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Positive and sign-changing solutions for a quasilinear Steklov nonlinear boundary problem with critical growth

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Abstract

In this work we study the existence of positive solutions and nodal solutions for the following p -laplacian problem with Steklov boundary conditions on a bounded regular domain $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = 0 & \text{in } \Omega; \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda a(x)|u|^{p-2}u + b(x)|u|^{p^*-2}u & \text{on } \partial\Omega; \end{cases}$$

with given numbers p, N satisfying $1 < p < N$, $p_* := \frac{p(N-1)}{N-p}$ the critical exponent for the Sobolev trace map $W^{1,p}(\Omega) \rightarrow L^q(\partial\Omega)$ and functions $b \geq 0$ and a, V possibly indefinite. By minimization on subsets of the associated Nehari manifold, we prove the existence of positive solutions if $N \geq \max\{2p-1, 3\}$ and the parameter λ close to the principal eigenvalues of the operator $-\Delta_p + V$ with weighted-Steklov boundary conditions. We also prove the existence on nodal solutions for a definite and $N > \max\{p^2, 2p, \frac{p}{p-1}, 2\}$. Our results show striking differences between the cases $p > 2, p = 2$ and $p < 2$.

Keywords: Critical growth, indefinite weights, Steklov boundary conditions, p -laplacian operator. 35D05; 35J60; 35J65; 35J70; 35J25; 35J35.

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1. Introduction

Consider the following problem of parameter λ

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = 0 & \text{in } \Omega; \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda a(x)|u|^{p-2}u + b(x)|u|^{p^*-2}u & \text{on } \partial\Omega; \end{cases} \quad (1.1)$$

for $1 < p < N$, a, b two given functions in $C^\gamma(\partial\Omega)$ for some $\gamma > 0$, $a \not\equiv 0$ with $b \geq 0$, $V \in L^\infty(\Omega)$ and $p_* := \frac{p(N-1)}{N-p}$. The domain Ω is a bounded subset of \mathbb{R}^N of class $C^{2,\alpha}$ for some $0 < \alpha < 1$ and $N \geq 3$. Our aim is to prove the existence of solutions for λ close to the principal eigenvalues of (1.5) (see below).

In the case $a \equiv 0$, $V \equiv 1$, $b \equiv 1$, the quasilinear problem (1.1) arises, for instance, when searching for functions $u \in W^{1,p}(\Omega)$ for which the norm of the Sobolev's trace immersion $i_{p_*,\Omega} : W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$ is achieved:

$$S_0 := \|i_{p_*,\Omega}\|^{-p} = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega (|\nabla u|^p + |u|^p) dx}{\left(\int_{\partial\Omega} |u|^{p^*} d\sigma\right)^{p/p_*}}, \quad (1.2)$$

where σ is the restriction to $\partial\Omega$ of the the $(N-1)$ -Hausdorff measure, which coincides with the usual Lebesgue surface measure as $\partial\Omega$ is regular enough. Due to the lack of compactness of $i_{p_*,\Omega}$, the existence of minimizers for (1.2) does not follows by standards methods. Following the ideas of [2], [8] and [5], Fernandez-Bonder and Rossi proved in [7] that a sufficient condition for the existence of minimizers for (1.2) is that $S_0 < K_{N,p}^{-1}$ where

$$K_{N,p}^{-1} \stackrel{\text{def}}{=} \inf \left\{ \int_{\mathbb{R}_+^N} |\nabla u|^p dx; |\nabla u| \in L^p(\mathbb{R}_+^N) \text{ and } \int_{\mathbb{R}^{N-1}} |u|^{p^*} dy = 1 \right\}. \quad (1.3)$$

In the linear case, i.e. $p = 2$, with $b \equiv 1$ and $V \equiv 0$, namely, for the problem

$$(Y) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \frac{N-2}{2}\beta u = u^{2^*-1} & \text{on } \partial\Omega, \end{cases}$$

which is related to the Yamabé problem when $\beta = cte = \text{mean curvature of } \partial\Omega$, Adimurthi-Yadava [1] proved that problem (Y) has solution when $\beta \in C^1(\partial\Omega)$, $N \geq 3$ and there exists a point $x_0 \in \partial\Omega$ such that

$$\beta(x_0) < h(x_0) := \frac{1}{N-1} \sum_{i=1}^{N-1} \nu_i, \quad (1.4)$$

where the ν_i are the principal curvatures at $x_0 \in \partial\Omega$ with respect to the unit outward normal. Finally, problem (1.1) in case $p = 2$ and $V = 0$ can also be related to well known λ - parameter problem of the Brézis-Nirenberg [4] with Dirichlet boundary condition

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Among the huge amount of improvements and generalization of this pioneering work we quote the work of Cerami-Solimini-Struwe [6] where they stated the existence of sign changing solutions of the Dirichlet problem for $\lambda \in (0, \lambda_1)$ and $N \geq 6$. We have adapted here their approach to our quasilinear problem with nonlinear boundary conditions.

Quasilinear elliptic problems with an indefinite potential V have attracted a lot of attention the last decade. After the work concerning the eigenvalue problem with Dirichlet boundary condition with an indefinite weights in [9] and the one for the eigenvalue problem with Steklov boundary conditions in [13], some others quasilinear problems with weights have been considered with sublinear, superlinear or concave-convex nonlinear terms. In the present work we would like to explore the effect of sign-changing weights a and V on the multiplicity of solutions for a rather simple critical-exponent quasilinear problem with a parameter λ . From the variational point of view, the geometry of the related functional associated, for example, to the eigenvalue problem

$$\begin{cases} -\Delta_p u + V|u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda a(x)|u|^{p-2}u & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

may take in consideration the disjoint subsets

$$\int_{\partial\Omega} a|u|^p d\sigma > 0 \text{ and } \int_{\partial\Omega} a|u|^p d\sigma < 0.$$

It is well known (see [13]) that, if a changes sign, there are two principal eigenvalues $\lambda_{-1} < \lambda_1$ for the above eigenvalue problem. We will prove in this work that positive and sign-changing solutions of problem (1.1) can also be found by minimizing the energy functional on the subset of the Nehari manifolds where $\int_{\partial\Omega} a|u|^p d\sigma \geq 0$. By considering indefinite weights, we improve and complete several existing results for similar problems.

This paper is organised as follows. In section 1 we study under which conditions the infimum of the associated energy functional along the Nehari manifold is achieved. We prove in Proposition 2.7 that this is the case where this infimum is less than

$$\left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_*-p}} \quad \text{where} \quad K_0 \stackrel{\text{def}}{=} \|b\|_{\infty, \partial\Omega}^{\frac{p}{p_*}} K_{N,p}^{-1}.$$

In order to assure this inequality we use the well known technique of *mass concentration* for the fundamental solutions, i.e. functions defined in \mathbb{R}_+^N realizing the infimum in (1.3). In section 2 we analyse the different Lebesgue norms of these functions and in section 3 we state our main existence result in Theorem 4.2. In section 4 we study the infimum of the associated energy Φ_λ along the so called *nodal subsets of the Nehari manifold*. Finally Theorem 6.2 state an existence result for positive weights a .

2. Minimization on the Nehari manifold

Let us define the following C^1 -functional on $W^{1,p}(\Omega)$ by

$$\begin{aligned} E_V(u) &\stackrel{\text{def}}{=} \int_{\Omega} (|\nabla u|^p + V(x)|u|^p) dx, & A(u) &\stackrel{\text{def}}{=} \int_{\partial\Omega} a |u|^p d\sigma, \\ B(u) &\stackrel{\text{def}}{=} \int_{\partial\Omega} b |u|^{p_*} d\sigma, & \Phi_\lambda(u) &\stackrel{\text{def}}{=} E_V(u) - \lambda A(u). \end{aligned}$$

The natural norm of $W^{1,p}(\Omega)$ will be denoted by $\|\cdot\|$, i.e.,

$$\forall u \in W^{1,p}(\Omega), \quad \|u\| = \left(\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^p dx \right)^{1/p}.$$

The Lebesgue norm of $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$ and the Lebesgue norm of $L^q(\partial\Omega, \rho)$ by $\|\cdot\|_{q, \partial\Omega}$, for any $q \in [q, +\infty[$. Solutions of problem (1.1) will be understood in the weak sense.

As in [6] we will make use of the Nehari manifold associated to our problem. For this end, we define the energy functional

$$I_\lambda(u) = \frac{1}{p} \Phi_\lambda(u) - \frac{1}{p_*} B(u)$$

and the Nehari manifold associated to I_λ

$$\mathcal{N} = \{u \in W^{1,p}(\Omega) \setminus \{0\}; \langle I'_\lambda(u), u \rangle = 0\} = \{u \in W^{1,p}(\Omega) \setminus \{0\}; \Phi_\lambda(u) = B(u)\}$$

that we split into three sets

$$\mathcal{A}_+ = \{u \in \mathcal{N}; A(u) > 0\}, \quad \mathcal{A}_- = \{u \in \mathcal{N}; A(u) < 0\}, \quad \mathcal{A}_0 = \{u \in \mathcal{N}; A(u) = 0\}.$$

It is well known that critical points of I_λ are solutions of problem (1.1) and belong to \mathcal{N} . Notice that I_λ restricted to \mathcal{N} is equal to

$$I_\lambda(u) = \left(\frac{1}{p} - \frac{1}{p_*}\right) B(u) = \left(\frac{1}{p} - \frac{1}{p_*}\right) \Phi_\lambda(u).$$

Minimizing the functional I_λ along \mathcal{A}_\pm provided us with positive solutions of our problem (1.1). Precisely, let us set

$$C_\lambda^\pm = \inf_{u \in \mathcal{A}_\pm} I_\lambda(u) \tag{2.1}$$

The following result is well known, we give the proof for the sake of completeness.

Lemma 2.1. *If C_λ^\pm is achieved and $C_\lambda^\pm > 0$ then C_λ^\pm is a critical value of I_λ associated to a positive solution of (1.1).*

Proof. Let $u \in \mathcal{A}_+$ such that $C_\lambda^+ = I_\lambda(u) = \inf_{v \in \mathcal{A}_+} I_\lambda(v)$. By taking $|u|$ instead of u we can assume that the infimum is achieved at some $u \geq 0$ in \mathcal{A}_+ . Furthermore, if we set $J_\lambda = \Phi_\lambda - B$, we have that $u \in \mathcal{N} \implies J_\lambda(u) = 0$, $u \neq 0$ and

$$\langle J'_\lambda(u), u \rangle = p\Phi_\lambda(u) - p_*B(u) = (p - p_*)B(u).$$

Observe that, since $0 < C_\lambda^+ = I_\lambda(u) = \left(\frac{1}{p} - \frac{1}{p_*}\right) B(u)$ then $B(u) \neq 0$ and therefore $\langle J'_\lambda(u), u \rangle \neq 0$. By Lagrange's Multipliers theorem there exists $\alpha \in \mathbb{R}$ such that $I'_\lambda(u) = \alpha J'_\lambda(u)$. Hence

$$0 = J_\lambda(u) = \langle I'_\lambda(u), u \rangle = \alpha(p - p_*)B(u) \implies \alpha = 0.$$

Thus $I'_\lambda(u) = 0$. □

The aim of this section is to prove that the previous infima are achieved and that they are strictly positive. The positivity of I_λ depend on whenever

$\lambda < \lambda_1$ or $\lambda > \lambda_{-1}$, where λ_1 and λ_{-1} are defined as follows. Let us recall the following results on the eigenvalue problem (1.5) associated to our problem (see [13]). By a principal eigenvalue we mean an eigenvalue having a positive eigenfunction.

