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# POSITIVE SOLUTIONS FOR SLIGHTY SUBCRITICAL ELLIPTIC PROBLEMS VIA ORLICZ SPACES

MABEL CUESTA AND ROSA PARDO

ABSTRACT. This paper concerns semilinear elliptic equations involving sign-changing weight function and a nonlinearity of subcritical nature understood in a generalized sens. Using an Orlicz-Sobolev space setting, we consider superlinear nonlinearities which do not have a polynomial growth, and state sufficient conditions guaranteeing the Palais-Smale condition. We study the existence of a bifurcated branch of classical positive solutions, containing a turning point, and providing multiplicity of solutions.

Positive solutions, subcritical nonlinearity, changing sign weight.  
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## 1. INTRODUCTION

In this paper we study the classical positive solutions to the Dirichlet problem for a class of semilinear elliptic equations whose nonlinear term is of subcritical nature in a generalized sens and involves indefinite nonlinearities. More precisely, given  $\Omega \subset \mathbb{R}^N$ ,  $N > 2$ , a bounded, connected open subset, with  $C^2$  boundary  $\partial\Omega$ , we look for positive solutions to:

$$(1.1) \quad -\Delta u = \lambda u + a(x)f(u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

where  $\lambda \in \mathbb{R}$  is a real parameter,  $a \in C^1(\bar{\Omega})$  changes sign in  $\Omega$ ,

$$(1.2) \quad f(s) := g(s) + h(s), \quad \text{with } h(s) := \frac{|s|^{2^*-2}s}{[\ln(e + |s|)]^\alpha},$$

$2^* = \frac{2N}{N-2}$  is the critical Sobolev exponent,  $\alpha > 0$  is a fixed exponent, and  $g \in C^1(\mathbb{R})$  satisfies

$$(H) \quad \begin{cases} (H)_0 & \lim_{s \rightarrow 0} \frac{f(s)}{|s|^{p-2}s} = L_1, & \text{for some } L_1 > 0, \text{ and } p \in \left(2, \frac{2N}{N-2}\right] \\ (H)_\infty & \lim_{s \rightarrow \infty} \frac{g(s)}{|s|^{q-2}s} = L_2, & \text{for some } L_2 \geq 0, \text{ and } q \in \left(2, \frac{2N}{N-2}\right) \\ (H)_{g'} & |g'(s)| \leq C(1 + |s|^{q-2}), & \text{for } s \in \mathbb{R}. \end{cases}$$

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We will say that  $g$  (or even  $f$ ) satisfies hypothesis (H) whenever  $(H)_0$ ,  $(H)_\infty$ , and  $(H)_{g'}$  are satisfied. Since we are interested in positive solutions, we

$$(1.3) \quad \text{redefine } f \text{ to be zero on } (-\infty, 0],$$

note that  $f(0) = 0$  and that

$$(1.4) \quad \lim_{s \rightarrow 0^+} \left( \frac{f(s)}{s} - L_1 |s|^{p-2} \right) = 0.$$

When  $\lambda = 0$ ,  $a(x) \equiv 1$  and  $g(s) \equiv 0$ , this kind of nonlinearity has been studied in [5, 6, 7, 16], and in [11] for the case of the  $p$ -laplacian operator, with  $\alpha > \frac{p}{N-p}$ . It is known the existence of uniform  $L^\infty$  a priori bounds for any positive classical solution, and as a consequence, the existence of positive solutions. When  $\alpha \rightarrow 0$ , there is a positive solution blowing up at a non-degenerate point of the Robin function as  $\alpha \rightarrow 0$ , see [9] for details.

From [10] it is known that  $(\lambda_1, 0)$  is a bifurcation point of positive solutions  $(\lambda, u_\lambda)$  to the equation (1.1).

For  $f$  behaving like  $|u|^{p-2}u$  at zero with  $2 \leq p \leq 2^*$ , the influence of the negative part of the weight  $a$  is displayed under the sign of  $\int_\Omega a(x)\varphi_1(x)^p dx$ , where  $\varphi_1$  is the first positive eigenfunction for  $-\Delta$  in  $H_0^1(\Omega)$ . Specifically, whenever

$$(1.5) \quad \int_\Omega a(x)\varphi_1(x)^p dx < 0$$

the bifurcation of positive solutions from the trivial solution set is 'on the right' of the first eigenvalue, in other words for values of  $\lambda > \lambda_1$ . And whenever

$$\int_\Omega a(x)\varphi_1(x)^p dx > 0$$

the bifurcation from the trivial solution set is 'on the left' of the first eigenvalue, in other words for values of  $\lambda < \lambda_1$ .

Inspired by the work of Alama and Tarantello in [1], we will focus our attention to the case  $a(x)$  changing sign and (1.5) is satisfied, and, among other things, we will prove the existence of a turning point for a value of the parameter  $\Lambda > \lambda_1$ , and in particular the existence of solutions when  $\lambda = \lambda_1$ . We will use local bifurcation and variational techniques.

All throughout the paper, for  $v : \Omega \rightarrow \mathbb{R}$ ,  $v = v^+ - v^-$  where

$$v^+(x) := \max\{v(x), 0\} \quad \text{and} \quad v^-(x) := \max\{-v(x), 0\}.$$

Let us also define

$$\Omega^\pm := \{x \in \Omega : \pm a(x) > 0\}, \quad \Omega^0 := \{x \in \Omega : a(x) = 0\},$$

and assume that both  $\Omega^+$ ,  $\Omega^-$  are non empty sets.

For this nonlinearity the Palais-Smale condition of the energy functional becomes a delicate issue, needing Orlicz spaces and a Orlicz-Sobolev embedding theorem.

In order to prove (PS) condition, Alama and Tarantello ([1]) assume that the zero set  $\Omega^0$  has a non empty interior. This is also a common hypothesis for other authors when dealing with changing sign superlinear nonlinearities [8, 21, 23]. But this is a technical hypothesis. (PS)-condition will be proved in Proposition 3.1 without assuming that hypothesis.

We neither use Ambrosetti-Rabinowitz.

Let us now denote

$$(1.6) \quad C_0 = \inf\{C \geq 0 : f'(s) + C \geq 0 \text{ for all } s \geq 0\},$$

and remark that hypothesis (H) implies that  $C_0 < +\infty$ . Observe also that

$$(1.7) \quad f(s) + C_0 s \geq 0, \text{ for all } s \geq 0; \quad f(s)s + C_0 s^2 \geq 0, \text{ for all } s \in \mathbb{R}.$$

Let  $u$  be a weak solution to (1.1). By a regularity result, see Lemma 2.1,  $u \in C^2(\Omega) \cap C^{1,\mu}(\bar{\Omega})$ . So by a *solution*, we mean a *classical solution*.

Assume that  $u$  is a non-negative nontrivial solution. It is easy to see that the solution is strictly positive. Indeed, adding  $\pm C_0 a(x)u$  to the r.h.s. of the equation, splitting  $a = a^+ - a^-$ , taking into account (1.4), and letting in each side the nonnegative terms, we can write

$$(1.8) \quad \left( -\Delta + a^-(x) \left[ \frac{f(u)}{u} + C_0 \right] + C_0 a(x)^+ \right) u \\ = \lambda u + a(x)^+ [f(u) + C_0 u] + C_0 a(x)^- u, \quad \text{in } \Omega.$$

Now, the strong Maximum Principle implies that  $u > 0$  in  $\Omega$ , and  $\frac{\partial u}{\partial \nu} < 0$  on  $\partial\Omega$ .

Our main result is the following theorem.

**Theorem 1.1.** *Assume that  $g \in C^1(\mathbb{R})$  satisfies hypothesis (H). Let  $C_0 > 0$  be defined by (1.6). If  $a$  changes sign in  $\Omega$ , and (1.5) holds, then there exists a  $\Lambda \in \mathbb{R}$ ,*

$$\lambda_1 < \Lambda < \min \left\{ \lambda_1(\text{int}(\Omega^0)), \quad \lambda_1(\text{int}(\Omega^+ \cup \Omega^0)) + C_0 \sup a^+ \right\}$$

*and such that (1.1) has a classical positive solution if and only if  $\lambda \leq \Lambda$ .*

Moreover, there exists a continuum (a closed and connected set)  $\mathcal{C}$  of classical positive solutions to (1.1) emanating from the trivial solution set at the bifurcation point  $(\lambda, u) = (\lambda_1, 0)$  which is unbounded.

Furthermore,

- (a) For every,  $\lambda \in (\lambda_1, \Lambda)$ , (1.1) admits at least two classical ordered positive solutions.
- (b) For  $\lambda = \Lambda$ , problem (1.1) admits at least one classical positive solution.
- (c) For every  $\lambda \leq \lambda_1$ , problem (1.1) admits at least one classical positive solution.

The paper is organized in the following way. Section 2 contains a regularity result and a non existence result. (PS)-condition and an existence of solutions result for  $\lambda < \lambda_1$  based in the Mountain Pass Theorem will be proved in Section 3. A bifurcation result for  $\lambda > \lambda_1$  is developed in Section 4. The main result is proved in Section 5. Appendix A contains some useful estimates. Orlicz spaces, a Orlicz-Sobolev embeddings theorems, and variational techniques, also including a (PS) condition in Orlicz-Sobolev spaces setting and the Mountain Pass Theorem, will be treated in Appendix B.

## 2. A REGULARITY RESULT AND A NON EXISTENCE RESULT

Next, we recall a regularity Lemma stating that any weak solution is in fact a classical solution.

**Lemma 2.1.** *If  $u \in H_0^1(\Omega)$  weakly solves (1.1) with a continuous function  $f$  with polynomial critical growth*

$$|f(x, s)| \leq C(1 + |s|^{2^*-1}),$$

then,  $u \in C^2(\Omega) \cap C^{1,\mu}(\overline{\Omega})$  and

$$\|u\|_{C^{1,\mu}(\overline{\Omega})} \leq C \left( 1 + \|u\|_{L^{(2^*-1)r}(\overline{\Omega})}^{2^*-1} \right),$$

for any  $r > N$  and  $\mu = 1 - N/r$ . Moreover, if  $\partial\Omega \in C^{2,\mu}$ , then  $u \in C^{2,\mu}(\overline{\Omega})$ .

*Proof.* Due to an estimate of Brézis-Kato [3], based on Moser's iteration technique [17],  $u \in L^r(\Omega)$  for any  $r > 1$ ; and by elliptic regularity  $u \in W^{2,r}(\Omega)$ , for any  $r > 1$  (see [22, Lemma B.3] and comments below).

Moreover, by Sobolev embeddings for  $r > N$  and interior elliptic regularity  $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ . Furthermore, if  $\partial\Omega \in C^{2,\alpha}$ , then  $u \in C^{2,\alpha}(\overline{\Omega})$ .  $\square$

**Proposition 2.2.** *Let  $f$  satisfy hypothesis (H) and let  $C_0$  be defined in (1.6). Assume that  $a$  changes sign in  $\Omega$ .*

- (1) *Problem (1.1) does not admit a positive solution  $u \in H_0^1(\Omega)$  for any*

$$\lambda \geq \lambda_1(\text{int}(\Omega^+ \cup \Omega^0)) + C_0 \sup a^+.$$

- (2) *If  $\text{int}(\Omega^0) \neq \emptyset$ , then  $\lambda_1(\text{int}(\Omega^0)) < +\infty$  and (1.1) does not admit a positive solution for any*

$$\lambda \geq \lambda_1(\text{int}(\Omega^0)).$$

*Proof.* 1. Let  $\lambda \geq \lambda_1(\text{int}(\Omega^+ \cup \Omega^0)) + C_0 \sup a^+$ , and assume by contradiction that there exists a non-negative non-trivial solution  $u \in H_0^1(\Omega)$  to (1.1) for the parameter  $\lambda$ . Since the Maximum Principle  $u > 0$  in  $\Omega$ , see (1.8).

