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Conformal anomaly and gravitational pair production

M. N. Chernodub¹

¹*Institut Denis Poisson UMR 7013, Université de Tours, 37200 Tours, France*

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We argue that the rate density of particle pair production Γ in background fields in conformal field theories is determined by the conformal anomaly and related to anomalous trace of the energy-momentum tensor as $\Gamma = (\pi/2)\langle T^\mu_\mu \rangle$ if the trace is positive (and $\Gamma = 0$ otherwise). This formula perfectly reproduces (presumably, non-Hawking) radiation generated by static gravitational fields in the absence of an event horizon via a new evaporation mechanism suggested recently. Our relation also correctly describes the one-loop Schwinger pair creation in massless (scalar and spinor) quantum electrodynamics. It also accurately points to the Savvidi instability of the gluonic vacuum towards the formation of the chromomagnetic condensate. Photon and neutrino pair production are also discussed.

Introduction. Signatures of vacuum instability in a strong electric field were first found in work by Sauter on the Klein paradox [1]. This effect has been recognized and developed further by Heisenberg and Euler [2] and later formalized in terms of a pair production process in QED by Schwinger [3, 4].

The physical interpretation of this instability, often called the Schwinger effect, is linked to the quantum vacuum fluctuations in which virtual pairs of electrons e^- and positrons e^+ are constantly created to be annihilated shortly later. In a sufficiently strong background electric field, the created e^+e^- particles are taken away in opposite directions by the field. As they are spatially separated, they cannot annihilate and become real particles. Thus, a sufficiently strong electric field creates matter (e^+e^- pairs) from the vacuum.

A similar phenomenon exists in gravitational fields near black holes. A black hole emits the Hawking radiation [5, 6], which can be associated with the particle tunneling process [7] in which one particle from the pair, created in the vicinity of the event horizon, gets swallowed by the hole while another particle has sufficient energy escape to infinity. The escaping particles form the outgoing energy flux, which diminishes the mass of the black hole and, therefore, leads to the black hole evaporation. Due to a nonlocality of the tunneling process, this effect operates in an extended vicinity above the black hole event horizon, thus creating the notion of the quantum atmosphere [8] (see also [9]). It was recently suggested that such quantum atmospheres could possess nontrivial thermodynamic features that can be probed in condensed matter experiments [10].

In 1+1 spacetime dimensions, the Hawking radiation can be related to a gravitational (Einstein) anomaly which implies a non-conservation of energy-momentum of a chiral particle in a curved spacetime [11]. The anomaly appears due to quantum fluctuations when classical symmetries are inconsistent with the quantization procedure [12]. In 1+1 and 3+1 spacetime dimensions, the Hawking effect can also be interpreted [13] in terms

of conformal (or trace) anomalies [14–16]¹.

In addition, one can argue that the particle creation in a static gravitational field can also produce particles even without the event horizon [18]. In this scenario, the virtual pairs of particles are separated by local tidal forces and become real particles, similar to what happens in the Schwinger effect. Some of these real particles will fall to the gravitating body and will later be recaptured, while other particles will escape to infinity and create, similarly to the Hawking effect, an outgoing flux of matter [18].

In our article, we argue that in the off-event-horizon mechanism of Ref. [18] of particle pair production, the creation rate in the background gravitational field can be directly related to the conformal anomaly. We will show that our approach works for the Schwinger pair-production mechanism in QED and is consistent with the Savvidi vacuum instability in non-Abelian gauge theories [19].

We set $\hbar = c = 1$ everywhere in the article and work in the mostly-plus metric convention.

Particle production and effective action. The rate density of particle production events $dN/dt = \Gamma$ is determined by the imaginary part, $\Gamma = 2 \text{Im} \mathcal{L}_{\text{eff}}$, of the Lagrangian \mathcal{L}_{eff} associated with the effective action [3],

$$W = \int d^4x \sqrt{-g} \mathcal{L}_{\text{eff}}, \quad (1)$$

which takes into account quantum corrections. The rate Γ has a sense of non-persistence of vacuum due to pair

¹ It is worth mentioning about terminology used in the paper. The terms “scale” and “conformal” in relation to the symmetries of the system and the quantum anomalies are often used interchangeably in the literature. Mathematically, these concepts correspond to different symmetries as the requirement of local conformal invariance is much stronger than the condition of global scale invariance. Physically, the distinction between the scale and conformal concepts is frequently ignored because all physically relevant scale-invariant field theories in four spacetime dimensions also exhibit conformal invariance [17]. Moreover, since these anomalies are seen as a non-zero trace of the energy-momentum tensor, they are also called “trace” anomalies.

creation [3, 4].