Proposition 2.2 ([13]). *Let*

$$\alpha_a \stackrel{\text{def}}{=} \inf\{E_V(u); \|u\|_p = 1, A(u) = 0\}. \quad (2.2)$$

Then problem (1.5) possesses a principal eigenvalue if and only if $\alpha_a > 0$. Precisely,

1. *if $\alpha_a > 0$ and a changes sign then (1.5) admits exactly two principal eigenvalues $\lambda_{-1} < \lambda_1$, with*

$$\lambda_1 := \min_{\mathcal{M}^+} E_V, \quad (2.3)$$

where $\mathcal{M}^+ := \{u \in W^{1,p}(\Omega); A(u) = 1\}$ and

$$\lambda_{-1} = -\min_{\mathcal{M}^-} E_V, \quad (2.4)$$

where $\mathcal{M}^- := \{u \in W^{1,p}(\Omega); A(u) = -1\}$;

2. *if $\alpha_a > 0$ and a is of definite sign then (1.5) admits exactly one principal eigenvalue, which are either λ_1 or λ_{-1} ;*
3. *if $\alpha_a = 0$ then (1.5) has a unique principal eigenvalue λ_* given by*

$$\lambda_* = \inf_{\mathcal{M}^+} = -\inf_{\mathcal{M}^-} E_V.$$

Moreover a function $u \in S$ is an eigenfunction associated to λ_ if and only if $A(u) = 0$ and $E_V(u) = \alpha_a = 0$.*

Remark 2.3. *Actually, the hypothesis of Theorem 3.3 of [13] are that both $\lambda_V^D = \text{first eigenvalue of } u \rightarrow -\Delta_p u + V|u|^{p-2}u \text{ with Dirichlet boundary condition and}$*

$$\beta(V, a) = \inf\{E_V(u); A(u) = 0, \|u\|_{p,\partial\Omega} = 1\}$$

are > 0 . These two hypothesis are equivalent to $\alpha_a > 0$.

As a straightforward consequence of the above proposition we have

Corollary 2.4. *Assume $\alpha_a > 0$. For any $\lambda < \lambda_1$ (resp. for any $\lambda > \lambda_{-1}$) there exists $c > 0$ such that, for all $u \in W^{1,p}(\Omega)$ satisfying $A(u) \geq 0$ (resp. $A(u) \leq 0$) it holds*

$$E_V(u) - \lambda A(u) \geq c\|u\|^p. \quad (2.5)$$

Remark 2.5. *Weak solutions of problem (1.1) and (1.5) belong to $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ according to [10]. Consequently weak solutions are of class $C^{1,\mu}(\bar{\Omega})$ for some $0 < \mu < 1$ (see [14]).*

Throughout the paper we will always assume $\alpha_a > 0$.

Let us now study the geometry of the fibering maps and the Nehari manifold.

Lemma 2.6.

1. *Assume either $\lambda < \lambda_1$ or $\lambda > \lambda_{-1}$. Then for any $u \in W^{1,p}(\Omega)$ such that $B(u) \neq 0$, the function $t \mapsto I_\lambda(tu)$ has a local maximum at*

$$0 < t_u := \left(\frac{\Phi_\lambda(u)}{B(u)} \right)^{\frac{1}{p^*-p}}, \quad (2.6)$$

$t_u u \in \mathcal{N}$ and

$$I_\lambda(t_u u) = \left(\frac{1}{p} - \frac{1}{p^*} \right) \left(\frac{\Phi_\lambda(u)}{B(u)^{p/p^*}} \right)^{\frac{p^*}{p^*-p}}.$$

2. *If $\lambda < \lambda_1$ then there exists a constant $c > 0$ such that*

$$\forall u \in \mathcal{A}_+ \cup \mathcal{A}_0 \implies \|u\| \geq c \text{ and } B(u) \geq c. \quad (2.7)$$

3. *If $\lambda > \lambda_{-1}$ then there exists a constant $c' > 0$ such that*

$$\forall u \in \mathcal{A}_- \cup \mathcal{A}_0 \implies \|u\| \geq c' \text{ and } B(u) \geq c'.$$

4. *All minimizing sequences for C_λ^\pm are bounded.*

5. $C_\lambda^\pm > 0$.

Proof. (1) For any $u \in W^{1,p}(\Omega)$ such that $B(u) > 0$ one easily proved that

$$g_u(t) = t^{p-1}\Phi_\lambda(u) - t^{p^*-1}B(u)$$

for $t > 0$, vanished at t_u and that the function $t \rightarrow I_\lambda(tu)$ has a global maximum at t_u . Clearly, $g_u(t) = 0 \Leftrightarrow tu \in \mathcal{N}$.

(2) We know from equation (2.5) that there exists a constant $c_1 > 0$ such that $\Phi_\lambda(u) \geq c_1\|u\|^p$. Moreover using Sobolev's embedding from the trace we have, for some constant $c_2 > 0$,

$$B(u) \leq c_2\|b\|_\infty\|u\|^{p^*}$$

and the conclusion follows using that $\Phi_\lambda(u) = B(u)$ because $u \in \mathcal{N}$. One can prove (3) in a similar way.

(4) Assume by contradiction that a minimizing $u_n \in \mathcal{A}_+$ is unbounded and take $v_n = \frac{u_n}{\|u_n\|}$. Thus, for a subsequence, there exists $v_0 \in W^{1,p}(\Omega)$ such that $v_n \rightharpoonup v_0$, strongly in $L^p(\Omega)$ and $L^p(\partial\Omega)$. Since

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\Phi_\lambda(v_n) = \frac{I_\lambda(u_n)}{\|u_n\|^p} \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

then

$$E_V(v_0) - \lambda A(v_0) \leq 0. \tag{2.8}$$

If $v_0 \equiv 0$ then

$$0 = E_V(v_0) - \lambda A(v_0) = \lim_{n \rightarrow \infty} \Phi_\lambda(v_n) \implies v_n \rightarrow 0 \text{ strongly in } W^{1,p}(\Omega),$$

what is in a contradiction with the fact that $\|v_n\| = 1$. Thus $v_0 \not\equiv 0$. Also we have $A(v_0) > 0$ because the possibility $A(v_0) = 0$ is ruled out from the condition $\alpha_a > 0$ and (2.8). If $\lambda < \lambda_1$ we then have a contradiction between (2.5) and (2.8)

(5) If for instance $C_\lambda^+ = 0$ and $(u_n)_n$ is a bounded minimizing sequence converging to some u_0 weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and also strongly in $L^p(\partial\Omega)$ hence $A(u_0) \geq 0$ and

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\Phi_\lambda(u_0) \leq \lim_{n \rightarrow \infty} I_\lambda(u_n) = C_\lambda^+ = 0. \tag{2.9}$$

If $u_0 \equiv 0$ then we will get from the last inequality that $u_n \rightarrow 0$ strongly in $W^{1,p}(\Omega)$, in contradiction with (2.7). Thus, $u_0 \not\equiv 0$ but now (2.9) contradicts (2.5). \square

In the next proposition we will prove that the values C_λ^\pm are achieved whenever they are smaller than a certain value involving $K_{N,p}$ if λ is close to λ_1 . This second constraint follows from the necessity to assure that the infimum is achieved at some point lying in the open set \mathcal{A}_+ . Precisely, let us consider

$$\gamma_{a,b} \stackrel{\text{def}}{=} \inf\{E_V(u); A(u) = 0, B(u) = 1\}. \quad (2.10)$$

Proposition 2.7. *One has*

1. $0 < \gamma_{a,b}$ and

$$C_\lambda^\pm \leq \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_*-p}}. \quad (2.11)$$

2. There exists $\delta_1 > 0$ (resp. $\delta_2 > 0$) such that

$$C_\lambda^+ < \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_*-p}} \quad \forall \lambda \in (\lambda_1 - \delta_1, \lambda_1),$$

$$(\text{ resp. } C_\lambda^- < \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_*-p}} \quad \forall \lambda \in (\lambda_{-1}, \lambda_{-1} + \delta_2)).$$

Proof. (1) It follows directly from $\alpha_a > 0$ that $\gamma_{a,b} \geq 0$. Assume by contradiction that $\gamma_{a,b} = 0$ and let $(u_n)_n$ be a minimizing sequence for $\gamma_{a,b}$. Assume furthermore that $(u_n)_n$ is an unbounded sequence and take $v_n = \frac{u_n}{\|u_n\|}$. Thus there exists a subsequence, still denoted v_n , and a function v_0 such that $v_n \rightarrow v_0$, strongly in $L^p(\Omega)$, in $L^p(\partial\Omega)$ and a.e. We have in one hand

$$\Phi_\lambda(v_0) \leq \liminf_{n \rightarrow +\infty} \Phi_\lambda(v_n) \leq 0, \quad (2.12)$$

and in other hand $A(v_0) = 0$. Besides $v_0 \not\equiv 0$ otherwise we will deduce from (2.12) that $v_n \rightarrow 0$ strongly in $W^{1,p}(\Omega)$, which is in contradiction with the fact that $\|v_n\| = 1$. Thus

$$\alpha_a \|v_0\|_p^p \leq E_V(v_0) = \Phi_\lambda(v_0) \leq 0,$$

which contradicts the hypothesis $\alpha_a > 0$. We conclude that the sequence $(u_n)_n$ is bounded. Hence, up to a subsequence, it converges weakly to some u_0 in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Hence $E_V(u_0) \leq 0$ and $A(u_0) = 0$. If $u_0 \not\equiv 0$ we have a contradiction with the hypothesis $\alpha_a > 0$. If $u_0 \equiv 0$ hence u_n converges strongly to 0, in contradiction with $B(u_n) = 1$.

Next, to prove for instance that $C_\lambda^+ \leq \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_*-p}}$, let u_n be a sequence in $W^{1,p}(\Omega)$ such that $A(u_n) = 0, B(u_n) = 1$ and $E_V(u_n) \rightarrow \gamma_{a,b}$. We can assume also that $u_n \geq 0$ by taking $|u_n|$ instead of u_n if necessary and, using the same argument as above, the sequence $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$. Let $\psi \in C^1(\bar{\Omega})$ be any positive function such that $\text{supp } \psi \cap \partial\Omega \subset \{x \in \partial\Omega; a(x) > 0\}$. Let us take $v_n = u_n + \frac{\psi}{n}$. Clearly $\|v_n - u_n\| \rightarrow 0$. Moreover

$$A(v_n) = \int_{\partial\Omega} a v_n^p = \int_{\partial\Omega} a^+(u_n + \frac{\psi}{n})^p - \int_{\partial\Omega} a^- u_n^p > A(u_n) = 0,$$

and clearly $B(v_n) \geq B(u_n) = 1$. Furthermore, using the following inequality

$$\|x + y\|^q - \|x\|^q - \|y\|^q \leq C \|xy\| (\|x\|^{q-2} + \|y\|^{q-2}), \quad (2.13)$$

valid for any $q \geq 1$ and any $x, y \in \mathbb{R}^N$ and using also that the sequence u_n is bounded we have

$$\Phi_\lambda(v_n) = E_V(u_n) + o(1) = \gamma_{a,b} + o(1).$$

Finally, if we consider $z_n := t_{v_n} v_n \in \mathcal{A}_+$, with t_{u_n} defined in Proposition (2.6), it comes

$$C_\lambda^+ \leq \left(\frac{1}{p} - \frac{1}{p_*}\right) \left(\frac{\Phi_\lambda(v_n)}{B(v_n)^{p/p_*}}\right)^{\frac{p_*}{p_*-p}} \rightarrow \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_*-p}}.$$