Let  $\hat{\varphi}$  be the positive eigenfunction of  $(-\Delta, H_0^1(\text{int}(\Omega^+ \cup \Omega^0)))$  of  $L^2$ -norm equal to 1. For simplicity we will denote also by  $\hat{\varphi}$  the extension by 0 of  $\hat{\varphi}$  in all  $\Omega$ . By Hopf's maximum principle we have  $\frac{\partial \hat{\varphi}}{\partial \nu} < 0$  on  $\partial(\text{int}(\Omega^+ \cup \Omega^0))$ , where  $\nu$  is the outward normal.

Again if we multiply the equation (1.1) by  $\hat{\varphi}$  and integrate along  $\text{int}(\Omega^+ \cup \Omega^0)$  we find, after integrating by parts,

$$\begin{aligned} 0 &> \int_{\partial(\text{int}(\Omega^+ \cup \Omega^0))} u \frac{\partial \hat{\varphi}}{\partial \nu} d\sigma \\ &\quad + \int_{\text{int}(\Omega^+ \cup \Omega^0)} \left[ \lambda_1(\text{int}(\Omega^+ \cup \Omega^0)) - \lambda + C_0 a^+(x) \right] u \hat{\varphi} dx \\ &= \int_{\Omega^+} a^+(x) [f(u) + C_0 u] \hat{\varphi} dx > 0, \end{aligned}$$

a contradiction.

2. Let  $\lambda \geq \lambda_1(\text{int}(\Omega^0))$  and assume by contradiction that there exists a positive solution  $u \in H_0^1(\Omega)$  of problem (1.1) for the parameter  $\lambda$ . Let  $\tilde{\varphi}$  be a positive eigenfunction associated to  $\lambda_1(\text{int}(\Omega^0)) < +\infty$ . For simplicity we will also denote by  $\tilde{\varphi}$  the extension by 0 in all  $\Omega$ . If we multiply equation (1.1) by  $\tilde{\varphi}$  and integrate along  $\Omega^0$  we find, after integrating by parts,

$$\int_{\text{int}(\Omega^0)} \nabla u \cdot \nabla \tilde{\varphi} dx = \lambda \int_{\text{int}(\Omega^0)} u \tilde{\varphi} dx.$$

On the other hand

$$\int_{\text{int}(\Omega^0)} \nabla u \cdot \nabla \tilde{\varphi} dx = \lambda_1(\text{int}(\Omega^0)) \int_{\text{int}(\Omega^0)} \tilde{\varphi} u dx + \int_{\partial(\text{int}(\Omega^0))} u \frac{\partial \tilde{\varphi}}{\partial \nu} d\sigma.$$

Hence

$$0 > \int_{\partial(\text{int}(\Omega^0))} u \frac{\partial \tilde{\varphi}}{\partial \nu} d\sigma = \left( \lambda - \lambda_1(\text{int}(\Omega^0)) \right) \int_{\text{int}(\Omega^0)} u \tilde{\varphi} dx \geq 0,$$

a contradiction.  $\square$

### 3. AN EXISTENCE RESULT FOR $\lambda < \lambda_1$

In this section we prove the existence of a nontrivial solution to equation (1.1) for  $\lambda < \lambda_1$ , through the Mountain Pass Theorem.

**3.1. On Palais-Smale sequences.** In this subsection, we define the framework for the functional  $J_\lambda$  associated to the problem (1.1) $_\lambda$ . Hereafter we denote by  $\|\cdot\|$  the usual norm of  $H_0^1(\Omega)$ :

$$\|u\| = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

Given  $f(s) = h(s) + g(s)$  defined by (1.2), let us denote by  $F(s) := \int_0^s f(t) dt$ . Observe that (1.7) implies the following

$$(3.1) \quad F(s) + \frac{1}{2}C_0s^2 \geq 0, \text{ for all } s \geq 0.$$

Consider the functional  $J_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$  given by

$$J_\lambda[v] := \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{\lambda}{2} \int_{\Omega} (v^+)^2 dx - \int_{\Omega} a(x)F(v^+) dx.$$

Observe that for all  $v \in H_0^1(\Omega)$ ,  $J_\lambda[v^+] \leq J_\lambda[v]$ .

The functional  $J_\lambda$  is well defined and belongs to the class  $C^1$  with

$$J'_\lambda[v] \psi = \int_{\Omega} \nabla v \nabla \psi dx - \lambda \int_{\Omega} v^+ \psi dx - \int_{\Omega} a(x)f(v^+)\psi dx,$$

for all  $\psi \in H_0^1(\Omega)$ . Consequently, non-negative critical points of the functional  $J_\lambda$  correspond to non-negative weak solutions to (1.1).

The next Proposition proves that *Palais-Smale sequences* are bounded whenever  $\lambda < \lambda_1(\text{int} \Omega^0)$ , where  $\lambda_1(\text{int} \Omega^0)$  may be infinite.

**Proposition 3.1.** *Assume that  $g \in C^1(\mathbb{R})$  satisfies hypothesis (H) and assume also that  $\lambda < \lambda_1(\text{int} \Omega^0) \leq +\infty$ .*

*Then any (PS) sequence, that is, a sequence satisfying*

$$(J_1) \quad J_\lambda[u_n] \leq C,$$

$$(J_2) \quad |J'_\lambda[u_n] \psi| \leq \varepsilon_n \|\psi\|, \text{ where } \varepsilon_n \rightarrow 0 \text{ as } n \rightarrow +\infty$$

*is a bounded sequence.*

*Proof.* 1. Let  $\{u_n\}_{n \in \mathbb{N}}$  be a (PS) sequence in  $H_0^1(\Omega)$  and assume by contradiction that  $\|u_n\| \rightarrow +\infty$ . Let us first prove the following claim:

*Claim.* Let  $v \in H_0^1(\Omega)$  be the weak limit of  $v_n = \frac{u_n}{\|u_n\|}$  and assume that  $v_n \rightarrow v$ , strongly in  $L^{2^*-1}(\Omega)$  and a.e. Then  $v = 0$  a.e. in  $\Omega$ .

Assume that  $v \neq 0$  and denote  $\gamma_n = \|u_n\|$ . Let  $\omega_n := \{x \in \Omega : v_n^+(x) > 1\}$ , then for any  $\psi \in C_0^1(\Omega)$ ,

$$\left| \frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^*-1}} \frac{(u_n^+(x))^{2^*-1}}{[\ln(e + \gamma_n v_n^+(x))]^\alpha} |\psi| \right| \leq |v_n^+(x)|^{2^*-1} \|\psi\|_\infty, \quad \forall x \in \omega_n.$$

Let  $x \in \Omega \setminus \omega_n$ , using the estimates (A.1),

$$\left| \frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^*-1}} \frac{(u_n^+)^{2^*-1}}{[\ln(e + \gamma_n v_n^+)]^\alpha} |\psi| \right| \leq (|v_n^+|^{2^*-2}) \|\psi\|_\infty \leq \|\psi\|_\infty$$

Besides, by the reverse of the Lebesgue dominated convergence theorem, see for instance [2, Theorem 4.9, p. 94], there exists  $h_i \in L^1(\Omega)$ ,  $1 \leq i \leq 3$  such that, up to a subsequence,

$$|v_n^+|^{2^*-1} \leq h_1, |v_n^+|^{p-1} \leq h_2 |v_n^+|^{2^*-2} \leq h_3, \quad , \quad a.e. \quad x \in \Omega,$$

for all  $n \in \mathbb{N}$ , and therefore

$$\left| \frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^*-1}} f(u_n^+) \psi \right| \leq C (h_1 + h_2 + h_3 + 1) \|\psi\|_\infty \in L^1(\Omega).$$

By Lebesgue dominated convergent theorem we have

$$\frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^*-1}} a(\cdot) f(u_n^+) \psi \rightarrow a(\cdot) (v^+)^{2^*-1} \psi \quad \text{strongly in } L^1(\Omega).$$

We have used here that if  $v^+(x) \neq 0$ , then

$$\lim_{n \rightarrow +\infty} \frac{\ln(e + \gamma_n)}{\ln(e + \gamma_n v_n^+(x))} = 1,$$

and if  $v^+(x) = 0$ , then

$$\lim_{n \rightarrow +\infty} \left( \frac{\ln(e + \gamma_n)}{\ln(e + \gamma_n v_n^+(x))} \right)^\alpha |v_n^+(x)|^{2^*-1} \leq \lim_{n \rightarrow +\infty} |v_n^+(x)|^{2^*-2} = 0.$$

On the other hand

$$\frac{\ln(e + \gamma_n)^\alpha}{\gamma_n^{2^*-1}} \int_{\Omega} \nabla u_n \cdot \nabla \psi \, dx \rightarrow 0.$$



Hence, using  $(J_2)$  for an arbitrary test function  $\psi$ , multiplying by  $\frac{\ln(e+\gamma_n)^\alpha}{\gamma_n^{2^*-1}}$  and passing to the limit we find

$$\int_{\Omega} a(x)(v^+)^{2^*-1}\psi \, dx = 0 \quad \forall \psi \in C_0^1(\Omega).$$

In particular  $v^+ = 0$  a.e. in  $\Omega \setminus \Omega^0$ .

Assume that  $\text{int } \Omega^0 \neq \emptyset$ , and that  $\lambda < \lambda_1(\text{int } \Omega^0)$ . Thus, for any  $\psi \in C_0^1(\text{int } \Omega^0)$  we have from  $(J_2)$

$$\int_{\text{int } \Omega^0} \nabla u_n \cdot \nabla \psi \, dx - \lambda \int_{\text{int } \Omega^0} u_n^+ \psi \, dx = o(1).$$

Dividing by  $\|u_n\|$  and passing to the limit we have

$$\int_{\text{int } \Omega^0} \nabla v \cdot \nabla \psi \, dx = \lambda \int_{\text{int } \Omega^0} v^+ \psi \, dx.$$

From the Maximum Principle,  $v \geq 0$  in  $\text{int } \Omega^0$ . Since  $\lambda < \lambda_1(\text{int } \Omega^0)$  then it must be  $v^+ \equiv 0$  in  $\text{int } \Omega^0$ . Hence  $v^+ \equiv 0$  in  $\Omega$ .

On the other hand, taking  $u_n^-$  as a test function in the condition  $(J_2)$ ,

$$\left| - \int_{\Omega} |\nabla u_n^-|^2 \, dx - \int_{\Omega} a(x) f(u_n^+) u_n^- \, dx \right| = \int_{\Omega} |\nabla u_n^-|^2 \, dx \leq \epsilon_n \|u_n^-\|$$

so  $\|u_n^-\| \rightarrow 0$  and then  $v^- \equiv 0$ , and we conclude the proof of the claim.

2. In order to achieve a contradiction, we use a Hölder inequality, and properties on convergence into an Orlicz space, cf. Appendix B.

