properties of the 3D PEs and is a source of many difficulties in different parts of the proof.

To formulate our second result, let us assume that η is a *Haar coloured noise* taking values in the same space $\mathcal{H}$ as above. This means that η has the form

$$\eta(t) = \sum_{i=1}^{10} \eta^i(t)\phi_i, \tag{0.11}$$

where $\{\eta^i\}$ are independent copies of a random process $\tilde{\eta}$ defined by

$$\tilde{\eta}(t) = \sum_{k=0}^{\infty} \xi_k \mathfrak{h}_0(t-k) + \sum_{j=1}^{\infty} j^{-q} \sum_{l=0}^{\infty} \xi_{jl} \mathfrak{h}_{jl}(t). \tag{0.12}$$

Here $q > 1$, $\{\xi_k, \xi_{jl}\}$ are independent identically distributed (i.i.d.) scalar random variables with Lipschitz-continuous density ρ, and $\{\mathfrak{h}_0, \mathfrak{h}_{jl}\}$ is the Haar system defined by

$$h_0(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ 0 & \text{for } t < 0 \text{ or } t \ge 1, \end{cases}$$

$$h_{jl}(t) = \begin{cases} 0 & \text{for } t < l2^{-j} \text{ or } t \ge (l+1)2^{-j}, \\ 1 & \text{for } l2^{-j} \le t < \left(l + \frac{1}{2}\right)2^{-j}, \\ -1 & \text{for } \left(l + \frac{1}{2}\right)2^{-j} \le t < (l+1)2^{-j} \end{cases}$$

with $j \geq 1$ and $0 \leq l \leq 2^j - 1$. Note that, for any $k \geq 1$, the functions

$$\big\{h_0(\cdot - k + 1),\, h_{jl},\, j \geq 1,\, (k-1)2^j \leq l \leq k2^j - 1\big\}$$

are supported on the interval $[k-1, k]$ and form an orthonormal basis in the space $L^2([k-1, k])$.

Let $(\tilde{v}, \tilde{\theta})$ be a trajectory of problem (0.7), (0.8) with process η defined by (0.11).

Theorem B. *Under the above conditions, the linearization of problem* (0.7), (0.8) *around the trajectory* $(\tilde{v}, \tilde{\theta})$ *(see system* (1.8)*) is almost surely approximately controllable by* $\mathcal{H}$*-valued controls.*

See Theorem 3.3 for a more precise formulation of the result. It is proved by showing that the kernel of the *random Gramian operator* is almost surely non-trivial. The latter is derived from the *observability property* of the Haar noise. Let us emphasize that on a non-empty, but zero-probability event (depending on the reference trajectory $(\tilde{v}, \tilde{\theta})$), the linearized problem is non-controllable. Indeed, assume that the source terms h_1 and h_2, the noise η, as well as the trajectory $(\tilde{v}, \tilde{\theta})$, are identically zero. Then the space $\mathcal{H}$ is invariant for the linearized problem, so the attainable set from the origin cannot be dense in H.

Controllability properties of nonlinear and linearized equations have applications to the study of randomly perturbed problems. Indeed, it is well known that approximate controllability implies, for example, irreducibility of the associated Markov process when the support of the law of the noise is the whole space $L^\infty([0, T], \mathcal{H})$ (see Section 6.3 in [KS12] for more details). As it is shown in the recent papers by Kuksin, Nersesyan, and Shirikyan [KNS20a, KNS20b], the controllability of the linearized system can be used in the analysis of the ergodicity problem when the system is perturbed by a bounded degenerate noise. In these papers the NS system, complex Ginzburg–Landau equations, and parabolic PDEs with polynomial nonlinearities are studied. See also the papers [KZ20] and [Ner19] for some related situations where the noise is non-degenerate. In our third result, we show that the approach of these papers can be extended to the more degenerate case of PEs. To formulate the result, let $((v_k, \theta_k), \mathbb{P}_{(v,\theta)})$ be the Markov family obtained by restricting the trajectories of system (0.7), (0.8), (0.11) to integer times. Recall that ρ is the density of the random variables $\{\xi_k, \xi_{jl}\}$ in (0.12).

Theorem C. *In addition to the above conditions, assume that* $(h_1, h_2) = 0$*, the support of the density* ρ *is bounded, and* $\rho(0) > 0$*. Then the family* $((v_k, \theta_k), \mathbb{P}_{(v,\theta)})$ *has a unique stationary measure on* V *which is exponentially mixing in the dual-Lipschitz metric.*

See Section 4 for more details. In the case of the 3D primitive equations with spatially regular white noise, existence of stationary measure is established by Glatt-Holtz, Kukavica, Vicol, and Ziane [GHKVZ14]. As far as we know, uniqueness of stationary measure for that situation is still an open problem due to rather weak tail estimates for solutions. The boundedness of the noise allows to

reduce the study of the system to a compact phase space. This naturally eliminates the problems coming from the tail estimates. On the other hand, bounded noises are well-accepted and commonly used in the physics literature (e.g., see [Ono13] and the references therein). In the case of *non-degenerate* bounded kick force, uniqueness and exponential mixing are proved by Chueshov [Chu14]. Let us also recall some previous results considering the problem of ergodicity for PDEs driven by a degenerate noise. Hairer and Mattingly [HM06, HM11] used Malliavin calculus to study the ergodicity for the NS system with a white noise which is degenerate in the Fourier space. Földes, Glatt-Holtz, Richards, and Thomann [FGRT15] considered a similar problem for the Boussinesq system. Using controllability methods, Shirikyan [Shi15, Shi20] studied the NS system with a noise that is localized in the physical space (distributed in a subdomain or on the boundary). For more results and references, we refer the reader to the book [KS12].

This paper is organized as follows. In Section 1, we recall the functional setting for the PEs and formulate perturbative results with respect to the initial condition and control. In Sections 2 and 3, we discuss the problems of controllability of nonlinear and linearized PEs and prove Theorems A and B. In Section 4, we consider the randomly forced PEs and prove Theorem C. Examples of saturating spaces are provided in Section 5. Finally, in Section 6, we establish a perturbative result formulated in Section 1.

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Notation

Throughout this paper, we use the following notation.

$\mathbb{Z}^d,\ d \geq 1$ is the integer lattice in $\mathbb{R}^d$, and $\mathbb{T}^d$ is the torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$.

$L^p(\mathbb{T}^d, \mathbb{R}^n),\ p \geq 1,\ n \geq 1$ and $H^k(\mathbb{T}^d, \mathbb{R}^n),\ k \geq 0$ are the usual Lebesgue and Sobolev spaces of functions $g : \mathbb{T}^d \to \mathbb{R}^n$ endowed with the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_k$, respectively. If $p = 2$, we write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$ and denote by $\langle\cdot,\cdot\rangle$ the corresponding scalar product. If $p = +\infty$, we write $\|\cdot\|_\infty$ instead of $\|\cdot\|_{L^\infty}$.

$C^\infty(\mathbb{T}^d, \mathbb{R}^n)$ is the space of infinitely differentiable functions $g : \mathbb{T}^d \to \mathbb{R}^n$.

Let X be a Banach space endowed with the norm $\|\cdot\|_X$.

$B_X(a, r)$ denotes the closed ball of radius $r > 0$ centred at $a \in X$.

$\mathcal{B}(X)$ is the Borel σ-algebra on X, and $\mathcal{P}(X)$ is the set of Borel probability measures on X.

$L^p(J_T, X)$, $1 \le p < \infty$ is the space of measurable functions $u : J_T \to X$ endowed with the norm

$$\|u\|_{L^p(J_T,X)} = \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t \right)^{1/p}, \quad J_T = [0, T].$$

$C(J_T, X)$ (resp. $L^\infty(J_T, X)$) is the space of continuous (resp. bounded measurable) functions $u : J_T \to X$ endowed with the norm

$$\|u\|_{C(J_T,X)} (\text{resp. } \|u\|_{L^\infty(J_T,X)}) = \sup_{t \in J_T} \|u(t)\|_X.$$

1 Preliminaries on primitive equations

We consider the system of PEs in the spaces H^k, $k \ge 0$ defined by

$$H^k = \text{closure of } \mathcal{V} \text{ in } H^k(\mathbb{T}^3, \mathbb{R}^3)$$

and endowed with the Sobolev norms $\|\cdot\|_k$ (with L^2-norm $\|\cdot\|$ if $k = 0$), where $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ and $\mathcal{V}_1$ and $\mathcal{V}_2$ are the spaces given by

$$\mathcal{V}_1 = \left\{ v \in C^\infty(\mathbb{T}^3, \mathbb{R}^2) : v \text{ is even in } z, \int_{\mathbb{T}} \operatorname{div} v \, \mathrm{d}z = 0, \int_{\mathbb{T}^3} v \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 0 \right\},$$
$$\mathcal{V}_2 = \left\{ \theta \in C^\infty(\mathbb{T}^3, \mathbb{R}) : \theta \text{ is odd in } z, \int_{\mathbb{T}^3} \theta \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 0 \right\}.$$

The condition $\int_{\mathbb{T}} \operatorname{div} v \, \mathrm{d}z = 0$ in the definition of $\mathcal{V}_1$ comes from equality (0.5) at $z = 2\pi$, the fact that w is 2π-periodic, and the boundary value $w|_{z=0} = 0$; see [CT07, PTZ09] for more details. We will mainly consider the spaces[2] $H = H_1 \times H_2 = H^0$, $V = V_1 \times V_2 = H^1$, and $U = U_1 \times U_2 = H^6$. For any $T > 0$, we set

$$\mathcal{X}_T = C(J_T, V) \cap L^2(J_T, H^2)$$

and endow this space with the norm

$$\|u\|_{\mathcal{X}_T} = \|u\|_{C(J_T,V)} + \|u\|_{L^2(J_T,H^2)}.$$

The Leray-type orthogonal projection onto H_1 in $L^2(\mathbb{T}^3, \mathbb{R}^2)$ is denoted by Π. Applying this projection to Eq. (0.7), we eliminate the pressure term and transform problem (0.7), (0.8) into an evolution system which can be written in the following dimensionless form:

$$\dot{u} + Lu + B(u) + Qu = h + \eta, \tag{1.1}$$

[2]The subscripts 1 and 2 are used with H, V, U to denote spaces of velocity fields and temperatures, respectively. The superscript $k \ge 0$ is used with H to indicate the Sobolev regularity.

where the unknown is the couple $u = (v, \theta)$, and the linear terms L and Q and the nonlinear term B are defined by[3]

$$
\begin{aligned}
Lu &= (L_1 v, L_2 \theta), \quad B(u) = (B_1(v), B_2(u)), \quad Qu = (Q_1 u, 0), \\
B_1(v) &= \Pi \left(\langle v, \nabla \rangle v - \int_0^z \operatorname{div} v(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \, \partial_z v \right), \\
B_2(u) &= \langle v, \nabla \rangle \theta - \int_0^z \operatorname{div} v(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \, \partial_z \theta, \qquad (1.2) \\
Q_1 u &= \Pi \left(f v^\perp - \int_0^z \nabla \theta(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \right). \qquad (1.3)
\end{aligned}
$$

Eq. (1.1) is supplemented with the initial condition

$$
u(0) = u_0. \qquad (1.4)
$$

Proposition 1.1. *For any $T > 0$, $u_0 \in V$, $\eta \in L^\infty(J_T, H)$, and $h \in H$, there is a unique solution u of problem (1.1), (1.4) belonging to $\mathcal{X}_T$. Let S be the mapping taking the couple[4] (u_0, η) to the solution u. For any $r > 0$, there is a constant $C = C(r, T) > 0$ such that*

$$
\|S(u_{0,1}, \eta_1) - S(u_{0,2}, \eta_2)\|_{\mathcal{X}_T} \leq C \left(\|u_{0,1} - u_{0,2}\|_1 + \|\eta_1 - \eta_2\|_{L^\infty(J_T, H)} \right),
$$

provided that $u_{0,i} \in V$, $\eta_i \in L^\infty(J_T, H)$, and $h \in H$ satisfy

$$
\|u_{0,i}\|_1 + \|\eta_i\|_{L^\infty(J_T, H)} + \|h\| \leq r, \quad i = 1, 2.
$$

Existence and uniqueness of solutions is established in [CT07, Pet06, KZ07], and the local Lipschitz property in [Ju07].

Inspired by ideas from [AS05, AS06, Shi06], together with Eq. (1.1), we will consider a more general equation with additional control ζ:

$$
\dot{u} + L(u + \zeta) + B(u + \zeta) + Q(u + \zeta) = h + \eta. \qquad (1.5)
$$

The well-posedness of problem (1.5), (1.4) with $\zeta \in V$ follows from that of problem (1.1), (1.4) using a change of unknown $u' = u + \zeta$. We denote by $S(u_0, \zeta, \eta)$ the corresponding solution and by $S_t(u_0, \zeta, \eta)$ its restriction at time $t \in J_T$. To avoid any ambiguity, in this and next sections, we write $S(u_0, 0, \eta)$ instead of $S(u_0, \eta)$ defined in Proposition 1.1. Let $\pi_1 : H \to H_1$ and $\pi_2 : H \to H_2$ be the projections $(v, \theta) \mapsto v$ and $(v, \theta) \mapsto \theta$. The following result is proved in Section 6.

[3] With a slight abuse of notation, instead of elements like $B_2(u)$, $Q(u)$, $Q_1(u)$ with $u = (v, \theta)$, we will often write $B_2(v, \theta)$, $Q(v, \theta)$, $Q_1(v, \theta)$. Note that the term B_1 does not depend on θ, so we write $B_1(v)$.

[4] In what follows, the source term h will be fixed, so we shall not indicate the dependence of S on it.

Proposition 1.2. *For any $u_0 \in H^4$ and $\zeta, \eta, \xi \in H^5$ with $\pi_1 \xi = 0$ and $\pi_2 \zeta = 0$, the following limits hold in V as $\delta \to 0^+$:*[5]

$$S_\delta(u_0, \delta^{-\frac{1}{2}}\zeta, \delta^{-1}\eta) \to u_0 + \eta - B(\zeta), \tag{1.6}$$

$$S_\delta(u_0, \delta^{-1}\xi, 0) \to u_0 - L\xi - (0, \Psi(u_0, \xi)) - Q\xi, \tag{1.7}$$

where $\Psi(u_0, \xi) = B_2(\pi_1 u_0 - \frac{1}{2} Q_1 \xi, \pi_2 \xi)$.

Now, let $\tilde{u} = (\tilde{v}, \tilde{\theta}) = S(u_0, 0, \eta)$ be a trajectory of Eq. (1.1) corresponding to initial condition $u_0 \in V$ and control $\eta \in L^\infty(J_T, H)$. The linearization of Eq. (1.1) around $\tilde{u}$ is given by

$$\dot{w} + Lw + b(\tilde{u}, w) + Qw = g, \tag{1.8}$$

where $w = (v, \theta)$ and the term $b(\tilde{u}, w) = (b_1(\tilde{v}, v), b_2(\tilde{u}, w))$ is defined by

$$\begin{aligned} b_1(\tilde{v}, v) = \Pi\Bigg(&\langle \tilde{v}, \nabla \rangle v + \langle v, \nabla \rangle \tilde{v} \\ &- \int_0^z \operatorname{div} \tilde{v}(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \, \partial_z v - \int_0^z \operatorname{div} v(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \, \partial_z \tilde{v} \Bigg), \\ b_2(\tilde{u}, w) = &\langle \tilde{v}, \nabla \rangle \theta + \langle v, \nabla \rangle \tilde{\theta} \\ &- \int_0^z \operatorname{div} \tilde{v}(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \, \partial_z \theta - \int_0^z \operatorname{div} v(t, x, y, \mathfrak{z}) \, \mathrm{d}\mathfrak{z} \, \partial_z \tilde{\theta}. \end{aligned}$$

Using standard techniques (e.g., see Chapter III in [Tem79]), one shows that, for any $w_0 \in V$ and $g \in L^2(J_T, H)$, the linear equation (1.8) has a unique solution $w \in \mathcal{X}_T$ issued from w_0.

2 Controllability of the nonlinear system

2.1 Saturation property and the result

In this section, we formulate a controllability result for Eq. (1.1) that is a generalization of Theorem A given in the Introduction. We start by introducing some definitions and notation.

Definition 2.1. Let $\mathcal{H}$ be a finite-dimensional subspace of U. Eq. (1.1) is said to be *approximately controllable in H* by $\mathcal{H}$-valued controls if for any $\varepsilon > 0$, any time $T > 0$, any initial point $u_0 \in V$, and any target $u_1 \in H$, there is a control $\eta \in L^\infty(J_T, \mathcal{H})$ such that

$$\|S_T(u_0, 0, \eta) - u_1\| < \varepsilon. \tag{2.1}$$

In a similar way, Eq. (1.1) is said to be *approximately controllable in V* if inequality (2.1) holds with respect to the H^1-norm $\|\cdot\|_1$ and the target u_1 is arbitrary in V.

[5] In (1.7), we denote $(0, \Psi)$ the element $\hat{\Psi} \in H$ such that $\pi_1 \hat{\Psi} = 0$ and $\pi_2 \hat{\Psi} = \Psi$. Likewise, in what follows, we will often have terms of the form $(v, 0)$ that denote an element $\hat{v} \in H$ with $\pi_1 \hat{v} = v$ and $\pi_2 \hat{v} = 0$.

Let us assume that $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, where $\mathcal{H}_i \subset U_i$, $i = 1, 2$ are finite-dimensional subspaces. We denote by $\mathcal{F}_1(\mathcal{H})$ the largest subspace of U_1 whose elements can be approximated, within any accuracy with respect to the H^1-norm, by elements of the form (cf. [AS05, AS06, Shi06])

$$Q_1(0, \zeta_0) + \zeta_1 - \sum_{i=2}^{m} B_1(\zeta_i), \tag{2.2}$$

where $m \geq 2$, $\zeta_0 \in \mathcal{H}_2$, and $\zeta_1, \ldots, \zeta_m \in \mathcal{H}_1$. As $\mathcal{H}$ is finite-dimensional, Q_1 is linear, and B_1 is bilinear, it is easy to see that $\mathcal{F}_1(\mathcal{H})$ is well defined and finite-dimensional.

Let $\mathcal{F}_2(\mathcal{H})$ be the subspace spanned by all the vectors of the form

$$\xi_0 + B_2(Q_1(0, \xi_1), \xi_2), \tag{2.3}$$

where $\xi_0, \xi_1, \xi_2 \in \mathcal{H}_2$ are such that $B_2(Q_1(0, \xi_1), \xi_2) \in U_2$. We denote by $\mathcal{F}(\mathcal{H})$ the product $\mathcal{F}_1(\mathcal{H}) \times \mathcal{F}_2(\mathcal{H})$, and define a non-decreasing sequence $\{\mathcal{H}(j)\}$ of finite-dimensional subspaces of U by

$$\mathcal{H}(0) = \mathcal{H}, \quad \mathcal{H}(j) = \mathcal{F}(\mathcal{H}(j-1)), \quad j \geq 1. \tag{2.4}$$

Let us set

$$\mathcal{H}(\infty) = \bigcup_{j=1}^{\infty} \mathcal{H}(j). \tag{2.5}$$

Definition 2.2. A subspace $\mathcal{H} \subset U$ is *H-saturating* (resp. *V-saturating*) if the following two conditions hold:

(a) $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, where $\mathcal{H}_i \subset U_i$, $i = 1, 2$ are finite-dimensional subspaces;

(b) the vector space $\mathcal{H}(\infty)$ is dense in[6] $H_1 \times V_2$ (resp. in V).

We are now ready to formulate the main result of this section.