In our paper, we argue that in *conformal* field theories, the pair-creation rate Γ in a background gravitational and gauge (electromagnetic or gluon) fields can be related to the conformal (trace) anomaly:

$$\Gamma = \frac{\pi}{2} \langle T^\mu_\mu \rangle, \quad (2)$$

where $\langle T^\mu_\mu \rangle \equiv \langle T^\mu_\mu \rangle_{\text{an}}$ is the anomalous trace of the energy-momentum tensor $T^{\mu\nu}$. Relation (2) is quantum because, in classical conformal theories in an even number of spacetime dimensions, the trace of the stress-energy tensor vanishes identically, $(T^\mu_\mu)_{\text{cl}} \equiv 0$. Quantum fluctuations can violate this identity, $\langle T^\mu_\mu \rangle \neq 0$, hence the term “trace anomaly” or “conformal anomaly”.

In order to keep Eq. (2) as simple as possible, we used the convention that this equation has a relation to the pair production if and only if $\langle T^\mu_\mu \rangle \geq 0$. Otherwise, $\Gamma < 0$ is equivalent to $\Gamma \equiv 0$ because a negative pair production rate does not lead to the production of pairs.

The particle production rate of N massless scalar degrees of freedom in the curved $d = 3 + 1$ dimensional spacetime (described by the metric $g_{\mu\nu}$) in the presence of the classical electromagnetic field (characterized by the field strength $F_{\mu\nu}$) has been found in the recent work [18]²:

$$\Gamma_{Nsc} = \frac{N}{32\pi} \left[\frac{1}{180} (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - R_{\mu\nu} R^{\mu\nu}) + \frac{1}{2} \left(\frac{1}{6} - \xi \right)^2 R^2 - \frac{q^2}{12} F_{\mu\nu} F^{\mu\nu} \right], \quad (3)$$

where the curved background is expressed via the Riemann tensor $R_{\mu\nu\alpha\beta}$, the Ricci tensor $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, and the scalar curvature $R \equiv R^\mu_\mu$. The subscript “Nsc” in Eq. (3) stands for N scalar degrees of freedom (for example, $N = 1$ for a neutral scalar field and $N = 2$ for a complex scalar field).

The quantity q in Eq. (3) is the electric charge of the scalar particle minimally coupled to electromagnetism. A neutral ($q = 0$) scalar field carrying one degrees of freedom ($N = 1$) is described by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \xi R \phi^2 - \frac{1}{2} m^2 \phi^2, \quad (4)$$

where the parameter ξ controls the local coupling of the Ricci curvature scalar R to the scalar field. The conformally invariant massless theory corresponds to $\xi = 1/6$. For consistency with previous studies, we also add to Eq. (4) the mass term, which will be set to zero at the

end of our considerations, $m = 0$. The charged (complex) scalar field carrying the elementary electric charge $q = e$ has $N = 2$ degrees of freedom with corresponding modifications of Eq. (4).

The remarkable feature of the pair-production effect (3) is that it can take place in static gravitational fields, which immediately suggests that this effect is a Hawking-type of radiation associated with the presence of an event horizon [5, 6]. However, the pair production (3) takes place even in the absence of an event horizon (that is, not only for a black hole), thus indicating that this phenomenon is either an addition or a generalization of Hawking radiation [18].

Effective action, trace anomaly, and pair production. It is instructive first to start from the simplest case of the scalar field for which the trace anomaly has been elaborated in great detail in Ref. [20]. Our relation (2) between the conformal (trace) anomaly and the off-event-horizon particle production can be deduced by matching the anomalous term in the one-loop effective action W represented as an integral over the proper time s of Ref. [20] with the representation of the same action in terms of the spectral parameter s in the heat-kernel approach of Ref. [18] based on the Barvinsky–Vilkovisky expansion [21].