To this end, the analysis of Lemma A.2 give us the existence of  $\alpha^* > 0$  such that the function  $s \rightarrow \frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$  is increasing along  $[0, +\infty[$  if  $\alpha \leq \alpha^*$ . In this case we will denote

$$(3.2) \quad m(s) = \frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$$

If  $\alpha > \alpha^*$  the function  $s \rightarrow \frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$  possesses a local maximum  $s_1$  in  $[0, +\infty[$ . Let us denote by  $\bar{s}_1$  the unique solution  $s > s_1$  such that

$$\frac{s_1^{2^*-1}}{[\ln(e+s_1)]^\alpha} = \frac{\bar{s}_1^{2^*-1}}{[\ln(e+\bar{s}_1)]^\alpha}$$

and define the non-decreasing function

$$(3.3) \quad m(s) := \begin{cases} \frac{s^{2^*-1}}{[\ln(e+s)]^\alpha} & \text{if } s \notin [s_1, \bar{s}_1], \\ \frac{s_1^{2^*-1}}{[\ln(e+s_1)]^\alpha} & \text{if } s \in [s_1, \bar{s}_1]. \end{cases}$$

It follows that

$$(3.4) \quad s \rightarrow M(s) = \int_0^s m(t) dt \quad \text{is a } N\text{-function in } [0, +\infty[.$$

By using

$$\lim_{s \rightarrow +\infty} \frac{\ln(e+s)}{\ln(e+2s)} = 1 \quad \text{and} \quad \lim_{s \rightarrow 0} \frac{\ln(e+s)}{\ln(e+2s)} = 1,$$

we get that

$$\lim_{s \rightarrow +\infty} \frac{m(2s)}{m(s)} < +\infty \quad \text{and} \quad \lim_{s \rightarrow 0^+} \frac{m(2s)}{m(s)} < +\infty,$$

which implies that there exists  $K > 0$  such that  $m(2s) \leq Km(s)$  for all  $s \geq 0$  and consequently  $M$  satisfies the  $\Delta_2$ -condition (B.1).

Since  $v_n \rightharpoonup 0$  in  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$ , it follows from  $(J_2)$  applied to  $\psi = u_n$  that

$$(3.5) \quad \lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n^+) u_n}{\|u_n\|^2} dx = \lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} v_n^+ dx = 1.$$

Since the Hölder inequality into Orlicz spaces, see Proposition B.11.(ii),

$$(3.6) \quad \int_{\Omega} \left| a(x) \frac{f(u_n^+)}{\|u_n\|} v_n^+ \right| dx \leq \frac{\|a\|_{\infty}}{\|u_n\|} \|f(u_n^+)\|_{M^*} \|v_n^+\|_M$$

By Theorem B.3 and Theorem B.12 we have

$$(3.7) \quad \|v_n - v\|_M \rightarrow 0.$$

Moreover, since there exists  $C > 0$  such that  $m(s) \leq Cs^{2^*-1}$  and  $M(s) \leq Cs^{2^*}$  for all  $s \geq 0$ , and the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset H_0^1(\Omega)$ , then, for each  $n \in \mathbb{N}$ , there exists a  $C_n$  such that

$$|u_n m(u_n)| \leq C_n, \quad |M(|u_n|)| \leq C_n.$$

By using definition B.8 of  $M^*$  and identities of Proposition B.9 we have

$$M^*(|m(u_n)|) = |u_n m(u_n)| - M(|u_n|)$$

then, for each  $n \in \mathbb{N}$ ,

$$\int_{\Omega} M^*(|m(u_n)|) dx \leq 2C_n.$$

Observe that  $f(s) \leq C(1+m(s))$ , that  $M^*(f(s)) \leq M^*(C(1+m(s)))$ , see Proposition B.11.(iii), and by convexity of  $M^*$ , that

$$\|f(u_n^+)\|_{M^*} \leq \int_{\Omega} M^*(C(1+m(u_n^+))) dx + 1 \leq C'_n,$$

see Proposition B.11.(i), and the r.h.s. is bounded for each  $n$ . Consequently,  $a(x)\frac{f(u_n^+)}{\|u_n\|} \in L_{M^*}(\Omega)$ , which is the dual of  $L_M(\Omega)$  (see [15], Theorem 14.2).

On the other hand, from  $J_2$ , for all  $\psi \in C_c^\infty(\Omega)$ ,

$$(3.8) \quad \left| \int_{\Omega} \nabla v_n \nabla \psi \, dx - \lambda_n \int_{\Omega} v_n \psi \, dx - \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} \psi \, dx \right| \leq \frac{\varepsilon_n}{\|u_n\|} \|\psi\|.$$

Taking the limit, and since  $C_c^\infty(\Omega)$  is dense in  $L_M(\Omega)$  (see [12]),

$$(3.9) \quad \lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} \psi \, dx = 0, \quad \text{for all } \psi \in L_M(\Omega).$$

Moreover, since (3.7),  $v_n \rightarrow v = 0$  in  $L_M(\Omega)$ . Hence [2, Proposition 3.13 (iv)], and (3.9) imply

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n^+)}{\|u_n\|} v_n \, dx = 0,$$

which contradicts (3.5), ending the proof.  $\square$

**Theorem 3.2.** *Assume the hypothesis of Proposition 3.1 and let  $\{u_n\}_{n \in \mathbb{N}}$  be a (PS) sequence in  $H_0^1(\Omega)$ .*

*Then, there exists a subsequence, denoted by  $\{u_n\}_{n \in \mathbb{N}}$ , such that*

$$u_n \rightarrow u \quad \text{in } H_0^1(\Omega).$$

*Proof.* From Proposition 3.1 we know that the sequence is bounded. Consequently, there exists a subsequence, denoted by  $\{u_n\}_{n \in \mathbb{N}}$ , and some  $u \in H_0^1(\Omega)$  such that

$$(3.10) \quad u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$(3.11) \quad \int_{\Omega} a(x) g(u_n) |u_n - u| \, dx \rightarrow 0,$$

$$(3.12) \quad u_n \rightarrow u \quad \text{a.e.}$$

By testing  $(J_2)$  against  $\psi = u_n - u$  and using (3.10), and (3.11) we get

$$\begin{aligned} \|u_n - u\|^2 &= \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) \, dx + o(1) \\ &\leq \|a\|_{\infty} \int_{\Omega} \frac{|u_n|^{2^*-1}}{[\ln(e + |u_n|)]^\alpha} |u_n - u| \, dx + o(1). \end{aligned}$$

*Claim.*

$$\int_{\Omega} \frac{|u_n|^{2^*-1}}{[\ln(e + |u_n|)]^\alpha} |u_n - u| \, dx = o(1),$$

In order to prove this claim, we use as in the above proposition, a Hölder inequality and a compact embedding into an Orlicz space, c.f. Appendix B.

By Theorem B.3 and Theorem B.12 we have

$$(3.13) \quad \|u_n - u\|_M \rightarrow 0,$$

where  $m$ , and  $M$  are defined by (3.2)-(3.4), as in the above proposition. On the other hand, since there exists  $C > 0$  such that  $m(s) \leq Cs^{2^*-1}$  and  $M(s) \leq Cs^{2^*}$  for all  $s \geq 0$ , and the sequence  $\{u_n\}_{n \in \mathbb{N}}$  is bounded in  $H_0^1(\Omega)$ , then

$$|u_n m(u_n)| \leq C, \quad |M(|u_n|)| \leq C \quad \text{for all } n \in \mathbb{N}$$

By using definition B.8 of  $M^*$  and identities of Proposition B.9 we have

$$M^*(|m(u_n)|) = |u_n m(u_n)| - M(|u_n|)$$

then

$$\int_{\Omega} M^*(|m(u_n)|) dx \leq C$$

for all  $n \in \mathbb{N}$ . Finally, by inequality (B.5) of Proposition B.12 we get

$$\sup \left\{ \|m(|u_n|)\|_{M^*}, n \in \mathbb{N} \right\} \leq C + 1.$$

Now, using Holder's inequality (B.6) and that  $\frac{s^{2^*-1}}{[\ln(e+s)]^\alpha} \leq m(s)$  for all  $s \geq 0$ , we get

$$\int_{\Omega} \frac{|u_n|^{2^*-1}}{[\ln(e+|u_n|)]^\alpha} |u_n - u| dx \leq \|u_n - u\|_M \|m(u_n)\|_{M^*} \leq (C+1) \|u_n - u\|_M$$

and it follows from (3.13) that  $\|u_n - u\| \rightarrow 0$ .  $\square$

**3.2. An existence result for  $\lambda < \lambda_1$ .** The next Theorem provides a solution to (1.1) for  $\lambda < \lambda_1$  based on the Mountain Pass Theorem.

**Theorem 3.3.** *Assume that  $\Omega \subset \mathbb{R}^N$  is a bounded domain with  $C^2$  boundary. Assume that the nonlinearity  $f$  defined by (1.2) satisfies (H), and that the weight  $a \in C^1(\overline{\Omega})$ . Then, the boundary value problem  $(1.1)_\lambda$  has at least one classical positive solution for any  $\lambda < \lambda_1$ .*

*Proof.* We verify the hypothesis of the Mountain Pass Theorem, see [13, Theorem 2, §8.5]. Observe that the derivative of the functional  $J'_\lambda : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is Lipschitz continuous on bounded sets of  $H_0^1(\Omega)$ ; also the (PS) condition is satisfied, see Proposition 3.1. Clearly  $J_\lambda[0] = 0$ .

1. Let now  $u \in H_0^1(\Omega)$  with  $\|u\| = r$ , for  $r > 0$  to be selected below. Then,

$$(3.14) \quad J_\lambda[u] = \frac{r^2}{2} - \frac{\lambda}{2} \int_\Omega (u^+)^2 dx - \int_\Omega a(x)F(u^+) dx.$$

From hypothesis (H) we have

$$\left| \int_\Omega a(x)G(u^+) dx \right| \leq C \int_\Omega (|u|^p + |u|^q) dx \leq C(r^p + r^q).$$

where  $G(s) := \int_0^s g(t) dt$ . Now, definition (1.2) implies that

$$\left| \int_\Omega a(x)F(u^+) dx \right| \leq C(r^p + r^q + r^{2^*}).$$

In view of (3.14), and thanks to the Poincaré inequality we get

$$J_\lambda[u] \geq \frac{1}{2} \left(1 - \frac{|\lambda|}{\lambda_1}\right) r^2 - C(r^p + r^q + r^{2^*}) \geq C_1 r^2,$$

taking  $|\lambda| < \lambda_1$ ,  $r > 0$  small enough, and using that  $p, q, 2^* > 2$ .

2. Now, fix some element  $0 \leq u_0 \in H_0^1(\Omega)$ ,  $u_0 > 0$  in  $\Omega^+$ ,  $u_0 \equiv 0$  in  $\Omega^-$ . Let  $v = tu_0$  for a certain  $t = t_0 > 0$  to be selected a posteriori. Since

$$(3.15) \quad f(tu_0) = |t|^{2^*-2} t f(u_0) \left( \frac{\ln(e + |u_0|)}{\ln(e + |tu_0|)} \right)^\alpha + g(tu_0),$$

then  $f(tu_0)/t \rightarrow +\infty$  as  $t \rightarrow +\infty$  in  $\Omega^+$ .

From definition, and integrating by parts,

$$\begin{aligned} F(s) &= \int_0^s \left( \frac{t^{2^*-1}}{\ln(e+t)^\alpha} + g(t) \right) dt \\ &= \frac{1}{2^*} sh(s) + G(s) + \frac{\alpha}{2^*} \int_0^s \left( \frac{1}{\ln(e+t)} \right)^{\alpha+1} \frac{t^{2^*}}{e+t} dt. \end{aligned}$$

It can be easily seen that  $\lim_{s \rightarrow +\infty} \frac{G(s)}{sf(s)} = 0$ .