Theorem 2.3. *If $\mathcal{H} \subset U$ is an H-saturating subspace, then Eq. (1.1) is approximately controllable in H by $\mathcal{H}$-valued controls. Moreover, if $\mathcal{H}$ is V-saturating, then the equation is approximately controllable in V.*

Examples of H and V-saturating subspaces are given in Section 5. When the control acts directly only on the temperature component (i.e., $\mathcal{H}_1 = \{0\}$ in (a) in Definition 2.2), we provide an H-saturating subspace—the ten-dimensional space considered in the Introduction. In particular, Theorem A is obtained as an immediate consequence of Theorems 2.3 and 5.1. We do not have an example of V-saturating subspace acting through the temperature component only. The example we provide is *less degenerate* and combines few modes from velocity and temperature components (see Theorem 5.5). Furthermore, in the case of the 2D and 3D NS systems, there are necessary and sufficient conditions on the Fourier modes to use in order to have approximate controllability (see [AS05, AS06, Ner15]). It would be interesting to obtain similar precise description in the case of 3D PEs.

[6] The reason why we take $H_1 \times V_2$ and not the space H is explained in Remark 2.8.

2.2 Proof of Theorem 2.3

The proof of Theorem 2.3 is divided into three steps. We first show that the temperature and velocity components can be separately controlled in small time. Then we derive simultaneous controllability of both components in arbitrary fixed time.

2.2.1 Controllability of θ-component

Let us set $\mathcal{H}_i(j) = \pi_i \mathcal{H}(j)$ for $j \geq 0$ and $i = 1, 2$. In this subsection, we prove the following proposition.

Proposition 2.4. *Let $\mathcal{H}_i \subset U_i$, $i = 1, 2$ be arbitrary finite-dimensional subspaces and $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. For any $u_0 \in V$ and $\eta \in \mathcal{H}_2(\infty)$, there is a family of controls $\{\zeta_\tau\}_{\tau>0} \subset L^\infty(J_1, \mathcal{H})$ such that*

$$S_\tau(u_0, 0, \zeta_\tau) \to u_0 + \hat{\eta} \quad \text{in } V \text{ as } \tau \to 0^+, \tag{2.6}$$

where $\hat{\eta} = (0, \eta) \in H$.

Proof. We first prove the result in the case $u_0 \in U$. It suffices to show that for any $N \geq 0$ and $\eta \in \mathcal{H}_2(N)$, there are controls $\{\zeta_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that limit (2.6) holds. We argue by induction on $N \geq 0$.

Step 1. Base case: $N = 0$. Let us check that limit (2.6) holds in the case $N = 0$, i.e., for any $\eta \in \mathcal{H}_2$. Indeed, by limit (1.6) with $\zeta = 0$ and $\eta = \hat{\eta}$, we have

$$S_\delta(u_0, 0, \delta^{-1}\hat{\eta}) \to u_0 + \hat{\eta} \quad \text{in } V \text{ as } \delta \to 0^+.$$

Taking $\delta = \tau$, we obtain the required limit with controls $\zeta_\tau = \tau^{-1}\hat{\eta}$.

Step 2. Inductive step. We assume that the limit is proved for $N - 1$, and take any $\eta \in \mathcal{H}_2(N)$ of the form

$$\eta = \xi_0 + B_2(Q_1(0, \xi_1), \xi_2) \tag{2.7}$$

with some $\xi_0, \xi_1, \xi_2 \in \mathcal{H}_2(N-1)$. Let us set $\hat{\xi}_i = (0, \xi_i) \in H$, $i = 1, 2, 3$. Using limit (1.7) with $\xi = \hat{\xi}_1$, we get

$$S_\delta(u_0, \delta^{-1}\hat{\xi}_1, 0) \to u_0 - L\hat{\xi}_1 - (0, \Psi(u_0, \hat{\xi}_1)) - Q\hat{\xi}_1. \tag{2.8}$$

By the uniqueness of the solution of the Cauchy problem, the following equality holds for any $t \geq 0$:

$$S_t(u_0 + \delta^{-1}\hat{\xi}_1, 0, 0) = S_t(u_0, \delta^{-1}\hat{\xi}_1, 0) + \delta^{-1}\hat{\xi}_1. \tag{2.9}$$

Taking here $t = \delta$ and using (2.8), we obtain the limit

$$\|S_\delta(u_0 + \delta^{-\frac{1}{2}}\hat{\xi}_1, 0, 0) - u_0 + L\hat{\xi}_1 + (0, \Psi(u_0, \hat{\xi}_1)) + Q\hat{\xi}_1 - \delta^{-1}\hat{\xi}_1\|_1 \to 0 \tag{2.10}$$

as $\delta \to 0^+$. The fact that $\xi_1 \in \mathcal{H}_2(N-1)$ and the induction hypothesis imply that, for any $\delta > 0$, there is a family of controls $\{\zeta^1_{\tau,\delta}\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$S_\tau(u_0, 0, \zeta^1_{\tau,\delta}) \to u_0 + \delta^{-\frac{1}{2}}\hat{\xi}_1 \quad \text{in } V \text{ as } \tau \to 0^+.$$

From Proposition 1.1 it follows

$$S_\delta(S_\tau(u_0, 0, \zeta^1_{\tau,\delta}), 0, 0) \to S_\delta(u_0 + \delta^{-\frac{1}{2}}\hat{\xi}_1, 0, 0) \quad \text{in } V \text{ as } \tau \to 0^+.$$

Combining this with (2.10), we find a family $\{\zeta^2_\delta\} \subset L^\infty(J_1, \mathcal{H})$ verifying

$$\|S_\delta(u_0, 0, \zeta^2_\delta) - u_0 + L\hat{\xi}_1 + (0, \Psi(u_0, \hat{\xi}_1)) + Q\hat{\xi}_1 - \delta^{-1}\hat{\xi}_1\|_1 \to 0$$

as $\delta \to 0^+$. Using one more time the assumption $\xi_1 \in \mathcal{H}_2(N-1)$, the induction hypothesis, and Proposition 1.1, we find $\{\zeta^3_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$S_\tau(u_0, 0, \zeta^3_\tau) \to u_0 - L\hat{\xi}_1 - (0, \Psi(u_0, \hat{\xi}_1)) - Q\hat{\xi}_1 \quad \text{in } V \text{ as } \tau \to 0^+. \tag{2.11}$$

Now we use the following lemma.

Lemma 2.5. *Let us denote*

$$F_{\xi_1}(u_0) = u_0 - L\hat{\xi}_1 - (0, \Psi(u_0, \hat{\xi}_1)) - Q\hat{\xi}_1.$$

Then

$$F_{-\xi_2}(F_{-\xi_1}(F_{\xi_2}(F_{\xi_1}(u_0)))) = u_0 + (0, B_2(Q_1(0, \xi_1 - \xi_2), \xi_1 + \xi_2)). \tag{2.12}$$

Using (2.12) and iterating four times the argument of the construction of the family $\{\zeta^3_\tau\}$, we find a family $\{\zeta^4_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$S_\tau(u_0, 0, \zeta^4_\tau) \to u_0 + (0, B_2(Q_1(0, \xi_1 - \xi_2), \xi_1 + \xi_2)) \quad \text{in } V \text{ as } \tau \to 0^+.$$

This and the induction hypothesis imply that, for any $\xi_0, \xi_1, \xi_2 \in \mathcal{H}_2(N-1)$, there are controls $\{\zeta^5_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$S_\tau(u_0, 0, \zeta^5_\tau) \to u_0 + (0, \xi_0 + B_2(Q_1(0, \xi_1), \xi_2)) \quad \text{in } V \text{ as } \tau \to 0^+.$$

Iterating this argument, we show that the system can be controlled in small time to any target of the form $u_0 + \hat{\eta}$ (in the sense of limit (2.6)), where $\hat{\eta}$ is now a linear combination of vectors of the form (2.7). This completes the proof of the proposition in the case of a regular initial condition u_0. In the case $u_0 \in V$, it suffices to take control equal to zero on a small time interval, to use the regularizing property of the PEs (e.g., see Theorem 3.1 in [Pet06]), and apply the already proved result for regular initial condition. □

Proof of Lemma 2.5. For any smooth u in H, we have

$$\begin{aligned} F_{-\xi_1}(F_{\xi_2}(u)) &= F_{\xi_2}(u) + L\hat{\xi}_1 + (0, \Psi(F_{\xi_2}(u), \hat{\xi}_1)) + Q\hat{\xi}_1 \\ &= u + L(\hat{\xi}_1 - \hat{\xi}_2) - (0, \Psi(u, \hat{\xi}_2)) + (0, \Psi(F_{\xi_2}(u), \hat{\xi}_1)) + Q(\hat{\xi}_1 - \hat{\xi}_2). \end{aligned} \tag{2.13}$$

Replacing in this equality u by $F_{\xi_1}(u_0)$, we obtain

$$\begin{aligned} F_{-\xi_1}(F_{\xi_2}(F_{\xi_1}(u_0))) &= u_0 - L\hat{\xi}_2 - Q\hat{\xi}_2 \\ &\quad - (0, \Psi(u_0, \hat{\xi}_1)) - (0, \Psi(F_{\xi_1}(u_0), \hat{\xi}_2)) + (0, \Psi(F_{\xi_2}(F_{\xi_1}(u_0)), \hat{\xi}_1)). \end{aligned} \tag{2.14}$$

Note that

$$\Psi(u,\xi) \text{ does not depend on } \pi_2 u \text{ and } \pi_1 L\hat{\xi}_1 = \pi_1 L\hat{\xi}_2 = 0, \tag{2.15}$$

so using (2.13), we get

$$\begin{aligned}\Psi(F_{\xi_1}(u_0),\hat{\xi}_2) &= \Psi(u_0 - Q\hat{\xi}_1,\hat{\xi}_2),\\ \Psi(F_{\xi_2}(F_{\xi_1}(u_0)),\hat{\xi}_1) &= \Psi(u_0 + Q(\hat{\xi}_1-\hat{\xi}_2),\hat{\xi}_1).\end{aligned}$$

Thus (2.14) can be rewritten as

$$\begin{aligned}F_{-\xi_1}(F_{\xi_2}(F_{\xi_1}(u_0))) &= u_0 - L\hat{\xi}_2 - Q\hat{\xi}_2\\ &\quad + (0, \Psi(u_0 + Q(\hat{\xi}_1-\hat{\xi}_2),\hat{\xi}_1) - \Psi(u_0 - Q\hat{\xi}_1,\hat{\xi}_2) - \Psi(u_0,\hat{\xi}_1)),\end{aligned}$$

hence

$$\begin{aligned}F_{-\xi_2}(F_{-\xi_1}(F_{\xi_2}(F_{\xi_1}(u_0)))) &= u_0 + (0, \Psi(F_{-\xi_1}(F_{\xi_2}(F_{\xi_1}(u_0))),\hat{\xi}_2))\\ &\quad + (0, \Psi(u_0 + Q(\hat{\xi}_1-\hat{\xi}_2),\hat{\xi}_1) - \Psi(u_0 - Q\hat{\xi}_1,\hat{\xi}_2) - \Psi(u_0,\hat{\xi}_1))\\ &= u_0 + (0, \Psi(u_0 - Q\hat{\xi}_2,\hat{\xi}_2))\\ &\quad + (0, \Psi(u_0 + Q(\hat{\xi}_1-\hat{\xi}_2),\hat{\xi}_1) - \Psi(u_0 - Q\hat{\xi}_1,\hat{\xi}_2) - \Psi(u_0,\hat{\xi}_1))\\ &= u_0 + (0, \mathfrak{b}_2(\xi_1-\xi_2,\xi_1+\xi_2)),\end{aligned}$$

where we used again (2.15) and the equality

$$\Psi(F_{-\xi_1}(F_{\xi_2}(F_{\xi_1}(u_0))),\hat{\xi}_2) = \Psi(u_0 - Q\hat{\xi}_2,\hat{\xi}_2).$$

□

2.2.2 Controllability of v-component

Here we prove the following version of Proposition 2.4 for the v-component.

Proposition 2.6. *Let $\mathcal{H}_i \subset U_i$, $i = 1,2$ be finite-dimensional subspaces, let $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, and assume that $\mathcal{H}_2(\infty)$ is dense in V_2. For any $u_0 \in V$ and $\eta \in \mathcal{H}_1(\infty)$, there is a family of controls $\{\xi_\tau\}_{\tau>0} \subset L^\infty(J_1,\mathcal{H})$ such that*

$$S_\tau(u_0,0,\xi_\tau) \to u_0 + \hat{\eta} \quad \text{in } V \text{ as } \tau \to 0^+, \tag{2.16}$$

where $\hat{\eta} = (\eta, 0) \in H$.

Proof. The argument is close to the one used in Proposition 2.4. Again, without loss of generality, we can assume that $u_0 \in U$. We prove limit (2.16) for any $\eta \in \mathcal{H}_1(N)$, arguing by induction on $N \geq 0$. The base case $N = 0$ follows from limit (1.6) with $\zeta = 0$ and $\eta = \hat{\eta}$:

$$S_\delta(u_0,0,\delta^{-1}\hat{\eta}) \to u_0 + \hat{\eta} \quad \text{in } V \text{ as } \delta \to 0^+.$$

Taking $\delta = \tau$, we obtain the required limit with $\xi_\tau = \tau^{-1}\hat{\eta}$.

Assume that the limit is proved in the case $N-1$, and let $\eta \in \mathcal{H}_1(N)$. By approximation, we can suppose that η is of the form

$$\eta = Q_1\hat{\zeta}_0 + \zeta_1 - \sum_{i=2}^{m} B_1(\zeta_i)$$

for some $m \geq 2$, $\hat{\zeta}_0 = (0, \zeta_0)$, $\zeta_0 \in \mathcal{H}_2(N-1)$, and $\zeta_1, \ldots, \zeta_m \in \mathcal{H}_1(N-1)$.

Step 1. Direction $\zeta_1 - \sum_{i=2}^{m} B_1(\zeta_i)$. Limit (1.6) with $\zeta = \hat{\zeta}_2 = (\zeta_2, 0)$ and $\eta = 0$ implies that

$$S_\delta(u_0, \delta^{-\frac{1}{2}}\hat{\zeta}_2, 0) \to u_0 - B(\hat{\zeta}_2) \quad \text{in } V \text{ as } \delta \to 0^+. \tag{2.17}$$

The equality

$$S_\delta(u_0 + \delta^{-\frac{1}{2}}\hat{\zeta}_2, 0, 0) = S_\delta(u_0, \delta^{-\frac{1}{2}}\hat{\zeta}_2, 0) + \delta^{-\frac{1}{2}}\hat{\zeta}_2$$

and limit (2.17) show that

$$\|S_\delta(u_0 + \delta^{-\frac{1}{2}}\hat{\zeta}_2, 0, 0) - u_0 + B(\hat{\zeta}_2) - \delta^{-\frac{1}{2}}\hat{\zeta}_2\|_1 \to 0 \quad \text{as } \delta \to 0^+.$$

The fact that $\zeta_2 \in \mathcal{H}_1(N-1)$ and the induction hypothesis imply that, for any $\delta > 0$, there is a family of controls $\{\xi^1_{\tau,\delta}\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$S_\tau(u_0, 0, \xi^1_{\tau,\delta}) \to u_0 + \delta^{-\frac{1}{2}}\hat{\zeta}_2 \quad \text{in } V \text{ as } \tau \to 0^+.$$

Then Proposition 1.1 implies that

$$S_\delta(S_\tau(u_0, 0, \xi^1_{\tau,\delta}), 0, 0) \to S_\delta(u_0 + \delta^{-\frac{1}{2}}\hat{\zeta}_2, 0, 0) \quad \text{in } V \text{ as } \tau \to 0^+.$$

Combining this with (2.17), we find a family $\{\xi^2_\delta\} \subset L^\infty(J_1, \mathcal{H})$ verifying

$$\|S_\delta(u_0, 0, \xi^2_\delta) - u_0 + B(\hat{\zeta}_2) - \delta^{-\frac{1}{2}}\hat{\zeta}_2\|_1 \to 0$$

as $\delta \to 0^+$. Using the assumption $\zeta_1, \zeta_2 \in \mathcal{H}_1(N-1)$, the induction hypothesis, and Proposition 1.1, we find a family of controls $\{\xi^3_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$\|S_\tau(u_0, 0, \xi^3_\tau) - u_0 - (\zeta_1 - B_1(\zeta_2), 0)\|_1 \to 0 \quad \text{as } \tau \to 0^+.$$

Iterating this argument with $\zeta_3, \ldots, \zeta_m$, we construct a family $\{\xi^{m+1}_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$\|S_\tau(u_0, 0, \xi^{m+1}_\tau) - u_0 - (\zeta_1 - \sum_{i=2}^{m} B_1(\zeta_i), 0)\|_1 \to 0 \quad \text{as } \tau \to 0^+. \tag{2.18}$$

Step 2. Direction $Q_1\hat{\zeta}_0$. Let $\hat{u}_0 \in H^4$. By limit (1.7) with $\xi = \hat{\zeta}_0$ and $\eta = 0$, we have

$$\begin{aligned} S_\delta(\hat{u}_0, \delta^{-1}\hat{\zeta}_0, 0) &\to \hat{u}_0 - L\hat{\zeta}_0 - (0, \Psi(\hat{u}_0, \hat{\zeta}_0)) - Q\hat{\zeta}_0 \\ &= \hat{u}_0 - (Q_1\hat{\zeta}_0, L_2\zeta_0 + \Psi(\hat{u}_0, \hat{\zeta}_0)). \end{aligned}$$

The equality

$$S_\delta(\hat{u}_0 + \delta^{-1}\hat{\zeta}_0, 0, 0) = S_\delta(\hat{u}_0, \delta^{-1}\hat{\zeta}_0, 0) + \delta^{-1}\hat{\zeta}_0$$

implies that

$$\|S_\delta(\hat{u}_0 + \delta^{-1}\hat{\zeta}_0, 0, 0) - \hat{u}_0 + (Q_1\hat{\zeta}_0, L_2\zeta_0 + \Psi(\hat{u}_0, \hat{\zeta}_0)) - \delta^{-1}\hat{\zeta}_0\|_1 \to 0$$

as $\delta \to 0^+$. Combining this with the assumption that $\mathcal{H}_2(\infty)$ is dense in V_2 and Propositions 1.1 and 2.4, we construct a family of controls $\{\xi_\tau^{m+2}\} \subset L^\infty(J_1, \mathcal{H})$ such that

$$S_\tau(\hat{u}_0, 0, \xi_\tau^{m+2}) \to \hat{u}_0 - (Q_1\hat{\zeta}_0, 0) \quad \text{in } V \text{ as } \tau \to 0^+.$$

Taking

$$\hat{u}_0 = u_0 + (\zeta_1 - \sum_{i=2}^{m} B_1(\zeta_i), 0)$$

and using (2.18), we find a family of controls $\{\xi_\tau\} \subset L^\infty(J_1, \mathcal{H})$ such that limit (2.16) holds. □

2.2.3 Completion of the proof

Assume that $\mathcal{H} \subset U$ is an H-saturating (resp. V-saturating) subspace, and let $\varepsilon > 0$, $T > 0$, $u_0 \in V$, and $u_1 \in H$ (resp. $u_1 \in V$) be arbitrary. Then there is $\eta = (\eta_1, \eta_2) \in \mathcal{H}(\infty)$ such that

$$\|S_T(u_0, 0, 0) + \eta - u_1\| < \frac{\varepsilon}{2} \quad \left(\text{resp. } \|S_T(u_0, 0, 0) + \eta - u_1\|_1 < \frac{\varepsilon}{2}\right). \tag{2.19}$$

Let us denote $\hat{u}_0 = S_T(u_0, 0, 0) + \eta$ and take $t_0 > 0$ and $r > 0$ so small that

$$\|S_t(u, 0, 0) - \hat{u}_0\|_1 < \frac{\varepsilon}{2} \quad \text{for } t \in [0, t_0] \text{ and } u \in B_V(\hat{u}_0, r). \tag{2.20}$$

This is possible by Proposition 1.1. Choosing, if necessary, t_0 smaller, we will also have

$$\|S_{T-t}(u_0, 0, 0) - S_T(u_0, 0, 0)\|_1 < \frac{r}{2} \quad \text{for } t \in [0, t_0]. \tag{2.21}$$

Now applying Propositions 2.4 and 2.6 with initial condition $S_{T-t_0}(u_0, 0, 0)$, we find a time $\tau \in (0, t_0)$ and a control $\xi \in L^\infty([0, \tau], \mathcal{H})$ such that

$$\|S_\tau(S_{T-t_0}(u_0, 0, 0), 0, \xi) - S_{T-t_0}(u_0, 0, 0) - (\eta_1, \eta_2)\|_1 < \frac{r}{2}.$$

In view of (2.21), this implies that $S_\tau(S_{T-t_0}(u_0, 0, 0), 0, \xi) \in B_V(\hat{u}_0, r)$. Finally, using (2.19) and (2.20), we conclude that

$$\|S_T(u_0, 0, \zeta) - u_1\| < \varepsilon \quad (\text{resp. } \|S_T(u_0, 0, \zeta) - u_1\|_1 < \varepsilon),$$

where $\zeta(t) = \mathbb{I}_{[T-t_0, T-t_0+\tau]}\xi(t - T + t_0)$, $t \in J_T$. This completes the proof of Theorem 2.3.