The one-loop action functional W is given by a formal divergent expression, $W = (i/2) \ln \det G^{-1}$, where $G(x, x') \equiv \langle iT(\phi(x)\phi(x')) \rangle$ represents the Green function associated with the quadratic Lagrangian (4):

$$\left(-\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu + \xi R + m^2 \right) G(x, x') = \delta(x - x') / \sqrt{-g}. \quad (5)$$

The functional W has a close relation to the expectation value of the energy-momentum tensor, $\langle T^{\mu\nu} \rangle$, in its response, $W \rightarrow W + \delta W$ to the metric variation, $g_{\mu\mu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ in (even) D spacetime dimensions:

$$\delta W = \frac{i}{2} \text{Tr} G \delta G^{-1} = \int d^D x \sqrt{-g} \langle T^{\mu\nu} \rangle \frac{1}{2} \delta g_{\mu\nu}, \quad (6)$$

thus giving access to the evaluation of the trace $\langle T^\mu_\mu \rangle$, allowing us to uncover an eventual conformal (trace) anomaly.

The variation of the effective action (6) can be expressed in the proper-time representation of Schwinger and DeWitt [3, 22] (in notations of [20]):

$$\delta W = -\frac{i}{2} \delta \text{Tr} \int_0^\infty \frac{ids}{is} e^{-isH}, \quad (7)$$

via a relativistic Hamiltonian-like operator $H = \Delta + \xi R + m^2$, where a second-order differential operator Δ represents the kinetic term, the coupling to the curvature R plays a role of an external potential, and m^2 gives the mass term. The correct analytical properties

² We have slightly re-arranged and combined the original expressions of Ref. [18] for further convenience.

of Eq. (7) and similar subsequent relations are maintained by an appropriate complex continuation of the mass term, $m^2 \rightarrow m^2(1 - i0^+)$, silently assumed here.

The effective Lagrangian (1) takes the following form:

$$L_{\text{eff}} = \frac{1}{2} \frac{1}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{id s}{(i s)^{1+\frac{D}{2}}} e^{-i m^2 s} F(x, x; i s; D), \quad (8)$$

where $F(x, x'; i s; D)$ is the weight bi-scalar in the proper-time Green's function $\langle x, s | x', 0 \rangle = \langle x | e^{-i s H} | x' \rangle$ defined in a manner similar to Eq. (8). The Green's function satisfies the Schrödinger-like equation: $-\frac{\partial}{\partial i s} \langle x, s | x', 0 \rangle = H \langle x, s | x', 0 \rangle$, which gives a quantum-mechanical flavor to the whole proper-time formalism.

We will not dwell on the precise definition of the bi-scalar F , which can be found in detail in Refs. [20, 22]. The key mathematical point of our arguments is that the function F allows for the power-series expansion in terms of the proper time s (omitting other arguments):

$$F = 1 + i s f_1 + (i s)^2 f_2 + \dots, \quad (9)$$

where, in four space-time dimensions, the $O(s^2)$ term captures the trace anomaly [20]. On the other hand, the $O(s^2)$ term in an identical³ expansion of the same 1-loop effective action over the proper time s has been shown in Ref. [18] to be associated with the (off-event-horizon) pair-production rate Γ . The mentioned equivalence of the $O(s^2)$ terms allows us to identify the trace-anomalous origin of the pair production and eventually leads us to Eq. (2) as we discuss below.

The renormalized energy-momentum tensor,

$$\langle T^{\mu\nu} \rangle_{\text{ren}} = \frac{1}{4} \mathcal{A}_4 g^{\mu\nu} + \text{non-anomalous part}, \quad (10)$$

contains the anomalous part given by an \mathcal{A}_4 function and a non-anomalous part (not shown explicitly). In $D = 4$ spacetime dimensions, the \mathcal{A}_4 function in stress-energy tensor (10) is related to the $O(s^2)$ prefactor in the power series expansion (9) of the bi-scalar F [20]:

$$\mathcal{A}_4 = \frac{1}{2} \frac{1}{(4\pi)^2} \left(\frac{\partial}{\partial i s} \right)^2 \left[e^{-i m^2 s} F(x, x; i s, 4) \right] \Big|_{s=0}. \quad (11)$$