(2) We only prove the estimate for C_λ^+ . By taking $t_{\varphi_1} \varphi_1$, where φ_1 is the unique positive eigenfunction associated to λ_1 such that $A(\varphi_1) = 1$ and t_{φ_1} has been defined in (2.6), by definition of C_λ^+ one has

$$\left(\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+\right)^{\frac{p_*-p}{p_*}} \leq \frac{\lambda_1 - \lambda}{(B(\varphi_1))^{p/p_*}}$$

Thus, if $\lambda \in \mathbb{R}$ is such that $\frac{\lambda_1 - \lambda}{(B(\varphi_1))^{p/p_*}} < \gamma_{a,b}$, i.e.,

$$\lambda > \lambda_1 - (B(\varphi_1))^{p/p_*} \gamma_{a,b}$$

then $\left(\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+\right)^{\frac{p_*-p}{p_*}} < \gamma_{a,b}$ as stated in (2.15). \square

Consequently, let us define

$$\lambda_*^\pm \stackrel{\text{def}}{=} \inf \left\{ \lambda \in \mathbb{R}; C_\lambda^\pm < \left(\frac{1}{p} - \frac{1}{p_*} \right) \gamma_{a,b}^{\frac{p_*}{p_*-p}} \right\} \quad (2.14)$$

As a consequence of (2) in Proposition 2.7 we have $\lambda_*^+ < \lambda_1$ and $\lambda_*^- > \lambda_{-1}$ and therefore

$$\lambda_*^\pm = \sup \left\{ \lambda \in \mathbb{R}; C_\lambda^\pm = \left(\frac{1}{p} - \frac{1}{p_*} \right) \gamma_{a,b}^{\frac{p_*}{p_*-p}} \right\}.$$

Proposition 2.8. *Let $\lambda \in \mathbb{R}$.*

1. *If $\lambda < \lambda_1$ and*

$$C_\lambda^+ < \left(\frac{1}{p} - \frac{1}{p_*} \right) K_0^{-\frac{p_*}{p_*-p}} \quad (2.15)$$

then there exists $u \in \mathcal{A}_+ \cup \mathcal{A}_0$ such that $I_\lambda(u) = C_\lambda^+$.

Similarly, if $\lambda > \lambda_{-1}$ and

$$C_\lambda^- < \left(\frac{1}{p} - \frac{1}{p_*} \right) K_0^{-\frac{p_*}{p_*-p}} \quad (2.16)$$

then there exists $u \in \mathcal{A}_- \cup \mathcal{A}_0$ such that $I_\lambda(u) = C_\lambda^-$.

2. *If furthermore $\lambda_*^+ < \lambda < \lambda_1$ and (2.15) holds then problem (1.1) with parameter λ possesses a positive solution u satisfying $A(u) > 0$ and $I_\lambda u = C_\lambda^+$.*

Similarly if $\lambda_{-1} < \lambda < \lambda_^-$ and (2.16) holds then problem (1.1) with parameter λ possesses a positive solution u satisfying $A(u) < 0$ and $I_\lambda u = C_\lambda^-$.*

Proof. We will only give the proof concerning C_λ^+ since the argument is similar for C_λ^- . Let $(u_n)_n$ be a minimizing sequence. By (4) of Lemma 2.6 the sequence $(u_n)_n$ is bounded so assume that $u_n \in \mathcal{A}_+$ converges weakly to some u_0 , strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Clearly $A(u_0) \geq 0$.

Claim: *We have*

$$\Phi_\lambda(u_0) \leq \left(\left(\frac{1}{p} - \frac{1}{p_*} \right)^{-1} C_\lambda^+ \right)^{\frac{p_*-p}{p_*}} B(u_0)^{\frac{p}{p_*}}. \quad (2.17)$$

Indeed, in one hand, using that $(u_n)_n$ is a minimizing sequence we have

$$\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+ = B(u_n) + o(1). \quad (2.18)$$

Besides, we also have by the Brézis-Lieb lemma ([3])

$$\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+ = \Phi_\lambda(u_0) + \|\nabla(u_n - u_0)\|_p^p + o(1) \quad (2.19)$$

In other hand, let us choose $\epsilon > 0$ such that

$$\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+ < (K_0 + \epsilon)^{\frac{-p_*}{p_*-p}}.$$

Using again Brézis-Lieb lemma and the fact that $p/p_* < 1$, we get

$$B(u_n)^{p/p_*} \leq B(u_0)^{\frac{p}{p_*}} + B(u_n - u_0)^{\frac{p}{p_*}} + o(1),$$

and hence it comes from (2.18) and Lemma 2.10 (see below) gives

$$\left(\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+\right)^{p/p_*} \leq B(u_0)^{\frac{p}{p_*}} + (K_0 + \epsilon) \|\nabla(u_n - u_0)\|_p^p + o(1). \quad (2.20)$$

Combining (2.20) with (2.19) and using that

$$\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+ < (K_0 + \epsilon)^{\frac{-p}{p_*-p}},$$

we obtain

$$\begin{aligned} \left(\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+\right)^{p/p_*} &\leq B(u_0)^{\frac{p}{p_*}} + \left(\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+\right)^{\frac{p}{p_*}} \\ &\quad - \Phi_\lambda(u_0) \left(\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+\right)^{\frac{p-p_*}{p_*}} \end{aligned}$$

and the proof of the claim follows. Notice that $u_0 \not\equiv 0$ since, otherwise, $u_n \rightarrow 0$ strongly in $W^{1,p}(\Omega)$ which contradicts (3) of Lemma 2.6. As a consequence of (2.17) and that $\Phi_\lambda(u_0) > 0$ we have $B(u_0) > 0$. Finally let

us prove that C_λ^+ is achieved at $t_{u_0}u_0 \in \mathcal{N}$. Indeed, again by (1) of Lemma 2.6 we have

$$\left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+ \leq \frac{\Phi_\lambda(u_0)^{\frac{p_*}{p_*-p}}}{B(u_0)^{\frac{p}{p_*-p}}}$$

while by the claim

$$\frac{\Phi_\lambda(u_0)^{\frac{p_*}{p_*-p}}}{B(u_0)^{\frac{p}{p_*-p}}} \leq \left(\frac{1}{p} - \frac{1}{p_*}\right)^{-1} C_\lambda^+$$

and the equality follows.

(2) Since $\lambda > \lambda_*^+$ we get $C_\lambda^+ < \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_*-p}}$ and therefore C_λ^+ is achieved at some $u \in \mathcal{A}_+$. By replacing u by $|u|$ if necessary, we can assume that $u \geq 0$. The result then comes from Lemma 2.1. By the regularity results (see Remark 2.9) and the strong maximum principle of [16], the solution u is strictly positive up to the boundary. \square

Remark 2.9. Notice that if $\lambda_*^+ < \lambda_*^-$ then, under the hypothesis of Proposition 2.8, we will obtain two positive solutions of problem (1.1) for any parameter $\lambda \in (\lambda_*^+, \lambda_*^-) \cap (\lambda_{-1}, \lambda_1)$: one in \mathcal{A}_+ and the other in \mathcal{A}_- . However, that $\lambda_*^+ < \lambda_*^-$ is not clear for general weights a, b and V .

We have used in the previous proposition the following *Cherrier's-type inequality* that has been proved by [5] in the case $b \equiv 1$ and can be trivially generalized for any positive bounded weight b :

Lemma 2.10. [5] For any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all $u \in W^{1,p}(\Omega)$ it holds

$$\left(\int_{\partial\Omega} b|u|^{p_*} d\sigma\right)^{p/p_*} \leq (K_0 + \epsilon) \int_{\Omega} |\nabla u|^p dx + C_\epsilon \int_{\Omega} |u|^p dx$$

where

$$K_0 := K_{N,p} \|b\|_\infty^{p/p_*} \tag{2.21}$$

and $K_{N,p}$ is defined in (1.3).

3. Estimates of the L^p -norms of fundamental solutions

We turn now our attention to the problem of finding the values λ for which we have $S_\lambda^\pm < K_0^{-1}$, where we denote here, for simplicity,

$$S_\lambda^\pm = \left(\left(\frac{1}{p} - \frac{1}{p_*} \right)^{-1} C_\lambda^+ \right)^{\frac{p_*-p}{p_*}}. \quad (3.1)$$

It is well known (see [12]), that the value $K_{N,p}^{-1}$ defined in (1.3) is achieved at functions of the form

$$U_{\epsilon, y_0}(y, t) = \epsilon^{-\frac{N-p}{p}} U \left(\frac{y - y_0}{\epsilon}, \frac{t}{\epsilon} \right),$$

with $y_0 \in \mathbb{R}^{N-1}$ arbitrary and $\epsilon \in]0 + \infty[$, where

$$U(y, t) = \frac{1}{((t+1)^2 + |y|^2)^{\frac{N-p}{2(p-1)}}}.$$

The functions U_{ϵ, y_0} are usually called *fundamental solutions*. The constant $K_{N,p}^{-1}$ can be computed explicitly (see [8, 12]) and it is equal to

$$K_{N,p}^{-1} = \left(\frac{N-p}{p-1} \right)^{p-1} \pi^{\frac{p-1}{2}} \left(\frac{\Gamma \left(\frac{N-1}{2(p-1)} \right)}{\Gamma \left(\frac{p(N-1)}{2(p-1)} \right)} \right)^{\frac{p-1}{N-1}}.$$

Let us assume for convenience that

$$x_0 = 0 \in \partial\Omega \quad \text{and} \quad |\Omega_a^+ \cap B_s(0)| > 0 \quad \forall 0 < s < r \quad (3.2)$$

for some $r > 0$, where $\Omega_a^+ = \{x \in \partial\Omega; a(x) > 0\}$. Let ϕ be a smooth radial function with compact support in the ball $B_{r/2}(0)$ satisfying $\phi \equiv 1$ in $B_{r/4}(0)$. For any $\epsilon > 0$ let us choose the following test functions:

$$u_\epsilon(y, t) = U_\epsilon(y, t)\phi(y, t) = \frac{\epsilon^{\frac{N-p}{p(p-1)}} \phi(y, t)}{((t+\epsilon)^2 + |y|^2)^{\frac{N-p}{2(p-1)}}}. \quad (3.3)$$

Notice that $A(u_\epsilon) > 0$. In order to give the asymptotic development with respect to the parameter ϵ of the quotient $\frac{\Phi(u_\epsilon)}{B(u_\epsilon)^{p/p_*}}$, we will compute each of

integrals involved. Much of the work have been done by [8] and we refer the reader to this paper for full details. To make the computations simpler, we will choose a special parametrization of the boundary $\partial\Omega$ around $0 \in \partial\Omega$. Since we are assuming that $\partial\Omega$ is of class C^2 , there exists $c > 0$ and a C^2 -function $\rho : \{y \in \mathbb{R}^{N-1}, |y| \leq c\} \rightarrow \mathbb{R}$ such that

$$\begin{aligned}\Omega \cap B_r(0) &= \{(y, t) \in Q_c; t > \rho(y)\} \\ \partial\Omega \cap B_r(0) &= \{(y, t) \in Q_c; t = \rho(y)\},\end{aligned}\tag{3.4}$$

where $Q_c := \{(y, t), |y| \leq c, 0 \leq t \leq c\}$ and

$$\rho(y) = \frac{1}{2} \sum_{i=1}^{N-1} \nu_i y_i^2 + O(|y|^3)\tag{3.5}$$

for some $\nu_i, i = 1, \dots, N-1$. We set hereafter

$$h_0 = \frac{1}{N-1} \sum_{i=1}^{N-1} \nu_i.\tag{3.6}$$

The value h_0 is known as the *mean curvature of $\partial\Omega$ at 0* with respect to the outward normal ν .