Remark 2.7. Note that the above proof gives approximate controllability to any target u_1 in $H_1 \times V_2$ with respect to the norm of that space.

Remark 2.8. The assumption that $\mathcal{H}_2(\infty)$ is dense in V_2 (see (b) in Definition 2.2) plays an important role in the above proof. We use it in Step 2 of the proof of Proposition 2.6. If $\mathcal{H}_2(\infty)$ was dense only in H, we would need a version of Proposition 1.1 with respect to the L^2-norm. The latter is an open problem.

3 Controllability of linearized system

3.1 Saturation for linearized system and the result

Before formulating the main result of this section, let us define a saturation property for linearized system (1.8), which is different from the one used in the nonlinear case (cf. Definition 2.2), and recall the concept of *observable measures* introduced in [KNS20a].

We assume that $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, where $\mathcal{H}_1 = \{0\} \subset U_1$ and $\mathcal{H}_2 \subset U_2$ is a finite-dimensional subspace. Let us define vector spaces $\mathcal{G}_1(\infty) \subset U_1$ and $\mathcal{G}_2(\infty) \subset U_2$ as follows:

- $\mathcal{G}_2(\infty) = \cup_{j=0}^{\infty} \mathcal{G}_2(j)$, where $\mathcal{G}_2(0) = \mathcal{H}_2$ and $\mathcal{G}_2(j)$, $j \geq 1$ is the space spanned by all the vectors of the form
$$\xi_0 + \mathfrak{b}_2(\xi_1, \xi_2),$$
where $\xi_0, \xi_1 \in \mathcal{G}_2(j-1)$ and $\xi_2 \in \mathcal{H}_2$ are such that
$$\mathfrak{b}_2(\xi_1, \xi_2) = B_2(Q_1(0, \xi_1), \xi_2) - B_2(Q_1(0, \xi_2), \xi_1) \in U_2;$$
- $\mathcal{G}_1(\infty)$ is the space spanned by all the vectors of the form
$$Q_1(0, \zeta_0) + b_1(Q_1(0, \zeta_1), Q_1(0, \zeta_2)),$$
where $\zeta_0, \zeta_1 \in \mathcal{G}_2(\infty)$ and $\zeta_2 \in \mathcal{H}_2$ are such that
$$Q_1(0, \zeta_0) + b_1(Q_1(0, \zeta_1), Q_1(0, \zeta_2)) \in U_1.$$

We set $\mathcal{G}(\infty) = \mathcal{G}_1(\infty) \times \mathcal{G}_2(\infty)$.

Definition 3.1. A finite-dimensional space $\mathcal{H}$ as above is said to be *saturating* for linearized system (1.8) if $\mathcal{G}(\infty)$ is dense in H.

We will see in Section 5 that the ten-dimensional subspace defined in the Introduction is saturating in the sense of Definition 3.1. Let us take any $T > 0$, denote $\mathcal{E} = L^\infty(J_T, \mathcal{H})$, and let $\{\varphi_i\}_{i=1}^d$ be a basis in $\mathcal{H}$.

Definition 3.2. A function $\zeta \in \mathcal{E}$ is said to be *observable* if for any continuously differentiable functions $a_i : J_T \to \mathbb{R}$, $i \in [\![1, d]\!]$ and any continuous function $a_0 : J_T \to \mathbb{R}$ the equality

$$\sum_{i=1}^{d} a_i(t)\langle \zeta(t), \varphi_i \rangle - a_0(t) = 0 \quad \text{for almost every } t \in J_T$$

implies that $a_i(t) = 0$ for any $t \in J_T$ and $i \in [\![0, d]\!]$. A measure $\ell \in \mathcal{P}(\mathcal{E})$ is said to be observable if ℓ-almost every trajectory $\eta \in \mathcal{E}$ is observable.

It is easy to see that the observability does not depend on the choice of the basis $\{\varphi_i\}$ in $\mathcal{H}$. See Section 5 in [KNS20a] for examples of observable measures. In particular, it is shown there that the law of the Haar noise defined by (0.11), (0.12) is observable.

Let $u \in U$, and let $D_\eta S_T(u, \eta)$ be the derivative of $S_T(u, \eta)$ with respect to $\eta \in \mathcal{E}$. Then the linear mapping

$$D_\eta S_T(u, \eta) : \mathcal{E} \to U, \quad g \mapsto w(T)$$

is the resolving operator for Eq. (1.8), where $\tilde{u}(t) = S_T(u, \eta)$, $(u, \eta) \in U \times \mathcal{E}$, and $t \in J_T$. Let $\mathcal{K}^u$ be a Borel set in $\mathcal{E}$ defined by

$$\mathcal{K}^u = \{\eta \in \mathcal{E} : \text{the image of } D_\eta S_T(u, \eta) \text{ is dense in } U\}. \tag{3.1}$$

The following theorem can be seen as a non-Gaussian extension of the non-degeneracy property of the Malliavin matrix. The latter is known to be an important ingredient in the study of ergodicity and existence of positive densities for stochastic equations driven by a white-in-time noise (see [MP06, HM06, HM11, FGRT15, HM15]).

Theorem 3.3. *Let $\ell \in \mathcal{P}(\mathcal{E})$, and let $\mathcal{H}$ be a saturating subspace in the sense of Definition 3.1. If there is $\tau \in (0, T)$ such that the restriction*[7] *ℓ' of ℓ to the interval J_τ is observable, then $\ell(\mathcal{K}^u) = 1$ for any $u \in U$.*

In other words, the conclusion of this theorem is that Eq. (1.8) is approximately controllable in V by $\mathcal{H}$-valued control g for any $u \in U$ and ℓ-a.e. $\eta \in \mathcal{E}$.

3.2 Proof of Theorem 3.3

We follow the scheme used in the case of the complex Ginzburg–Landau equation considered in [KNS20a]. Let $w(t; w_0, g)$ be the solution of Eq. (1.8) corresponding to initial condition $w_0 \in H$, control $g \in \mathcal{E}$, and reference trajectory $\tilde{u}(t) = S_t(u, \eta)$, $t \in J_T$. Our goal is to prove that the vector space $\Lambda = \{w(T; 0, g), g \in \mathcal{E}\}$ is dense in U for any $u \in U$ and ℓ-a.e. $\eta \in \mathcal{E}$.

A well-known property of approximate controllability by initial condition,[8] applied to Eq. (1.8) with $g \equiv 0$, shows that the vector space $\{w(s; w_0, 0), w_0 \in H\}$

[7] This means that ℓ' is the image of ℓ by the mapping $\pi_{J_T} : \mathcal{E} \to L^\infty(J_\tau, \mathcal{H})$, $\eta \mapsto \eta|_{J_\tau}$.

[8] In the case of Eq. (1.8), this can be proved by literally repeating the arguments of Section 7.2 in [KNS20a], where a similar result is proved for linear parabolic equations.

is dense in U for any $s \in [0,T]$. Let us apply this result for the interval $[\tau, T]$, where τ is as in Theorem 3.3. Furthermore, the resolving operator for Eq. (1.8) on the interval $[\tau, T]$ with $g \equiv 0$ is continuous from H to U. Hence, to show that Λ is dense in U, it suffices to prove the density of the vector space $\{w(\tau; 0, g), g \in \mathcal{E}\}$ in H.

For any $0 \le s \le t \le \tau$, we denote by $R^{\tilde{u}}(t,s) : H \to H$ the two-parameter resolving operator for the homogeneous problem

$$\dot{w} + Lw + b(\tilde{u}, w) + Qw = 0, \quad w(s) = w_0. \tag{3.2}$$

Let $G^{\tilde{u}}$ be the *controllability Gramian* for Eq. (1.8) (see Chapter 1 in [Cor07]):

$$G^{\tilde{u}} = \int_0^\tau R^{\tilde{u}}(\tau, t) \mathsf{P}_{\mathcal{H}} R^{\tilde{u}}(\tau, t)^* \mathrm{d}t,$$

where $R^{\tilde{u}}(\tau, t)^* : H \to H$ is the adjoint of $R^{\tilde{u}}(\tau, t)$, and $\mathsf{P}_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H}$ in H. It is easy to see that the required assertion will be established if we show that $\operatorname{Ker} G^{\tilde{u}}$ is trivial for ℓ-a.e. $\eta \in \mathcal{E}$.

It is easily seen that $p(t) = R^{\tilde{u}}(\tau, t)^* w_0$ is the solution of the dual of problem (3.2) given by

$$\dot{p} - Lp - b(\tilde{u})^* p - Q^* p = 0, \quad p(\tau) = w_0, \tag{3.3}$$

where $b(\tilde{u})^*$ and Q^* are the adjoints of $b(\tilde{u}, \cdot)$ and Q in H.

Let us fix any observable $\eta \in \mathcal{E}$ and show that $\operatorname{Ker}(G^{\tilde{u}}) = \{0\}$. For any $w_0 \in \operatorname{Ker}(G^{\tilde{u}})$, we have

$$\langle G^{\tilde{u}} w_0, w_0 \rangle = \int_0^\tau \|\mathsf{P}_{\mathcal{H}} R^{\tilde{u}}(\tau, t)^* w_0\|^2 \mathrm{d}t = \int_0^\tau \|\mathsf{P}_{\mathcal{H}} p(t)\|^2 \mathrm{d}t = 0,$$

which implies that $\mathsf{P}_{\mathcal{H}} p(t) = 0$ for $t \in J_\tau$. Thus,

$$\langle \zeta, p(t) \rangle = 0, \quad t \in J_\tau \tag{3.4}$$

for any $\zeta \in \mathcal{H}$. From this we are going to derive that $\pi_1 w_0$ and $\pi_2 w_0$ are zero.

Step 1. Proof of $\pi_2 w_0 = 0$. Let us denote $p_i(t) = \pi_i p(t)$, $i = 1, 2$. In this step, we show that

$$p_2(t) = 0 \quad \text{for } t \in J_\tau. \tag{3.5}$$

Choosing $t = \tau$ in this equality, we get $\pi_2 w_0 = 0$. To prove (3.5), let us take $\zeta = \hat{\xi} = (0, \xi) \in \{0\} \times \mathcal{H}_2$ in (3.4). Then[9]

$$\langle \xi, p_2(t) \rangle = 0, \quad t \in J_\tau. \tag{3.6}$$

This shows that $p_2(t)$ is orthogonal to $\mathcal{H}_2$ for any $t \in J_\tau$. In what follows, we prove that $p_2(t)$ is orthogonal to all subspaces $\mathcal{G}_2(j), j \ge 1$. By the saturation assumption, the subspace $\mathcal{G}_2(\infty)$ is dense in H_2, so we get (3.5).

[9] We shall use the same notation $\langle \cdot, \cdot \rangle$ for the scalar products in the spaces H, H_1, and H_2.

We proceed by induction on $j \geq 0$. The case $j = 0$ is already considered above. Assuming that (3.6) holds for any $\xi \in \mathcal{G}_2(j-1)$, let us prove it for any $\xi \in \mathcal{G}_2(j)$. Taking $\zeta = \hat{\xi} = (0, \xi) \in \{0\} \times \mathcal{H}_2$ in (3.3), differentiating the resulting equality in time, and using (3.3), we obtain

$$\langle Q_1\hat{\xi}, p_1(t)\rangle + \langle L_2\xi + b_2(\tilde{u}(t), \hat{\xi}), p_2(t)\rangle = 0, \quad t \in J_\tau. \tag{3.7}$$

Note that, as $\pi_1\hat{\xi} = 0$, we have $b_2(\tilde{u}(t), \hat{\xi}) = B_2(\tilde{v}(t), \xi)$, where $\tilde{v}(t) = \pi_1\tilde{u}(t)$. Thus (3.7) becomes

$$\langle Q_1\hat{\xi}, p_1(t)\rangle + \langle L_2\xi + B_2(\tilde{v}(t), \xi), p_2(t)\rangle = 0, \quad t \in J_\tau.$$

Taking the derivative in time of this equality, we get

$$\langle Q_1\hat{\xi}, \dot{p}_1(t)\rangle + \big\langle B_2(\dot{\tilde{v}}(t), \xi), p_2(t)\big\rangle + \langle L_2\xi + B_2(\tilde{v}(t), \xi), \dot{p}_2(t)\rangle = 0, \quad t \in J_\tau.$$

From the equations for $\tilde{v}$ and p and the fact that $\pi_1\eta = 0$ we derive

$$\begin{aligned}\langle L_1(Q_1\hat{\xi}) + b_1(\tilde{v}(t), Q_1\hat{\xi}) + Q_1\hat{q}(t), p_1(t)\rangle + \langle L_2q(t) + b_2(\tilde{u}(t), \hat{q}(t)), p_2(t)\rangle \\ - \langle B_2(L_1\tilde{v}(t) + B_1(\tilde{v}(t)) + Q_1\tilde{u}(t) - h_1, \xi), p_2(t)\rangle = 0,\end{aligned} \tag{3.8}$$

where $q(t) = L_2\xi + B_2(\tilde{v}(t), \xi)$ and $\hat{q}(t) = (Q_1\hat{\xi}, q(t)) \in H$, $t \in J_\tau$. Setting

$$y(t) = \tilde{u}(t) - \int_0^t \eta(s)\,\mathrm{d}s = \tilde{u}(t) - \sum_{i=1}^d \varphi_i \int_0^t \eta^i(s)\,\mathrm{d}s, \tag{3.9}$$

where $\eta^i(t) = \langle \eta(t), \varphi_i\rangle$, and using the equalities

$$\begin{aligned} b_2(\tilde{u}(t), \hat{q}(t)) &= B_2(\tilde{v}(t), q(t)) + B_2(Q_1\hat{\xi}, \pi_2\tilde{u}(t)) \\ &= B_2(\tilde{v}(t), q(t)) + B_2(Q_1\hat{\xi}, \pi_2 y(t)) + \sum_{i=1}^d B_2(Q_1\hat{\xi}, \pi_2\varphi_i)\int_0^t \eta^i(s)\,\mathrm{d}s, \end{aligned} \tag{3.10}$$

we rewrite (3.8) as

$$\begin{aligned} &\langle L_1(Q_1\hat{\xi}) + b_1(\tilde{v}(t), Q_1\hat{\xi}) + Q_1\hat{q}(t), p_1(t)\rangle \\ &+\langle L_2q(t) + B_2(\tilde{v}(t), q(t)) + B_2(Q_1\hat{\xi}, \pi_2 y(t)), p_2(t)\rangle \\ &-\langle B_2(L_1\tilde{v}(t) + B_1(\tilde{v}(t)) + Q_1 y(t) - h_1, \xi), p_2(t)\rangle \\ &+\sum_{i=1}^d \langle B_2(Q_1\hat{\xi}, \pi_2\varphi_i) - B_2(Q_1\varphi_i, \xi), p_2(t)\rangle \int_0^t \eta^i(s)\,\mathrm{d}s = 0. \end{aligned}$$

Taking the derivative in time of this equality and setting

$$
\begin{aligned}
a_i(t) &= \langle B_2(Q_1\hat{\xi}, \pi_2\varphi_i) - B_2(Q_1\varphi_i, \xi), p_2(t)\rangle, \quad i \in [\![1, d]\!], \\
a_0(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle L_1(Q_1\hat{\xi}) + b_1(\tilde{v}(t), Q_1\hat{\xi}) + Q_1\hat{q}(t), p_1(t)\rangle \\
&\qquad + \langle L_2 q(t) + B_2(\tilde{v}(t), q(t)) + B_2(Q_1\hat{\xi}, \pi_2 y(t)), p_2(t)\rangle \\
&\qquad - \langle B_2(L_1\tilde{v}(t) + B_1(\tilde{v}(t)) + Q_1 y(t) - h_1, \xi), p_2(t)\rangle\Big) \\
&\quad + \sum_{i=1}^{d} \langle B_2(Q_1\hat{\xi}, \pi_2\varphi_i) - B_2(Q_1\varphi_i, \xi), \dot{p}_2(t)\rangle \int_0^t \eta^i(s)\,\mathrm{d}s,
\end{aligned}
$$

we obtain

$$
a_0(t) + \sum_{i=1}^{d} a_i(t)\eta^i(t) = 0.
$$

The functions $\{a_i\}_{i=1}^d$ are continuously differentiable and a_0 is continuous. The observability of η implies that $a_i(t) = 0$ for $t \in J_\tau$ and $i \in [\![0, d]\!]$. Thus (3.6) holds with ξ replaced by

$$
B_2(Q_1\hat{\xi}, \pi_2\varphi_i) - B_2(Q_1\varphi_i, \xi) = \mathfrak{b}_2(\xi, \pi_2\varphi_i). \tag{3.11}
$$

We conclude that (3.6) holds with any $\xi \in \mathcal{G}_2(j)$.

Step 2. Proof of $\pi_1 w_0 = 0$. In this step, we show that

$$
p_1(t) = 0 \quad \text{for } t \in J_\tau. \tag{3.12}
$$

Choosing $t = \tau$, we get $\pi_1 w_0 = 0$, which will complete the proof of the theorem.