³ Taking into account the signs and i -th prefactors arising from the difference between Minkowski/Euclidean spacetimes employed in Refs. [18, 20] one finds that $f_1 = (\frac{1}{6} - \xi)R$ term in Eq. (A20) of the proper-time approach of Ref. [20] coincides precisely with the second, $O(s)$ term under the integral in Eq. (S.17) of the heat-kernel expansion of Ref. [18]. Analogously, f_2 in Eq. (A24) of Ref. [20] coincides precisely with purely gravitational contribution to the third, $O(s^2)$ term under the integral in Eq. (S.17) of Ref. [18]. The f_2 term in series (9) is also reproduced, up to irrelevant contact term $\square R$, by the $m = 0$ expression in the square brackets of our Eq. (13) below. Notice that our functions f_a in Eq. (9) correspond to f_a of Ref. [20] and not to f_a of Ref. [18].

In the massless theory ($m = 0$), the last term in Eq. (10) reduces to a traceless tensor, and the trace of the energy-momentum tensor (10) is fully determined by the trace (scale) anomaly (11):

$$\langle T^\mu_\mu \rangle \equiv g_{\mu\nu} \langle T^{\mu\nu} \rangle_{\text{ren}} = \mathcal{A}_4, \quad [\text{for } m = 0], \quad (12)$$

where the short-hand notation $\langle T^\mu_\mu \rangle$ is used for representational convenience (see also the discussion of Ref. [14] on non-commutativity of the regularization operation and the trace operation).

Finally, combining Eqs. (9), (11), (12) and matching them with the $O(s^2)$ term in the effective action of Ref. [18] leads us to our main result (2). In the rest of the paper, we ensure that Eq. (2) is valid for physical environments where both sides of this equation are known. We also discuss photon and neutrino pair production.

A neutral scalar field in curved spacetime. As the first check of our result (2), we consider a single-component neutral scalar field of mass m and generic non-conformal coupling ξ to gravity described by Lagrangian (4). It is well known that quantum fluctuations in this theory produce the following trace anomaly [20, 23]:

$$\langle T^\mu_\mu \rangle_{1\text{sc}} = \frac{1}{(4\pi)^2} \left[\frac{1}{180} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{6} \left(\frac{1}{5} - \xi \right) \square R + \frac{1}{2} \left(\frac{1}{6} - \xi \right)^2 R^2 + \frac{1}{2} m^4 \right]. \quad (13)$$

We substitute Eq. (13) to our formula (2) and recover exactly the result of Ref. [18] given in Eq. (3) for the off-horizon pair creation rate for a single ($N = 1$) neutral ($q = 0$) massless ($m = 0$) scalar field⁴. The subscript “1sc” in Eq. (13) stresses that this expression is derived for a one-component scalar field.

General case. A quantum field theory of N_S scalar degrees of freedom, N_F Dirac fermions (a single Majorana or Weyl fermion contributes half of a Dirac fermion, $N_F = 1/2$) and N_V species of massless vector fields, the trace anomaly gets the following form [14, 24, 25]:

$$\langle T^\mu_\mu \rangle = b C^2 + b' E_4 + c F_{\mu\nu} F^{\mu\nu}, \quad (14)$$

where

$$C^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{R^2}{3}, \quad (15)$$

is the Weyl tensor squared and

$$E_4 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2, \quad (16)$$

⁴ These equations correspond to the same physical result since the $\square R$ term, present in Eq. (13) and absent in Eq. (3), can be removed by a finite local counterterm during the renormalization procedure and is, therefore, physically irrelevant.

is the Euler density (the integrand of the topological, Gauss–Bonnet term) in $D = 4$ dimensions. In Eq. (14), the physically irrelevant $\square R$ term is omitted⁵ and the conformal coupling ($\xi = 1/6$ for scalars) is assumed. The parameters are as follows [13–16, 20, 26]:

$$b = \frac{1}{120} \frac{1}{(4\pi)^2} (N_S + 6N_F + 12N_V), \quad (17)$$

$$b' = -\frac{1}{360} \frac{1}{(4\pi)^2} (N_S + 11N_F + 62N_V). \quad (18)$$

As an immediate check, one finds that in a pure gravitational background ($F_{\mu\nu} = 0$), $N_S = N$ species of neutral scalar fields (with $N_D = N_V = 0$), Eq. (14) expectedly reduces to Eq. (13) with a factor N and leads us, via Eq. (2), to the recent result (3).

