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► **To cite this version:**

Blaise Colle, Jérôme Lohéac, Takéo Takahashi. Flatness approach for the boundary controllability of a system of heat equations. *SIAM Journal on Control and Optimization*, 2024, 62 (3), pp.1766-1782. 10.1137/23M1577833 . hal-04119834

**HAL Id: hal-04119834**

**<https://hal.science/hal-04119834v1>**

Submitted on 6 Jun 2023

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# Flatness approach for the boundary controllability of a system of heat equations

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June 6, 2023

## Abstract

We study the boundary controllability of 2 × 2 system of heat equations by using a flatness approach. According to the relation between the diffusion coefficients of the heat equation, it is known that the system can be not null-controllable or null-controllable for any  $T > T_0$  where  $T_0 \in [0, \infty]$ . Here we recover this result in the case that  $T_0 \in [0, \infty)$  by using the flatness method, and we obtain an explicit formula for the control and for the corresponding solutions. In particular, the state and the control have Gevrey regularity in time and in space.

## 1 Introduction

We consider the boundary controllability of the following system coupling two heat equations:

$$\left\{ \begin{array}{l} \partial_t y_1 - \partial_x^2 y_1 = 0 \quad \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \frac{1}{d} \partial_x^2 y_2 = y_1 \quad \text{in } (0, T) \times (0, 1), \\ y_1(t, 0) = 0, \quad y_1(t, 1) = u(t) \quad \text{in } (0, T), \\ y_2(t, 0) = y_2(t, 1) = 0 \quad \text{in } (0, T), \\ y_1(0, \cdot) = y_1^0, \quad y_2(0, \cdot) = y_2^0 \quad \text{in } (0, 1), \end{array} \right. \quad (1.1)$$

where  $d > 0$ . This is a “cascade” system: the control  $u$  acts on  $y_1$  (at  $x = 1$ ) whereas  $y_2$  is controlled through  $y_1$ . The controllability of coupled heat equations has been studied by several authors, and we refer the reader to the survey paper [3] for results until 2011. In the case of distributed controls, it is possible to obtain general results such as Kalman criterion to show the null-controllability of the corresponding system for any time  $T > 0$  (see [1] and also [24] in the case of Stokes systems instead of heat equations). In the case of boundary controls, we can quote for instance [4–6] where the authors obtain in particular that the minimal time of controllability can be positive.

For system (1.1), the controllability properties depend on the coefficient  $d$ : if  $d = 1$ , then the system is null-controllable for any  $T > 0$  (see [2] and [12]), whereas if  $d \neq 1$  and  $\sqrt{d} \in \mathbb{Q}$ , then the system is not null-controllable for any  $T > 0$ . In the case  $\sqrt{d} \notin \mathbb{Q}$ , the system is null-controllable at any time  $T > c(\Lambda_d)$  and not null-controllable for any  $T < c(\Lambda_d)$  (see [4, Theorem 6.17]), where  $c(\Lambda_d)$  is the index of condensation of the sequence

$$\Lambda_d := \left\{ \frac{(\pi n)^2}{d}, (\pi n)^2 \right\}_{n \geq 1}.$$

In that case, one can show (see [4, Relation (6.16)]) that

$$c(\Lambda_d) = \max(l_1, l_2), \quad l_1 := \limsup_{n \rightarrow \infty} \frac{-\ln \left| \sin \left( \frac{\pi n}{\sqrt{d}} \right) \right|}{\frac{\pi^2 n^2}{d}}, \quad l_2 := \limsup_{n \rightarrow \infty} \frac{-\ln \left| \sin \left( \pi n \sqrt{d} \right) \right|}{\pi^2 n^2}. \quad (1.2)$$

Using the differential flatness, our aim is to show that the system is null-controllable, as soon as  $d = 1$  or  $\sqrt{d} \notin \mathbb{Q}$  with  $c(\Lambda_d) < \infty$ . The flatness approach comes from the study of finite-dimensional control systems and has already been applied to several partial differential equations: the heat equation [16, 18], general parabolic equations [17], the Schrödinger equation [19], the linear Korteweg–de Vries equation [15] and a class of semi-linear heat equations [14]. The authors of the present paper have used this method to show some controllability properties of some free boundary models: the Stefan problem, in [9] and a cross-diffusion problem in [10].