We prove (3.12) by repeating the arguments of Step 1. In view of (3.5) and (3.7), we have

$$
\langle Q_1\hat{\xi}, p_1(t)\rangle = 0, \qquad t \in J_\tau \tag{3.13}
$$

for any $\hat{\xi} = (0, \xi) \in \{0\} \times G_2(\infty)$. Taking the derivative in time, we obtain

$$
\langle L_1(Q_1\hat{\xi}) + b_1(\tilde{v}(t), Q_1\hat{\xi}) + Q_1\hat{q}(t), p_1(t)\rangle = 0, \qquad t \in J_\tau,
$$

where $q(t) = L_2\xi + B_2(\tilde{v}(t), \xi)$ and $\hat{q}(t) = (Q_1\hat{\xi}, q(t)) \in H$, $t \in J_\tau$. Taking another derivative and using the equality

$$
\frac{\mathrm{d}}{\mathrm{d}t} Q_1\hat{q}(t) = Q_1(0, B_2(\dot{\tilde{v}}(t), \xi)),
$$

we get

$$
\langle b_1(\dot{\tilde{v}}(t), Q_1\hat{\xi}) + Q_1(0, B_2(\dot{\tilde{v}}(t), \xi)), p_1(t)\rangle + \langle r(t), \dot{p}_1(t)\rangle = 0, \qquad t \in J_\tau,
$$

where $r(t) = L_1(Q_1\hat{\xi}) + b_1(\tilde{v}(t), Q_1\hat{\xi}) + Q_1\hat{q}(t)$. Now, we use the equations for $\tilde{v}$ and p_1:

$$
\begin{aligned}
-\langle b_1(L_1\tilde{v}(t) + B_1(\tilde{v}(t)) + Q_1\tilde{u}(t) - h_1, Q_1\hat{\xi}))& \\
- Q_1(0, B_2(L_1\tilde{v}(t) + B_1(\tilde{v}(t)) + Q_1\tilde{u}(t) - h_1, \xi)), p_1(t)\rangle& \\
+ \langle L_1 r(t) + b_1(\tilde{v}(t), r(t)) + Q_1\hat{r}(t), p_1(t)\rangle = 0, \qquad t \in J_\tau,&
\end{aligned}
$$

where $\hat{r}(t) = (r(t), L_2 q(t) + b_2(\tilde{u}(t), \hat{q}(t)))$. Combining this with (3.9), (3.10), and (3.11), we arrive at

$$\begin{aligned}
-\langle b_1(L_1\tilde{v} + B_1(\tilde{v}) + Q_1 y - h_1, Q_1\hat{\xi}))& \\
&- Q_1(0, B_2(L_1\tilde{v} + B_1(\tilde{v}) + Q_1 y - h_1, \xi)), p_1\rangle \\
&+ \langle L_1 r + b_1(\tilde{v}, r) + Q_1(r, L_2 q + B_2(\tilde{v}, q) + B_2(Q_1\hat{\xi}, \pi_2 y)), p_1\rangle \\
&+ \sum_{i=1}^{d} \langle -b_1(Q_1\varphi_i, Q_1\hat{\xi}) + Q_1(0, \mathfrak{b}_2(\xi, \pi_2\varphi_i)), p_1\rangle \int_0^t \eta^i(s)\,\mathrm{d}s = 0.
\end{aligned}$$

Taking the derivative in this equality and denoting

$$\begin{aligned}
\tilde{a}_i(t) &= \langle -b_1(Q_1\varphi_i, Q_1\hat{\xi}) + Q_1(0, \mathfrak{b}_2(\xi, \pi_2\varphi_i)), p_1\rangle, \quad i \in [\![1, d]\!], \\
\tilde{a}_0(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(- \langle b_1(L_1\tilde{v} + B_1(\tilde{v}) + Q_1 y - h_1, Q_1\hat{\xi})) \\
&\qquad - Q_1(0, B_2(L_1\tilde{v} + B_1(\tilde{v}) + Q_1 y - h_1, \xi)), p_1\rangle \\
&\qquad + \langle L_1 r + b_1(\tilde{v}, r) + Q_1(r, L_2 q + B_2(\tilde{v}, q) + B_2(Q_1\hat{\xi}, \pi_2 y)), p_1\rangle\Big) \\
&\quad + \sum_{i=1}^{d} \langle -b_1(Q_1\varphi_i, Q_1\hat{\xi}) + Q_1(0, \mathfrak{b}_2(\xi, \pi_2\varphi_i)), \dot{p}_1\rangle \int_0^t \eta^i(s)\,\mathrm{d}s,
\end{aligned}$$

we obtain

$$\tilde{a}_0(t) + \sum_{i=1}^{d} \tilde{a}_i(t)\eta^i(t) = 0.$$

Again the functions $\{\tilde{a}_i\}_{i=1}^d$ are continuously differentiable and $\tilde{a}_0$ is continuous, so the observability of η implies that $\tilde{a}_i(t) = 0$ for $t \in J_\tau$ and $i \in [\![0, d]\!]$. We have $\mathfrak{b}_2(\xi, \pi_2\varphi_i) \in \mathcal{G}_2(\infty)$ for any $\xi \in \mathcal{G}_2(\infty)$. Hence, $Q_1(0, \mathfrak{b}_2(\xi, \pi_2\varphi_i)) \in \mathcal{G}_1(\infty)$, and from (3.13) it follows that

$$\langle Q_1(0, \mathfrak{b}_2(\xi, \pi_2\varphi_i)), p_1(t)\rangle = 0, \qquad t \in J_\tau.$$

Combining this with the equality $\tilde{a}_i(t) = 0$ for $t \in J_\tau$, we derive that

$$\langle b_1(Q_1\varphi_i, Q_1\hat{\xi}), p_1(t)\rangle = 0, \qquad t \in J_\tau.$$

Thus $\langle \zeta, p_1(t)\rangle = 0$ for any $\zeta \in \mathcal{G}_1(\infty)$ and $t \in J_\tau$. The saturation assumption implies that (3.12) holds.

4 Ergodicity of primitive equations

4.1 Abstract result

Here we formulate an abstract sufficient condition for exponential mixing which is applied in the next section to the randomly forced primitive equations. It is derived from Theorem 1.1 in [KNS20a].

Let H and E be separable Hilbert spaces, let $\mathcal{E}$ be a dense Banach subspace of E, and let X and $\mathcal{K} \subset \mathcal{E}$ be compact sets in H and E, respectively. Assume that $S : X \times \mathcal{K} \to X$ is a continuous mapping, $\{\eta_k\}$ is a sequence of i.i.d. random variables in $\mathcal{E}$ with common law ℓ and $\mathcal{K} = \operatorname{supp} \ell$, and consider a random sequence defined by

$$u_k = S(u_{k-1}, \eta_k), \ k \geq 1, \quad u_0 = u \in X.$$

Then $(u_k, \mathbb{P}_u)$, $u \in X$ is a Markov family in X, let $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ be the associated Markov operators. A measure $\mu \in \mathcal{P}(X)$ is said to be stationary for $(u_k, \mathbb{P}_u)$ if $\mathfrak{P}_1^* \mu = \mu$. Recall that the dual-Lipschitz metric on $\mathcal{P}(X)$ is defined by

$$\|\mu_1 - \mu_2\|_L^* = \sup_{f \in L(X),\, \|f\|_L \leq 1} |(f, \mu_1) - (f, \mu_2)|,$$

where $(f, \mu) = \int_X f(u)\, \mu(\mathrm{d}u)$ and $L(X)$ is the space of functions $f : X \to \mathbb{R}$ such that

$$\|f\|_L = \sup_{u \in X} |f(u)| + \sup_{0 < \|u - v\|_H \leq 1} \frac{|f(u) - f(v)|}{\|u - v\|_H} < \infty.$$

Theorem 4.1. *Assume that the following conditions hold.*

($\mathbf{H}_1$) *There is a Banach space V compactly embedded into the space H such that $X \subset V$. There is an open set $\mathcal{O} = \mathcal{O}_H \times \mathcal{O}_E$ in $H \times E$ containing $X \times \mathcal{K}$ and an extension $\tilde{S} : \mathcal{O} \to V$ of S that is twice continuously differentiable with derivatives that are bounded on bounded subsets of $\mathcal{O}$. Moreover, for any $u \in \mathcal{O}_H$, the mapping $\eta \mapsto \tilde{S}(u, \eta)$, $\mathcal{O}_E \to H$ is analytic, and all the derivatives $(D_\eta^j \tilde{S})(u, \eta)$ are continuous in (u, η) and are bounded on bounded subsets of $\mathcal{O}$.*

($\mathbf{H}_2$) *There are $a \in (0, 1)$, $\hat{\eta} \in \mathcal{K}$, and $\hat{u} \in X$ such that*

$$\|S(u, \hat{\eta}) - \hat{u}\|_H \leq a \|u - \hat{u}\|_H \quad \text{for any } u \in X.$$

($\mathbf{H}_3$) *For any $u \in X$ and ℓ-a.e. $\eta \in E$, the image of the linear mapping $(D_\eta \tilde{S})(u, \eta) : E \to H$ is dense in H.*

($\mathbf{H}_4$) *The random variables η_k are of the form $\eta_k = \sum_{j=1}^\infty b_j \xi_{jk} e_j$, where $\{e_j\}$ is an orthonormal basis in E such that $e_j \in \mathcal{E}$ and $\sup_{j \geq 1} \|e_j\|_{\mathcal{E}} < \infty$, $\{b_j\}$ are non-zero numbers satisfying $\sum_{j=1}^\infty b_j^2 < \infty$, and $\{\xi_{jk}\}$ are independent scalar random variables with Lipschitz-continuous density ρ_j such that $\operatorname{supp} \rho_j \subset [-1, 1]$.*

Then the family $(u_k, \mathbb{P}_u), u \in X$ is exponentially mixing, i.e. it has a unique stationary measure $\mu \in \mathcal{P}(X)$, and there are numbers $C > 0$ and $c > 0$ such that

$$\|\mathfrak{P}_k^* \lambda - \mu\|_L^* \leq C e^{-ck}, \quad k \geq 0 \tag{4.1}$$

for any initial measure $\lambda \in \mathcal{P}(X)$.

This theorem is a slight modification of Theorem 1.1 in [KNS20a]. The difference is in Condition (H_1) which is a localized version of the condition used in [KNS20a]. Indeed, it is not clear whether in the case of primitive equations this regularity condition holds with $\mathcal{O} = H \times E$ (see Theorem 3 in [Bou20]). Condition (H_2) is usually satisfied with $\hat{u} = 0$ if the origin is an exponentially stable equilibrium for the unforced equation and $0 \in \mathcal{K}$. Condition (H_3) is a Hörmander-type condition, and (H_4) is quite usual decomposability assumption. We refer the reader to Section 1 in [KNS20a] for a detailed discussion of these conditions and for a short scheme of the proof of the original version of the theorem.

Proof of Theorem 4.1. Truncating the mapping $\tilde{S}$, we easily obtain an extension $\hat{S} : H \times E \to V$ of S satisfying (H_1) with $\mathcal{O} = H \times E$. Note that the family $(u_k, \mathbb{P}_u)$, $u \in X$ does not change if we replace S by its extension $\hat{S}$. In view of Conditions (H_1)-(H_4), the hypotheses of Theorem 1.1 in [KNS20a] are satisfied for the random dynamical system $u_k = \hat{S}(u_{k-1}, \eta_k)$. Applying that theorem, we prove the mixing (4.1) for $(u_k, \mathbb{P}_u)$, $u \in X$. □

4.2 Application

In this section, we combine Theorems 3.3 and 4.1 to prove the exponential mixing for the randomly forced 3D primitive equations. More precisely, we consider Eq. (1.1) with $h = 0$ and random process η of the form

$$\eta(t) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1,k)}(t)\eta_k(t-k+1), \quad t \geq 0,$$

where $\mathbb{I}_{[k-1,k)}$ is the indicator function of the interval $[k-1, k)$, $\{\eta_k\}$ is a sequence of i.i.d. random variables in the space $\mathcal{E} = L^\infty(J, \mathcal{H})$, $J = [0, 1]$, and $\mathcal{H} \subset U$ is a finite-dimensional subspace. In what follows, we denote by ℓ the law of the random variable η_k and assume that $\mathcal{K} = \operatorname{supp} \ell$ is compact in $\mathcal{E}$. The restriction to integer times of the solution of Eq. (1.1) satisfies the relation $u_k = S_1(u_{k-1}, \eta_k)$, $k \geq 1$ and defines a family of Markov processes $(u_k, \mathbb{P}_u)$ parametrised by the initial condition $u_0 = u \in V$. The following lemma is proved by using standard arguments based on dissipative and regularizing properties of PEs.

Lemma 4.2. *The family $(u_k, \mathbb{P}_u)$ admits a closed invariant absorbing set X in U in the sense that, for any $R > 0$, there is an integer $k_0 = k_0(R) \geq 0$ such that*

$$\mathbb{P}_u\{u_k \in X,\ k \geq k_0\} = 1 \quad \textit{for } u \in B_V(0, R),$$
$$\mathbb{P}_u\{u_k \in X,\ k \geq 0\} = 1 \quad \textit{for } u \in X.$$

The following theorem is a more detailed version of Theorem C formulated in the Introduction.

Theorem 4.3. *Let a finite-dimensional subspace $\mathcal{H} \subset U$ be saturating in the sense of Definition 3.1, and assume that the following two conditions are fulfilled.*

Decomposability. *The random variables η_k are of the form $\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j$, where $\{e_j\}$ is an orthonormal basis in the Hilbert space $E = L^2(J, \mathcal{H})$ such that $\sup_{j\geq 1} \|e_j\|_{L^\infty(J,\mathcal{H})} < \infty$, $\{b_j\}$ are non-zero numbers satisfying $\sum_{j=1}^{\infty} b_j^2 < \infty$, and $\{\xi_{jk}\}$ are independent scalar random variables with Lipschitz-continuous density ρ_j such that $\operatorname{supp} \rho_j \subset [-1, 1]$ and $\rho_j(0) > 0$.*

Observability. *There is $\tau \in (0, 1)$ such that the law ℓ' of the restriction of the random variable η_k to the interval J_τ is observable.*

Then the family $(u_k, \mathbb{P}_u), u \in X$ is exponentially mixing.

Proof. By Theorem 3 in [Bou20], there is an open set $\mathcal{O}_E$ in E containing $\mathcal{K}$ and an extension $\tilde{S} : H^2 \times \mathcal{O}_E \to H^3$ of S_1 that is twice continuously differentiable with derivatives that are bounded on bounded subsets of $H^2 \times \mathcal{O}_E$. Moreover, for any $u \in H^2$, the mapping $\eta \mapsto \tilde{S}(u, \eta)$, $\mathcal{O}_E \to H^2$ is analytic and the derivatives $(D_\eta^j \tilde{S})(u, \eta)$ are continuous in (u, η) and bounded on bounded subsets of $H^2 \times \mathcal{O}_E$. Thus Condition (H_1) in Theorem 4.1 is verified with $H = H^2$, $V = H^3$, and $\mathcal{O} = H^2 \times \mathcal{O}_E$.

Next, for any $\delta > 0$, we define a norm on H^2 by $|u|_\delta = \left(\|u\|^2 + \delta \|u\|_2^2\right)^{1/2}$. Then for any bounded set $B \subset H^2$, there are numbers $\delta > 0$ and $a \in (0, 1)$ such that

$$|S_1(u)|_\delta \leq a|u|_\delta \quad \text{for } u \in B, \tag{4.2}$$

where $S_1(u) = S_1(u, 0)$. This inequality with $B = X$ shows that Condition (H_2) is verified with $\hat{u} = 0$, $\hat{\eta} = 0$, and the norm $|\cdot|_\delta$. To prove (4.2), we use the following inequalities (see [CT07, Ju07, Pet06, Bou20]):

$$\|S_1(u)\| \leq q\|u\| \quad \text{for } u \in H,$$
$$\|S_1(u)\|_2 \leq C_B \|u\| \quad \text{for } u \in B,$$

where $q \in (0, 1)$ and $C_B > 0$. These inequalities imply that

$$|S_1(u)|_\delta^2 = \|S_1(u)\|^2 + \delta \, \|S_1(u)\|_2^2 \leq q^2 \|u\|^2 + \delta \, C_B^2 \|u\|^2 \leq \left(q^2 + \delta C_B^2\right) |u|_\delta^2.$$

Choosing $\delta > 0$ so small that $a^2 = q^2 + \delta C_B^2 < 1$, we obtain (4.2). Finally, Condition (H_3) is established in Theorem 3.3, and Condition (H_4) is verified by the decomposability hypothesis. Applying Theorem 4.1, we complete the proof. □

Theorem 4.3 is formulated for initial measures λ supported by X. As a consequence of Lemma 4.2 and Theorem 4.3, we obtain the following result.

Corollary 4.4. *Under the conditions of Theorem 4.3, the measure μ is the unique stationary measure for the family $(u_k, \mathbb{P}_u)$ in $\mathcal{P}(V)$. Moreover, inequality (4.1) holds for any $R > 0$, $\lambda \in \mathcal{P}(V)$ with $\operatorname{supp} \lambda \subset B_V(0, R)$, and $k \geq k_0$.*

5 Saturating subspaces

In this section, we show that the ten-dimensional subspace defined in the Introduction is saturating in the sense of both Definitions 2.2 and 3.1. We also give an example of V-saturating subspace.

5.1 H-saturating subspace

Let us consider the subspace

$$\mathcal{H} = \operatorname{span}\{(0, \phi_i) : i \in [\![1, 10]\!]\} \subset H, \tag{5.1}$$

where ϕ_i are the eigenfunctions of the operator L_2 given by

$$\begin{gathered}\phi_1 = \cos x \sin z,\ \phi_2 = \sin x \sin z,\ \phi_3 = \cos y \sin z,\ \phi_4 = \sin y \sin z,\\ \phi_5 = \sin z,\ \phi_6 = \cos 2x \sin z,\ \phi_7 = \sin 2x \sin z,\ \phi_8 = \cos 2y \sin z,\\ \phi_9 = \sin 2y \sin z,\ \phi_{10} = \sin 2z.\end{gathered}$$

Theorem 5.1. *The subspace $\mathcal{H}$ is H-saturating in the sense of Definition 2.2.*

To prove this theorem, we introduce the following two orthogonal bases:

- in H_1, composed of eigenfunctions of the operator L_1:

$$\begin{gathered}m\, c_m(x,y)\cos pz,\ m\, s_m(x,y)\cos pz,\ m^\perp c_m(x,y)\cos pz,\ m^\perp s_m(x,y)\cos pz,\\ m^\perp c_m(x,y),\ m^\perp s_m(x,y),\ \imath\cos pz,\ \jmath\cos pz \quad m \in \mathbb{Z}^2_*,\ p \geq 1;\end{gathered}$$

- in H_2, composed of eigenfunctions of the operator L_2:

$$c_m(x,y)\sin pz, \quad s_m(x,y)\sin pz, \quad \sin pz \quad m \in \mathbb{Z}^2_*,\ p \geq 1,$$

where we denote $m^\perp = (-m_2, m_1)$, $\imath = (1,0)$, $\jmath = (0,1) \in \mathbb{R}^2$, and

$$c_m(x,y) = \cos(m_1 x + m_2 y), \qquad s_m(x,y) = \sin(m_1 x + m_2 y).$$

The following two propositions are established in the next two subsections.

Proposition 5.2. *Any vector of the basis in H_1 belongs*[10] *to $\overline{\mathcal{H}_1(\infty)}^{H_1}$.*

Proposition 5.3. *Any vector of the basis in H_2 belongs to $\mathcal{G}_2(\infty)$.*

These propositions and the inclusion $\mathcal{G}_2(\infty) \subset \mathcal{H}_2(\infty)$ readily imply that $\mathcal{H}(\infty)$ is dense in $H_1 \times V_2$ and prove Theorem 5.1.

[10] Here $\overline{\mathcal{H}_1(\infty)}^{H_1}$ is the closure of $\mathcal{H}_1(\infty)$ in H_1.

5.1.1 Saturation in θ-component

In this subsection, we give a proof of Proposition 5.3.

Proof of Proposition 5.3. Step 1. We first show that $\sin x \sin pz \in \mathcal{G}_2(\infty)$ for any integer $p \geq 1$. Indeed, as $\phi_2, \phi_5 \in \mathcal{G}_2$, we have $\mathfrak{b}_2(\phi_2, \phi_5) \in \mathcal{G}_2(1)$. The equalities

$$Q_1(0, \phi_2) = \imath \cos x \cos z, \quad Q_1(0, \phi_5) = 0,$$
$$\mathfrak{b}_2(\phi_2, \phi_5) = B_2(Q_1(0, \phi_2), \phi_5) = \frac{1}{2} \sin x \sin 2z$$

imply that $\sin x \sin 2z \in \mathcal{G}_2(1)$. A simple computation shows that

$$\begin{aligned} Q_1(0, \sin x \sin 2z) &= \frac{1}{2} \imath \cos x \cos 2z, \\ \mathfrak{b}_2(\sin x \sin 2z, \phi_5) &= B_2(Q_1(0, \sin x \sin 2z), \phi_5) \\ &= \frac{1}{4} \sin x \sin 2z \cos z \\ &= \frac{1}{8} (\sin x \sin z + \sin x \sin 3z) \in \mathcal{G}_2(2). \end{aligned}$$

This implies that $\sin x \sin 3z \in \mathcal{G}_2(2)$. Iterating this argument, we see that $\sin x \sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$. In a similar way, we can prove that $\cos x \sin pz$, $\cos y \sin pz$, $\sin y \sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$.

Repeating the above arguments and using the fact that $\sin 2x \sin z$, $\cos 2x \sin z$, $\sin 2y \sin z$, $\cos 2y \sin z \in \mathcal{G}_2$, we can obtain also that $\sin 2x \sin pz$, $\cos 2x \sin pz$, $\sin 2y \sin pz$, $\cos 2y \sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$.