The last term (14) represents a non-universal (“matter”) part which accounts for renormalization effects related to the scale dependence of the couplings of the theory. While Eq. (14) gives the matter term for a vector-field background, a nontrivial scalar background in an interacting scalar field theory can also generate a matter-type contribution to the conformal anomaly which can be found in Ref. [27].

For gauge vector fields coupled minimally with matter fields via the electric coupling e , the prefactor

$$c = -\frac{\beta(e)}{2e}, \quad (19)$$

of the last term in Eq. (14) depends the beta function $\beta(e) = \mu de/d\mu$ associated with the running of the coupling e . A nonvanishing beta function expresses the fact that radiative corrections make the electric charge $e = e(\mu)$ dependent on the renormalization energy scale μ . This effect appears as a result of vacuum polarization which implies, for example, that the electric charges of a particle probed at a large distance (by a low-energy photon) does not match the charge of the same particle at a short distance (probed by a high-energy photon). Therefore, radiative corrections can break the scale invariance of the system and naturally contribute to the scale anomaly (14).

Notice that the gravitational part of the anomaly represented by the first two terms in Eq. (14) is exact in one loop implying that the higher-order corrections to this expression vanish. This statement is not true for the third, matter term since radiative corrections exist, generally, in all loops [28].

Scalar QED in flat spacetime. A similarity between the gravitational particle production and the Schwinger pair production in flat spacetime in the background electric field has been noticed in Ref. [18] on the

basis of equation (3). Here we show that the conformal anomaly plays an essential role in this relation.

Consider a theory of N_S species of massless complex scalars coupled to electromagnetism with the same electric coupling e (a “scalar Quantum Electrodynamics” or sQED). Since we already established the relation with the gravitational part of the trace anomaly in this theory, we consider below a flat spacetime where first two (gravitational) terms in the trace (14) vanish. However, this theory still possesses the trace anomaly because its one-loop beta function is non-zero [29, 30],

$$\beta_{\text{sQED}}^{\text{1loop}} = \frac{N_S e^3}{48\pi^2}. \quad (20)$$

Equations (19) and (20) imply that the coefficient in the last term of the trace anomaly (14) is $c = N_S e^2 / (96\pi^2)$. Then Eq. (2) gives us $\Gamma_{\text{sQED}} = -N_S e^2 F_{\mu\nu} F^{\mu\nu} / (192\pi)$, which exactly coincides with the pair production rate (3) of Ref. [18] if one takes into account that each complex field carries two degrees of freedom: $N = 2N_S$.

According to our convention in Eq. (2), there is no particle production for a negative production rate. Since $F_{\mu\nu} F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2)$, implies the absence of particle creation in a pure magnetic field because $\Gamma_{\text{sQED}} < 0$. However, in the electric-field background, one gets the following well-known result the complex scalar field (reproduced also in Ref. [18] for $N_S = 1$):

$$\Gamma_{\text{sQED}} = N_S \frac{e^2 \mathbf{E}^2}{96\pi}. \quad (21)$$

This equivalence further supports the validity of Eq. (2) in one loop.

Spinor QED. The pair creation rate in the (spinor) QED with a single flavor of massless Dirac fermion and a single gauge (electromagnetic) field can be derived with the use of correspondence (2) together with the anomaly relations (14)–(19) by setting $N_S = 0$, $N_F = 1$, and $N_V = 1$. Taking into account that the one-loop beta function of the spinor QED is four times bigger (per particle) than its scalar analogue (22) [30]:

$$\beta_{\text{QED}}^{\text{1loop}} = \frac{e^3}{12\pi^2}, \quad (22)$$

one gets the following prediction for the particle production rate:

$$\Gamma_{\text{QED}}^{(m=0)} = \frac{1}{11520\pi} \left(-19R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + 184R_{\mu\nu} R^{\mu\nu} - 55R^2 \right) - \frac{e^2}{48} F_{\mu\nu} F^{\mu\nu}, \quad (23)$$

Notice that in flat background, the particle production rate in the massless QED (23) reduces exactly to the well-known QED result [3, 30] in the limit $m \rightarrow 0$:

$$\Gamma_{\text{QED}}^{(m=0)} = \frac{e^2 \mathbf{E}^2}{24\pi} \quad [\text{electromagnetic}]. \quad (24)$$

⁵ See also a relevant discussion on $\square R$ in Ref. [14].