Therefore, using l'Hôpital's rule we can write

$$(3.16) \quad \lim_{s \rightarrow +\infty} \frac{F(s)}{sf(s)} = \frac{1}{2^*} \in \left(0, \frac{1}{2}\right),$$

hence

$$(3.17) \quad \lim_{t \rightarrow +\infty} \frac{F(tu_0)}{tu_0 f(tu_0)} = \frac{1}{2^*} \in \left(0, \frac{1}{2}\right) \quad \text{in } \Omega^+.$$

Let  $C_0 \geq 0$  be such that  $F(s) + \frac{1}{2}C_0s^2 \geq 0$  for all  $s \geq 0$  (see (1.7)), and let

$$(3.18) \quad \tilde{\Omega}_\delta^+ := \{x \in \Omega^+ : a(x) = a^+(x) > \delta\}.$$

By definition,  $u_0 \equiv 0$  in  $\Omega^-$ , so, introducing  $\pm \frac{1}{2}C_0(tu_0)^2$ , splitting the integral, and using (3.17)-(3.18) we obtain

$$\begin{aligned} - \int_{\Omega} a(x)F(tu_0) dx &= - \int_{\Omega^+} a^+(x)F(tu_0) dx \\ &\leq \frac{C_0t^2}{2} \int_{\Omega^+} a^+(x)u_0^2 dx - \int_{\tilde{\Omega}_\delta^+} a^+(x) \left[ \frac{1}{2}C_0(tu_0)^2 + F(tu_0) \right] dx \\ &\leq C + \frac{C_0t^2}{2} \int_{\Omega^+} a^+(x)u_0^2 dx - \frac{\delta t^2}{2} \int_{\tilde{\Omega}_\delta^+} \left[ C_0u_0^2 + \frac{u_0f(tu_0)}{2^*t} \right] dx. \end{aligned}$$

Hence, there exists a positive constant  $C > 0$  such that

$$\begin{aligned} J_\lambda[tu_0] &= \frac{t^2}{2} \|u_0\|^2 - t^2 \frac{\lambda}{2} \|u_0\|_{L^2(\Omega)}^2 - \int_{\Omega^+} a^+(x)F(tu_0) \\ &\leq C(1+t^2) - \frac{\delta t^2}{2} \int_{\tilde{\Omega}_\delta^+} \left[ C_0(u_0)^2 + \frac{u_0f(tu_0)}{2^*t} \right] dx < 0 \end{aligned}$$

for  $t = t_0 > 0$  big enough.

*Step 3.* We have at last checked that all the hypothesis of the Mountain Pass Theorem are accomplished. Let

$$\Gamma := \{\mathbf{g} \in C([0, 1]; H_0^1(\Omega)) : \mathbf{g}(0) = 0, \mathbf{g}(1) = t_0u_0\},$$

then, there exists  $c \geq C_1 r^2 > 0$  such that

$$c := \inf_{\mathbf{g} \in \Gamma} \max_{0 \leq t \leq 1} J_\lambda[\mathbf{g}(t)]$$

is a critical value of  $J_\lambda$ , that is, the set  $\mathcal{H}_c := \{v \in H_0^1(\Omega) : J_\lambda[v] = c, J'_\lambda[v] = 0\} \neq \emptyset$ . Thus there exists  $u \in H_0^1(\Omega)$ ,  $u \geq 0$ ,  $u \neq 0$  such that for each  $\psi \in H_0^1(\Omega)$ , we have

$$(3.19) \quad \int_{\Omega} \nabla u \cdot \nabla \psi dx = \int_{\Omega} [\lambda u^+ + a(x)f(u^+)] \psi dx.$$

and thereby  $u$  is a nontrivial weak solution to (3.19). By Lemma 2.1,  $u$  is a classical solution, and by (1.8),  $u > 0$  in  $\Omega$ .  $\square$

#### 4. A BIFURCATION RESULT FOR $\lambda > \lambda_1$

Next Proposition uses Crandall - Rabinowitz's local bifurcation theory, see [10], and Rabinowitz's global bifurcation theory, see [19].

**Proposition 4.1.** *Let us define*

$$\Lambda := \sup\{\lambda > 0 : (1.1) \text{ admits a positive solution}\}.$$

If (1.5) holds then,

$$\lambda_1 < \Lambda < \min \left\{ \lambda_1(\text{int}(\Omega^0)), \quad \lambda_1(\text{int}(\Omega^+ \cup \Omega^0)) + C_0 \sup a^+ \right\}$$

where  $C_0 > 0$  is such that  $f(s) + C_0 s \geq 0$  for all  $s \geq 0$ , (see definition (1.6)).

Moreover, there exists an unbounded continuum (a closed and connected set)  $\mathcal{C}$  of classical positive solutions to (1.1) emanating from the trivial solution set at the bifurcation point  $(\lambda, u) = (\lambda_1, 0)$ .

*Proof.* Proposition 2.2 proves the upper bounds for  $\Lambda$ . Next we concentrate our attention in proving that  $\Lambda > \lambda_1$ . Choosing  $\lambda$  as the bifurcation parameter, we check that the conditions of Crandall - Rabinowitz's Theorem [10] are satisfied. For  $r > N$ , we define the set  $W_+^{2,r} := \{u \in W^{2,r}(\Omega) : u > 0 \text{ in } \Omega\}$ , and consider  $W_+^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$  endowed with the topology of  $W^{2,r}(\Omega)$ . If  $r > N$ , we have that  $W_+^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \hookrightarrow C_0^{1,\mu}(\Omega)$  for  $\mu = 1 - \frac{N}{r} \in (0, 1)$ . Moreover, from Hopf's lemma, we know that if  $\tilde{u}$  is a positive solution to (1.1) then  $\tilde{u}$  lies in the interior of  $W_+^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ .

We consider the map  $\mathcal{F} : \mathbb{R} \times W_+^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \rightarrow L^r(\Omega)$  for  $r > N$ ,

$$\mathcal{F} : (\lambda, u) \rightarrow -\Delta u - \lambda u - a(x)f(u)$$

The map  $\mathcal{F}$  is a continuously differentiable map. Since hypothesis (i),  $g(0) = 0$ , and so  $a(x)F(0) = 0$ ,  $\mathcal{F}(\lambda, 0) = 0$  for all  $\lambda \in \mathbb{R}$ , and since  $F_u(x, 0) = 0$ ,

$$D_u \mathcal{F}(\lambda_1, 0)w := -\Delta w - \lambda_1 w,$$

$$D_{\lambda, u} \mathcal{F}(\lambda_1, 0)w := -w.$$

Observe that

$$N(D_u \mathcal{F}(\lambda_1, 0)) = \text{span}[\varphi_1], \quad \text{codim } R(D_u \mathcal{F}(\lambda_1, 0)) = 1,$$

$$D_{\lambda, u} \mathcal{F}(\lambda_1, 0)\varphi_1 = -\varphi_1 \notin R(D_u \mathcal{F}(\lambda_1, 0)),$$

where  $N(\cdot)$  is the kernel, and  $R(\cdot)$  denotes the range of a linear operator.

Hence, the hypotheses of Crandall-Rabinowitz theorem are satisfied and  $(\lambda_1, 0)$  is a bifurcation point. Thus, decomposing

$$C_0^{1,\mu}(\overline{\Omega}) = \text{span}[\varphi_1] \oplus Z,$$

where  $Z = \text{span}[\varphi_1]^\perp$ , there exists a neighbourhood  $\mathcal{U}$  of  $(\lambda_1, 0)$  in  $\mathbb{R} \times C_0^{1,\mu}(\overline{\Omega})$ , and continuous functions  $\lambda(s), \tilde{w}(s)$ ,  $s \in (-\varepsilon, \varepsilon)$ ,  $\lambda : (-\varepsilon, \varepsilon) \rightarrow$

$\mathbb{R}$ ,  $\tilde{w} : (-\varepsilon, \varepsilon) \rightarrow Z$  such that  $\lambda(0) = \lambda_1$ ,  $\tilde{w}(0) = 0$ , with  $\int_{\Omega} \tilde{w} \varphi_1 dx = 0$ , and the only nontrivial solutions to (1.1) in  $\mathcal{U}$ , are

$$(4.1) \quad \{(\lambda(s), s\varphi_1 + s\tilde{w}(s)) : s \in (-\varepsilon, \varepsilon)\}.$$

Set  $u = u(s) = s\varphi_1 + s\tilde{w}(s)$ . Note that by continuity  $\tilde{w}(s) \rightarrow 0$  as  $s \rightarrow 0$ , which guarantees that  $u(s) > 0$  in  $\Omega$  for all  $s \in (0, \varepsilon)$  small enough.

Next, we show that  $\lambda(s) > \lambda_1$  for all  $s$  small enough. Since (3.15), and hypothesis  $(H)_0$  on  $f$ , note that  $\frac{a(x)f(su)}{s^{p-1}u^{p-1}} \rightarrow L_1 a(x)$  as  $s \rightarrow 0$ . In fact, as  $\tilde{w}(s) \rightarrow 0$  uniformly as  $s \rightarrow 0$ , hypothesis  $(H)_0$  yields

$$\frac{a(x)f(s\varphi_1 + s\tilde{w}(s))}{s^{p-1}(\varphi_1 + \tilde{w}(s))^{p-1}} \rightarrow L_1 a(x) \text{ uniformly in } \Omega \quad \text{as } s \rightarrow 0.$$

Hence, multiplying and dividing by  $(\varphi_1 + \tilde{w}(s))^{p-1}$ , we deduce

$$\frac{1}{s^{p-1}} \int_{\Omega} a(x)f(u(s))\varphi_1 \xrightarrow{s \rightarrow 0} L_1 \int_{\Omega} a(x)\varphi_1^p.$$

Now we prove that  $\lambda(s) > \lambda_1$  arguing by contradiction. Assume that there is a sequence  $(\lambda_n, u_n) = (\lambda(s_n), u(s_n))$  of bifurcated solutions to (1.1) in  $\mathcal{U}$ , with  $\lambda(s_n) \leq \lambda_1$ . Multiplying (1.1) $_{\lambda_n}$  by  $\varphi_1$  and integrating by parts

$$0 \leq \frac{(\lambda_1 - \lambda(s_n))}{s_n^{p-1}} \int_{\Omega} u(s_n)\varphi_1 = \frac{1}{s_n^{p-1}} \int_{\Omega} a(x)f(u(s_n))\varphi_1 \rightarrow L_1 \int_{\Omega} a(x)\varphi_1^p < 0$$

which yields a contradiction, and consequently,  $\Lambda > \lambda_1$ . Finally, Rabinowitz's global bifurcation Theorem [19] states that in fact, the set  $\mathcal{C}$  of positive solutions to (1.1) emanating from  $(\lambda_1, 0)$  is a continuum (a closed and connected set) which is either unbounded, or contains another bifurcation point, or contains a pair of points  $(\lambda, u)$ ,  $(\lambda, -u)$  with  $u \neq 0$ . Since (1.8), any non-negative non-trivial solution is strictly positive, moreover  $(\lambda_1, 0)$  is the only bifurcation point to positive solutions, so  $\mathcal{C}$  can not reach another bifurcation point. Since (1.3), neither  $\mathcal{C}$  contains a pair of points  $(\lambda, u)$ ,  $(\lambda, -u)$  with  $u \neq 0$ , which state that  $\mathcal{C}$  is unbounded, ending the proof.  $\square$

## 5. PROOF OF THEOREM 1.1

First we prove an auxiliary result.