Step 2. Let us show that $\sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$. The equalities

$$\begin{aligned} \imath \cos x \cos nz &= Q_1(0, n \sin x \sin nz), \\ \imath \sin x \cos nz &= Q_1(0, -n \cos x \sin nz) \end{aligned}$$

and the fact that $\cos x \sin nz$, $\sin x \sin nz \in \mathcal{G}_2(\infty)$ imply that

$$\begin{aligned} &\mathfrak{b}_2(n \cos x \sin nz, \phi_1) + \mathfrak{b}_2(n \sin x \sin nz, \phi_2) \\ &\qquad = \frac{n^2 - 1}{2n} ((n-1) \sin(n-1)z - (n+1) \sin(n+1)z) \in \mathcal{G}_2(\infty). \end{aligned}$$

Thus $n \sin nz - p \sin pz \in \mathcal{G}_2(\infty)$ for any integers $n > p \geq 1$ that are both even or both odd. As $\sin z$, $\sin 2z \in \mathcal{G}_2$, we obtain $\sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$.

Step 3. Now we prove the following property $P(m)$ for any $m \geq 1$:

$P(m)$: for any $p \geq 1$ and $q \in [\![1, m]\!]$, we have $\sin qx \sin pz \in \mathcal{G}_2(\infty)$.

We argue by induction on m. The cases $m = 1, 2$ are considered in Step 1.

Assuming that $P(m)$ is true for $m \geq 2$, we prove it for $m+1$. Note that

$$\begin{aligned}
\imath \cos mx \cos nz &= \frac{n}{m} Q_1(0, \sin mx \sin nz), \\
\mathfrak{b}_2\left(\frac{n}{m} \sin mx \sin nz, \phi_1\right) &= \frac{(m-n)(m+n^2)}{4mn} \sin(m+1)x \sin(n+1)z \\
&\quad + \frac{(m+n)(m+n^2)}{4mn} \sin(m+1)x \sin(n-1)z \\
&\quad + \frac{(m+n)(m-n^2)}{4mn} \sin(m-1)x \sin(n+1)z \\
&\quad + \frac{(m-n)(m-n^2)}{4mn} \sin(m-1)x \sin(n-1)z \in \mathcal{G}_2(\infty)
\end{aligned} \tag{5.2}$$

for any $n \geq 1$. By the induction hypothesis, we have $\sin(m-1)x \sin(n+1)z$, $\sin(m-1)x \sin(n-1)z \in \mathcal{G}_2(\infty)$. Thus (5.2) implies that

$$(m-n) \sin(m+1)x \sin(n+1)z + (m+n) \sin(m+1)x \sin(n-1)z \in \mathcal{G}_2(\infty) \tag{5.3}$$

Taking here $n = 1$, we obtain $\sin(m+1)x \sin 2z \in \mathcal{G}_2(\infty)$. It follows that

$$\mathfrak{b}_2\left(\frac{2}{m+1} \sin(m+1)x \sin 2z, \phi_5\right) = \frac{(m+1)}{4} \sin(m+1)x(\sin 3z + \sin z) \in \mathcal{G}_2(\infty).$$

Thus

$$\sin(m+1)x \sin 3z + \sin(m+1)x \sin z \in \mathcal{G}_2(\infty). \tag{5.4}$$

Taking $n = 2$ in (5.3), we get

$$(m-2) \sin(m+1)x \sin 3z + (m+2) \sin(m+1)x \sin z \in \mathcal{G}_2(\infty).$$

This and (5.4) imply that $\sin(m+1)x \sin z \in \mathcal{G}_2(\infty)$. Repeating the argument of Step 1, we show that $\sin(m+1)x \sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$. Thus $P(m)$ is true for any $m \geq 1$

In a similar way, $\cos mx \sin pz$, $\cos my \sin pz$, $\sin my \sin pz$ belong to $\mathcal{G}_2(\infty)$ for any $m, p \geq 1$.

Step 4. In this step, we show that $s_m(x,y) \sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$ and $m = (m_1, m_2) \in \mathbb{Z}_*^2$. We confine ourselves to the case $m_2 \geq 0$, the case $m_2 < 0$ being similar. Arguing by induction on m_2, we prove the following property:

$P'(m_2)$: for any $p \geq 1$, $m_1 \in \mathbb{Z}$, and $q \in [\![0, m_2]\!]$, we have $s_m(x,y) \sin pz \in \mathcal{G}_2(\infty)$, where $m = (m_1, q)$.

The case $m_2 = 0$ is considered in Step 3. Assuming that $P'(m_2)$ is true for $m_2 \geq 0$, let us prove it for $m_2 + 1$. We first consider the case $m_1 \neq \pm 1$. Let

$$\begin{aligned}
\theta_1 &= -n \cos(m_1 x + m_2 y) \sin nz, \\
\theta_2 &= n \sin(m_1 x + m_2 y) \sin nz.
\end{aligned}$$

Using the equalities

$$Q_1(0,\theta_1) = ms_m(x,y)\cos nz,$$
$$Q_1(0,\theta_2) = mc_m(x,y)\cos nz,$$

we get

$$\begin{aligned}\mathfrak{b}_2(\theta_1, \sin y \sin z) - \mathfrak{b}_2(\theta_2, \cos y \sin z) = a_1 \sin(m_1 x + (m_2+1)y)\sin(n+1)z \\ + a_2 \sin(m_1 x + (m_2+1)y)\sin(n-1)z \in \mathcal{G}_2(\infty),\end{aligned} \tag{5.5}$$

where

$$a_1 = \frac{1}{2n}\left(n^3 - n(n-1)m_2 - m_1^2 - m_2^2\right),$$
$$a_2 = -\frac{1}{2n}\left(n^3 + n(n+1)m_2 + m_1^2 + m_2^2\right).$$

Taking $n = 1$ in (5.5), we obtain

$$(1 - m_1^2 - m_2^2)\sin(m_1 x + (m_2+1)y)\sin 2z \in \mathcal{G}_2(\infty).$$

As $m_1 \neq \pm 1$, we have $1 - m_1^2 - m_2^2 \neq 0$, so $\sin(m_1 x + (m_2+1)y)\sin 2z \in \mathcal{G}_2(\infty)$. The latter implies that

$$\begin{aligned}\mathfrak{b}_2(2\sin(m_1 x + (m_2+1)y)\sin 2z, \phi_5) = &\frac{m_1^2 + (m_2+1)^2}{4}\sin(m_1 x + (m_2+1)y) \\ &\times (\sin 3z + \sin z) \in \mathcal{G}_2(\infty).\end{aligned} \tag{5.6}$$

On the other hand, taking $n = 2$ in (5.5), we get

$$a_1 \sin(m_1 x + (m_2+1)y)\sin 3z + a_2 \sin(m_1 x + (m_2+1)y)\sin z \in \mathcal{G}_2(\infty). \tag{5.7}$$

When $n = 2$, we have $a_1 - a_2 = 4 + m_2 \neq 0$, since $m_2 \geq 0$. Combining (5.6) and (5.7), we see that $\sin(m_1 x + (m_2+1)y)\sin z \in \mathcal{G}_2(\infty)$. Now applying the argument of Step 1, we infer that $\sin(m_1 x + (m_2+1)y)\sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$ and $m_1 \neq \pm 1$.

Finally, computing the term $\mathfrak{b}_2(\sin(\pm 2x + (m_2+1)y)\sin nz, \phi_1)$, one easily shows that $\sin(\pm x + (m_2+1)y)\sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$. Thus $P'(m_2)$ holds for any $m_2 \geq 0$, and we conclude that that $s_m(x,y)\sin pz \in \mathcal{G}_2(\infty)$ for any $p \geq 1$. The proof of $c_m(x,y)\sin pz \in \mathcal{G}_2(\infty)$ is similar.

Thus all the vectors of the basis in H_2 belong to $\mathcal{G}_2(\infty)$. This completes the proof of Proposition 5.3. □

5.1.2 Saturation in v-component

Here we prove Proposition 5.2. We first establish the following lemma.

Lemma 5.4. *For any $m \in \mathbb{Z}_*^2$, $p \geq 1$, and $i \in [\![1,4]\!]$, the following properties hold:*

(a) the functions $m\, c_m(x,y)\cos pz$, $m\, s_m(x,y)\cos pz$ belong to $\mathcal{H}_1(\infty)$;

(b) the functions

$$\psi^c_{m,p,i} = b_1(m\, c_m(x,y)\cos pz, \psi_i),\ \ \psi^s_{m,p,i} = b_1(m\, s_m(x,y)\cos pz, \psi_i)$$

belong to $\mathcal{H}_1(\infty)$*, where*

$$\psi_1 = \imath\cos x\cos z,\ \ \psi_2 = \imath\sin x\cos z,\ \ \psi_3 = \jmath\cos y\cos z,\ \ \psi_4 = \jmath\sin y\cos z.$$

Proof. By Proposition 5.3, we know that $s_m(x,y)\sin pz$, $c_m(x,y)\sin pz \in \mathcal{H}_2(\infty)$. Recall that $Q_1(0,\theta) \in \mathcal{H}_1(\infty)$ for any $\theta \in \mathcal{H}_2(\infty)$. So property (a) follows from the equalities

$$\begin{aligned} &m\, c_m(x,y)\cos pz = Q_1(0, ps_m(x,y)\sin pz),\\ &m\, s_m(x,y)\cos pz = Q_1(0, -pc_m(x,y)\sin pz), \quad m \in \mathbb{Z}^2_*,\ p \geq 1. \end{aligned}$$

To prove (b), we take any $\varepsilon > 0$ and $a \in \mathbb{R}$ and note that

$$\begin{aligned} B_1(\varepsilon\, m\, c_m(x,y)\cos pz + a\varepsilon^{-1}\psi_i) = \varepsilon^{-2}B_1(a\psi_i) + \varepsilon^2 B_1(m\, c_m(x,y)\cos pz)\\ + b_1(m\, c_m(x,y)\cos pz, a\psi_i). \end{aligned}$$

Using the fact that $B_1(\psi_i) = 0$, we obtain the following limit in V_1 as $\varepsilon \to 0^+$:

$$B_1(\varepsilon m\, c_m(x,y)\cos pz + a\varepsilon^{-1}\psi_i) \to b_1(m\, c_m(x,y)\cos pz, a\psi_i) = a\psi^c_{m,i}.$$

As $a \in \mathbb{R}$ is arbitrary, this shows that $\psi^c_{m,p,i} \in \mathcal{H}_1(\infty)$. Repeating these arguments with $s_m(x,y)$ instead of $c_m(x,y)$, we prove that $\psi^s_{m,p,i} \in \mathcal{H}_1(\infty)$. □

Proof of Proposition 5.2. Step 1. Let us show that $\imath\cos pz \in \overline{\mathcal{H}_1(\infty)}^{H_1}$ for any $p \geq 1$. To this end, we take any $n \geq 2$ and compute the term

$$\begin{aligned} b_1(\imath\cos x\cos nz, \psi_2) = &\frac{1}{2}\imath\cos 2x\,(\cos(n+1)z + \cos(n-1)z)\\ &+ \frac{1+n^2}{2n}\imath\cos 2x\,(\cos(n-1)z - \cos(n+1)z)\\ &+ \frac{n^2-1}{2n}\imath\,(\cos(n-1)z - \cos(n+1)z)\,. \end{aligned}$$

By property (a) in Lemma 5.4, we have $\imath\cos 2x\cos(n\pm 1)z \in \mathcal{H}_1(\infty)$, and by property (b), that $b_1(\imath\cos x\cos nz, \psi_2) \in \mathcal{H}_1(\infty)$. It follows that

$$\imath\,(\cos(n-1)z - \cos(n+1)z) \in \mathcal{H}_1(\infty) \quad \text{for any } n \geq 2,$$

so

$$\imath\,(\cos pz - \cos qz) \in \mathcal{H}_1(\infty), \tag{5.8}$$

provided that $p, q \geq 1$ are both odd or both even. Passing to the limit as $q \to \infty$, we see that, for any $p \geq 1$, the function $\imath\cos pz$ is in the L^2-weak closure of $\mathcal{H}_1(\infty)$, hence in $\overline{\mathcal{H}_1(\infty)}^{H_1}$, since $\mathcal{H}_1(\infty)$ is a vector space.

Computing the term $b_1(\jmath \cos y \cos pz, \psi_4)$ and repeating the above arguments, we infer that $\jmath \cos pz \in \overline{\mathcal{H}_1(\infty)}^{H_1}$.

Step 2. In this step, we show that $m^\perp s_m(x,y)\cos pz$ belongs to $\overline{\mathcal{H}_1(\infty)}^{H_1}$ for any $m \in \mathbb{Z}_*^2$ and $p \geq 0$. Let us take any $m = (m_1, m_2) \in \mathbb{Z}_*^2$ and $n \geq 1$, and use the equality

$$\begin{aligned} b_1(s_{m+\jmath}(x,y)\cos nz, \psi_4) + b_1(c_{m+\jmath}(x,y)\cos nz, \psi_3) \\ = -(m_2+1)\Pi\left(s_m(x,y)\cos nz \cos z\right) \\ + \Pi\left(A_\jmath(m,n)\sin(m_1x+m_2y)\sin nz \sin z\right), \end{aligned} \tag{5.9}$$

where $A_\jmath(m,n) = n^{-1}(m_1n^2, (m_2+1)n^2 - m_1^2 - (m_2+1)^2)$. By Lemma 5.4, we have that the functions $\Pi\left(s_m(x,y)\cos nz\cos z\right)$, $b_1(s_{m+\jmath}(x,y)\cos nz, \psi_4)$, and $b_1(c_{m+\jmath}(x,y)\cos nz, \psi_3)$ belong to $\mathcal{H}_1(\infty)$. Hence,

$$\Pi\left(A_\jmath(m,n)\sin(m_1x+m_2y)\sin nz\sin z\right) \in \mathcal{H}_1(\infty). \tag{5.10}$$

The vector $A_\jmath(m,n)$ is parallel to m if and only if one of the following two conditions hold:

- $m_1 = 0$ and $m_2 \neq 0$;
- $m_1 \neq 0$ and $n^2 = m_1^2 + (m_2+1)^2$.

Let us denote by $\mathcal{A}_\jmath$ the set of couples (m,n) such that $A_\jmath(m,n)$ is non-parallel to m. From (5.10) we derive that

$$m^\perp s_m(x,y)(\cos(n+1)z - \cos(n-1)z) \in \mathcal{H}_1(\infty) \tag{5.11}$$

for any $(m,n) \in \mathcal{A}_\jmath$. In a similar way, we compute the sum

$$\begin{aligned} b_1(s_{m+\iota}(x,y)\cos nz, \psi_2) + b_1(c_{m+\iota}(x,y)\cos nz, \psi_1) \\ = -(m_1+1)\Pi\left(s_m(x,y)\cos nz\cos z\right) \\ + \Pi\left(A_\iota(m,n)\sin(m_1x+m_2y)\sin nz\sin z\right), \end{aligned} \tag{5.12}$$

where $A_\iota(m,n) = n^{-1}((m_1+1)n^2 - m_2^2 - (m_1+1)^2, m_2n^2)$. As above, $A_\iota(m,n)$ is parallel to m if and only if one of the following conditions hold:

- $m_2 = 0$ and $m_1 \neq 0$;
- $m_2 \neq 0$ and $n^2 = (m_1+1)^2 + m_2^2$.

Let $\mathcal{A}_\iota$ be the set of (m,n) such that $A_\iota(m,n)$ is non-parallel to m. From (5.12) it follows that (5.11) holds for any $(m,n) \in \mathcal{A}_\iota$.

Let us go back to (5.9), and replace m by $-m$. We see that

$$\Pi\left(A_\jmath(-m,n)\sin(m_1x+m_2y)\sin nz\sin z\right) \in \mathcal{H}_1(\infty),$$

and $A_\jmath(-m,n)$ is parallel to m if one of the following conditions hold:

- $m_1 = 0$ and $m_2 \neq 0$;
- $m_1 \neq 0$ and $n^2 = m_1^2 + (m_2 - 1)^2$.

We denote by $\mathcal{A}_\jmath^-$ the set of (m, n) such that $A_\jmath(m, n)$ is non-parallel to m. Again (5.11) holds for any $(m, n) \in \mathcal{A}_\jmath^-$. The set $\mathcal{A}_\iota^-$ is defined in a similar way, by replacing m by $-m$ in (5.12). Then (5.11) holds for any $(m, n) \in \mathcal{A}_\iota^-$.

It is easy to see that the union of the sets $\mathcal{A}_\jmath^\pm$, $\mathcal{A}_\iota^\pm$ is $\mathbb{Z}_*^2 \times \mathbb{N}_*$, so (5.11) holds for *any* $m \in \mathbb{Z}_*^2$ and $n \geq 1$. Iterating (5.11), we obtain

$$m^\perp s_m(x, y)(\cos pz - \cos qz) \in \mathcal{H}_1(\infty), \tag{5.13}$$

provided that the integers $p, q \geq 0$ are both even or odd. Passing to the limit as $q \to \infty$, we conclude that $m^\perp s_m(x, y) \cos pz$ belongs to the L^2-weak closure of $\mathcal{H}_1(\infty)$, hence to $\overline{\mathcal{H}_1(\infty)}^{H_1}$ for any $m \in \mathbb{Z}_*^2$ and $p \geq 0$.

A similar argument shows that $m^\perp c_m(x, y) \cos pz \in \overline{\mathcal{H}_1(\infty)}^{H_1}$ for any $m \in \mathbb{Z}_*^2$ and $p \geq 0$. This completes the proof of Proposition 5.2. □

5.2 V-saturating subspace

We do not know whether the subspace $\mathcal{H}$ defined in (5.1) is V-saturating. The main difficulty comes from Proposition 5.2, which only implies that $\mathcal{H}_1(\infty)$ is dense in H_1. In this section, we add new vectors to $\mathcal{H}$ with non-zero v-components. This results in a larger $\mathcal{H}_1(\infty)$-space that contains all the vectors of the basis in H_1. As a consequence, we get V-saturation property. More precisely, we define the space

$$\tilde{\mathcal{H}} = \operatorname{span}\{(\tilde{\phi}_i, 0), (0, \phi_j) : i = 1, \ldots, 6,\ j = 1, \ldots, 10\} \subset H,$$

where the functions ϕ_j are as in Section 5.1 and

$$\tilde{\phi}_1 = \jmath \cos z,\ \tilde{\phi}_2 = \jmath \cos 2z,\ \tilde{\phi}_3 = \imath \cos z,\ \tilde{\phi}_4 = \imath \cos 2z,$$
$$\tilde{\phi}_5 = \jmath \cos x,\ \tilde{\phi}_6 = \jmath \sin x.$$

Theorem 5.5. *The subspace $\tilde{\mathcal{H}}$ is V-saturating in the sense of Definition 2.2.*

Proof. Let $\tilde{\mathcal{H}}(j)$ be the subspaces defined by (2.4) and (2.5) with $\mathcal{H} = \tilde{\mathcal{H}}$, and let $\tilde{\mathcal{H}}_i(j) = \pi_i \tilde{\mathcal{H}}(j)$, $i = 1, 2$. From Proposition 5.3 it follows that $\tilde{\mathcal{H}}_2(\infty)$ is dense in V_2. The proposition will be proved if we show that any vector of the basis in H_1 belongs to $\tilde{\mathcal{H}}_1(\infty)$ (cf. Proposition 5.2). By Lemma 5.4, we have $m\, c_m(x, y) \cos pz$, $m\, s_m(x, y) \cos pz \in \tilde{\mathcal{H}}_1(\infty)$ for any $m \in \mathbb{Z}_*^2$ and $p \geq 1$. Combining (5.8), the version of (5.8) with $\jmath$ instead of ι, and the assumption that $(\tilde{\phi}_i, 0) \in \tilde{\mathcal{H}}$, $i = 1, \ldots, 4$, we obtain that $\imath \cos pz$, $\jmath \cos pz \in \tilde{\mathcal{H}}_1(\infty)$ for any $p \geq 1$. For any $m \in \mathbb{Z}_*^2$ and $p \geq 1$, the following equality holds:

$$\begin{aligned} b_1(s_{m+\iota}(x, y) \cos pz, \tilde{\phi}_6) + b_1(c_{m+\iota}(x, y) \cos pz, \tilde{\phi}_5) \\ = \Pi\left(A(m) \sin(m_1 x + m_2 y) \cos pz\right), \end{aligned} \tag{5.14}$$

where $A(m) = (-(m_1+1)m_2, m_1+1-m_2^2)$. Note that the vector $A(m)$ is parallel to m if and only if $A(m) = 0$, i.e., $m = (-1,0)$. Assume that $m \neq (-1,0)$. Since $B_1(\tilde{\phi}_5) = B_1(\tilde{\phi}_6) = 0$, as in the proof of (b) in Lemma 5.4, we show that $b_1(s_{m+\iota}(x,y)\cos pz,\ \tilde{\phi}_6), b_1(c_{m+\iota}(x,y)\cos pz, \tilde{\phi}_5) \in \tilde{\mathcal{H}}_1(\infty)$. As $A(m)$ is non-parallel to m, from (5.14) we derive that $m^\perp s_m(x,y)\cos pz \in \tilde{\mathcal{H}}_1(\infty)$ for any $p \geq 1$. From (5.13) it follows also that $m^\perp s_m(x,y) \in \tilde{\mathcal{H}}_1(\infty)$. Finally, if $m = (-1,0)$, then $m^\perp s_m(x,y) = \tilde{\phi}_6(x) \in \tilde{\mathcal{H}}$, and (cf. (b) in Lemma 5.4)

$$b_1((1,1)\cos pz, \tilde{\phi}_5(x)) = -m^\perp s_m(x,y)\cos pz \in \tilde{\mathcal{H}}_1(\infty), \quad p \geq 1.$$

With similar arguments one proves also that $m^\perp c_m(x,y)\cos pz \in \tilde{\mathcal{H}}_1(\infty)$ for any $m \in \mathbb{Z}_*^2$ and $p \geq 0$. Thus any vector of the basis in H_1 belongs to $\tilde{\mathcal{H}}_1(\infty)$. We conclude that $\tilde{\mathcal{H}}_1(\infty)$ is dense in V_1 and $\tilde{\mathcal{H}}(\infty)$ is dense in V. □

5.3 Saturation for linearized system

Now we turn to the saturation property for the linearized system.