To estimate the contribution of the gravitational part to the production rate, consider now a purely gravitational background given by the static Schwarzschild spacetime of a body with the mass M :

$$ds^2 = - \left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (25)$$

Given Ricci flatness ($R_{\mu\nu} = 0$) of this metric, the gravitational contribution to the pair-creation rate (23) is provided only by the Kretschmann scalar:

$$K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 48 \frac{G^2 M^2}{r^6}. \quad (26)$$

Thus, it appears that the purely gravitational contribution to the pair production rate (23) is always negative

$$\delta\Gamma_{\text{QED}}^{(m=0)} = -\frac{19}{240\pi} \frac{G^2 M^2}{r^6} \quad [\text{gravitational}], \quad (27)$$

implying that our conformal anomaly mechanism alone cannot create particles outside of the event horizon even in the presence of the strong gravitational field and even for massless QED. Moreover, our result imply that the ordinary Schwinger pair production due to background electric field (24) will be inhibited by the gravitational contribution (27) in curved spacetime.

It is worth here mentioning that the considerations of Ref. [14] on the sense of the conformal anomaly in the context of renormalization of quantum field theories suggest that in conformally-non-invariant theories (for example, for massive fields), the right-hand-side of Eq. (2) should be modified: $\langle T^\mu_\mu \rangle \rightarrow g_{\mu\nu} \langle T^{\mu\nu} \rangle_{\text{ren}} - \langle g_{\mu\nu} T^{\mu\nu} \rangle_{\text{ren}}$. This conjecture implies, in particular, that the explicitly non-conformal mass term m^4 will not enter Eq. (2).

Coming back to the massless case in flat spacetime, the proportionality of the pair-creation rates for scalar (21) and spinor (23) QED to their beta functions, Eqs. (20) and (22), respectively, is not surprising given an intimate relation between the effective Euler-Heisenberg Lagrangian and the beta function (for an excellent review, see Ref. [30]). Since the beta function also contributes to the trace anomaly, the relation of the trace anomaly to the pair-creation rate closes the logical triangle, thus qualitatively supporting Eq. (2) on physical grounds.

Our results (2) suggest that in the flat spacetime, the creation rate of pairs of massless particles in classical (electromagnetic) background is related to the respective beta function:

$$\Gamma_{\text{flat}} = -\frac{\pi\beta(e)}{4e} F_{\mu\nu} F^{\mu\nu}. \quad (28)$$

This result should be valid at least in one-loop order with the already mentioned reservation that a negative production rate implies no production.

No photon production. In realistic QED in weak background electromagnetic fields with the strength below the Schwinger limit, the four-photon scattering can be neglected [30], so that photon propagation can be described by free Maxwell theory with simple Lagrangian, $\mathcal{L}_{\text{ph}} = -(1/4)F_{\mu\nu}F^{\mu\nu}$. The particle creation rate corresponds to $N_F = 0$, $N_S = 0$ and $N_V = 1$ and gives us the following discouraging result $\Gamma_{\text{ph}} = -13G^2M^2/(120\pi r^4) < 0$ implying that no photons can be created in the gravitational field due to this conformal anomaly mechanism.

Neutrino–anti-neutrino pairs. Similar considerations can also be applied to neutrino–anti-neutrino pair creation with the appropriate replacement of the spinor degrees of freedom by the sum of Dirac and Majorana/Weyl neutrino species: $N_D \rightarrow N_\nu = N_D + (1/2)N_M$ and taking $N_V = N_S = 0$ in the above expressions. One gets $\Gamma_\nu = 7G^2M^2/(240\pi r^4) > 0$, so that the pairs of light neutrino can potentially be created by sufficiently strong background gravitational field. For a Dirac neutrino, the pair production rate Γ_ν is 7/2 times bigger than the rate of pair production for scalar particles which is about two times bigger than the one for the Hawking radiation. The relevant estimations for scalar particles in physically interesting gravitational fields can be found in Ref. [18].