**Proposition 5.1.** *For each  $\lambda \in (\lambda_1, \Lambda)$ , the following holds:*

- (i) *Problem (1.1) $_{\lambda}$  admits a positive solution*

$$u_{\lambda} = \inf \{u(x) : u > 0 \text{ solving } (1.1)_{\lambda}\},$$



in other words  $u_\lambda$  is minimal.

- (ii) Moreover, the map  $\lambda \rightarrow u_\lambda$  is strictly monotone increasing, that is, if  $\lambda < \mu < \Lambda$ , then  $u_\lambda(x) < u_\mu(x)$  for all  $x \in \Omega$ , and  $\frac{\partial u_\lambda}{\partial \nu}(x) > \frac{\partial u_\mu}{\partial \nu}(x)$  for all  $x \in \partial\Omega$ .
- (iii) Furthermore,  $u_\lambda$  is a local minimum of the functional  $J_\lambda$ .

*Proof. (i.a) Step 1. Existence of positive solutions for any  $\lambda \in (\lambda_1, \Lambda)$ .*

Let  $\lambda \in (\lambda_1, \Lambda)$  be fixed. By definition of  $\Lambda$ , there exists a  $\lambda_0 \in (\lambda, \Lambda)$  such that the problem  $(1.1)_{\lambda_0}$  admits a positive solution  $u_0$ . It is easy to verify that  $u_0 > 0$  is a supersolution to  $(1.1)_\lambda$ . Indeed, for any  $\psi \in H_0^1(\Omega)$  with  $\psi \geq 0$  in  $\Omega$

$$\int_{\Omega} \nabla u_0 \nabla \psi \, dx - \lambda \int_{\Omega} u_0 \psi \, dx - \int_{\Omega} a(x) f(u_0) \psi \, dx = (\lambda_0 - \lambda) \int_{\Omega} u_0 \psi \, dx \geq 0.$$

Moreover, for every  $\delta > 0$  satisfying

$$(5.1) \quad 0 < \delta < \left( \frac{\lambda - \lambda_1}{2L_1 \|a^-\|_\infty} \right)^{\frac{1}{p-2}} \frac{1}{\|\varphi_1\|_\infty}$$

the function  $\underline{u} = \delta\varphi_1$  is a subsolution for  $(1.1)_\lambda$  whenever  $\lambda > \lambda_1$ . Let  $\delta > 0$  satisfying (5.1) and such that  $g(s) \geq 0$  for any  $s \in [0, \delta\|\varphi_1\|_{L^\infty(\Omega)}]$ . For any  $\psi \in H_0^1(\Omega)$ ,  $\psi > 0$  with in  $\Omega$  we deduce

$$\begin{aligned} & \delta \int_{\Omega} \nabla \varphi_1 \nabla \psi \, dx - \lambda \delta \int_{\Omega} \varphi_1 \psi \, dx - \int_{\Omega} a(x) f(\delta\varphi_1) \psi \, dx \\ &= -(\lambda - \lambda_1) \delta \int_{\Omega} \varphi_1 \psi \, dx - \int_{\Omega} a(x) f(\delta\varphi_1) \psi \, dx \\ &= -(\lambda - \lambda_1) \delta \int_{\Omega} \varphi_1 \psi \, dx \\ &\quad - \int_{\Omega} a(x) \left[ \frac{(\delta\varphi_1)^{2^*-1}}{[\ln(e + \delta\varphi_1)]^\alpha} + g(\delta\varphi_1) \right] \psi \, dx \\ &\leq -(\lambda - \lambda_1) \delta \int_{\Omega} \varphi_1 \psi \, dx \\ &\quad + \|a^-\|_\infty \int_{\Omega} [h(\delta\varphi_1) + g(\delta\varphi_1)] \psi \, dx < 0. \end{aligned}$$

This allow us to take  $\underline{u} = \delta\varphi_1$  as a subsolution for  $(1.1)_\lambda$  with  $\underline{u} < u_0$ . The sub- and supersolution method now guarantees a positive solution  $u$  to  $(1.1)_\lambda$ , with  $\underline{u} \leq u \leq u_0$ .

*(i.b) Step 2. Existence of a minimal positive solution  $u_\lambda$  for any  $\lambda \in (\lambda_1, \Lambda)$ .*

To show that there is in fact a minimal solution, for each  $x \in \Omega$  we define

$$\underline{u}_\lambda(x) := \inf \{u(x) : u > 0 \text{ solving } (1.1)_\lambda\}.$$

Firstly, we claim that  $\underline{u}_\lambda \geq 0$ ,  $\underline{u}_\lambda \not\equiv 0$ . Assume by contradiction that  $\underline{u}_\lambda \equiv 0$ . This would yield a sequence  $u_n$  of positive solutions to  $(1.1)_\lambda$  such that  $\|u_n\|_{C(\overline{\Omega})} \rightarrow 0$  as  $n \rightarrow \infty$ , or in other words,  $(\lambda, 0)$  is a bifurcation point from the trivial solution set to positive solutions. Set  $v_n := \frac{u_n}{\|u_n\|_{C(\overline{\Omega})}}$ . Observe that  $v_n$  is a weak solution to the problem

$$(5.2) \quad -\Delta v_n = \lambda v_n + a(x)f(u_n)/\|u_n\|_{C(\overline{\Omega})} \text{ in } \Omega; \quad v_n = 0 \text{ on } \partial\Omega.$$

It follows from  $(H)_0$  that  $\frac{a(x)f(u_n)}{\|u_n\|_{C(\overline{\Omega})}} \rightarrow 0$  in  $C(\overline{\Omega})$  as  $n \rightarrow \infty$ . Therefore, the right-hand side of (5.2) is bounded in  $C(\overline{\Omega})$ . Hence, by the elliptic regularity,  $v_n \in W^{2,r}(\Omega)$  for any  $r > 1$ , in particular for  $r > N$ . Then, the Sobolev embedding theorem implies that  $\|v_n\|_{C^{1,\alpha}(\overline{\Omega})}$  is bounded by a constant  $C$  that is independent of  $n$ . Then, the compact embedding of  $C^{1,\mu}(\overline{\Omega})$  into  $C^{1,\beta}(\overline{\Omega})$  for  $0 < \beta < \mu$  yields, up to a subsequence,  $v_n \rightarrow \Phi \geq 0$  in  $C^{1,\beta}(\overline{\Omega})$ . Since  $\|v_n\|_{C(\overline{\Omega})} = 1$ , we have that  $\|\Phi\|_{C(\overline{\Omega})} = 1$ . Hence,  $\Phi \geq 0$ ,  $\Phi \not\equiv 0$ .

Using the weak formulation of equation (5.2), passing to the limit, and taking into account that  $\lambda$  is fixed and  $v_n \rightarrow \Phi$ , we obtain that  $\Phi \geq 0$ ,  $\Phi \not\equiv 0$ , is a weak solution to the equation

$$-\Delta\Phi = \lambda\Phi \text{ in } \Omega, \quad \Phi = 0 \text{ on } \partial\Omega.$$

Then, by the maximum principle it follows that  $\Phi = \varphi_1 > 0$ , the first eigenfunction, and  $\lambda = \lambda_1$  is its corresponding eigenvalue, which contradicts that  $\lambda > \lambda_1$ .

Secondly, we show that  $\underline{u}_\lambda$  solves  $(1.1)_\lambda$ . We argue on the contrary. Observe that the minimum of any two positive solutions to  $(1.1)_\lambda$  furnishes a supersolution to  $(1.1)_\lambda$ . Assume that there are a finite number of solutions to  $(1.1)_\lambda$ , then  $\underline{u}_\lambda(x) := \min \{u(x) : u > 0 \text{ solves } (1.1)_\lambda\}$  and  $\underline{u}_\lambda$  is a supersolution. Choosing  $\varepsilon_0$  small enough so that  $\varepsilon_0\varphi_1 < \underline{u}_\lambda$ , the sub-supersolution method provides a solution  $\varepsilon_0\varphi_1 \leq v \leq \underline{u}_\lambda$ . Since  $v$  is a solution and  $\underline{u}_\lambda$  is not, then  $v \not\leq \underline{u}_\lambda$ , contradicting the definition of  $\underline{u}_\lambda$ , and achieving this part of the proof.

Assume now that there is a sequence  $u_n$  of positive solutions to  $(1.1)_\lambda$  such that, for each  $x \in \Omega$ ,  $\inf u_n(x) = \underline{u}_\lambda(x) \geq 0$ ,  $\underline{u}_\lambda \not\equiv 0$ . Let  $\underline{u}_1 := \min\{u_1, u_2\}$ . Choosing  $\varepsilon_1$  small enough so that  $\varepsilon_1\varphi_1 < \underline{u}_1$ , the sub-supersolution method provides a solution  $\varepsilon_1\varphi_1 \leq v_1 \leq \underline{u}_1$ . We reason by induction.

Let  $\underline{u}_n := \min\{v_{n-1}, u_{n+1}\}$ . Choosing  $\varepsilon_n$  small enough so that  $\varepsilon_n \varphi_1 < \underline{u}_n$ , the sub- supersolution method provides a solution  $\varepsilon_n \varphi_1 \leq v_n \leq \underline{u}_n \leq v_{n-1}$ . With this induction procedure, we build a monotone sequence of solutions  $v_n$ , such that

$$(5.3) \quad 0 < v_n \leq \underline{u}_n \leq v_{n-1} \leq \underline{u}_{n-1} \leq \cdots \leq v_1.$$

Since monotonicity and Lemma 2.1,  $\|v_n\|_{C(\overline{\Omega})} \leq \|v_1\|_{C(\overline{\Omega})}$ , by elliptic regularity,  $\|v_n\|_{C^{1,\mu}(\overline{\Omega})} \leq C$  for any  $\mu < 1$ , and by compact embedding  $v_n \rightarrow v$  in  $C^{1,\beta}(\overline{\Omega})$  for any  $\beta < \alpha$ . Using the weak formulation of equation  $(1.1)_\lambda$ , passing to the limit, and taking into account that  $\lambda$  is fixed, we obtain that  $v$  is a weak solution to the equation  $(1.1)_\lambda$ . Hence  $v(x) \geq \underline{u}_\lambda > 0$ . Moreover, since (5.3),  $v_n(x) \downarrow v(x)$  pointwise for  $x \in \Omega$ , so  $\inf v_n(x) = v(x)$ . Also, and due to (5.3),  $\underline{u}_n(x) \downarrow v(x)$  pointwise for  $x \in \Omega$ , and  $\inf \underline{u}_n(x) = v(x)$ .

On the other hand, by construction  $\underline{u}_n \leq u_{n+1}$ , so, for each  $x \in \Omega$ ,  $v(x) = \inf \underline{u}_n(x) \leq \inf u_n(x) = \underline{u}_\lambda(x)$ . Therefore, and by definition of  $\underline{u}_\lambda$ , necessarily  $v = \underline{u}_\lambda$ , proving that  $\underline{u}_\lambda$  solves  $(1.1)_\lambda$ , and achieving the proof of step 2.