Theorem 5.6. *The subspace $\mathcal{H}$ defined by (5.1) is saturating for linearized system (1.8) in the sense of Definition 3.1.*

Proof. This theorem follows from the proof of Theorem 5.1. Indeed, by Proposition 5.3, $\mathcal{G}_2(\infty)$ is dense in H_2. The computations in Section 5.1.2 show that any vector of the basis in H_1 belongs to $\overline{\mathcal{G}_1(\infty)}^{H_1}$. We conclude that $\mathcal{G}(\infty)$ is dense in H, so $\mathcal{H}$ is saturating in the sense of Definition 3.1. □

6 Proof of Proposition 1.2

We confine ourselves to the proof of limit (1.7), which is relatively more complicated; see Remark 6.1. Let us take any $u_0 = (v_0, \theta_0) \in H^4$ and $\xi \in H^5$ such that $\pi_1\xi = 0$ and consider the function $w(t) = u(\delta t) - q(t)$, where $u(t) = S_t(u_0, \delta^{-1}\xi, 0)$, $q(t) = (q_1(t), q_2(t))$,

$$q_1(t) = v_0 - tQ_1\xi, \tag{6.1}$$

$$q_2(t) = \theta_0 - t\,(L_2\zeta + B_2(v_0,\zeta)) + \frac{t^2}{2}B_2(Q_1\xi,\zeta), \tag{6.2}$$

and $\zeta = \pi_2\xi$. For any $r > 0$, we show that

$$w(1) \to 0 \quad \text{in } V \text{ as } \delta \to 0^+ \tag{6.3}$$

uniformly with respect to u_0 and ζ satisfying

$$\|u_0\|_4 + \|\zeta\|_5 \leq r. \tag{6.4}$$

Note that $v = \pi_1 w$ is a solution of the following equation:

$$\partial_t v + \delta L_1(v+q_1) + \delta\langle v+q_1, \nabla\rangle(v+q_1) - \delta\int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(v+q_1)$$
$$+\delta f(v+q_1)^\perp + \delta\nabla p_s - \delta\int_0^z \nabla(\theta + q_2 + \delta^{-1}\zeta)\,\mathrm{d}\mathfrak{z} + \partial_t q_1 = \delta h_1.$$

Using (6.1) and (1.3), we see that this equation is equivalent to

$$\partial_t v + \delta L_1(v+q_1) + \delta\langle v+q_1,\nabla\rangle(v+q_1) - \delta\int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(v+q_1)$$
$$+\delta f(v+q_1)^\perp + \delta\nabla p_s - \delta\int_0^z \nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z} = \delta h_1. \tag{6.5}$$

In a similar way, $\theta = \pi_2 w$ is a solution of the equation

$$\partial_t\theta + \delta L_2(\theta+q_2+\delta^{-1}\zeta) + \delta\langle v+q_1,\nabla\rangle(\theta+q_2+\delta^{-1}\zeta)$$
$$-\delta\int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\theta+q_2+\delta^{-1}\zeta) + \partial_t q_2 = \delta h_2,$$

which, in view of (6.2) and (1.2), can be rewritten as follows:

$$\partial_t\theta + \delta L_2(\theta+q_2) + \delta\langle v+q_1,\nabla\rangle(\theta+q_2)$$
$$-\delta\int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\theta+q_2) + \langle v,\nabla\rangle\zeta - \int_0^z \operatorname{div} v\,\mathrm{d}\mathfrak{z}\,\partial_z\zeta = \delta h_2. \tag{6.6}$$

The initial conditions are $v(0)=0$ and $\theta(0)=0$. Using Eqs. (6.5) and (6.6), we estimate the norms of v and θ. We begin with estimates for L^2-norms of v and θ, where we use standard energy methods. Then, following the ideas of Cao and Titi [CT07], we estimate the L^6-norm of v and H^1-norms of v and θ using specific barotropic-baroclinic formulation of PEs. The latter consists in representing the velocity field v as follows $v = \bar{v} + \tilde{v}$, where $\bar{v}$ is the vertical average of v and $\tilde{v}$ is the remaining part. The average $\bar{v}$ is two-dimensional, so it is treated by usual 2D NS methods, and the advantage of the reminder $\tilde{v}$ is that it satisfies an equation without pressure. We provide a very detailed proof of limit (6.3) that is divided into nine steps.

Step 1. L^2-estimate for v. The goal of the first two steps is to show the limit $\|v(1)\| + \|\theta(1)\| \to 0$ as $\delta \to 0^+$. We start by taking the scalar product in L^2 of Eq. (6.5) with v and integrating by parts:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \delta\nu_1\|\nabla v\|^2 + \delta\mu_1\|\partial_z v\|^2 = -\delta\,\langle L_1 q_1, v\rangle - \delta\,\langle\langle v+q_1,\nabla\rangle(v+q_1), v\rangle$$
$$+\delta\left\langle\int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(v+q_1), v\right\rangle - \delta\,\langle f(v+q_1)^\perp, v\rangle$$
$$-\delta\,\langle\nabla p_s, v\rangle + \delta\left\langle\int_0^z \nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}, v\right\rangle + \delta\,\langle h_1, v\rangle = \sum_{i=1}^7 I_i. \tag{6.7}$$

To estimate the terms I_1, I_4, I_6, I_7, we integrate by parts and use[11] the Cauchy–Schwarz and Young inequalities and the assumption that u_0 and ζ satisfy (6.4)

[11] These inequalities and assumptions are used in almost all the estimates below, so we will not mention them every time. The same letter C is used to denote constants which may change from line to line.

and $t \in [0,1]$:

$$
\begin{aligned}
&|I_1| \le \delta \|L_1 q_1\| \, \|v\| \le C\delta \|v\|, \\
&|I_4| = \delta |\langle f q_1^{\perp}, v\rangle| \le \delta \|f q_1\| \, \|v\| \le C\delta \|v\|, \\
&|I_6| \le C\delta \|\theta + q_2\| \, \|\nabla v\| \le C\delta \left(\|\theta\|^2 + 1\right) + \frac{\delta \nu_1}{4} \|\nabla v\|^2, \\
&|I_7| \le \delta \|h_1\| \, \|v\| \le C\delta \|v\|.
\end{aligned}
$$

Integrating by parts and using the condition $\int_{\mathbb{T}} \operatorname{div} v \, \mathrm{d}z = 0$, we get

$$
\begin{aligned}
I_5 &= -\delta \int_{\mathbb{T}^2} \nabla p_s(x,y) \left(\int_{\mathbb{T}} v(x,y,z) \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y \\
&= \delta \int_{\mathbb{T}^2} p_s(x,y) \left(\int_{\mathbb{T}} \operatorname{div} v(x,y,z) \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y = 0.
\end{aligned}
$$

To estimate I_2 and I_3, we note that

$$
\begin{aligned}
\left\langle \int_0^z \operatorname{div}(v + q_1) \, \mathrm{d}\mathfrak{z} \, \partial_z v, v \right\rangle &= \frac{1}{2} \int_{\mathbb{T}^3} \int_0^z \operatorname{div}(v + q_1) \, \mathrm{d}\mathfrak{z} \, \partial_z |v|^2 \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= -\frac{1}{2} \int_{\mathbb{T}^3} \operatorname{div}(v + q_1) \, |v|^2 \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \langle \langle (v + q_1), \nabla \rangle v, v \rangle .
\end{aligned}
$$

Thus

$$
\begin{aligned}
|I_2 + I_3| &\le \delta \left(\|v\| + \|q_1\|\right) \|\nabla q_1\|_\infty \|v\| + C\delta \left(\|\nabla v\| + \|\nabla q_1\|\right) \|\partial_z q_1\|_\infty \|v\| \\
&\le C\delta \left(\|v\|^2 + 1\right) + \frac{\delta \nu_1}{4} \|\nabla v\|^2.
\end{aligned}
$$

Combining the estimates for I_i with inequality (6.7), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|v\|^2 + \delta \nu_1 \|\nabla v\|^2 + \delta \mu_1 \|\partial_z v\|^2 \le C\delta \left(\|v\|^2 + \|\theta\|^2 + 1\right). \tag{6.8}
$$

Step 2. L^2-estimate for θ. Now we take the scalar product in L^2 of Eq. (6.6) with θ:

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\theta\|^2 + \delta \nu_2 \|\nabla \theta\|^2 + \delta \mu_2 \|\partial_z \theta\|^2 &= -\delta \langle L_2 q_2, \theta \rangle - \delta \langle \langle v + q_1, \nabla \rangle (\theta + q_2), \theta \rangle \\
&\quad + \delta \left\langle \int_0^z \operatorname{div}(v + q_1) \, \mathrm{d}\mathfrak{z} \, \partial_z (\theta + q_2), \theta \right\rangle - \langle \langle v, \nabla \rangle \zeta, \theta \rangle \\
&\quad + \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z \zeta, \theta \right\rangle + \delta \langle h_2, \theta \rangle = \sum_{i=1}^{6} J_i.
\end{aligned} \tag{6.9}
$$

We start with the terms J_1, J_4, J_5, J_6:

$$
\begin{aligned}
&|J_1| \le \delta \|L_2 q_2\| \, \|\theta\| \le C\delta \|\theta\|, \\
&|J_4| \le \|v\| \, \|\nabla \zeta\|_\infty \|\theta\| \le C \|v\| \, \|\theta\|, \\
&|J_5| \le C \|\nabla v\| \, \|\partial_z \zeta\|_\infty \|\theta\| \le C \|\nabla v\| \, \|\theta\|, \\
&|J_6| \le \delta \|h_2\| \, \|\theta\| \le C\delta \|\theta\|.
\end{aligned}
$$

To estimate J_2 and J_3, we use the equality

$$\begin{aligned}\left\langle \int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z\theta, \theta \right\rangle &= \frac{1}{2}\int_{\mathbb{T}^3}\int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\theta^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= -\frac{1}{2}\int_{\mathbb{T}^3} \operatorname{div}(v+q_1)\,(\theta^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \langle\langle (v+q_1), \nabla\rangle\theta, \theta\rangle\,.\end{aligned}$$

Then

$$\begin{aligned}|J_2+J_3| &\le \delta\left(\|v\|+\|q_1\|\right)\|\nabla q_2\|_\infty\|\theta\| + C\delta\left(\|\nabla v\|+\|\nabla q_1\|\right)\|\partial_z q_2\|_\infty\|\theta\| \\ &\le C\delta\left(\|v\|^2+\|\theta\|^2+1\right)+\delta\nu_1\|\nabla v\|^2.\end{aligned}$$

The estimates for J_i and (6.9) imply that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2+\delta\nu_2\|\nabla\theta\|^2+\delta\mu_2\|\partial_z\theta\|^2 &\le C\left(\|v\|^2+\|\nabla v\|^2+\|\theta\|^2+1\right) \\ &\quad + C\delta\left(\|v\|^2+\|\theta\|^2+1\right)+\delta\nu_1\|\nabla v\|^2.\end{aligned}$$

Combining this with (6.8), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2+\left(C(\nu_1\delta)^{-1}+1\right)\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\delta\nu_2\|\nabla\theta\|^2+\delta\mu_2\|\partial_z\theta\|^2 \\ \le C(1+\delta)\left(\|v\|^2+\|\theta\|^2+1\right).\end{aligned} \tag{6.10}$$

Integrating in time, we obtain

$$\|\theta(t)\|^2+\|v(t)\|^2 \le C(1+\delta)\int_0^t\left(\|v\|^2+\|\theta\|^2+1\right)\mathrm{d}s.$$

The Gronwall inequality implies

$$\sup_{t\in[0,1],\,\delta\in(0,1]}\left(\|\theta(t)\|^2+\|v(t)\|^2\right)<\infty.$$

Going back to (6.8) and (6.10), we see that

$$\|v(t)\|^2+\|\theta(t)\|^2+\int_0^t\left(\|v\|_1^2+\|\theta\|_1^2\right)\mathrm{d}s \le C\delta \quad \text{for } t\in[0,1]. \tag{6.11}$$

Step 3. Barotropic-baroclinic formulation. Following [CT07], next we estimate the L^6-norm of v and H^1-norms of v and θ by using barotropic-baroclinic formulation of PEs. More precisely, we denote

$$\bar\phi = (2\pi)^{-1}\int_{\mathbb{T}}\phi(x,y,z)\,\mathrm{d}z, \quad \tilde\phi = \phi-\bar\phi.$$

Then the barotropic mode $\bar v$ satisfies the following system of equations:

$$\begin{aligned}\partial_t\bar v-\delta\nu_1\Delta(\bar v+\bar q_1)+\delta\langle\bar v+\bar q_1,\nabla\rangle(\bar v+\bar q_1) \\ +\delta\overline{\langle\tilde v+\tilde q_1,\nabla\rangle(\tilde v+\tilde q_1)+\operatorname{div}(\tilde v+\tilde q_1)(\tilde v+\tilde q_1)}+\delta f(\bar v+\bar q_1)^\perp \\ +\delta\nabla p_s-\delta\overline{\int_0^z\nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}}=\delta\bar h_1, \quad \operatorname{div}\bar v=0,\end{aligned} \tag{6.12}$$

and the baroclinic mode $\tilde{v}$ the following one:

$$\begin{aligned}\partial_t\tilde{v}+\delta L_1(\tilde{v}+\tilde{q}_1)+\delta\langle\tilde{v}+\tilde{q}_1,\nabla\rangle(\tilde{v}+\tilde{q}_1)-\delta\int_0^z \operatorname{div}(\tilde{v}+\tilde{q}_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\tilde{v}+\tilde{q}_1)\\ +\delta\langle\bar{v}+\bar{q}_1,\nabla\rangle(\tilde{v}+\tilde{q}_1)+\delta\langle\tilde{v}+\tilde{q}_1,\nabla\rangle(\bar{v}+\bar{q}_1)\\ -\delta\overline{\langle\tilde{v}+\tilde{q}_1,\nabla\rangle(\tilde{v}+\tilde{q}_1)+\operatorname{div}(\tilde{v}+\tilde{q}_1)(\tilde{v}+\tilde{q}_1)}+\delta f(\tilde{v}+\tilde{q}_1)^{\perp}\\ -\delta\int_0^z\nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}+\delta\overline{\int_0^z\nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}}=\delta\tilde{h}_1\end{aligned}\tag{6.13}$$

(see [CT07] for details). The advantage of this representation is that there is no pressure term in Eq. (6.13) and the barotropic mode depends only on horizontal variables (x,y) (its properties are similar to the ones of 2D NS system).

Step 4. L^6-estimate for $\tilde{v}$. We take the scalar product in L^2 of Eq. (6.13) with $\tilde{v}|\tilde{v}|^4$:

$$\begin{aligned}\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}\|_{L^6}^6+\delta\nu_1\||\nabla\tilde{v}|\,|\tilde{v}|^2\|^2+\delta\nu_1\|\tilde{v}\,|\nabla|\tilde{v}|^2|\|^2\\ +\delta\mu_1\||\partial_z\tilde{v}|\,|\tilde{v}|^2\|^2+\delta\mu_1\|\tilde{v}\,|\partial_z|\tilde{v}|^2|\|^2=-\delta\left\langle L_1\tilde{q}_1,\tilde{v}|\tilde{v}|^4\right\rangle\\ -\delta\left\langle\langle\tilde{v}+\tilde{q}_1,\nabla\rangle(\tilde{v}+\tilde{q}_1)-\int_0^z\operatorname{div}(\tilde{v}+\tilde{q}_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\tilde{v}+\tilde{q}_1),\tilde{v}|\tilde{v}|^4\right\rangle\\ -\delta\left\langle\langle\bar{v}+\bar{q}_1,\nabla\rangle(\tilde{v}+\tilde{q}_1),\tilde{v}|\tilde{v}|^4\right\rangle-\delta\left\langle\langle\tilde{v}+\tilde{q}_1,\nabla\rangle(\bar{v}+\bar{q}_1),\tilde{v}|\tilde{v}|^4\right\rangle\\ +\delta\left\langle\overline{\langle\tilde{v}+\tilde{q}_1,\nabla\rangle(\tilde{v}+\tilde{q}_1)+\operatorname{div}(\tilde{v}+\tilde{q}_1)(\tilde{v}+\tilde{q}_1)},\tilde{v}|\tilde{v}|^4\right\rangle\\ +\delta\left\langle\int_0^z\nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}-\overline{\int_0^z\nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}},\tilde{v}|\tilde{v}|^4\right\rangle\\ -\delta\left\langle f(\tilde{v}+\tilde{q}_1)^{\perp},\tilde{v}|\tilde{v}|^4\right\rangle+\delta\left\langle h_1,\tilde{v}|\tilde{v}|^4\right\rangle=\sum_{i=1}^8 I_i.\end{aligned}\tag{6.14}$$

Then

$$\begin{aligned}&|I_1|\le\delta\|L_1\tilde{q}_1\|_{L^6}\|\tilde{v}\|_{L^6}^5\le C\delta\|\tilde{v}\|_{L^6}^5,\\ &|I_7|=\delta|\langle f\tilde{q}_1^{\perp},\tilde{v}|\tilde{v}|^4\rangle|\le C\delta\|\tilde{q}_1\|_{L^6}\|\tilde{v}\|_{L^6}^5\le C\delta\|\tilde{v}\|_{L^6}^5,\\ &|I_8|\le\delta\|\tilde{h}_1\|_{L^6}\|\tilde{v}\|_{L^6}^5\le C\delta\|\tilde{v}\|_{L^6}^5.\end{aligned}$$