Proposition 3.1. *Let $N \geq 2p - 1$. Assume for convenience the hypothesis (3.2) and let u_ϵ be as in (3.3). Then*

1. $\int_{\Omega} |\nabla u_\epsilon|^p dx = A_1 + f_1(\epsilon)$, where

$$f_1(\epsilon) := \begin{cases} A_2 \epsilon + O(\epsilon^{\frac{N-p}{p-1}}) & \text{if } p < \frac{N+1}{2} \\ -\frac{h_0}{2} \omega_{N-2} \epsilon \ln(1/\epsilon) + O(\epsilon) & \text{if } p = \frac{N+1}{2} \end{cases}$$

and

$$\begin{aligned}A_1 &= \frac{1}{2} \left(\frac{N-p}{p-1} \right)^{p-1} \beta \left(\frac{N-1}{2}, \frac{N-1}{2(p-1)} \right) \omega_{N-2}; \\ A_2 &= -\frac{h_0}{4} \left(\frac{N-p}{p-1} \right)^p \beta \left(\frac{N+1}{2}, \frac{N-2p+1}{2(p-1)} \right) \omega_{N-2}.\end{aligned}$$

2. $\int_{\Omega} V(x)|u_{\epsilon}|^p dx = f_2(\epsilon),$

$$f_2(\epsilon) := \begin{cases} O(\epsilon^p) & \text{if } p^2 < N \\ O(\epsilon^p \ln(\frac{1}{\epsilon})) & \text{if } p^2 = N \\ O(\epsilon^{\frac{N-p}{p-1}}) & \text{if } p^2 > N \end{cases}$$

3. Assume that $b(0) = \|b\|_{\infty}$ and

$$b(y, \rho(y)) - b(0) = O(|y|^{\gamma+1}) \quad \forall |y| \leq c$$

for some $\gamma > 0$. Then

$$\int_{\partial\Omega} b|u_{\epsilon}|^{p^*} d\sigma = B_1 + B_2\epsilon + o(\epsilon)$$

where

$$\begin{aligned} B_1 &= \frac{1}{2}\|b\|_{\infty}\beta \left(\frac{N-1}{2}, \frac{N-1}{2(p-1)} \right) \omega_{N-2}, \\ B_2 &= -\frac{1}{2}\|b\|_{\infty}(N-1)h_0\beta \left(\frac{N-1}{2}, \frac{N-1}{2(p-1)} \right) \omega_{N-2}. \end{aligned}$$

4. Assume that $a \in C^{\gamma}$ close to 0 for some $\gamma > 0$. Then

$$\int_{\partial\Omega} a(x)|u_{\epsilon}|^p d\sigma = f_3(\epsilon)$$

where

$$f_3(\epsilon) = \begin{cases} C_1\epsilon^{p-1} + o(\epsilon^{p-1}) & \text{if } N > p^2 - p + 1 \\ a(0)\omega_{N-2}\epsilon^{p-1} \ln(1/\epsilon) + O(\epsilon^{p-1}) & \text{if } N = p^2 - p + 1, \\ O(\epsilon^{\frac{N-p}{p-1}}) & \text{if } N < p^2 - p + 1, \end{cases} \quad (3.7)$$

and

$$C_1 = \frac{1}{2}a(0)\beta \left(\frac{N-1}{2}, \frac{N-p^2+p-1}{2(p-1)} \right) \omega_{N-2}.$$

We recall that

$$\omega_{N-1} = \text{measure of the unit sphere } S^{N-1} \text{ of } \mathbb{R}^N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

and

$$\beta(x, y) := \int_0^\infty \frac{t^{x-1} dt}{(1+t)^{x+y}} = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y > 0$.

Proof. (1) -(2) These estimates can be found in [8].

(3) To estimate $\int_{\partial\Omega} b(x)|u_\epsilon|^{p^*} d\sigma$ we write

$$\begin{aligned} \int_{\partial\Omega} b(x)|u_\epsilon|^{p^*} d\sigma &= b(0) \int_{\partial\Omega \cap B_r(0)} |u_\epsilon|^{p^*} d\sigma \\ &\quad + O\left(\int_{\partial\Omega \cap B_r(0)} |y|^{\gamma+1} |u_\epsilon|^{p^*} d\sigma\right) + \epsilon^{\frac{N-1}{p-1}} O(1). \end{aligned}$$

From basic integration, we deduce for any $a > -1$ and $b > 0$ that

$$\int_0^{c/\epsilon} \frac{t^a}{(1+t^2)^b} dt = \begin{cases} \frac{1}{2}\beta\left(\frac{a+1}{2}, \frac{2b-a-1}{2}\right) + O(\epsilon^{2b-a-1}) & \text{if } 2b-a-1 > 0, \\ \ln(1/\epsilon) + O(1) & \text{if } 2b-a-1 = 0, \\ O(\epsilon^{2b-a-1}) & \text{if } 2b-a-1 < 0. \end{cases} \quad (3.8)$$

Thus for any $a, b, \epsilon \in \mathbb{R}^+$ we have

$$\begin{aligned} &\int_{|y| \leq c} \frac{|y|^a}{(\epsilon^2 + |y|^2)^b} dy \\ &= \begin{cases} \omega_{N-2} \frac{1}{2} \epsilon^{N-1+a-2b} \beta\left(\frac{a+N-1}{2}, \frac{2b-a-N+1}{2}\right) + O(1) & \text{if } 2b-a-N+1 > 0, \\ \omega_{N-2} \ln(1/\epsilon) + O(1) & \text{if } 2b-a-N+1 = 0, \\ O(1) & \text{if } 2b-a-N+1 < 0. \end{cases} \quad (3.9) \end{aligned}$$

By expanding $((\epsilon + \rho(y))^2$ and using Taylor's theorem we find

$$\begin{aligned}
\epsilon^{\frac{N-1}{p-1}} \int_{\partial\Omega \cap B_r(0)} |y|^{\gamma+1} |u_\epsilon|^{p^*} d\sigma &= \int_{|y| \leq c} \frac{|y|^{\gamma+1} \sqrt{1 + |\nabla \rho(y)|^2} dy}{((\epsilon + \rho(y))^2 + |y|^2)^{\frac{p(N-1)}{2(p-1)}}} \\
&= \int_{|y| \leq c} \frac{|y|^{\gamma+1} dy}{(\epsilon^2 + |y|^2)^{\frac{p(N-1)}{2(p-1)}}} \\
&\quad + O\left(\int_{|y| \leq c} \frac{|y|^{\gamma+3} dy}{(\epsilon^2 + |y|^2)^{\frac{p(N-1)}{2(p-1)}}} \right) \\
&= \begin{cases} O(\epsilon^{\gamma+N-\frac{p}{p-1}(N-1)}) & \text{if } N > p(\gamma+1) - \gamma \\ O(\ln(1/\epsilon)) & \text{if } N = p(\gamma+1) - \gamma \\ O(1) & \text{if } N < p(\gamma+1) - \gamma. \end{cases}
\end{aligned}$$

Since $\frac{N-1}{p-1} + N + \gamma - \frac{p}{p-1}(N-1) = \gamma + 1$ then

$$\int_{\partial\Omega \cap B_r(0)} |y|^{\gamma+1} |u_\epsilon|^{p^*} d\sigma = o(\epsilon)$$

and the result follows.

(4) First of all we use the fact that $a \in C^\gamma(\partial\Omega)$ and write

$$a(y, \rho(y)) = a(0) + O(|y|^\gamma); \quad |y| \leq c$$

so

$$\begin{aligned}
\int_{\partial\Omega} a(x) |u_\epsilon|^p d\sigma &= a(0) \int_{\partial\Omega \cap B_r(0)} |u_\epsilon|^p d\sigma \\
&\quad + O\left(\int_{\partial\Omega \cap B_r(0)} |y|^\gamma |u_\epsilon|^p d\sigma \right) + \epsilon^{\frac{N-p}{p-1}} O(1).
\end{aligned}$$

Let us then compute separately

$$I := \int_{\partial\Omega \cap B_r(0)} |u_\epsilon|^p d\sigma \quad \text{and} \quad II := \int_{\partial\Omega \cap B_r(0)} |y|^\gamma |u_\epsilon|^p d\sigma.$$

$$\begin{aligned}
\epsilon^{-\frac{N-p}{p-1}} I &= \int_{|y| \leq c} \frac{1 + \frac{1}{2} |\nabla \rho(y)|^2 + O(|y|^4)}{(\epsilon^2 + \rho(y)^2 + 2\epsilon\rho(y) + |y|^2)^{\frac{p(N-p)}{2(p-1)}}} dy \\
&= \int_{|y| \leq c} \frac{dy}{(\epsilon^2 + |y|^2)^{\frac{p(N-p)}{2(p-1)}}} \\
&\quad - \frac{p(N-p)}{p-1} \epsilon \int_{|y| \leq c} \frac{\rho(y) dy}{(\epsilon^2 + |y|^2)^{\frac{p(N-p)}{2(p-1)} + 1}} \\
&\quad + O\left(\int_{|y| \leq c} \frac{|y|^2}{(\epsilon^2 + |y|^2)^{\frac{p(N-p)}{2(p-1)}}} \right) \\
&= I^1 - \frac{p(N-p)}{p-1} \epsilon I^2 + O(I^3)
\end{aligned}$$

where

$$I^1 = \omega_{N-2} \times \begin{cases} \frac{1}{2} \epsilon^{p-\frac{N-1}{p-1}} \beta \left(\frac{N-1}{2}, \frac{N-p^2+p-1}{2(p-1)} \right) + O(1) & \text{if } N > p^2 - p + 1 \\ \ln(1/\epsilon) + O(1) & \text{if } N = p^2 - p + 1 \\ C_2 & \text{if } N < p^2 - p + 1 \end{cases}$$

and $C_2 = \frac{1}{p^2-p+1-N} |c|^{p^2-p+1-N} \omega_{N-2} > 0$. Clearly $I^2 = O(I^1)$ and $I^3 = \epsilon^2 O(I^1)$. Consequently

$$I = \begin{cases} \frac{1}{2} \omega_{N-2} \epsilon^{p-1} \beta \left(\frac{N-1}{2}, \frac{N-p^2+p-1}{2(p-1)} \right) + O\left(\frac{N-p}{p-1}\right) & \text{if } N > p^2 - p + 1, \\ O\left(\epsilon^{\frac{N-p}{p-1}} \ln(1/\epsilon)\right) & \text{if } N = p^2 - p + 1; \\ O\left(\epsilon^{\frac{N-p}{p-1}}\right) & \text{if } N < p^2 - p + 1. \end{cases}$$

Similar computations for II give

$$\begin{aligned}
II &= \epsilon^{\frac{N-p}{p-1}} \int_{|y| \leq c} \frac{|y|^\gamma dy}{(\epsilon^2 + |y|^2)^{\frac{p(N-p)}{2(p-1)}}} \\
&= \begin{cases} O(\epsilon^{p-1+\gamma}) & \text{if } N > p^2 - (p-1)(1-\gamma); \\ O\left(\epsilon^{\frac{N-p}{p-1}} \ln(1/\epsilon)\right) & \text{if } N = p^2 - (p-1)(1-\gamma); \\ O(\epsilon^{\frac{N-p}{p-1}}) & \text{if } N < p^2 - (p-1)(1-\gamma). \end{cases}
\end{aligned}$$

Since by hypothesis $\gamma > 0$ then $II = I + o(I)$ and we conclude. \square

In section 5 we will need the ϵ -asymptotic of several L^q -norm of the fundamental solution u_ϵ defined in (3.3).