In order to state our mains results, we first recall the notion of Gevrey functions. A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Gevrey function of order  $\sigma \geq 1$  if it is  $C^\infty$  and if for any compact subset  $K$  of  $\mathbb{R}$ , there exist  $M, R \in \mathbb{R}_+^*$  such that its derivatives satisfy

$$\left| f^{(\ell)}(t) \right| \leq M \frac{\ell!^\sigma}{R^\ell} \quad (\ell \in \mathbb{N}, t \in K).$$

Similarly,  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a Gevrey function of order  $(\sigma_1, \sigma_2)$  if it is  $C^\infty$  and if for any compact subset  $K$  of  $\mathbb{R}^2$ , there exist  $M, R_1, R_2 \in \mathbb{R}_+^*$  such that

$$\left| \partial_t^\ell \partial_x^k g(t, x) \right| \leq M \frac{\ell!^{\sigma_1}}{R_1^\ell} \frac{k!^{\sigma_2}}{R_2^k} \quad (\ell, k \in \mathbb{N}, (t, x) \in K).$$

One of the interests in such functions for controllability properties comes from the fact that one can construct Gevrey function with compact support or which are constant on some intervals. For instance, for any  $\sigma > 1$ , the function

$$\theta_\sigma(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{e^{-(t^{-c_\sigma})}}{e^{-(1-t)^{-c_\sigma}} + e^{-(t^{-c_\sigma})}} & \text{if } t \in (0, 1), \\ 1 & \text{if } t \geq 1, \end{cases} \quad (1.3)$$

with  $c_\sigma = (\sigma - 1)^{-1}$  is a Gevrey function of order  $\sigma$  (see, for instance [11, Appendix B]). In particular, this function satisfies

$$\theta_\sigma^{(\ell)}(0) = \theta_\sigma^{(\ell)}(1) = 0 \quad (\ell \geq 1),$$

and one can show the existence of  $M, R > 0$  such that

$$\left| \theta_\sigma^{(\ell)}(t) \right| \leq M \frac{\ell!^\sigma}{R^\ell} \quad (\ell \in \mathbb{N}, t \in \mathbb{R}).$$

For more information on the Gevrey functions, one can refer for instance to [22, 23, 25].

We are now in position to state our main results.

**Theorem 1.1.** *Assume  $d \neq 1$  and  $\sqrt{d} \notin \mathbb{Q}$  with  $c(\Lambda_d) < \infty$ . For any  $\sigma \in (1, 2)$ ,  $T > c(\Lambda_d)$  and  $(y_1^0, y_2^0) \in L^2(0, 1)^2$ , there exists a control  $u$  Gevrey of order  $\sigma$  in  $[0, T]$  such that the solution  $(y_1, y_2)$  of (1.1) is Gevrey of order  $(\sigma, 1)$  in  $(0, T] \times [0, 1]$  and such that*

$$y_1(T, \cdot) = y_2(T, \cdot) = 0.$$

**Theorem 1.2.** *Assume  $d = 1$ . For any  $\sigma \in (1, 2)$ ,  $T > 0$  and  $(y_1^0, y_2^0) \in L^2(0, 1)^2$ , there exists a control  $u$  Gevrey of order  $\sigma$  in  $[0, T]$  such that the solution  $(y_1, y_2)$  of (1.1) is Gevrey of order  $(\sigma, 1)$  in  $(0, T] \times [0, 1]$  and such that*

$$y_1(T, \cdot) = y_2(T, \cdot) = 0.$$

*Remark 1.3.* We recall that if  $\sqrt{d}$  is irrational and algebraic, then  $c(\Lambda_d) = 0$ . Moreover, for any  $T_0 > 0$ , there exists  $d$  such that  $c(\Lambda_d) = T_0$  (see [4, Proposition 6.20]).