Integrating by parts, we see that

$$\left\langle\langle\tilde{v}+\tilde{q}_1,\nabla\rangle\tilde{v}-\int_0^z\operatorname{div}(\tilde{v}+\tilde{q}_1)\,\mathrm{d}\mathfrak{z}\,\partial_z\tilde{v},\tilde{v}|\tilde{v}|^4\right\rangle=0,$$

which implies (again by integrating by parts)

$$\begin{aligned}|I_2|&\le\delta\left(\|\tilde{v}\|_{L^6}+\|\tilde{q}_1\|_{L^6}\right)\|\nabla\tilde{q}_1\|_\infty\|\tilde{v}\|_{L^6}^5\\ &\quad+C\delta\left(\|\tilde{v}\|_{L^6}+\|\tilde{q}_1\|_{L^6}\right)\|\nabla\tilde{q}_1\|_\infty\|\tilde{v}\|_{L^6}^5\\ &\quad+C\delta\left(\|\tilde{v}\|_{L^6}+\|\tilde{q}_1\|_{L^6}\right)\|\tilde{v}\|_{L^6}^2\|\partial_z\tilde{q}_1\|_\infty\||\nabla\tilde{v}|\,|\tilde{v}|^2\|\\ &\le C\delta\left(\|\tilde{v}\|_{L^6}^6+1\right)+\frac{\delta\nu_1}{9}\||\nabla\tilde{v}|\,|\tilde{v}|^2\|^2.\end{aligned}$$

As $\operatorname{div}\bar{v} = \operatorname{div}\bar{q}_1 = 0$, we have that

$$\left\langle \langle \bar{v} + \bar{q}_1, \nabla \rangle \tilde{v}, \tilde{v}|\tilde{v}|^4 \right\rangle = 0,$$

hence

$$|I_3| \leq \delta \left(\|\bar{v}\|_{L^6} + \|\bar{q}_1\|_{L^6} \right) \|\nabla \tilde{q}_1\|_\infty \|\tilde{v}\|_{L^6}^5 \leq C\delta \left(\|\bar{v}\|_{L^6} + 1 \right) \|\tilde{v}\|_{L^6}^5.$$

To estimate I_4, we first integrate by parts:

$$I_4 = \delta \left\langle (\bar{v} + \bar{q}_1) \operatorname{div} (\tilde{v} + \tilde{q}_1), \tilde{v}|\tilde{v}|^4 \right\rangle + \delta \left\langle \langle \tilde{v} + \tilde{q}_1, \nabla \rangle (\tilde{v}|\tilde{v}|^4), \bar{v} + \bar{q}_1 \right\rangle.$$

We decompose I_4 as $I_4 = I_4^1 + I_4^2$, where

$$I_4^1 = -\delta \left\langle \langle \tilde{v}, \nabla \rangle \bar{v}, \tilde{v}|\tilde{v}|^4 \right\rangle = \delta \left\langle \bar{v} \operatorname{div} \tilde{v}, \tilde{v}|\tilde{v}|^4 \right\rangle + \delta \left\langle \langle \tilde{v}, \nabla \rangle (\tilde{v}|\tilde{v}|^4), \bar{v} \right\rangle.$$

It is proved on pages 255–257 in [CT07] that

$$\begin{aligned} |I_4^1| &\leq C\delta \left(\|\bar{v}\|^{\frac{1}{2}} \|\nabla \bar{v}\|^{\frac{1}{2}} \|\tilde{v}\|_{L^6}^{\frac{3}{2}} \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \|^{\frac{3}{2}} + \|\bar{v}\|^{\frac{1}{2}} \|\nabla \bar{v}\|^{\frac{1}{2}} \|\tilde{v}\|_{L^6}^6 \right) \\ &\leq C\delta \left(\|\bar{v}\|^2 \|\nabla \bar{v}\|^2 + 1 \right) \|\tilde{v}\|_{L^6}^6 + \frac{\delta \nu_1}{9} \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \|^2. \end{aligned}$$

Then, integrating by parts, we estimate the term I_4^2 as follows:

$$\begin{aligned} |I_4^2| \leq\ & C\delta \big(\|\bar{v}\|_{L^6} \|\nabla \tilde{q}_1\|_\infty \|\tilde{v}\|_{L^6}^5 + \|\bar{q}_1\|_\infty \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \| \, \|\tilde{v}\|_{L^6}^3 \\ & + \|\bar{q}_1\|_\infty \|\nabla \tilde{q}_1\|_{L^6} \|\tilde{v}\|_{L^6}^5 + \|\tilde{q}_1\|_\infty \|\tilde{v}\|_{L^6}^2 \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \| \left(\|\bar{v}\|_{L^6} + \|\bar{q}_1\|_{L^6} \right) \big) \\ \leq\ & C\delta \left(\|\bar{v}\|_{L^6} \|\tilde{v}\|_{L^6}^5 + \|\bar{v}\|_{L^6}^2 \|\tilde{v}\|_{L^6}^4 + \|\tilde{v}\|_{L^6}^6 + 1 \right) + \frac{\delta \nu_1}{9} \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \|^2. \end{aligned}$$

To estimate I_5, we first integrate by parts:

$$\begin{aligned} I_5 &= \delta \left\langle \overline{\langle \tilde{v} + \tilde{q}_1, \nabla \rangle (\tilde{v} + \tilde{q}_1) + \operatorname{div}(\tilde{v} + \tilde{q}_1)(\tilde{v} + \tilde{q}_1)}, \tilde{v}|\tilde{v}|^4 \right\rangle \\ &= -\delta \sum_{k,j=1}^{2} \int_{\mathbb{T}^3} \overline{(\tilde{v} + \tilde{g}_1)_k (\tilde{v} + \tilde{g}_1)_j} \partial_k (\tilde{v}_j |\tilde{v}|^4), \end{aligned}$$

where $\partial_1 = \partial_x$ and $\partial_2 = \partial_y$. We write $I_5 = I_5^1 + I_5^2$, where

$$I_5^1 = -\delta \sum_{k,j=1}^{2} \int_{\mathbb{T}^3} \overline{\tilde{v}_k \tilde{v}_j} \partial_k (\tilde{v}_j |\tilde{v}|^4).$$

By the computations on pages 255–257 in [CT07], we have

$$\begin{aligned} |I_5^1| &\leq C\delta \|\tilde{v}\|_{L^6}^3 \left(\|\tilde{v}\| + \|\nabla \tilde{v}\| \right) \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \| \\ &\leq C\delta \|\tilde{v}\|_{L^6}^6 \left(\|\tilde{v}\|^2 + \|\nabla \tilde{v}\|^2 \right) + \frac{\delta \nu_1}{9} \| |\nabla \tilde{v}| \, |\tilde{v}|^2 \|^2. \end{aligned}$$

Then we estimate I_5^2 as follows:

$$
\begin{aligned}
|I_5^2| &\le C\delta \int_{\mathbb{T}^2} \left(\left(\int_{\mathbb{T}} (|\tilde{v}|+1)\ \mathrm{d}z \right)^2 \int_{\mathbb{T}} |\nabla\tilde{v}||\tilde{v}|^4\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y \\
&\le C\delta \int_{\mathbb{T}^2} \left(\int_{\mathbb{T}} |\tilde{v}|^2\,\mathrm{d}z \int_{\mathbb{T}} |\nabla\tilde{v}||\tilde{v}|^4\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y + C\delta \int_{\mathbb{T}^3} |\nabla\tilde{v}||\tilde{v}|^4 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}
$$

The first term on the right-hand side is estimated exactly in the same way as I_5^1 (see [CT07]), and the second term by

$$
C\delta \int_{\mathbb{T}^3} |\nabla\tilde{v}||\tilde{v}|^4 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le C\delta \| |\nabla\tilde{v}|\,|\tilde{v}|^2 \|\,\|\tilde{v}\|_{L^4}^2 \le C\delta \|\tilde{v}\|_{L^4}^4 + \frac{\delta\nu_1}{9} \| |\nabla\tilde{v}|\,|\tilde{v}|^2 \|^2.
$$

It remains to estimate I_6. To this end, we write $I_6 = I_6^1 + I_6^2$, where

$$
I_6^1 = \delta \left\langle \int_0^z \nabla\theta\,\mathrm{d}\mathfrak{z} - \overline{\int_0^z \nabla\theta\,\mathrm{d}\mathfrak{z}}, \tilde{v}|\tilde{v}|^4 \right\rangle.
$$

Again we refer to pages 255–257 in [CT07] for the proof of the following inequality:

$$
\begin{aligned}
|I_6^1| &\le C\delta \|\bar{\theta}\|^{\frac{1}{2}} \|\nabla\bar{\theta}\|^{\frac{1}{2}} \|\tilde{v}\|_{L^6}^{\frac{3}{2}} \left(\|\tilde{v}\|^{\frac{1}{2}} + \|\nabla\tilde{v}\|^{\frac{1}{2}} \right) \| |\nabla\tilde{v}|\,|\tilde{v}|^2 \| \\
&\le C\delta \left(\|\bar{\theta}\|^2 \|\nabla\bar{\theta}\|^2 + \|\tilde{v}\|_{L^6}^6 \|\tilde{v}\|^2 + \|\tilde{v}\|_{L^6}^6 \|\nabla\tilde{v}\|^2 \right) + \frac{\delta\nu_1}{9} \| |\nabla\tilde{v}|\,|\tilde{v}|^2 \|^2.
\end{aligned}
$$

Then we estimate I_6^2 as follows:

$$
|I_6^2| \le C\delta \|\nabla q_2\|_\infty \|\tilde{v}\|_{L^5}^5 \le C\delta \|\tilde{v}\|_{L^5}^5.
$$

Combining all the above estimates for the terms I_i with (6.14), (6.11), the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, and the Gronwall inequality, we conclude that

$$
\|\tilde{v}(t)\|_{L^6} + \int_0^t \| |\nabla\tilde{v}|\,|\tilde{v}|^2 \|^2\,\mathrm{d}s \le C\delta \quad \text{for } t \in [0,1]. \tag{6.15}
$$

Step 5. Estimate for $\nabla\bar{v}$. Here we take the scalar product in L^2 of the first equation in (6.12) with $-\Delta\bar{v}$:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{v}\|^2 + \delta\nu_1\|\Delta\bar{v}\|^2 &= -\delta\nu_1 \langle \Delta\bar{q}_1, \Delta\bar{v} \rangle + \delta \left\langle \langle \bar{v}+\bar{q}_1, \nabla \rangle (\bar{v}+\bar{q}_1), \Delta\bar{v} \right\rangle \\
&\quad + \delta \left\langle \overline{\langle \tilde{v}+\tilde{q}_1, \nabla \rangle (\tilde{v}+\tilde{q}_1) + \mathrm{div}(\tilde{v}+\tilde{q}_1)(\tilde{v}+\tilde{q}_1)}, \Delta\bar{v} \right\rangle \\
&\quad + \delta \left\langle f(\bar{v}+\bar{q}_1)^\perp, \Delta\bar{v} \right\rangle + \delta \left\langle \nabla p_s, \Delta\bar{v} \right\rangle \\
&\quad - \delta \left\langle \overline{\int_0^z \nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z}}, \Delta\bar{v} \right\rangle - \delta \left\langle \bar{h}_1, \Delta\bar{v} \right\rangle = \sum_{i=1}^7 I_i.
\end{aligned} \tag{6.16}
$$

We estimate[12] I_1, I_4, I_7 as follows:

$$
\begin{aligned}
|I_1| &\le \delta\nu_1 \|\Delta\bar{q}_1\| \, \|\Delta\bar{v}\| \le C\delta + \frac{\delta\nu_1}{10}\|\Delta\bar{v}\|^2, \\
|I_4| &= \delta|\langle f\bar{q}_1^{\perp}, \Delta\bar{v}\rangle| \le C\delta + \frac{\delta\nu_1}{10}\|\Delta\bar{v}\|^2, \\
|I_7| &\le \delta\|\bar{h}_1\| \, \|\Delta\bar{v}\| \le C\delta + \frac{\delta\nu_1}{10}\|\Delta\bar{v}\|^2.
\end{aligned}
$$

Integrating by parts and using the fact that $\operatorname{div}\bar{v} = 0$, we get $I_5 = I_6 = 0$. Next we use the Hölder inequality, the Sobolev embedding $H^1(\mathbb{T}^2) \subset L^4(\mathbb{T}^4)$, and the interpolation inequality to estimate I_2:

$$
\begin{aligned}
|I_2| &\le \delta\|\bar{v} + \bar{q}_1\|_{L^4}\|\nabla(\bar{v} + \bar{q}_1)\|_{L^4}\|\Delta\bar{v}\| \le \delta\|\bar{v} + \bar{q}_1\|^{\frac{1}{2}}\|\nabla(\bar{v} + \bar{q}_1)\| \, \|\Delta\bar{v}\|^{\frac{3}{2}} \\
&\le C\delta\left(\|\nabla\bar{v}\|^6 + 1\right) + \frac{\delta\nu_1}{10}\|\Delta\bar{v}\|^2.
\end{aligned}
$$

Finally, we use the Hölder inequality to estimate I_3:

$$
\begin{aligned}
|I_3| &\le C\delta \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}} |\tilde{v} + \tilde{q}_1| \, |\nabla(\tilde{v} + \tilde{q}_1)| \, \mathrm{d}z\right) |\Delta\bar{v}| \, \mathrm{d}x \, \mathrm{d}y \\
&\le C\delta \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}} (|\tilde{v}| + 1)\,(|\nabla\tilde{v}| + 1) \, \mathrm{d}z\right) |\Delta\bar{v}| \, \mathrm{d}x \, \mathrm{d}y \\
&\le C\delta \int_{\mathbb{T}^2}\left(\int_{\mathbb{T}} (|\nabla\tilde{v}| + 1) \, \mathrm{d}z\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}} (|\tilde{v}| + 1)^2 (|\nabla\tilde{v}| + 1) \, \mathrm{d}z\right)^{\frac{1}{2}} |\Delta\bar{v}| \, \mathrm{d}x \, \mathrm{d}y \\
&\le C\delta\left(\|\nabla\tilde{v}\|^2 + \|\tilde{v}\|_{L^4}^2 + \||\nabla\tilde{v}| \, |\tilde{v}|^2\|^2 + 1\right) + \frac{\delta\nu_1}{10}\|\Delta\bar{v}\|^2.
\end{aligned}
$$

Combining the estimates for I_i and inequalities (6.16), (6.15), and (6.11), we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{v}\|^2 + \delta\nu_1\|\Delta\bar{v}\|^2 \le C\delta\left(\|\nabla\bar{v}\|^6 + 1\right). \tag{6.17}
$$

This inequality implies that

$$
\|\nabla\bar{v}(t)\|^2 + \int_0^t \|\Delta\bar{v}\|^2 \mathrm{d}s \le C\delta \quad \text{for } t \in [0, 1], \tag{6.18}
$$

provided that $\delta > 0$ is sufficiently small. Indeed, to see this, let $A_\delta = C\delta$ and

$$
\Phi(t) = A_\delta + A_\delta \int_0^t \|\nabla\bar{v}\|^6 \, \mathrm{d}s.
$$

By inequality (6.17), we have

$$
\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi\right)^{\frac{1}{3}} \le A_\delta^{\frac{1}{3}}\Phi,
$$

[12] In the estimate for I_4, we use the equality $\langle\bar{v}^{\perp}, \Delta\bar{v}\rangle = 0$ which is easily verified by integration by parts in x and y.

or, equivalently,

$$\frac{1}{\Phi^3}\frac{\mathrm{d}}{\mathrm{d}t}\Phi \le A_\delta.$$

Integrating this, we obtain

$$\Phi(t) \le A_\delta \left(1 - 2A_\delta^3 t\right)^{-\frac{1}{2}} \quad \text{for } 0 \le t < 1 \wedge \left(\frac{1}{2A_\delta^3}\right).$$

Thus

$$\Phi(t) \le 2A_\delta \quad \text{for } 0 \le t < 1 \wedge \left(\frac{3}{8A_\delta^3}\right).$$

Choosing $\delta_0 > 0$ so small that

$$\frac{3}{8A_\delta^3} > 1 \quad \text{for } \delta \in (0, \delta_0),$$

we arrive at

$$\Phi(t) \le 2A_\delta \quad \text{for } t \in [0,1],\ \delta \in (0,\delta_0).$$

Combining this with (6.17), we prove (6.18). Below everywhere we shall assume that $\delta \in (0, \delta_0)$.

Step 6. Estimate for $\partial_z v$. The function $\omega = \partial_z v$ is a solution of the equation

$$\begin{aligned}\partial_t \omega + \delta L_1(\omega + \hat{q}_1) + \delta\langle v + q_1, \nabla\rangle(\omega + \hat{q}_1) - \delta \int_0^z \operatorname{div}(v + q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\omega + \hat{q}_1)\\ +\delta\langle \omega + \hat{q}_1, \nabla\rangle(v + q_1) - \delta \operatorname{div}(v + q_1)\,(\omega + \hat{q}_1)\\ +\delta f(\omega + \hat{q}_1)^\perp - \delta\nabla(\theta + q_2) = \delta\hat{h}_1,\end{aligned} \tag{6.19}$$

where we denote $\hat{q}_1 = \partial_z q_1$ and $\hat{h}_1 = \partial_z h_1$. Let us take the scalar product in L^2 of Eq. (6.19) with ω:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2 + \delta\nu_1\|\nabla\omega\|^2 + \delta\mu_1\|\partial_z\omega\|^2 = -\delta\,\langle L_1\hat{q}_1, \omega\rangle\\ &\quad - \delta\left\langle \langle v + q_1, \nabla\rangle(\omega + \hat{q}_1) - \int_0^z \operatorname{div}(v + q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(\omega + \hat{q}_1), \omega\right\rangle\\ &\quad - \delta\,\langle \langle \omega + \hat{q}_1, \nabla\rangle(v + q_1) - \operatorname{div}(v + q_1)\,(\omega + \hat{q}_1), \omega\rangle\\ &\quad - \delta\left\langle f(\omega + \hat{q}_1)^\perp, \omega\right\rangle + \delta\,\langle \nabla(\theta + q_2), \omega\rangle + \delta\langle \hat{h}_1, \omega\rangle = \sum_{i=1}^6 I_i.\end{aligned} \tag{6.20}$$

Then I_1, I_4, I_5, I_6 are estimated as follows:

$$\begin{aligned}&|I_1| \le \delta\|L_1 q_1\|\,\|\omega\| \le C\delta\|\omega\|,\\ &|I_4| = \delta|\langle f\hat{q}_1^\perp, \omega\rangle| \le \delta\|f\hat{q}_1\|\,\|\omega\| \le C\delta\|\omega\|,\\ &|I_5| \le C\delta\|\theta + q_2\|\,\|\nabla\omega\| \le C\delta\left(\|\theta\|^2 + 1\right) + \frac{\delta\nu_1}{4}\|\nabla\omega\|^2,\\ &|I_6| \le \delta\|\hat{h}_1\|\,\|\omega\| \le C\delta\|\omega\|.\end{aligned}$$

Integrating by parts in z, we get

$$\left\langle \langle v+q_1,\nabla\rangle\omega - \int_0^z \operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z\omega,\omega\right\rangle = 0,$$

so we can estimate I_2 by

$$\begin{aligned}|I_2| &\le \delta\left(\|v\|+\|q_1\|\right)\|\nabla\hat{q}_1\|_\infty\|\omega\| + C\delta\left(\|\nabla v\|+\|\nabla q_1\|\right)\|\partial_z\hat{q}_1\|_\infty\|\omega\|\\ &\le C\delta\left(\|v\|_1+1\right)\|\omega\|.\end{aligned}$$

Integrating by parts in x and y and using the Hölder, Gagliardo–Nirenberg, and Sobolev inequalities, we obtain

$$\begin{aligned}|I_3| &\le C\delta\int_{\mathbb{T}^3}|v+q_1|\left(|\nabla(\omega+\hat{q}_1)||\omega|+|\omega+\hat{q}_1||\nabla\omega|\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\le C\delta\int_{\mathbb{T}^3}\left(|v|+1\right)\left(|\omega|+1\right)\left(|\nabla\omega|+1\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\le C\delta\left(\|v\|_{L^6}+1\right)\left(\|\omega\|_{L^3}+1\right)\left(\|\nabla\omega\|+1\right)\\ &\le C\delta\left(\|v\|_{L^6}+1\right)\left(\|\omega\|^{\frac{1}{2}}\|\omega\|_1^{\frac{1}{2}}+1\right)\left(\|\nabla\omega\|+1\right)\\ &\le C\delta\left(\|v\|_{L^6}^4\|\omega\|^2+\|v\|_{L^6}^2+\|\omega\|^2+1\right)+\frac{\delta\nu_1}{2}\|\nabla\omega\|^2+\frac{\delta\mu_1}{2}\|\partial_z\omega\|^2\\ &\le C\delta\left(\left(\|\nabla\bar{v}\|^4+\|\tilde{v}\|_{L^6}^4\right)\|\omega\|^2+\|\nabla\bar{v}\|^2+\|\tilde{v}\|_{L^6}^2+\|\omega\|^2+1\right)\\ &\quad+\frac{\delta\nu_1}{4}\|\nabla\omega\|^2+\frac{\delta\mu_1}{2}\|\partial_z\omega\|^2.\end{aligned}$$