(ii) The monotonicity of the minimal solutions is concluded from a sub-supersolution method. Reasoning as in step 1,  $u_\mu$  is a strict supersolution to  $(1.1)_\lambda$ , so  $w := u_\mu(x) - u_\lambda(x) \geq 0$ ,  $w \not\equiv 0$ . Moreover,  $w = 0$  on  $\partial\Omega$ , and we can always choose  $c_0 := C_0 \|a\|_\infty > 0$  where  $C_0$  is defined by (1.6), so that  $a^-(x)f'(s) + c_0 \geq 0$  and  $a^+(x)f'(s) + c_0 \geq 0$  for all  $s \geq 0$ , then

$$\begin{aligned} \left( -\Delta + a^-(x)f'(\theta u_\mu + (1-\theta)u_\lambda) + c_0 \right) w &= (\mu - \lambda)u_\mu + \lambda w \\ &+ [a^+(x)f'(\theta u_\mu + (1-\theta)u_\lambda) + c_0] w > 0 \text{ in } \Omega, \end{aligned}$$

finally, the Maximum Principle implies that  $w > 0$  in  $\Omega$ , and  $\frac{\partial w}{\partial \nu} < 0$  on  $\partial\Omega$ , ending the proof of step 3.

(iii) Since [4, Theorem 2] if there exists an ordered pair of  $L^\infty$  bounded sub and super-solution  $\underline{u} \leq \bar{u}$  to  $(1.1)_\lambda$ , and neither  $\underline{u}$  nor  $\bar{u}$  is a solution to  $(1.1)_\lambda$ , then there exist a solution  $\underline{u} < u < \bar{u}$  to  $(1.1)_\lambda$  such that  $u$  is a local minimum of  $J_\lambda$  at  $H_0^1(\Omega)$ .

Reasoning as in (i),  $\bar{u} := u_\mu$  with  $\mu > \lambda$  is a strict super-solution to  $(1.1)_\lambda$ , and  $\underline{u} := \delta \varphi_1$  is a strict sub-solution for  $\delta > 0$  small enough, such that  $\underline{u}(x) < \bar{u}(x)$  for each  $x \in \Omega$ . This achieves the proof.  $\square$

*Proof of Theorem 1.1.* Theorem 3.3 provides the existence of positive solutions for  $\lambda < \lambda_1$ , and Proposition 5.1 provide the existence of minimal positive solutions for  $\lambda \in (\lambda_1, \Lambda)$ .

(a) *Step 1. Existence of a second positive solution for  $\lambda \in (\lambda_1, \Lambda)$ .*

Fix an arbitrary  $\lambda \in (\lambda_1, \Lambda)$ , and let  $u_\lambda$  be the minimal solution to  $(1.1)_\lambda$  given by Proposition 5.1, minimizing  $J_\lambda$ . A second solution follows seeking a solution through variational arguments [14, Theorem 5.10] and the Mountain Pass procedure shown below.

First, reasoning as in Proposition 5.1(iii), we get a local minimum  $\tilde{u}_\lambda > 0$  of  $J_\lambda$ . If  $\tilde{u}_\lambda \neq u_\lambda$ , then  $\tilde{u}_\lambda$  is the second positive solution, ending the proof. Assume that  $\tilde{u}_\lambda = u_\lambda$ .

Now we reason as in [14, Theorem 5.10] on the nature of local minima. Thus, either

(i) there exists  $\varepsilon_0 > 0$ , such that  $\inf \{J_\lambda(u) : \|u - \tilde{u}_\lambda\| = \varepsilon_0\} > J_\lambda(\tilde{u}_\lambda)$ , in other words,  $\tilde{u}_\lambda$  is a strict local minimum, or

(ii) for each  $\varepsilon > 0$ , there exists  $u_\varepsilon \in H_0^1(\Omega)$  such that  $J_\lambda$  has a local minimum at a point  $u_\varepsilon$  with  $\|u_\varepsilon - \tilde{u}_\lambda\| = \varepsilon$  and  $J_\lambda(u_\varepsilon) = J_\lambda(\tilde{u}_\lambda)$ .

Let us assume that (i) holds, since otherwise case (ii) implies the existence of a second solution.

Consider now the functional  $I_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$  given by  $I_\lambda[v] = J_\lambda[u_\lambda + v] - J_\lambda[u_\lambda]$ , more specifically

$$I_\lambda[v] := \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{\lambda}{2} \int_\Omega (v^+)^2 dx - \int_\Omega \tilde{G}_\lambda(x, v^+) dx.$$

where

$$\begin{aligned} \tilde{G}_\lambda(x, s) &:= a(x) [F(u_\lambda(x) + s) - F(u_\lambda(x)) - f(u_\lambda(x))s] \\ &= a(x) \left[ \frac{1}{2} f'(u_\lambda(x)) s^2 + o(s^2) \right]. \end{aligned}$$

Obviously  $I_\lambda[v^+] \leq I_\lambda[v]$ , and observe that  $I'_\lambda[v] = 0 \iff J'_\lambda[u_\lambda + v] = 0$ .

Fix now some element  $0 \leq v_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ ,  $v_0 > 0$  in  $\Omega^+$ ,  $v_0 \equiv 0$  in  $\Omega^-$ . Let  $v = tv_0$  for a certain  $t = t_0 > 0$  to be selected a posteriori, and evaluate

$$I_\lambda[tv_0] = \frac{1}{2} t^2 \left( \|\nabla v_0\|_{L^2(\Omega)}^2 - \lambda \|v_0\|_{L^2(\Omega)}^2 \right) - \int_\Omega \tilde{G}_\lambda(x, tv_0) dx.$$

Reasoning as in the proof of Theorem 3.3 for large positive  $t$ , since  $F(t)/t^2 \rightarrow \infty$  as  $t \rightarrow \infty$ , and using also (3.1) we obtain that

$$\begin{aligned} I_\lambda[tv_0] &\leq C(1+t+t^2) - \int_{\Omega^+} a^+(x) \left[ F(u_\lambda + tv_0) + \frac{1}{2}C_0(u_\lambda + tv_0)^2 \right] \\ &\leq C(1+t+t^2) - \delta \int_{\tilde{\Omega}_\delta^+} \left[ F(u_\lambda + tv_0) + \frac{1}{2}C_0(u_\lambda + tv_0)^2 \right] dx, \end{aligned}$$

so

$$I_\lambda[tv_0] < 0$$

for  $t = t_0$  big enough, and where  $\tilde{\Omega}_\delta^+$  is defined by (3.18). Thus, the Mountain Pass Theorem implies that if

$$\Gamma := \{\mathbf{g} \in C([0, 1]; H_0^1(\Omega)) : \mathbf{g}(0) = 0, I_\lambda[\mathbf{g}(1)] < 0\},$$

then, there exists  $c > 0$  such that

$$c := \inf_{\mathbf{g} \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda[\mathbf{g}(t)]$$

is a critical value of  $I_\lambda$ , and thereby  $\mathcal{K}_c := \{v \in H_0^1(\Omega) : I_\lambda[v] = c, I'_\lambda[v] = 0\}$  is non empty.

Since for any  $\mathbf{g} \in \Gamma$  we have  $I_\lambda[\mathbf{g}^+(t)] \leq I_\lambda[\mathbf{g}(t)]$  for all  $t \in [0, 1]$ , it follows that  $\mathbf{g}^+ \in \Gamma$ , and we derive the existence of a sequence  $v_n$  such that

$$I_\lambda[v_n] \rightarrow c, \quad \|I'_\lambda[v_n]\| \rightarrow 0, \quad v_n \geq 0.$$

On the other hand,  $w_n := u_\lambda + v_n$  is a (PS) sequence for the original functional  $J_\lambda$ . Since Theorem 3.2, if  $\lambda < \lambda_1(\text{int } \Omega^0)$ ,  $v_n \rightarrow v_\lambda$  en  $H_0^1(\Omega)$ , so  $I'_\lambda[v] = 0$  and  $I_\lambda[v] = c > 0$ , hence  $v_\lambda \geq 0$  is a nontrivial critical point of  $I_\lambda$ . Consequently,  $w_\lambda := u_\lambda + v_\lambda$  is a positive critical point of  $J_\lambda$ , such that, for each  $\psi \in H_0^1(\Omega)$ , we have

$$\int_{\Omega} \nabla w_\lambda \cdot \nabla \psi \, dx = \int_{\Omega} \left( \lambda w_\lambda + a(x)f(w_\lambda) \right) \psi \, dx,$$

and thereby  $w_\lambda := u_\lambda + v_\lambda \geq u_\lambda$ ,  $w_\lambda \neq u_\lambda$  is a second positive solution to (1.1) $_\lambda$ .

(b) *Step 2. Existence of a classical positive solution for  $\lambda = \Lambda$ .*

We prove the existence of a solution for  $\lambda = \Lambda$ . For each  $\lambda \in (\lambda_1, \Lambda)$ , problem (1.1) admits a minimal positive weak solution  $u_\lambda$  and  $\lambda \rightarrow u_\lambda$  is increasing, see Proposition 5.1. Taking the monotone pointwise limit, let us define

$$u_\Lambda(x) := \lim_{\lambda \uparrow \Lambda} u_\lambda(x).$$

We next see that  $\|u_\Lambda\| < +\infty$ , reasoning on the contrary. Assume that there exists a sequence of solutions  $u_n := u_{\lambda_n}$  such that  $\|u_{\lambda_n}\| \rightarrow +\infty$  as  $\lambda_n \rightarrow \Lambda$ . Set  $v_n := u_n/\|u_n\|$ , then there exists a subsequence, again denoted by  $v_n$  such that  $v_n \rightharpoonup v$  in  $H_0^1(\Omega)$ , and  $v_n \rightarrow v$  in  $L^p(\Omega)$  for any  $p < 2^*$  and a.e. Arguing as in the claim of Proposition 3.1,  $v \equiv 0$ . Moreover

$$(5.4) \quad \lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n)}{\|u_n\|} v_n dx = 1.$$

On the other hand, from the weak formulation, for all  $\psi \in C_c^\infty(\Omega)$ ,

$$(5.5) \quad \int_{\Omega} \nabla v_n \nabla \psi dx = \lambda_n \int_{\Omega} v_n \psi dx + \int_{\Omega} a(x) \frac{f(u_n)}{\|u_n\|} \psi dx.$$

Taking the limit, and since  $C_c^\infty(\Omega)$  is dense in  $L^2(\Omega)$

$$(5.6) \quad \lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n)}{\|u_n\|} \psi dx = 0, \quad \text{for all } \psi \in L^2(\Omega).$$

Since Lemma 2.1,  $u \in C^2(\Omega) \cap C^{1,\mu}(\bar{\Omega})$  and so  $a(x) \frac{f(u_n)}{\|u_n\|} \in L^2(\Omega)$ . Moreover  $v_n \rightarrow v = 0$  in  $L^2(\Omega)$ . Hence [2, Proposition 3.13 (iv)], and (5.6) imply

$$\lim_{n \rightarrow \infty} \int_{\Omega} a(x) \frac{f(u_n)}{\|u_n\|} v_n dx = 0,$$

which contradicts (5.4) and yields  $\|u_\Lambda\| < +\infty$ .

By Sobolev embedding and the Lebesgue dominated convergence theorem,  $u_n \rightarrow u_\Lambda$  in  $L^{2^*}(\Omega)$ .