Proposition 3.2.

1. $\|\nabla u_\epsilon\|_1 = f_4(\epsilon)$, where

$$f_4(\epsilon) = \begin{cases} O\left(\epsilon^{\frac{N-p}{p(p-1)}}\right) & \text{if } p > \frac{2N-1}{N}, \\ O\left(\epsilon^{\frac{N-p}{p(p-1)}} \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } p = \frac{2N-1}{N}, \\ O(\epsilon^{N-\frac{N}{p}}) & \text{if } p \leq \frac{2N-1}{N}. \end{cases}$$

2. $\|\nabla u_\epsilon\|_{p-1}^{p-1} = O(\epsilon^{\frac{N}{p}-1})$.

3. $\|u_\epsilon\|_1 = f_5(\epsilon)$, where

$$f_5(\epsilon) = \begin{cases} O\left(\epsilon^{\frac{N-p}{p(p-1)}}\right) & \text{if } p > \frac{2N}{N+1}, \\ O\left(\epsilon^{\frac{N-p}{p(p-1)}} \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } p = \frac{2N}{N+1}, \\ O(\epsilon^{N+1-\frac{N}{p}}) & \text{if } p < \frac{2N}{N+1}. \end{cases}$$

4. $\|u_\epsilon\|_{p-1}^{p-1} = O(\epsilon^{\frac{N}{p}-1})$.

5. $\|u_\epsilon\|_{1,\partial\Omega} = f_4(\epsilon)$.

6. $\|u_\epsilon\|_{p-1,\partial\Omega}^{p-1} = O(\epsilon^{\frac{N}{p}-1})$.

$$7. \|u_\epsilon\|_{p^*-1, \partial\Omega}^{p^*-1} = O(\epsilon^{\frac{N}{p}-1}).$$

Proof. Let us denote by $\alpha = \frac{N-p}{p(p-1)}$.

(1) We have

$$|\nabla u_\epsilon| = \frac{N-p}{p-1} \epsilon^\alpha [(t+\epsilon)^2 + |y|^2]^{-\frac{N-1}{2(p-1)}} \quad \text{in } B_{r/4}(0).$$

so

$$\epsilon^{-\alpha} \int_{\Omega} |\nabla u_\epsilon| \, dx = \frac{N-p}{p-1} \int_{\Omega \cap B_{r/4}(0)} [(t+\epsilon)^2 + |y|^2]^{-\gamma} \, dydt + O(1),$$

where $\gamma = \frac{N-1}{2(p-1)}$. Notice that the integral on the right hand side goes to a constant as ϵ goes to 0 if $2\gamma - N < 0$, that is, if $p > \frac{2N-1}{N}$. In the case $1 < p \leq \frac{2N-1}{N}$ let us compute this integral as follows. We write

$$\begin{aligned} \int_{\Omega \cap B_{r/4}(0)} [(t+\epsilon)^2 + |y|^2]^{-\gamma} \, dydt &= \int_{Q_c} \frac{dt dy}{[(t+\epsilon)^2 + |y|^2]^\gamma} \\ &\quad - \int_{Q_c \setminus \Omega} \frac{dt dy}{[(t+\epsilon)^2 + |y|^2]^\gamma} \\ &= I_1 - I_2 + O(1) \end{aligned}$$

After firstly changing the variables t and y by $t\epsilon$ and $\epsilon.y$ respectively and secondly changing y by $(t+1).z$, one gets

$$\begin{aligned} I_1 &= C \epsilon^{N-2\gamma} \left(\int_0^{+\infty} (1+t)^{N-1-2\gamma} \, dt \right) \left(\int_0^{+\infty} \frac{r^{N-2}}{(1+r^2)^\gamma} \, dt \right) + O(1) \\ &= O(\epsilon^{N-2\gamma}) \end{aligned}$$

since $N - 2\gamma = \frac{Np-2N+1}{p-1} < 0$ in the case $1 < p < \frac{2N-1}{N}$. When $p = \frac{2N-1}{N}$ we have

$$I_1 = C \ln \left(\frac{1}{\epsilon} \right) + O(1) = -C \ln(\epsilon) + O(1).$$

For I_2 we have

$$\begin{aligned}
I_2 &= \int_{|y| \leq c} dy \int_0^{\rho(y)} \frac{dt}{((t + \epsilon)^2 + |y|^2)^\gamma} \\
&= \int_{|y| \leq c} \frac{\rho(y) dy}{(\epsilon^2 + |y|^2)^\gamma} + \epsilon O\left(\int_{|y| \leq c} \frac{|y|^4 dy}{(\epsilon^2 + |y|^2)^{\gamma+1}}\right) \\
&= O\left(\int_{|y| \leq c} \frac{|y|^2 dy}{(\epsilon^2 + |y|^2)^\gamma}\right) + \epsilon O\left(\int_{|y| \leq c} \frac{|y|^4 dy}{(\epsilon^2 + |y|^2)^{\gamma+1}}\right) \\
&= o(\epsilon^{N-2\gamma}).
\end{aligned}$$

Thus

$$\int_{\Omega} |\nabla u_\epsilon| dx = \epsilon^\alpha \begin{cases} O(1) & \text{if } p > \frac{2N-1}{N}, \\ C \ln\left(\frac{1}{\epsilon}\right) + O(1) & \text{if } p = \frac{2N-1}{N}, \\ O(\epsilon^{N-2\gamma}) & \text{if } p < \frac{2N-1}{N}, \end{cases}$$

and the conclusion follows.

(2) We have

$$\begin{aligned}
&\epsilon^{-\alpha(p-1)} \int_{\Omega} |\nabla u_\epsilon|^{p-1} dx \\
&= \left(\frac{N-p}{p-1}\right)^{p-1} \int_{B_{r/4}(0)} [(t + \epsilon)^2 + |y|^2]^{-\gamma(p-1)} dy dt + O(1).
\end{aligned}$$

Since in this case $2\gamma(p-1) - N = -1 < 0$ the integral on the right handside converges to a constant as $\epsilon \rightarrow 0$ and the result follows.

(3) Set now $\gamma_1 = \frac{N-p}{2(p-1)}$. We have, by letting $\epsilon \rightarrow 0$ in the integral below,

$$\epsilon^{-\alpha} \int_{\Omega} u_\epsilon(y, t) dt dy = \int_{\Omega \cap B_{r/4}(0)} \frac{dy dt}{[(t + \epsilon)^2 + |y|^2]^{\gamma_1}} = O(1)$$

if $N - 2\gamma_1 > 0$. Notice that $N - 2\gamma_1 > 0 \iff p > \frac{2N}{N+1}$. If $p \leq \frac{2N}{N+1}$ we write

$$\epsilon^{-\alpha} \int_{\Omega} u_\epsilon(y, t) dt dy = I_1 - I_2 + O(1)$$

with

$$I_1 = \int_{Q_c} \frac{dy dt}{[(t + \epsilon)^2 + |y|^2]^{\gamma_1}} \quad \text{and} \quad I_2 = \int_{Q_c \setminus \Omega} \frac{dy dt}{[(t + \epsilon)^2 + |y|^2]^{\gamma_1}}.$$

Following the computations of (1) (with γ_1 instead of γ) we will have

$$I_1 = \begin{cases} C \ln(1/\epsilon) + O(1) & \text{if } p = \frac{2N}{N+1}, \\ O(\epsilon^{N-2\gamma_1}) & \text{if } p < \frac{2N}{N+1}, \end{cases}$$

and besides

$$I_2 = O(\epsilon^{N-2\gamma_1}).$$

Thus

$$\int_{\Omega} |u_{\epsilon}| dx = \epsilon^{\alpha} \begin{cases} O(1) & \text{if } p > \frac{2N}{N+1}, \\ C \ln(\frac{1}{\epsilon}) + O(1) & \text{if } p = \frac{2N}{N+1}, \\ O(\epsilon^{N-2\gamma_1}) & \text{if } p < \frac{2N}{N+1}, \end{cases}$$

and the conclusion follows.

(4) We have

$$\epsilon^{-\alpha(p-1)} \int_{\Omega} |u_{\epsilon}|^{p-1} dx = \int_{\Omega \cap B_{r/4}(0)} [(t + \epsilon)^2 + |y|^2]^{-\gamma_1(p-1)} dy dt + O(1).$$

In this case $2\gamma_1(p-1) - N = -p < 0$ the the integral on the right converges to a constant as $\epsilon \rightarrow 0$ and the result follows.

(5) In this case we have

$$\epsilon^{-\alpha} \int_{\partial\Omega} |u_{\epsilon}| d\sigma = \int_{|y| \leq c} \frac{\sqrt{1 + |\nabla\rho(y)|^2} dy}{((\epsilon + \rho(y))^2 + |y|^2)^{\gamma_1}} + O(1) = I_1 + O(1)$$

and

$$\begin{aligned} I_1 &= \int_{|y| \leq c} \frac{dy}{((\epsilon + \rho(y))^2 + |y|^2)^{\gamma_1}} + O\left(\int_{|y| \leq c} \frac{|y|^2 dy}{((\epsilon + \rho(y))^2 + |y|^2)^{\gamma_1}}\right) \\ &= I_2 + I_3. \end{aligned}$$

By expanding $(\epsilon + \rho(y))^2$ and using Taylor's theorem we have

$$\begin{aligned} I_2 &= \int_{|y| \leq c} \frac{dy}{(\epsilon^2 + |y|^2)^{\gamma_1}} - 2\gamma_1\epsilon \int_{|y| \leq c} \frac{\rho(y) dy}{(\epsilon^2 + |y|^2)^{\gamma_1+1}} \\ &\quad + O\left(\int_{|y| \leq c} \frac{|y|^4 dy}{(\epsilon^2 + |y|^2)^{\gamma_1+1}}\right) \\ &= I_2^1 + I_2^2 \end{aligned}$$

If $2\gamma_1 - N + 1 < 0$, i.e., if $p > \frac{2N-1}{N}$ then I_2^1 converges to a constant as ϵ goes to 0 and I_2^1 is $O(\ln(\frac{1}{\epsilon}))$ if $p = \frac{2N-1}{N}$. In the case $p < \frac{2N-1}{N}$ we have

$$I_2^1 = \epsilon^{N-1-2\gamma_1} \int_0^{c/\epsilon} \frac{r^{N-2}}{(1+r^2)^{\gamma_1}} dt = O(\epsilon^{N-1-2\gamma_1}).$$

Finally one can easily check that the remaining terms can be neglected when compared with I_2^1 and the result follows.

(6) By expanding $(\epsilon + \rho(y))^2$ and using Taylor's theorem as above we have

$$\begin{aligned} \epsilon^{-\alpha(p-1)} \|u_\epsilon\|_{p-1, \partial\Omega}^{p-1} &= \int_{|y| \leq c} \frac{\sqrt{1 + |\nabla \rho(y)|^2} dy}{((\epsilon + \rho(y))^2 + |y|^2)^{\gamma_1(p-1)}} + O(1) \\ &= \int_{|y| \leq c} \frac{dy}{(\epsilon^2 + |y|^2)^{\frac{N-p}{2}}} \\ &\quad - 2(N-p)\epsilon \int_{|y| \leq c} \frac{\rho(y) dy}{(\epsilon^2 + |y|^2)^{\frac{N-p}{2}}} \\ &\quad + O\left(\int_{|y| \leq c} \frac{|y|^2 dy}{(\epsilon^2 + |y|^2)^{\frac{N-p}{2}}}\right) \end{aligned}$$

and, since $(N-p) - N + 1 = -p + 1 < 0$, the first integral on the right converges as ϵ goes to 0 and we get the result.