Combining the estimates for I_i with (6.20), the Gronwall inequality, and inequalities (6.11), (6.15), and (6.18), we obtain

$$\|\partial_z v(t)\|^2+\int_0^t\|\partial_z v\|_1^2\mathrm{d}s \le C\delta \quad \text{for } t\in[0,1]. \tag{6.21}$$

Step 7. Estimate for ∇v. We take the scalar product in L^2 of Eq. (6.5) with $-\Delta v$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2+\delta\nu_1\|\Delta v\|^2+\delta\mu_1\|\nabla\partial_z v\|^2 &= \delta\langle L_1 q_1,\Delta v\rangle\\ &\quad+\delta\left\langle\langle v+q_1,\nabla\rangle(v+q_1),\Delta v\right\rangle\\ &\quad-\delta\left\langle\int_0^z\operatorname{div}(v+q_1)\,\mathrm{d}\mathfrak{z}\,\partial_z(v+q_1),\Delta v\right\rangle\\ &\quad+\delta\left\langle f(v+q_1)^\perp,\Delta v\right\rangle+\delta\left\langle\nabla p_s,\Delta v\right\rangle\\ &\quad-\delta\left\langle\int_0^z\nabla(\theta+q_2)\,\mathrm{d}\mathfrak{z},\Delta v\right\rangle-\delta\left\langle h_1,\Delta v\right\rangle=\sum_{i=1}^7 I_i.\end{aligned} \tag{6.22}$$

Again the terms I_1, I_4, I_6, I_7 are easier to estimate:[13]

$$
\begin{aligned}
&|I_1| \le \delta \|L_1 q_1\| \, \|\Delta v\| \le C\delta + \frac{\delta \nu_1}{16} \|\Delta v\|^2, \\
&|I_4| = \delta |\langle f q_1^\perp, \Delta v\rangle| \le \delta \|f q_1\| \, \|\Delta v\| \le C\delta + \frac{\delta \nu_1}{16} \|\Delta v\|^2, \\
&|I_6| \le C\delta \|\theta + q_2\|_1 \, \|\Delta v\| \le C\delta \left(\|\theta\|_1^2 + 1\right) + \frac{\delta \nu_1}{16} \|\Delta v\|^2, \\
&|I_7| \le \delta \|h_1\| \, \|\Delta v\| \le C\delta + \frac{\delta \nu_1}{16} \|\Delta v\|^2.
\end{aligned}
$$

Integrating by parts in x and y and using the condition $\int_{\mathbb{T}} \operatorname{div} v \, \mathrm{d}z = 0$, we get

$$
\begin{aligned}
I_5 &= \delta \int_{\mathbb{T}^2} \nabla p_s(x,y) \left(\int_{\mathbb{T}} \Delta v(x,y,z) \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y \\
&= \delta \int_{\mathbb{T}^2} \nabla p_s(x,y) \, \Delta \left(\int_{\mathbb{T}} v(x,y,z) \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y \\
&= -\delta \int_{\mathbb{T}^2} p_s(x,y) \, \Delta \left(\int_{\mathbb{T}} \operatorname{div} v(x,y,z) \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y = 0.
\end{aligned}
$$

We decompose the terms I_2 and I_3 as follows:

$$
\begin{aligned}
&I_2 = P_1 + P_2, \quad P_1 = \delta \left\langle \langle v, \nabla \rangle v, \Delta v \right\rangle, \\
&I_3 = Q_1 + Q_2, \quad Q_1 = -\delta \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z v, \Delta v \right\rangle,
\end{aligned}
$$

and estimate the quadratic in v terms P_2 and Q_2 in the following way:

$$
\begin{aligned}
|P_2| &\le \delta \left(\|v\| \, \|\nabla q_1\|_\infty + \|\nabla v\| \, \|q_1\|_\infty + \|q_1\| \, \|\nabla q_1\|_\infty \right) \|\Delta v\| \\
&\le C\delta \left(\|v\|_1^2 + 1 \right) + \frac{\delta \nu_1}{16} \|\Delta v\|^2, \\
|Q_2| &\le C\delta \left(\|\partial_z v\| \, \|\nabla q_1\|_\infty + \|\nabla v\| \, \|\partial_z q_1\|_\infty + \|\nabla q_1\| \, \|\partial_z q_1\|_\infty \right) \|\Delta v\| \\
&\le C\delta \left(\|v\|_1^2 + 1 \right) + \frac{\delta \nu_1}{16} \|\Delta v\|^2.
\end{aligned}
$$

For P_1, we use the Hölder, Sobolev, and Gagliardo–Nirenberg inequalities:

$$
\begin{aligned}
|P_1| &\le C\delta \int_{\mathbb{T}^3} |v| |\nabla v| |\Delta v| \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \le C\delta \|v\|_{L^6} \|\nabla v\|_{L^3} \|\Delta v\| \\
&\le C\delta \|v\|_{L^6} \|\nabla v\|^{\frac{1}{2}} \|\nabla v\|_1^{\frac{1}{2}} \|\Delta v\| \\
&\le C\delta \left(\|\nabla \bar{v}\|^4 + \|\tilde{v}\|_{L^6}^4 \right) \|\nabla v\|^2 + C\delta \|\nabla \partial_z v\|^2 + \frac{\delta \nu_1}{16} \|\Delta v\|^2.
\end{aligned}
$$

Next, we use the following inequality which is proved in Proposition 2.2 in [CT03]:

$$
\left| \left\langle \int_0^z \operatorname{div} \phi \, \mathrm{d}\mathfrak{z} \, \varphi, \psi \right\rangle \right| \le C \|\phi\|_1^{\frac{1}{2}} \|L_1 \phi\|^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}} \|\varphi\|_1^{\frac{1}{2}} \|\psi\| \tag{6.23}
$$

[13] In the estimate for I_4, we use the equality $\langle v^\perp, \Delta v \rangle = 0$.

for any $\phi \in H^2(\mathbb{T}^3, \mathbb{R}^2)$, $\varphi \in H^1(\mathbb{T}^3, \mathbb{R})$, and $\psi \in L^2(\mathbb{T}^3, \mathbb{R})$. Applying (6.23), we obtain

$$
\begin{aligned}
|Q_1| &\le C\delta \|v\|_1^{\frac{1}{2}} \|L_1 v\|^{\frac{1}{2}} \|\partial_z v\|^{\frac{1}{2}} \|\partial_z v\|_1^{\frac{1}{2}} \|\Delta v\| \\
&\le C\delta \left(\|\nabla v\|^2 + \|\partial_z v\|^2 + \|v\|^2 \right) \|\partial_z v\|^2 \|\nabla \partial_z v\|^2 + C\delta \|\partial_{zz} v\|^2 + \frac{\delta \nu_1}{16} \|\Delta v\|^2.
\end{aligned}
$$

The estimates for P_1, P_2, Q_1, Q_2 and I_i, together with (6.22), the Gronwall inequality, and inequalities (6.11), (6.15), (6.18), and (6.21) imply that

$$
\|\nabla v(t)\|^2 + \int_0^t \|\Delta v\|^2 \mathrm{d}s \le C\delta \quad \text{for } t \in [0,1].
$$

Combining this with (6.21), we get

$$
\|v(t)\|_1^2 + \int_0^t \|v\|_2^2 \, \mathrm{d}s \le C\delta \quad \text{for } t \in [0,1], \tag{6.24}
$$

which implies, in particular that $\|v(1)\|_1 \to 0$ as $\delta \to 0^+$.

Step 8. Estimate for $\partial_z \theta$. Now we turn to the H^1-estimates for θ. To estimate $\partial_z \theta$, we take the scalar product in L^2 of Eq. (6.6) with $-\partial_{zz}\theta$:

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_z \theta\|^2 + \delta \nu_2 \|\nabla \partial_z \theta\|^2 + \delta \mu_2 \|\partial_{zz} \theta\|^2 &= \delta \langle L_2 q_2, \partial_{zz} \theta \rangle \\
&+ \delta \langle \langle v + q_1, \nabla \rangle (\theta + q_2), \partial_{zz} \theta \rangle + \langle \langle v, \nabla \rangle \zeta, \partial_{zz} \theta \rangle \\
&- \delta \left\langle \int_0^z \operatorname{div}(v + q_1) \, \mathrm{d}\mathfrak{z} \, \partial_z (\theta + q_2), \partial_{zz} \theta \right\rangle - \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z \zeta, \partial_{zz} \theta \right\rangle \\
&- \delta \langle h_2, \partial_{zz} \theta \rangle = \sum_{i=1}^{6} J_i.
\end{aligned} \tag{6.25}
$$

We start with the terms J_1 and J_6:

$$
\begin{aligned}
|J_1| &\le \delta \|L_2 q_2\| \, \|\partial_{zz} \theta\| \le C\delta + \frac{\delta \mu_2}{12} \|\partial_{zz} \theta\|^2, \\
|J_6| &\le \delta \|h_2\| \, \|\partial_{zz} \theta\| \le C\delta + \frac{\delta \mu_2}{12} \|\partial_{zz} \theta\|^2.
\end{aligned}
$$

To estimate J_3 and J_5, we integrate by parts in z and use the Cauchy–Schwarz and Sobolev inequalities:

$$
\begin{aligned}
|J_3| &\le |\langle \langle \partial_z v, \nabla \rangle \zeta, \partial_z \theta \rangle| + |\langle \langle v, \nabla \rangle \partial_z \zeta, \partial_z \theta \rangle| \\
&\le \|\partial_z v\| \, \|\nabla \zeta\|_\infty \|\partial_z \theta\| + \|v\| \, \|\nabla \partial_z \zeta\|_\infty \|\partial_z \theta\| \le C \|v\|_1 \|\partial_z \theta\|, \\
|J_5| &\le |\langle \operatorname{div} v \, \partial_z \zeta, \partial_z \theta \rangle| + \left| \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_{zz} \zeta, \partial_z \theta \right\rangle \right| \\
&\le C \|\nabla v\| \, \|\partial_z \zeta\|_\infty \|\partial_z \theta\| + C \|\nabla v\| \, \|\partial_{zz} \zeta\|_\infty \|\partial_z \theta\| \le C \|v\|_1 \|\partial_z \theta\|.
\end{aligned}
$$

We write the terms J_2 and J_4 as follows:

$$J_2 = P_1 + P_2, \quad P_1 = \delta \left\langle \langle v, \nabla \rangle \theta, \partial_{zz}\theta \right\rangle,$$
$$J_4 = Q_1 + Q_2, \quad Q_1 = -\delta \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z \theta, \partial_{zz}\theta \right\rangle,$$

and estimate P_2 and Q_2 as in the previous steps:

$$\begin{aligned}
|P_2| &\le \delta \left(\|v\| \, \|\nabla q_2\|_\infty + \|\nabla \theta\| \, \|q_1\|_\infty + \|q_1\| \, \|\nabla q_2\|_\infty \right) \|\partial_{zz}\theta\| \\
&\le C\delta \left(\|v\|_1^2 + \|\theta\|_1^2 + 1 \right) + \frac{\delta \mu_2}{12} \|\partial_{zz}\theta\|^2, \\
|Q_2| &\le C\delta \left(\|\nabla v\| \, \|\partial_z q_2\|_\infty + \|\partial_z \theta\| \, \|\nabla q_1\|_\infty + \|\nabla q_1\| \, \|\partial_z q_2\|_\infty \right) \|\partial_{zz}\theta\| \\
&\le C\delta \left(\|v\|_1^2 + \|\theta\|_1^2 + 1 \right) + \frac{\delta \mu_2}{12} \|\partial_{zz}\theta\|^2.
\end{aligned}$$

To estimate P_1, we use the Hölder, Sobolev, and Gagliardo–Nirenberg inequalities:

$$\begin{aligned}
|P_1| &\le \delta \|v\|_{L^6} \|\nabla \theta\|_{L^3} \|\partial_{zz}\theta\| \le C\delta \|v\|_1 \|\nabla \theta\|^{\frac{1}{2}} \|\nabla \theta\|_1^{\frac{1}{2}} \|\partial_{zz}\theta\| \\
&\le C\delta \|v\|_1^4 \|\theta\|_1^2 + \delta \nu_2 \|\Delta \theta\|^2 + \frac{\delta \mu_2}{12} \|\partial_{zz}\theta\|^2.
\end{aligned}$$

To estimate Q_1, we first integrate by parts in z:

$$\begin{aligned}
\left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z \theta, \partial_{zz}\theta \right\rangle &= \frac{1}{2} \int_{\mathbb{T}^3} \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z ((\partial_z \theta)^2) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= -\frac{1}{2} \int_{\mathbb{T}^3} (\partial_z \theta)^2 \, \operatorname{div} v \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,
\end{aligned}$$

then we use the Cauchy–Schwarz and Gagliardo–Nirenberg inequalities:

$$\begin{aligned}
|Q_1| &\le C\delta \|\nabla v\| \, \|\partial_z \theta\|_{L^4}^2 \le C\delta \|\nabla v\| \, \|\partial_z \theta\|^{\frac{1}{2}} \|\partial_z \theta\|_1^{\frac{3}{2}} \\
&\le C\delta \|v\|_1^4 \|\theta\|_1^2 + \frac{\delta \nu_2}{2} \|\nabla \partial_z \theta\|^2 + \frac{\delta \mu_2}{12} \|\partial_{zz}\theta\|^2.
\end{aligned}$$

Combining the estimates for P_1, P_2, Q_1, Q_2 and J_i with inequalities (6.11), (6.24), and (6.25), we obtain

$$\|\partial_z \theta(t)\|^2 + \delta \mu_2 \int_0^t \|\partial_{zz}\theta\|^2 \mathrm{d}s \le C\delta + \delta \nu_2 \int_0^t \|\Delta \theta\|^2 \mathrm{d}s. \tag{6.26}$$

Step 9. Estimate for $\nabla \theta$. Finally, to estimate $\nabla \theta$, we take the scalar product in L^2 of Eq. (6.6) with $-\Delta \theta$:

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \theta\|^2 + \delta \nu_2 \|\Delta \theta\|^2 + \delta \mu_2 \|\nabla \partial_z \theta\|^2 &= \delta \left\langle L_2 q_2, \Delta \theta \right\rangle \\
&+ \delta \left\langle \langle v + q_1, \nabla \rangle (\theta + q_2), \Delta \theta \right\rangle + \left\langle \langle v, \nabla \rangle \zeta, \Delta \theta \right\rangle \\
&- \delta \left\langle \int_0^z \operatorname{div}(v + q_1) \, \mathrm{d}\mathfrak{z} \, \partial_z (\theta + q_2), \Delta \theta \right\rangle - \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z \zeta, \Delta \theta \right\rangle \\
&- \delta \left\langle h_2, \Delta \theta \right\rangle = \sum_{i=1}^6 J_i.
\end{aligned} \tag{6.27}$$

The terms J_1 and J_6 are estimated as follows:

$$|J_1| \le \delta \|L_2 q_2\| \, \|\Delta\theta\| \le C\delta + \frac{\delta\nu_2}{12}\|\Delta\theta\|^2,$$
$$|J_6| \le \delta \|h_2\| \, \|\Delta\theta\| \le C\delta + \frac{\delta\nu_2}{12}\|\Delta\theta\|^2.$$

To estimate J_3 and J_5, we integrate by parts and use the Cauchy–Schwarz and Sobolev inequalities:

$$|J_3| \le C\|v\|_1 \|\theta\|_1,$$
$$|J_5| \le C\|\Delta v\| \, \|\theta\|_1.$$

The terms J_2 and J_3 are decomposed as follows:

$$J_2 = P_1 + P_2, \qquad P_1 = \delta \left\langle \langle v, \nabla\rangle\theta, \Delta\theta \right\rangle,$$
$$J_3 = Q_1 + Q_2, \quad Q_1 = -\delta \left\langle \int_0^z \operatorname{div} v \, \mathrm{d}\mathfrak{z} \, \partial_z\theta, \Delta\theta \right\rangle,$$

and P_2 and Q_2 are estimated by

$$\begin{aligned}
|P_2| &\le \delta \left(\|v\| \, \|\nabla q_2\|_\infty + \|\nabla\theta\| \, \|q_1\|_\infty + \|q_1\| \, \|\nabla q_2\|_\infty \right) \|\Delta\theta\| \\
&\le C\delta \left(\|v\|_1^2 + \|\theta\|_1^2 + 1 \right) + \frac{\delta\nu_2}{12}\|\Delta\theta\|^2, \\
|Q_2| &\le C\delta \left(\|\nabla v\| \, \|\partial_z q_2\|_\infty + \|\partial_z\theta\| \, \|\nabla q_1\|_\infty + \|\nabla q_1\| \, \|\partial_z q_1\|_\infty \right) \|\Delta\theta\| \\
&\le C\delta \left(\|v\|_1^2 + \|\theta\|_1^2 + 1 \right) + \frac{\delta\nu_2}{12}\|\Delta\theta\|^2.
\end{aligned}$$

Using the Hölder, Sobolev, and Gagliardo–Nirenberg inequalities, we obtain

$$\begin{aligned}
|P_1| &\le \delta \|v\|_{L^6} \|\nabla\theta\|_{L^3} \|\Delta\theta\| \le C\delta \|v\|_1 \|\nabla\theta\|^{\frac12} \|\nabla\theta\|_1^{\frac12} \|\Delta\theta\| \\
&\le C\delta \|v\|_1^4 \|\theta\|_1^2 + \frac{\delta\mu_2}{4}\|\partial_{zz}\theta\|^2 + \frac{\delta\nu_2}{12}\|\Delta\theta\|^2.
\end{aligned}$$

By (6.23), we have

$$\begin{aligned}
|Q_1| &\le C\delta \|v\|_1^{\frac12} \|L_1 v\|^{\frac12} \|\partial_z\theta\|^{\frac12} \|\partial_z\theta\|_1^{\frac12} \|\Delta\theta\| \\
&\le C\delta \|v\|_1^2 \|L_1 v\|^2 \|\partial_z\theta\|^2 + \frac{\delta\mu_2}{4}\|\partial_{zz}\theta\|^2 + \frac{\delta\nu_2}{12}\|\Delta\theta\|^2.
\end{aligned}$$

The estimates of P_1, P_2, Q_1, Q_2 and I_i and the inequalities (6.11), (6.24), and (6.27) imply that

$$\|\nabla\theta(t)\|^2 + \delta\nu_2 \int_0^t \|\Delta\theta\|^2 \, \mathrm{d}s \le C\delta + \delta\mu_2 \|\partial_{zz}\theta\|^2.$$

From this and (6.26) we derive that $\|\theta(t)\|_1 \le C\delta$, so $\|\theta(1)\|_1 \to 0$ as $\delta \to 0^+$. This completes the proof of limit (1.7).

Remark 6.1. Limit (1.6) can be established by repeating the arguments of the proof of limit (1.7), by considering the function

$$w(t) = u(\delta t) - q(t),$$

where $u(t) = S_t(u_0, \delta^{-\frac{1}{2}}\zeta, \delta^{-1}\eta)$, $q(t) = (q_1(t), q_2(t))$,

$$\begin{aligned} q_1(t) &= v_0 + t(\eta_1 - B_1(\zeta_1)), \\ q_2(t) &= \theta_0 + t\eta_2, \end{aligned}$$

$\eta_i = \pi_i\eta$, and $\zeta_1 = \pi_1\zeta$. See Proposition 2.4 in [Ner20] for a proof of a limit similar to (1.6) in the case of parabolic equations with polynomially growing nonlinearities.

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