Now, by substituting  $\psi = u_n$  in (5.5), using Hölder inequality and Sobolev embeddings we obtain

$$\left[ \|u_n\| \leq \Lambda \|v_n\|_{L^2(\Omega)} \|u_n\| + C, \quad \text{with } \|v_n\|_{L^2(\Omega)} \rightarrow 0 \right] \Rightarrow \|u_n\| \leq C.$$

By compactness, for a subsequence again denoted by  $u_n$ ,  $u_n \rightharpoonup u^*$  in  $H_0^1(\Omega)$ ,  $u_n \rightarrow u^*$  in  $L^p(\Omega)$  for any  $p < 2^*$  and a.e. By uniqueness of the limit,  $u_\Lambda = u^*$ . Finally, by taking limits in the weak formulation of  $u_n$  as  $\lambda_n \rightarrow \Lambda$ , we get

$$\int_{\Omega} \nabla u_\Lambda \nabla \psi = \Lambda \int_{\Omega} u_\Lambda \psi + \int_{\Omega} a(x) f(u_\Lambda) \psi.$$

Hence  $u_\Lambda$  is a positive weak solution to  $(1.1)_\Lambda$ . Lemma 2.1 yields that  $u_\Lambda \in C^2(\Omega) \cap C^{1,\mu}(\bar{\Omega})$  is a classical solution.

(c) *Step 3. Existence of a classical positive solution for  $\lambda \leq \lambda_1$ .*

The existence of a classical positive solution for  $\lambda < \lambda_1$  is done in Theorem 3.3. Let's look for a solution when  $\lambda = \lambda_1$ .

Since step 1, for any  $\lambda \in (\lambda_1, \Lambda)$  there exists a second positive solution to (1.1) $_\lambda$ . Let's denote it by  $\tilde{u}_\lambda \neq u_\lambda$ . Now, define the pointwise limit

$$(5.7) \quad \tilde{u}_{\lambda_1}(x) := \limsup_{\lambda \rightarrow \lambda_1} \tilde{u}_\lambda(x).$$

Reasoning as in step 2,  $\|\tilde{u}_{\lambda_1}\| < +\infty$  and  $\tilde{u}_{\lambda_1} \in C^2(\Omega) \cap C^{1,\mu}(\bar{\Omega})$  is a classical solution to (1.1) $_{\lambda_1}$ .

Moreover,  $\tilde{u}_{\lambda_1} > 0$ . Assume on the contrary that  $\tilde{u}_{\lambda_1} = 0$ . By the Crandall-Rabinowitz's Theorem [10], the only nontrivial solutions to (1.1) in a neighborhood of the bifurcation point  $(\lambda_1, 0)$  are given by (4.1). Since Proposition 5.1, those are the minimal solutions  $u_\lambda$ , and due to  $\tilde{u}_\lambda \neq u_\lambda$ ,  $\tilde{u}_\lambda$  are not in a neighbourhood of  $(\lambda_1, 0)$ , contradicting the definition of  $\tilde{u}_{\lambda_1}(x)$ , (5.7)

Hence,  $\tilde{u}_{\lambda_1} \geq 0$ , and reasoning as in (1.8), the Maximum Principle implies that  $\tilde{u}_{\lambda_1} > 0$ .  $\square$

#### APPENDIX A. SOME ESTIMATES

First, we prove an useful estimate of  $\frac{\ln(e+s)}{\ln(e+as)}$ .

**Lemma A.1.** *Let  $0 < a \leq 1$  be fixed. Then for all  $s \geq 0$ ,*

$$(A.1) \quad \frac{\ln(e+s)}{\ln(e+as)} \leq \ln\left(\frac{e}{a}\right) \leq \frac{1}{a}.$$

*Proof.* Denote  $\ell(s) = \frac{\ln(e+s)}{\ln(e+as)}$  for all  $s \geq 0$ . Then  $1 \leq \ell(s) \leq \ell(s_0)$  where  $s_0 > 0$  is the unique value where  $\ell'(s) = 0$ . When computing  $s_0$  we find

$$\ell'(s_0) = 0 \iff (e + as_0) \ln(e + as_0) - a(e + s_0) \ln(e + s_0) = 0$$

and therefore

$$\max \ell = \ell(s_0) = \frac{\ln(e + s_0)}{\ln(e + as_0)} = \frac{e + as_0}{a(e + s_0)}.$$

Notice that we have  $\ell(s_0) \leq \frac{1}{a}$ . In order to find a better upper bound of  $\ln\left(\frac{e+as_0}{e+s_0}\right)$  let us denote for all  $s \geq 0$

$$\theta(s) = (e + as) \ln(e + as) - a(e + s) \ln(e + s).$$

Then, there exists  $\chi \in (0, s_0)$  such that

$$0 - e(1 - a) = \theta(s_0) - \theta(0) = \theta'(\chi)s_0 \implies \frac{e(1 - a)}{s_0} = -\theta'(\chi).$$

Then

$$-\theta'(s) = a \ln\left(\frac{e + s}{e + as}\right) \leq a \ln\left(\frac{1}{a}\right),$$

and

$$\frac{e(1-a)}{s_0} \leq a \ln\left(\frac{1}{a}\right) \implies s_0 \geq \frac{e(1-a)}{a \ln\left(\frac{1}{a}\right)}.$$

Since  $\frac{e+as}{a(e+s)}$  is decreasing,

$$\begin{aligned} \max_{s \geq 0} \ell(s) &= \ell(s_0) = \frac{e + as_0}{a(e + s_0)} \leq \frac{e + \frac{e(1-a)}{\ln\left(\frac{1}{a}\right)}}{ae + \frac{e(1-a)}{\ln\left(\frac{1}{a}\right)}} \\ &= \frac{\ln(1/a) + 1 - a}{a \ln(1/a) + 1 - a} \leq \ln(1/a) + 1, \end{aligned}$$

and the first inequality of (A.1) is achieved. The second one is obvious.  $\square$

Next lemma is about the variations of  $h(s) = \frac{s^{2^*-1}}{[\ln(e+s)]^\alpha}$  for  $s \geq 0$ .

**Lemma A.2.** *There exists  $\alpha^* > 2(2^* - 1)$  such that  $h$  is an increasing function on  $]0, +\infty[$  if and only if  $\alpha \leq \alpha^*$ . Moreover, if  $\alpha > \alpha^*$  there exists  $s_1 < s_2$  such that  $h$  is increasing in  $[0, +\infty[ \setminus ]s_1, s_2[$ .*

*Proof.* We have

$$h'(s) = \frac{s^{2^*-2}}{[\ln(e+s)]^{\alpha+1}} \left( (2^* - 1) \ln(e+s) - \frac{\alpha s}{s+e} \right),$$

so

$$h'(s) \geq 0 \iff \ln(e+s) \geq \frac{\alpha}{2^* - 1} \frac{s}{s+e}.$$

Let us define for  $s \geq 0$ ,

$$\theta(s) := \ln(e+s) - \frac{\alpha}{2^* - 1} \frac{s}{s+e}.$$

We have:

$$\begin{cases} \theta(0) = 1, \\ \theta(s) \rightarrow +\infty \quad \text{as } s \rightarrow +\infty, \\ \theta'(s) = \frac{s+e(1-\frac{\alpha}{2^*-1})}{(e+s)^2}. \end{cases}$$

Hence:

(1) If  $\frac{\alpha}{2^*-1} \leq 1$  then  $\theta'(s) \geq 0$  for all  $s \geq 0$  and in particular  $\theta(s) \geq 0$  and therefore  $h'(s) \geq 0$  for all  $s \geq 0$ ;

(2) if  $\frac{\alpha}{2^*-1} > 1$  then

$$\theta'(s_0) = 0 \quad \text{for } s_0 = e \left( \frac{\alpha}{2^* - 1} - 1 \right).$$



Let us compute  $\theta(s_0)$ :

$$\theta(s_0) = \ln \left( \frac{\alpha}{2^* - 1} \right) - \frac{\alpha}{2^* - 1} + 2,$$

and hence:

(i) if  $\theta(s_0) \geq 0$  then  $\theta(s) \geq 0$  for all  $s \geq 0$  and therefore  $h'(s) \geq 0$  for all  $s \geq 0$ ;

(ii) if  $\theta(s_0) < 0$  then there exists  $s_1 < s_2$  such that

$$\theta(s) > 0 \quad \forall s \in [0, +\infty[ \setminus ]s_1, s_2[ \implies h'(s) > 0 \quad \forall s \in [0, +\infty[ \setminus ]s_1, s_2[.$$

Notice that  $t \rightarrow \ln t$  is greater than  $t \rightarrow t - 2$  somewhere between some  $t_1 < 1$  and the value  $t^*$  = the unique solution  $> 2$  of the equation

$$\ln t^* = t^* - 2.$$

Finally the statement of the lemma holds for  $\alpha^* = t^*(2^* - 1)$ .  $\square$

#### APPENDIX B. A COMPACT EMBEDDING USING ORLICZ SPACES

For references on Orlicz spaces see [15, 20]. Throughout  $\Omega \subset \mathbb{R}^N$  is an bounded open set. We will denote

$$\mathcal{L}(\Omega) = \{\varphi : \Omega \rightarrow \mathbb{R} : \varphi \text{ is Lebesgue measurable}\}.$$

**Definition B.1.** *We will say that a function  $M : [0, +\infty[ \rightarrow [0, +\infty[$  is a **N-function** if and only if*

(N1)  *$M$  is convex, increasing and continuous,*

(N2)  $\lim_{s \rightarrow 0^+} \frac{M(s)}{s} = 0,$

(N3)  $\lim_{s \rightarrow +\infty} \frac{M(s)}{s} = +\infty.$

The proof of the following property is trivial, we just quoted it for the sake of completeness.

**Proposition B.2.** *Any N-function  $M$  admits a representation of the form*

$$M(s) = \int_0^s m(t) dt$$

where  $m : [0, +\infty[ \rightarrow [0, +\infty[$  is a non-decreasing right-continuous function satisfying  $m(0) = 0$  and

$$\lim_{s \rightarrow +\infty} m(s) = +\infty.$$

Thus,  $m$  is the right-derivative of  $M$ .

Our first aim is to prove the following result:

**Theorem B.3.** *Let  $M : [0, +\infty[ \rightarrow \mathbb{R}$  be a  $N$ -function such that*

$$\lim_{s \rightarrow +\infty} \frac{s^{2^*}}{M(s)} = +\infty.$$

*Assume also that  $M$  satisfies the  $\Delta_2$ -condition, that is,*

$$(B.1) \quad \exists K > 0, \quad \forall s \in [0, +\infty[, \quad M(2s) \leq KM(s).$$

*Let  $\{u_n\}_{n \in \mathbb{N}}$  in  $H_0^1(\Omega)$  be a sequence satisfying*

- (1)  $\sup_{n \in \mathbb{N}} \|u_n\|_{2^*} < \infty,$
- (2) *there exists  $u \in H_0^1(\Omega)$  such that  $\lim_{n \rightarrow +\infty} u_n(x) = u(x)$  a.e.*

*Then there exists a subsequence  $\{u_{n_k}\}_{k \in \mathbb{N}}$  such that*

$$(B.2) \quad \lim_{k \rightarrow \infty} \int_{\Omega} M(|u_{n_k}(x) - u(x)|) dx = 0.$$

In order to proof this theorem we need some definitions.