Finally, one could ask whether these results, derived for massless spinors, are applicable to particles with mass m_ν . For the pure electromagnetic contribution to the pair creation rate (23), the condition is well known [30]: the electric field strength should substantially exceed the critical Schwinger field, $E \gg E_c^{\text{Sch}} = m_e^2/e \simeq 1.3 \times 10^{18}$ V/m. Likewise, the same condition can be obtained by demanding that the gravitational contribution should exceed the anticipated⁶ $\propto m_e^4$ term generated by the explicit breaking of the conformal symmetry. For neutrinos in the field of a gravitating body with mass M , the applicability condition then reads as $r \ll r_c$ with the critical field $r_c = \sqrt{GM}/m_\nu$ up to a $O(1)$ factor.

Savvidi magnetic instability in QCD. In the pure magnetic field, both in scalar QED and spinor QED, the right-hand of Eq. (2) is a negative quantity and, therefore, no instability associated with the particle production can occur. Of course, this natural conclusion is supported by the fact that their beta functions, Eqs. (20) and (22), are positively defined. But what happens if the beta function is negative?

Consider, for example, Yang-Mills (YM) theory which determines non-perturbative properties of Quantum Chromodynamics (QCD). The beta function of N_c -color YM theory, $\beta_{\text{YM}}(g) = -11N_c g^3/(48\pi^2)$, is a negative function of the strong coupling constant g . Adopting Eq. (28) to non-Abelian fields possessing the field-

⁶ Cf. the last term in Eq. (13).

strengths $F_{\mu\nu}^a$, one gets the following *formal perturbative expression* for the gluon production rate:

$$\begin{aligned}\Gamma_{\text{YM}}^{\text{pert}} &= \frac{11N_c g^2}{192\pi} F_{\mu\nu}^a F^{a,\mu\nu} \\ &\equiv \frac{11N_c}{96\pi} [(g\mathbf{B}^a)^2 - (g\mathbf{E}^a)^2],\end{aligned}\quad (29)$$

where the sum over the gluon species, $a = 1, \dots, N_c^2 - 1$, is implicitly assumed.

Equation (29) represents a formal expression which is not applicable to the ground state of YM theory because Eq. (29) corresponds to the anomalous breaking of scale symmetry associated with the perturbative renormalization of couplings – hence the superscript “pert” in Eq. (29) – while in YM theory, the conformal symmetry is broken dynamically and non-perturbatively⁷. Despite this fact, Eq. (29) still allows us to make another interesting relationship with already known effect: the instability of the perturbative gluonic vacuum. Indeed, since $\Gamma_{\text{YM}} = 11N_c(g\mathbf{B}^a)^2/(96\pi) > 0$, even the weakest background gluomagnetic field leads to the creation of gluon pairs and makes the gluonic vacuum unstable. This observation matches well with the instability of the perturbative gluon vacuum [31] which drives creation of the magnetic condensate (the Savvidi vacuum [19]) and the formation of the suggested magnetic-spaghetti vacuum state [32, 33] precisely due to the negativeness of the YM beta function, $\beta_{\text{YM}}(g) < 0$ (see also a related critical discussion in Ref. [34]).

Conclusions. We suggested the simple formula (2) for the off-horizon particle production rate in curved spacetime proposed in Ref. [18] and argued that its underlying mechanism is based on the anomalous breaking of the conformal symmetry. The anomalous particle production can occur in static gravitational fields and can operate, in particular, above the event horizons of black holes. These two properties discriminate the anomalous production from the dynamical Casimir effect in time-dependent backgrounds and the Hawking mechanism of particle production, which occurs near the black-hole horizons.

Our formula (2) agrees with known results for the pair production rate in the gravitational background for a scalar field presented recently in Ref. [18] where the off-horizon pair production has been suggested first. We also reproduce the known expressions for the Schwinger pair production in QCD with scalar and spinor particles. Our approach also supports instability in the perturbative gluonic vacuum in the chromomagnetic field, thus suggesting the formation of the magnetic condensate in

accordance with widely accepted ideas about the nature of the QCD vacuum.

Our mechanism suggests that the photon pairs cannot be produced in a static Schwarzschild spacetime. However, our estimations show that a sufficiently strong gravitational field can create pairs of neutrinos and anti-neutrinos (as well as other light spinors), thus providing us with another channel for the evaporation of black holes and other gravitating objects.

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⁷ In QCD, the magnitude of the dynamical breaking of conformal symmetry, $\langle T^\mu_\mu \rangle \simeq \Lambda_{\text{QCD}}^4$, is determined by an intrinsic mass scale Λ_{QCD} of the order of a few hundred MeV.

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