(7) As previously and, using that $2\gamma_1(p_* - 1) = N$,

$$\begin{aligned} \epsilon^{-\alpha(p_*-1)} \int_{\partial\Omega} |u_\epsilon|^{p_*-1} d\sigma &= \int_{|y| \leq c} \frac{\sqrt{1 + |\nabla \rho(y)|^2} dy}{((\epsilon + \rho(y))^2 + |y|^2)^{\gamma_1(p_*-1)}} + O(1) \\ &= \int_{|y| \leq c} \frac{dy}{((\epsilon + \rho(y))^2 + |y|^2)^{N/2}} \\ &\quad + O\left(\int_{|y| \leq c} \frac{|y|^2 dy}{((\epsilon + \rho(y))^2 + |y|^2)^{N/2}}\right). \end{aligned}$$

By expanding $(\epsilon + \rho(y))^2$ and using Taylor's theorem the first integral is now

$$I_1 = \epsilon^{-1} \int_0^{c/\epsilon} \frac{r^{N-2}}{(1+r^2)^{N/2}} dt = O(\epsilon^{-1})$$

and all the other integral are negligible when compared with I_1 . Finally, since $\alpha(p_* - 1) = \frac{N}{p}$ the result follows. \square

4. Existence of positive solutions

We can give now sufficient conditions on V, a, b et λ to fulfil the condition $S_\lambda^\pm < K_0^{-1}$. By taking $-a$ instead of a , one can prove similar results in order to have the inequality $S_\lambda^- < K_0^{-1}$. We will assume here the following hypothesis (\mathcal{B}): there exists a point $x_0 \in \partial\Omega$ such that

$$(\mathcal{B}) \quad \begin{cases} \|b\|_\infty \text{ is achieved at } x_0, \\ b(x_0) - b(x) = O(|x - x_0|^{\gamma+1}) \text{ for some } \gamma > 0, \\ a \in C^\gamma \text{ close to } x_0 \text{ for some } \gamma > 0, \\ a(x_0) > 0. \end{cases} \quad (4.1)$$

Proposition 4.1. *Let $N \geq 2p - 1$. Assume that there exists a point $x_0 \in \partial\Omega$ satisfying hypothesis (\mathcal{B}) in (4.1). Then*

$$C_\lambda^+ < \left(\frac{1}{p} - \frac{1}{p}\right) K_0^{-\frac{p^*}{p^*-p}}$$

holds in the following cases:

1. for any $\lambda \in \mathbb{R}$ if $p > 2$ and the mean curvature h_0 at x_0 is positive;
2. for $\lambda, a(x_0)$ and h_0 satisfying

$$\frac{N-2}{2}h_0 + \lambda a(x_0) > 0 \quad (4.2)$$

if $p = 2$;

3. for any $\lambda > 0$ if $1 < p < 2$.

Proof. By definition of S_λ^\pm in (3.1) we have

$$S_\lambda^+ \leq \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p^*}}.$$

Notice that $A(u_\epsilon) > 0$ if ϵ and r are small enough as a consequence of the estimate (4) in Proposition 3.1. We are going to prove that there exists a positive constant Λ such that

$$K_0 \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p_*}} = 1 - \Lambda g(\epsilon) \quad (4.3)$$

with

$$g(\epsilon) \stackrel{\text{def}}{=} \begin{cases} \epsilon & \text{if } p \geq 2, N > 2p - 1; \\ \epsilon \ln(\frac{1}{\epsilon}) & \text{if } p \geq 2, N = 2p - 1; \\ \epsilon^{p-1} & \text{if } 1 < p < 2. \end{cases} \quad (4.4)$$

First case: If $p > 2$ and $N > 2p - 1$ then the integrals (c) and (d) in Proposition 3.1 are $o(\epsilon)$ and therefore

$$\begin{aligned} \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p_*}} &= \frac{A_1 + A_2\epsilon + o(\epsilon)}{(B_1 + B_2\epsilon + o(\epsilon))^{p/p_*}} \\ &= \frac{A_1}{B_1^{p/p_*}} \left(1 + \left(\frac{A_2}{A_1} - \frac{p}{p_*} \frac{B_2}{B_1} \right) \epsilon \right) + o(\epsilon). \end{aligned}$$

We have

$$\frac{A_2}{A_1} = -\frac{1}{2} \frac{N-p}{N-2p+1} (N-1)h_0, \quad \frac{B_2}{B_1} = -\frac{1}{2} (N-1)h_0,$$

and hence we define Λ as

$$\Lambda := - \left(\frac{A_2}{A_1} - \frac{p}{p_*} \frac{B_2}{B_1} \right) = \frac{(N-p)(p-1)}{N-2p+1} h_0 > 0.$$

Notice that

$$\frac{A_1}{B_1^{p/p_*}} = K_0^{-1}.$$

Second case: If $p > 2$ and $N = 2p - 1$ then

$$\begin{aligned} \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p_*}} &= \frac{A_1 - \frac{h_0\omega_{N-2}}{2}\epsilon \ln(1/\epsilon) + O(\epsilon)}{(B_1 + O(\epsilon))^{p/p_*}} \\ &= \frac{A_1}{B_1^{p/p_*}} \left(1 - \left(\frac{h_0\omega_{N-2}}{2A_1} \right) \epsilon \ln(1/\epsilon) \right) + O(\epsilon) \end{aligned}$$

Here we define

$$\Lambda := \frac{h_0\omega_{N-2}}{2A_1} > 0.$$

Third case: If $1 < p < 2$ then

$$K_0 \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p_*}} = \frac{A_1 - \lambda C_1 \epsilon^{p-1} + o(\epsilon^{p-1})}{(B_1 + o(\epsilon^{p-1}))^{p/p_*}} = \frac{A_1}{B_1^{p/p_*}} - \lambda \frac{C_1}{B_1^{p/p_*}} \epsilon^{p-1} + o(\epsilon^{p-1})$$

Notice that if $1 < p < 2$ one has $\epsilon \ln(1/\epsilon) = o(\epsilon^{p-1})$. Here

$$\Lambda := \lambda \frac{C_1}{A_1} > 0$$

in the case $\lambda > 0$ and $a(x_0) > 0$.

Fourth case: If $p = 2$ and $N > 2p - 1 = 3$ then

$$\begin{aligned} \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p_*}} &= \frac{A_1 + (A_2 - \lambda C_1) \epsilon + o(\epsilon)}{(B_1 + B_2 \epsilon + o(\epsilon))^{p/p_*}} \\ &= \frac{A_1}{B_1^{p/p_*}} \left(1 + \left(\frac{A_2 - \lambda C_1}{A_1} - \frac{p}{p_*} \frac{B_2}{B_1} \right) \epsilon \right) + o(\epsilon). \end{aligned}$$

Now we have (in this case)

$$-\Lambda := \frac{A_2 - \lambda C_1}{A_1} - \frac{p}{p_*} \frac{B_2}{B_1} = -\frac{N-2}{N-3} h_0 - \lambda \frac{2}{N-3} a(x_0)$$

and therefore, if (4.2) holds, $\Lambda > 0$.

Fifth case: If $p = 2$ and $N = 2p - 1 = 3$ then

$$\begin{aligned} \frac{\Phi_\lambda(u_\epsilon)}{B(u_\epsilon)^{p/p_*}} &= \frac{A_1 + \left(-\frac{h_0 \omega_1}{2} - \lambda a(x_0) \omega_1\right) \epsilon \ln(1/\epsilon) + O(\epsilon)}{(B_1 + O(\epsilon))^{p/p_*}} = \\ &= \frac{A_1}{B_1^{p/p_*}} \left(1 + \left(\frac{-\frac{h_0}{2} - \lambda a(x_0)}{A_1} \right) \omega_1 \epsilon \ln(1/\epsilon) \right) + O(\epsilon). \end{aligned}$$

Here we have

$$\Lambda := \left(\frac{\frac{h_0}{2} + \lambda a(x_0)}{A_1} \right) \omega_1.$$

Thus, if (4.2) holds, $\Lambda > 0$. □

As a direct consequence of Proposition 4.1 and Proposition 2.8 we can now formulate the main result of this section.

Theorem 4.2. *Let $N \geq 2p - 1$. Assume that there exists a point $x_0 \in \partial\Omega$ such that hypothesis (\mathcal{B}) of (4.1) are satisfied. Let $\lambda \in \mathbb{R}$ satisfying*

$$\lambda_*^+ < \lambda < \lambda_1 \quad (4.5)$$

with λ_^+ defined in (2.14). Then problem (1.1) possesses a positive solution u satisfying $A(u) > 0$ in the following cases:*

1. *if $p > 2$ and the mean curvature at x_0 satisfies $h_0 > 0$,*
2. *if $1 < p < 2$, $\lambda > 0$,*
3. *if $p = 2$ and $a(x_0), h_0$ satisfy (4.2).*

We also have the analogous result when considering $-a$ instead of a :

Theorem 4.3. *Let $N \geq 2p - 1$ and assume that there exists a point $x_0 \in \partial\Omega$ such that hypothesis (\mathcal{B}) of (4.1) are satisfied for b and $-a$. Let $\lambda \in \mathbb{R}$ satisfying*

$$\lambda_{-1} < \lambda < \lambda_*^- \quad (4.6)$$

with λ_^- defined in (2.14) Then the problem (1.1) possesses a positive solution u satisfying $A(u) < 0$ in the following cases:*

1. *if $p > 2$ and the mean curvature at x_0 satisfies $h_0 > 0$,*
2. *if $1 < p < 2$, $\lambda < 0$ and $a(x_0) < 0$,*
3. *if $p = 2$ and $a(x_0), h_0$ satisfy (4.2).*

Remark 4.4.

1. *Notice that if $p = 2$, $V \equiv 0, a = \beta$ then $\lambda_1 = 0$, $\lambda_*^+ = -\infty$. Condition (4.2) for $\lambda = 1$ is condition (1.4) of [1] for the Yamabe problem (Y).*
2. *In order to have $\lambda > 0$ in the case $1 < p < 2$, one can required that $\lambda_1 > 0$ (resp. $\lambda_{-1} < 0$). Thus, it will be enough to ask for instance that*

$$\inf\{E_V(u); \|u\|_p = 1\} > 0,$$

a condition weaker than $\alpha_a > 0$.

5. Minimisation along nodal subsets of the Nehari manifold

In order to find nodal solutions we introduce the following nodal subsets of the Nehari manifold:

$$\mathcal{N}^+ = \{u \in W^{1,p}(\Omega); u^\pm \in \mathcal{A}_+\}, \quad \mathcal{N}^- = \{u \in W^{1,p}(\Omega); u^\pm \in \mathcal{A}_-\} \quad (5.1)$$

and let us define

$$D_\lambda^\pm = \inf_{u \in \mathcal{N}^\pm} I_\lambda(u). \quad (5.2)$$

Clearly $2C_\lambda^\pm \leq D_\lambda^\pm$ for any $\lambda \in \mathbb{R}$ since $I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) \geq C_\lambda^+ + C_\lambda^+$.