**Definition B.4.** *Let  $\mathcal{K} \subset \mathcal{L}(\Omega)$ . We say that  $\mathcal{K}$  has **equi-absolutely continuous integrals** if and only if  $\forall \varepsilon > 0$  there exists  $h > 0$  such that*

$$\forall \varphi \in \mathcal{K}, \forall A \subset \Omega \text{ measurable}, |A| < h \implies \int_A |\varphi(x)| dx < \varepsilon.$$

**Lemma B.5.** *Let  $M : [0, +\infty[ \rightarrow \mathbb{R}$  be a  $N$ -function satisfying the  $\Delta_2$  condition (B.1). Let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence of measurable functions converging a.e. to some function  $u$  and such that the set*

$$\left\{ M(|u_n|) : n \in \mathbb{N} \right\}$$

*has equi-absolutely continuous integrals. Then (B.2) holds.*

*Proof.* Let fix  $\varepsilon > 0$  and let  $\delta > 0$  be such that

$$\forall n \in \mathbb{N}, \forall A \subset \Omega \text{ measurable}, |A| < \delta \implies \int_A M(|u_n|) dx \leq \varepsilon.$$

Using Fatou's lemma we infer that also

$$\forall A \subset \Omega \text{ measurable}, |A| < \delta \implies \int_A M(|u|) dx \leq \varepsilon.$$

Let  $\Omega_n = \{x \in \Omega : |u_n(x) - u(x)| > M^{-1}(\varepsilon)\}$ . As a consequence of Egoroff's theorem, the sequence  $(u_n)_{n \in \mathbb{N}}$  converge in measure to  $u$  so there exists  $n_0 \in \mathbb{N}$  such that

$$|\Omega_n| < \delta.$$

Then, using the convexity of  $M$  and (B.1) it comes

$$\begin{aligned} \int_{\Omega} M(|u_n - u|) dx &= \int_{\Omega_n} M(|u_n - u|) dx + \int_{\Omega \setminus \Omega_n} M(|u_n - u|) dx \\ &\leq \frac{1}{2} \left( \int_{\Omega_n} (M(2|u_n|) + M(2|u|)) dx \right) + |\Omega| M(M^{-1}(\varepsilon)) \\ &\leq \frac{K}{2} \left( \int_{\Omega_n} (M(|u_n|) + M(|u|)) dx \right) + |\Omega| \varepsilon \leq (K + |\Omega|) \varepsilon. \end{aligned}$$

□

In order to prove that, for the sequence of our theorem, the set

$$\left\{ M(|u_n|) : n \in \mathbb{N} \right\}$$

has equi-absolutely continuous integrals we are going to use the following lemma :

**Lemma B.6.** *Let  $\mathcal{K} \subset \mathcal{L}(\Omega)$  and let  $\Phi : [0, +\infty[ \rightarrow [0, +\infty[$  be an increasing function satisfying*

$$(B.3) \quad \lim_{s \rightarrow +\infty} \frac{\Phi(s)}{s} = +\infty.$$

*Suppose that there exists  $D > 0$  such that*

$$(B.4) \quad \sup_{u \in \mathcal{K}} \int_{\Omega} \Phi(|u|) dx \leq D.$$

*Then all the functions  $u \in \mathcal{K}$  are integrable and  $\mathcal{K}$  has equi-absolutely continuous integrals (Valle Poussin's theorem).*

*Moreover, if  $M : [0, +\infty[ \rightarrow [0, +\infty[$  is a continuous increasing function satisfying*

$$\lim_{s \rightarrow +\infty} \frac{M(s)}{s} = +\infty \text{ and } \lim_{s \rightarrow +\infty} \frac{\Phi(s)}{M(s)} = +\infty,$$

*then the family  $\mathcal{K}_1 = \{M(|u|) : u \in \mathcal{K}\}$  has equi-absolutely continuous integrals.*

*Proof.* For the Valle Poussin's theorem see [18] page 159. To prove the second statement remark that the function  $\tilde{\Phi} = \Phi \circ M^{-1}$  satisfies (B.3). Here  $M^{-1}$  stand for the right-hand inverse. □

*Proof of theorem B.3.* Let us take  $\Phi(s) = |s|^{2^*}$ . From hypothesis (1) of the theorem, the set  $\mathcal{K} = \{u_n : n \in \mathbb{N}\}$  satisfies (B.4) for some  $D > 0$ . Then the conclusion follows from lemma B.5 and Lemma B.6 . □

**Remark B.7.** Whenever (B.2) is satisfied we say that the sequence  $\{u_{n_k}\}_{k \in \mathbb{N}}$  converges in  $M$ -mean to  $u$ .

One can formulate Theorem B.3 as a compact embedding of  $H_0^1(\Omega)$  in some vector space endowed of the *Luxembourg norm associate to  $M$*  (see [15, 20]). Instead, we are going to use the Orlicz-norm which is more suitable to our purposes. We will see later in Theorem B.12 that the convergence in  $M$ -mean implies the convergence with respect to the Orlicz-norm, provided that the  $\Delta_2$ -condition is satisfied.

**Definition B.8.** Let  $M$  be a  $N$ -function. The **complementary of  $M$**  defined for all  $s \geq 0$  is the function

$$M^*(s) := \max \{st - M(t) : t \geq 0\}.$$

As before, we give the following trivial result for the sake of completeness:

**Proposition B.9.** If  $m$  is the right derivative of  $M$  then

$$m^*(s) = \sup \{t : m(t) \leq s\}$$

is the right derivative of  $M^*$  and  $M^*$  is a  $N$ -function. Furthermore, for all  $s \geq 0$  we have

$$sm(s) = M(s) + M^*(m(s)), \quad sm^*(s) = M(m^*(s)) + M^*(s).$$

Next, let us introduce the Orlicz norm associated to  $M$  :

**Definition B.10.** Let  $M$  be a  $N$ -function and let  $M^*$  be its complementary. Let us denote for any  $v \in \mathcal{L}(\Omega)$

$$\rho(v, M^*) = \int_{\Omega} M^*(|v|) dx$$

and define the **Orlicz norm** of any  $u \in \mathcal{L}(\Omega)$  by

$$\|u\|_M := \sup \left\{ \int_{\Omega} uv dx : v \in \mathcal{L}(\Omega), \rho(v, M^*) \leq 1 \right\}.$$

$\|\cdot\|_M$  is a norm in the real vector space

$$L_M(\Omega) = \{u \in \mathcal{L}(\Omega) : \|u\|_M < +\infty\}.$$

(see [15] for the details). Let us prove the following less trivial properties:

**Proposition B.11.**

(i) For all  $u \in \mathcal{L}(\Omega)$ ,

$$(B.5) \quad \|u\|_M \leq \int_{\Omega} M(|u|) dx + 1.$$

(ii) For any  $u$  and  $v$  in  $\mathcal{L}(\Omega)$  it holds

$$(B.6) \quad \left| \int_{\Omega} uv \, dx \right| \leq \|u\|_M \|v\|_{M^*} \quad (\text{Holder's inequality}).$$

(iii) For any  $u$  and  $v$  in  $\mathcal{L}(\Omega)$  we have  $\|u\|_M \leq \|v\|_M$  if  $|u| \leq |v|$  a.e.

*Proof.* (i) This follows from the definition of  $\|\cdot\|_M$  and the inequality  $|uv| \leq M(|u|) + M^*(|v|)$ .

(ii) We divide the proof in 3 steps.

*Step 1:* For all  $v \in \mathcal{L}(\Omega)$ ,

$$\left| \int_{\Omega} uv \, dx \right| \leq \begin{cases} \|u\|_M & \text{if } \rho(v, M^*) \leq 1 \\ \rho(v, M^*) \|u\|_M & \text{if } \rho(v, M^*) > 1 \end{cases}$$

Indeed, the first case follows directly from the definition. If  $\rho(v, M^*) > 1$  then by convexity

$$M^* \left( \frac{|v|}{\rho(v, M^*)} \right) \leq \frac{M^*(|v|)}{\rho(v, M^*)}$$

and therefore

$$\rho \left( \frac{|v|}{\rho(v, M^*)}, M^* \right) \leq \frac{1}{\rho(v, M^*)} \int_{\Omega} M^*(|v|) \, dx = 1$$

and

$$\left| \int_{\Omega} u \frac{v}{\rho(v, M^*)} \, dx \right| \leq \|u\|_M.$$

*Step 2:* If  $\|u\|_M \leq 1$  then  $\rho(m(|u|), M^*) \leq 1$ .

Set  $u_n = u \chi_{\{|u| \leq n\}}$  for all  $n \in \mathbb{N}$ . Since  $u_n$  is bounded then  $\rho((m(|u_n|)), M^*) < +\infty$ . Assume by contradiction that  $\int_{\Omega} M^*(m(|u|)) \, dx > 1$  and let  $n_0 \in \mathbb{N}$  be such that  $\int_{\Omega} M^*(m(|u_{n_0}|)) \, dx > 1$ . We have

$$M^*(m(|u_{n_0}|)) < M(|u_{n_0}|) + M^*(m(|u_{n_0}|)) = |u_{n_0}| m(|u_{n_0}|)$$

and therefore, by (i),

$$\rho(m(|u_{n_0}|), M^*) < \int_{\Omega} |u_{n_0}| m(|u_{n_0}|) \, dx \leq \|u_{n_0}\|_M \rho(m(|u_{n_0}|), M^*)$$

which contradicts  $\|u_{n_0}\|_M \leq \|u\|_M \leq 1$ .

This is trivial from the definition of  $\|u\|_M$ , step 1 and the fact that  $|u|m(|u|) = M(|u|) + M^*(m(|u|))$ .

*Step 3:* If  $\|u\|_M \leq 1$  then  $\rho(u, M) \leq \|u\|_M$ .

Let us remark that for all  $s \geq 0$

$$M^*(m(s)) + M(s) = sm(s).$$

Set  $v_0 = m(|u|)$ . From step 2,  $\rho(v_0, M^*) \leq 1$  and then

$$\rho(u, M) \leq \rho(u, M) + \rho(v_0, M^*) = \int_{\Omega} uv_0 dx \leq \|u\|_M.$$

Now we prove Holder's inequality. From step 2 applied to  $M^*$  and  $\frac{v}{\|v\|_{M^*}}$  we have  $\rho\left(\frac{v}{\|v\|_{M^*}}, M^*\right) \leq 1$ , so then

$$\left| \int_{\Omega} u \frac{v}{\|v\|_{M^*}} dx \right| \leq \|u\|_M$$

and Holder's inequality follows.

The proof of (iii) is trivial.  $\square$

Finally, we give the following compact embedding result:

**Theorem B.12.** *Let  $M$  be a  $N$ -function satisfying the  $\Delta_2$ -condition (B.1) and let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{L}(\Omega)$  such that*

$$\lim_{n \rightarrow \infty} \rho(u_n, M) = 0.$$

*Then*

$$\lim_{n \rightarrow \infty} \|u_n\|_M = 0.$$

*Thus, the convergence in  $M$ -mean implies the converge with respect to the  $\|\cdot\|_M$  norm.*

*Proof.* Let  $\varepsilon > 0$  and take  $m \in \mathbb{N}$  such that  $\frac{1}{2^{m-1}} < \varepsilon$ . Using condition (B.1) we also have

$$\lim_{n \rightarrow \infty} \int_{\Omega} M(2^m |u_n|) dx = 0.$$

Let  $n_0 \in \mathbb{N}$  be such that for all  $n \geq n_0$  we have

$$\int_{\Omega} M(2^m |u_n|) dx < 1.$$

From step 1 of the proof in the previous proposition we have that for all  $n \geq n_0$

$$\|2^m u_n\|_M \leq \rho(2^m |u_n|, M) + 1 < 2,$$

which implies that

$$\|u_n\|_M < \frac{1}{2^{m-1}} < \varepsilon.$$

$\square$

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