In what follows we are going to show, under certain conditions on λ , p , a and b , similar to those of the first section, that both D_λ^\pm are achieved, providing us with a pair of nodal solutions of problem (1.1). Our intention is to prove now that the Palais-Smale condition is satisfied. For any $u \in \mathcal{N}^+$ we denote

$$T_{\mathcal{N}^+}(u) = \{v \in W^{1,p}(\Omega); \langle \Phi'_\lambda(u^\pm), v \rangle = \langle B'(u^\pm), v \rangle\}$$

the tangent subspace to \mathcal{N}^+ at u . If $L \in W^{-1,p'}(\Omega)$, by $\|L\|_{T_{\mathcal{N}^+}(u)}$ we mean the norm of the restriction of L to the subspace $T_{\mathcal{N}^+}(u)$.

Proposition 5.1. *Let $\lambda < \lambda_1$ and $c \in \mathbb{R}$ satisfy $c < C_\lambda^+ + \left(\frac{1}{p} - \frac{1}{p^*}\right) K_0^{-\frac{p^*}{p^*-p}}$. Then I_λ satisfies the Palais-Smale condition at level c on \mathcal{N}^+ , i.e., any sequence $(u_n) \in \mathcal{A}^+$ satisfying*

$$\begin{aligned} (PS1) \quad & I_\lambda(u_n) \rightarrow c, \\ (PS2) \quad & \|I'_\lambda(u_n)\|_{T_{\mathcal{N}^+}(u_n)} = o(1) \end{aligned} \quad (5.3)$$

possesses a convergent subsequence.

Similarly, if $\lambda > \lambda_{-1}$ and $c < C_\lambda^- + \left(\frac{1}{p} - \frac{1}{p^}\right) K_0^{-\frac{p^*}{p^*-p}}$ then I_λ satisfies the Palais-Smale condition at level c on \mathcal{N}^- .*

Proof. We only prove the first case. Let $(u_n)_n$ be a sequence in \mathcal{N}^+ satisfying (5.3). Using that $I_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{p^*}\right)\Phi_\lambda(u_n)$, we can prove as in ((4) of Lemma 2.6 that the sequence is bounded. Let u be such that, up to a subsequence, $u_n \rightharpoonup u$, strongly in $L^p(\Omega)$, in $L^p(\partial\Omega)$ and a.e. Then we also have $u_n^\pm \rightharpoonup u^\pm$, strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Let us assume by contradiction that, for

instance, $u_n^+ \not\rightarrow u^+$ strongly in $W^{1,p}(\Omega)$. Let us denote, for each $v \in W^{1,p}(\Omega)$ and each $u \in \mathcal{N}$, the real number

$$s_u(v) = \frac{\langle \Phi'_\lambda(u), v \rangle - \langle B'(u), v \rangle}{(p - p_*)B(u)}$$

that satisfies $v - s_u(v)u \in T_{\mathcal{N}}(u)$. Let us choose $v = u_n^+ - u^+$ and apply (PS2) to $w = v - s_{u_n^+}(v)u_n^+ - s_{u_n^-}(v)u_n^- \in T_{\mathcal{N}^+}(u_n)$. We have $I'_\lambda(u_n)(w) = o(1)\|w\|$, that is

$$\langle I'_\lambda(u_n), u_n^+ - u^+ \rangle = o(1) \left\| u_n^+ - u^+ - s_{u_n^+}(u_n^+ - u^+)u_n^+ - s_{u_n^-}(u_n^+ - u^+)u_n^- \right\|.$$

Using estimate (2.7) we have that the sequence $B(u_n^\pm)$ is bounded away from 0 and therefore $s_{u_n^\pm}(u_n^+ - u^+)$ is bounded. Hence $\langle I'_\lambda(u), u_n^+ - u^+ \rangle = o(1)$ and, because we also have that $u_n^+ \rightarrow u^+$, it comes also that

$$\langle I'_\lambda(u_n) - I'_\lambda(u), u_n^+ - u^+ \rangle = o(1).$$

Using Brézis-Lieb identity we deduce

$$\langle I'_\lambda(u_n) - I'_\lambda(u), u_n^+ - u^+ \rangle = \|\nabla u_n^+ - \nabla u^+\|_p^p - \int_{\partial\Omega} b|u_n^+ - u^+|^{p_*} + o(1).$$

Thus

$$\|\nabla u_n^+ - \nabla u^+\|_p^p = \int_{\partial\Omega} b|u_n^+ - u^+|^{p_*} + o(1). \quad (5.4)$$

Using the fact that $\Phi_\lambda(u_n^+) = B(u_n^+)$ and Brézis-Lieb Lemma we get

$$\begin{aligned} & \left(\frac{1}{p} - \frac{1}{p_*}\right)B(u^+) + \left(\frac{1}{p} - \frac{1}{p_*}\right) \int_{\partial\Omega} b|u_n^+ - u^+|^{p_*} + o(1) = I_\lambda(u_n^+) \\ & = \left(\frac{1}{p} - \frac{1}{p_*}\right)\Phi_\lambda(u^+) + \left(\frac{1}{p} - \frac{1}{p_*}\right)\|\nabla u_n^+ - \nabla u^+\|_p^p + o(1) \end{aligned}$$

and letting $n \rightarrow \infty$ we see that $\Phi_\lambda(u^+) = B(u^+)$. In particular $I_\lambda(u^+) \geq 0$. Let now $\epsilon > 0$. We have, by using Lemma 2.10 and (5.4),

$$\begin{aligned} \|\nabla u_n^+ - \nabla u^+\|_p^p - \int_{\partial\Omega} b|u_n^+ - u^+|^{p_*} & \geq \|\nabla u_n^+ - \nabla u^+\|_p^p + o(1) \\ & \quad - (K_0 + \epsilon)^{p_*/p} \|\nabla u_n^+ - \nabla u^+\|_p^{p_*} \end{aligned}$$

and, since we are assuming that $\|\nabla u_n^+ - \nabla u^+\|_p \rightarrow 0$, then

$$\|\nabla u_n^+ - \nabla u^+\|_p \geq (K_0 + \epsilon)^{-\frac{p_*}{p(p_*-p)}} + o(1). \quad (5.5)$$

Besides, using again Brézis-Lieb Lemma and (5.5) we have

$$\begin{aligned} I_\lambda(u_n^+) &= I_\lambda(u_n^+ - u^+) + I_\lambda(u^+) + o(1) \\ &= \left(\frac{1}{p} - \frac{1}{p_*}\right) \|\nabla u_n^+ - \nabla u^+\|_p^p + I_\lambda(u^+) + o(1) \\ &\geq \left(\frac{1}{p} - \frac{1}{p_*}\right) (K_0 + \epsilon)^{-\frac{p_*}{p_*-p}} + o(1), \end{aligned}$$

since $I_\lambda(u^+) \geq 0$. Also one has $I_\lambda(u_n^-) \geq C_\lambda^+$. Finally we have the estimate

$$\begin{aligned} I_\lambda(u_n) &= I_\lambda(u_n^+) + I_\lambda(u_n^-) \\ &\geq \left(\frac{1}{p} - \frac{1}{p_*}\right) (K_0 + \epsilon)^{-\frac{p_*}{p_*-p}} + I_\lambda(u_n^-) + o(1) \\ &\geq \left(\frac{1}{p} - \frac{1}{p_*}\right) (K_0 + \epsilon)^{-\frac{p_*}{p_*-p}} + C_\lambda^+ + o(1). \end{aligned}$$

Since by hypothesis (PS1) we have $I_\lambda(u_n) \rightarrow c < C_\lambda^+ + \left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_*-p}}$, we get a contradiction by choosing $\epsilon > 0$ small enough. \square

As in the previous section, we need to assure that the infima D_λ^\pm are not achieved for any $u \in \mathcal{N}^\pm$ satisfying either $A(u^+) = 0$ or $A(u^-) = 0$. Let us introduce the values

$$\begin{aligned} \eta_\lambda^+ &\stackrel{\text{def}}{=} \inf\{I_\lambda(u); u \in \mathcal{N}^+, A(u^+) = 0, A(u^-) \geq 0\}; \\ \eta_\lambda^- &\stackrel{\text{def}}{=} \inf\{I_\lambda(u); u \in \mathcal{N}^-, A(u^+) = 0, A(u^-) \leq 0\}. \end{aligned} \quad (5.6)$$

Clearly $D_\lambda^\pm \leq \eta_\lambda^\pm$. Now, we prove that the infima D_λ^\pm are achieved provided they are sufficiently smaller.

Proposition 5.2. *Let $\lambda < \lambda_1$ and assume that*

$$D_\lambda^+ < \min \left\{ \eta_\lambda^+, C_\lambda^+ + \left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_*-p}} \right\}.$$

Then there exists $u \in \mathcal{N}^+$ solution of problem (1.1) satisfying $I_\lambda(u) = D_\lambda^+$. Similarly, if $\lambda > \lambda_{-1}$ and

$$D_\lambda^- < \min \left\{ \eta_\lambda^-, C_\lambda^- + \left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_*-p}} \right\}$$

then exists $v \in \mathcal{N}^-$ solution of the problem (1.1) satisfying $I_\lambda(v) = D_\lambda^-$.

Proof. First we are going to prove that we can find a minimizing sequence for the infimum D_λ^+ that satisfies the hypothesis (PS1) and (PS2) of the previous proposition. The idea is to apply Ekeland's variational principle to the complete metric space $X = \overline{\mathcal{N}^+}$ inherited with the distance of $W^{1,p}(\Omega)$. Notice that

$$\overline{\mathcal{N}^+} = \{u \in W^{1,p}(\Omega); u^\pm \in \mathcal{N}, A(u^\pm) \geq 0\} \cap \{\|u^\pm\| \geq c \text{ and } B(u^\pm) \geq c\}$$

according to the estimates (2.7). For any $\epsilon > 0$ let $u_\epsilon \in \mathcal{N}^+$ such that $I_\lambda(u_\epsilon) \leq D_\lambda^+ + \epsilon^2$. We can assume that $\epsilon > 0$ such that $0 < \epsilon^2 < \eta_\lambda^+ - D_\lambda^+$. By Ekeland's variational principle (see [11]) there exists $v_\epsilon \in X$ such that

$$\begin{aligned} (E1) \quad & I_\lambda(v_\epsilon) < I_\lambda(u_\epsilon), \\ (E2) \quad & \text{dist}(v_\epsilon, u_\epsilon) < \epsilon, \\ (E3) \quad & I_\lambda(v_\epsilon) \leq I_\lambda(w) + \epsilon\|v_\epsilon - w\| \quad \forall w \in X, w \neq v_\epsilon. \end{aligned}$$

Using the fact that $D_\lambda^+ < \eta_\lambda^+$, we can assume that $A(v_\epsilon^\pm) > 0$ otherwise we will have $D_\lambda^+ + \epsilon^2 \geq I_\lambda(v_\epsilon) \geq \eta_\lambda^+$, which is a contradiction. For any $w \in W^{1,p}(\Omega)$ consider $w_t = v_\epsilon + tw$ for t small enough to have $B(w_t^\pm) > 0$ and $A(w_t^\pm) > 0$. Put

$$s_1(t) = \left(\frac{\Phi_\lambda(w_t^+)}{B(w_t^+)} \right)^{\frac{1}{p_* - p}}; \quad s_2(t) = \left(\frac{\Phi_\lambda(w_t^-)}{B(w_t^-)} \right)^{\frac{1}{p_* - p}}$$

so $s_1(t)w_t^+ - s_2(t)w_t^- \in \mathcal{N}^+$. Hence, using (E3),

$$\left(\frac{I_\lambda(v_\epsilon) - I_\lambda(s_1(t)w_t^+ - s_2(t)w_t^-)}{t} \right) \left(\frac{t}{\|v_\epsilon - s_1(t)w_t^+ - s_2(t)w_t^-\|} \right) \leq \epsilon. \quad (5.7)$$

If we write $h(t) = I_\lambda(s_1(t)w_t^+ - s_2(t)w_t^-)$ then $h(0) = I_\lambda(v_\epsilon)$ and by elementary computations

$$\begin{aligned} h'(0) &= \langle I'_\lambda(v_\epsilon), s'_1(0)v_\epsilon^+ - s'_2(0)v_\epsilon^- + w \rangle = \langle I'_\lambda(v_\epsilon), w \rangle, \\ \lim_{t \rightarrow 0} \frac{\|v_\epsilon - s_1(t)(v_\epsilon + tw)^+ + s_2(t)(v_\epsilon + tw)^-\|}{t} &= \| -v_\epsilon^+ s'_1(0) + v_\epsilon^- s'_2(0) - w \|, \\ s'_1(0) &= \frac{1}{p_* - p} \left(\frac{\langle \Phi'_\lambda(v_\epsilon^+), w \rangle - \langle B'(v_\epsilon^+), w \rangle}{B(v_\epsilon^+)} \right)^{\frac{1}{p_* - p} - 1}, \end{aligned}$$

$$s'_2(0) = \frac{1}{p_* - p} \left(\frac{\langle \Phi'_\lambda(v_\epsilon^-), w \rangle - \langle B'(v_\epsilon^-), w \rangle}{B(v_\epsilon^-)} \right)^{\frac{1}{p_* - p} - 1}$$

and therefore $s'_1(0) = s'_2(0) = 0$ if $w \in T_{\mathcal{N}^+}(v_\epsilon)$. Letting $t \rightarrow 0$ in (5.7) we get

$$\langle I_\lambda(v_\epsilon), w \rangle \leq \epsilon \|w\| \quad \forall w \in T_{\mathcal{N}^+}(v_\epsilon).$$

Choosing $\epsilon = 1/n$ we have that $v_n = v_{1/n}$ provides a minimizing sequence in \mathcal{N}^+ that satisfies both (PS1) and (PS2) of the previous proposition. Then there exists a converging subsequence and we will conclude from (E1) that D_λ^+ is achieved at some $u \in \mathcal{N}^+$. Since the possibility that $A(u^\pm) = 0$ is excluded from the hypothesis $D_\lambda^+ < \eta_\lambda^+$ the conclusion comes finally from Lemma 2.1. \square

Remark 5.3. *The condition $D_\lambda^+ < \eta_\lambda^+$ is needed here to avoid the minimizing sequences to converge to some u satisfying $A(u^+) = 0$ or $A(u^-) = 0$. Notice that we required the similar condition (i.e. $C_\lambda^- < \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_* - p}}$) in order to prove that C_λ^+ is achieved and we have given in Proposition 2.7 a condition on λ to assure that $C_\lambda^+ < \left(\frac{1}{p} - \frac{1}{p_*}\right) \gamma_{a,b}^{\frac{p_*}{p_* - p}}$. We speculate that also $D_\lambda^+ < \eta_\lambda^+$ for λ close to λ_1 , but we have been unable to prove it. Notice that if $a > 0$ (or $a < 0$) then*

$$\gamma_{a,b} = \eta_\lambda^\pm = +\infty.$$

6. Existence of nodal solutions

Proposition 6.1. *Assume hypothesis (B) in (4.1), condition (4.6) and the additional constraint*

$$N > \max\{p^2, 2p, \frac{p}{p-1}, 2\}. \quad (6.1)$$

Then

$$D_\lambda^+ < C_\lambda^+ + \left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_* - p}}$$

for λ as in cases (1),(2) and (3) of Proposition 4.1.

Proof. Let $u_0 > 0$ be a critical point of I_λ with critical value C_λ^+ . By hypothesis we assume that $a > 0$ on $\partial\Omega \cap B_{r/4}(x_0)$ for some $r > 0$ satisfying furthermore

$$\int_{\partial\Omega \setminus B_{r/4}(x_0)} a|u_0|^p > 0.$$

Let u_ϵ be defined as in (3.3) and define the map $\sigma : [0, 1]^2 \rightarrow W^{1,p}(\Omega)$ by

$$\sigma(s, t) = Kt(su_0 - (1 - s)u_\epsilon)$$

for some $K > 0$ to be fixed later.

First we claim that

$$D_\lambda^+ \leq \max_{(s,t) \in [0,1]^2} I_\lambda(\sigma(s, t)). \quad (6.2)$$

To see that, consider the map $\varrho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as

$$\varrho(s, t) = (f_\lambda(\sigma(s, t)^+) - f_\lambda(\sigma(s, t)^-), f_\lambda(\sigma(s, t)^+) + f_\lambda(\sigma(s, t)^-) - 2),$$

where

$$f_\lambda(u) = \begin{cases} 0 & \text{if } u = 0, \\ \frac{B(u)}{\Phi_\lambda(u)} & \text{if } u \neq 0. \end{cases}$$

Notice that the estimate (2.5) implies that f_λ is a continuous map. Moreover we have

$$\begin{aligned} f_\lambda(\sigma(0, t)^+) - f_\lambda(\sigma(0, t)^-) &\leq 0 \\ f_\lambda(\sigma(1, t)^+) - f_\lambda(\sigma(1, t)^-) &\geq 0 \\ f_\lambda(\sigma(s, 0)^+) + f_\lambda(\sigma(s, 0)^-) - 2 &\leq 0 \end{aligned}$$

and we choose $K > 0$ big enough to have

$$f_\lambda(\sigma(s, 1)^+) + f_\lambda(\sigma(s, 1)^-) - 2 \geq 0.$$

We can apply Miranda's theorem [15] to get the existence of some $(\bar{s}, \bar{t}) \in [0, 1]^2$ such that $\varrho(\bar{s}, \bar{t}) = (0, 0)$, i.e.,

$$f_\lambda(\sigma(\bar{s}, \bar{t})^+) = f_\lambda(\sigma(\bar{s}, \bar{t})^-) = 1. \quad (6.3)$$

Thus $\bar{u} = \sigma(\bar{s}, \bar{t})$ is such that $\bar{u}^\pm \in \mathcal{N}$. It remains to proof that $A(\bar{u}^\pm) > 0$ to conclude (6.2). We have

$$\begin{aligned} A(\bar{u}^+) &= K^p \bar{t}^p \int_{\partial\Omega} a |(\bar{s}u_0 - (1 - \bar{s})u_\epsilon)^+|^p = \\ &= K^p \bar{t}^p \left(\int_{\partial\Omega \setminus B_{r/4}(x_0)} a |\bar{s}u_0|^p + \int_{\partial\Omega \cap B_{r/4}(x_0)} a^+ |\bar{u}^+|^p \right) > 0; \end{aligned}$$

and also

$$A(\bar{u}^-) = K^p \bar{t}^p \int_{\partial\Omega \cap B_{r/4}(x_0)} a^+ |\bar{u}^-|^p > 0,$$

otherwise $\bar{u} \geq 0$ on $\partial\Omega$ and therefore $B(\bar{u}^-) = 0$, in contradiction with (6.3).
(2) Next we prove that

$$\max_{(s,t) \in [0,1]^2} I_\lambda(\sigma(s,t)) < C_\lambda^+ + \left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_*-p}}. \quad (6.4)$$

We write for simplicity the functions $u \in \sigma([0,1]^2)$ as $u = \alpha u_0 + \beta u_\epsilon$, with $|\alpha|, |\beta| \leq K$. Then using the inequality (2.13) we have for some positive constants K_1, K_2, K_3

$$\begin{aligned} & I_\lambda(\alpha u_0 + \beta u_\epsilon) - I_\lambda(|\alpha|u_0) - I_\lambda(|\beta|v_\epsilon) \leq \\ & + K_1 \left(\|\nabla \alpha u_0\|_{L^\infty(B_r(0))}^{p-1} \|\nabla \beta v_\epsilon\|_1 + \|\nabla \alpha u_0\|_{L^\infty(B_r(0))} \|\nabla \beta v_\epsilon\|_{p-1}^{p-1} \right) \\ & + K_2 \left(\|\alpha u_0\|_{L^\infty(B_r(0))} \|\beta v_\epsilon\|_{p-1}^{p-1} + \|\alpha u_0\|_{L^\infty(B_r(0))}^{p-1} \|\beta v_\epsilon\|_1 \right) \\ & + K_3 \left(\|\alpha u_0\|_{L^\infty(\partial\Omega \cap B_r(0))} \|\beta v_\epsilon\|_{p-1, \partial\Omega}^{p-1} + \|\alpha u_0\|_{L^\infty(\partial\Omega \cap B_r(0))}^{p-1} \|\beta v_\epsilon\|_{1, \partial\Omega} \right) \\ & + K_4 \left(\|\alpha u_0\|_{L^\infty(\partial\Omega \cap B_r(0))} \|\beta v_\epsilon\|_{p_*-1, \partial\Omega}^{p_*-1} + \|\alpha u_0\|_{L^\infty(\partial\Omega \cap B_r(0))}^{p_*-1} \|\beta v_\epsilon\|_{1, \partial\Omega} \right). \end{aligned} \quad (6.5)$$

Using (1) of Lemma 2.6 and (4.3) in the proof of Proposition 4.1 we have

$$I_\lambda(|\alpha|u_0) \leq I_\lambda(u_0) = C_\lambda^+;$$

$$\begin{aligned} I_\lambda(|\beta|v_\epsilon) & \leq \left(\frac{1}{p} - \frac{1}{p_*}\right) \left(\frac{\Phi_\lambda(v_\epsilon)}{B(v_\epsilon)^{p/p_*}}\right)^{\frac{p_*}{p_*-p}} \\ & = o(\epsilon) + \left(\frac{1}{p} - \frac{1}{p_*}\right) K_0^{-\frac{p_*}{p_*-p}} - p K_0^{-\frac{p_*}{p_*-p}} \Lambda g(\epsilon) \end{aligned}$$

with $g(\epsilon)$ defined in (4.4). Besides, the remaining terms in (6.5) are $o(\epsilon)$ if (6.1) is satisfied. Indeed, notice that in the estimate of the norm $\|\nabla u_\epsilon\|_1$, all the powers of ϵ are > 1 if either $p \geq 2$ and $N > p^2$ or $1 < p < 2$ and $N > \max\{2p, \frac{p}{p-1}\}$. The other terms ($\|\nabla u_\epsilon\|_{p-1}^{p-1}, \|u_\epsilon\|_1, \dots$) the power of ϵ is > 1 if $p < \frac{N}{2}$. \square

As a corollary of Proposition 5.2 and Proposition 6.1 we have the following existence result. In order to assure that the condition $D_\lambda^\pm < \eta_\lambda^\pm$ is satisfied, we only consider now weights a with definite sign.

Theorem 6.2. *Assume hypothesis (\mathcal{B}) and that the conditions of Theorem 4.2 hold. Let N and p satisfy (6.1).*

1. *If $a > 0$ then there exists a nodal solution $u \in W^{1,p}(\Omega)$ to the problem (1.1) for any $\lambda < \lambda_1$.*
2. *If $a < 0$ then there exists a nodal solution $u \in W^{1,p}(\Omega)$ to the problem (1.1) for any $\lambda > \lambda_{-1}$.*

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