We also consider the particular case where the initial and final conditions of our control problem are stationary states, that is, solutions of

$$\begin{cases} -\partial_x^2 \bar{y}_1 = 0 & \text{in } (0, 1), \\ -\frac{1}{d} \partial_x^2 \bar{y}_2 = \bar{y}_1 & \text{in } (0, 1), \\ \bar{y}_1(0) = 0, \quad \bar{y}_1(1) = \bar{u}, \\ \bar{y}_2(0) = 0, \quad \bar{y}_2(1) = 0, \end{cases}$$

for  $\bar{u} \in \mathbb{R}$ . The explicit solutions of the above system are

$$\bar{y}_1(x) = \bar{u}x, \quad \bar{y}_2(x) = \frac{d}{6}\bar{u}(x - x^3) \quad (x \in [0, 1]). \quad (1.4)$$

For such particular control problem, one obtain the exact controllability for any  $d > 0$  and any  $T > 0$ :

**Theorem 1.4.** *Assume  $d > 0$ ,  $T > 0$  and  $\sigma \in (1, 2)$ . For any stationary states  $(\bar{y}_1^{(1)}, \bar{y}_2^{(1)})$  and  $(\bar{y}_1^{(2)}, \bar{y}_2^{(2)})$ , there exists a control  $u$  Gevrey of order  $\sigma$  in  $[0, T]$  such that the solution  $(y_1, y_2)$  of (1.1) is Gevrey of order  $(\sigma, 1)$  in  $(0, T] \times [0, 1]$  and such that*

$$y_1(0, \cdot) = \bar{y}_1^{(1)}, \quad y_2(0, \cdot) = \bar{y}_2^{(1)} \quad \text{in } (0, 1)$$

and

$$y_1(T, \cdot) = \bar{y}_1^{(2)}, \quad y_2(T, \cdot) = \bar{y}_2^{(2)} \quad \text{in } (0, 1).$$

The outline of the paper is the following: in Section 2, we recall the flatness method that we use in our main results. Then we show Theorem 1.1 in Section 3, Theorem 1.2 in Section 4 and Theorem 1.4 in Section 5. Finally, the Section 6 conclude this paper with some open problems.

## 2 Flatness approach

### 2.1 The general method

Following the approach developed in [16] for a single heat equation, we look for solutions of (1.1) of the form

$$y_1(t, x) = \sum_{k=0}^{\infty} a_k(t) \frac{x^{2k+1}}{(2k+1)!}, \quad y_2(t, x) = \sum_{k=0}^{\infty} b_k(t) \frac{x^{2k+1}}{(2k+1)!}.$$

Inserting these expressions into the two first equations of (1.1) leads to the following induction formulae:

$$a_{k+1} = a'_k, \quad b_{k+1} = d(b'_k - a_k) \quad (k \geq 0)$$

so that the sequences  $(a_k)$  and  $(b_k)$  can be expressed through  $a := a_0$  and  $b := b_0$ . More precisely, we obtain

$$y_1(t, x) = \sum_{k=0}^{\infty} a^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}, \quad y_2(t, x) = \sum_{k=0}^{\infty} \left( d^k b^{(k)}(t) - d \frac{d^k - 1}{d - 1} a^{(k-1)}(t) \right) \frac{x^{2k+1}}{(2k+1)!}, \quad \text{if } d \neq 1, \quad (2.1)$$

$$y_1(t, x) = \sum_{k=0}^{\infty} a^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}, \quad y_2(t, x) = \sum_{k=0}^{\infty} \left( b^{(k)}(t) - k a^{(k-1)}(t) \right) \frac{x^{2k+1}}{(2k+1)!}, \quad \text{if } d = 1, \quad (2.2)$$

where we denote by  $a^{(k)}$  the  $k$ -th derivative of  $a$ .

*Remark 2.1.* We could also add in the solutions a series for the even powers of  $x$ , but, as in the case of the heat equation alone (see, [16]), one can see that the corresponding coefficients are 0 due to the homogeneous Dirichlet boundary conditions at  $x = 0$ .

In order to have that (2.1) or (2.2) are well-defined, we will search  $a$  and  $b$  as Gevrey functions and use the following result.

**Lemma 2.2.** *Assume that  $a$  and  $b$  are Gevrey functions of order  $\sigma \in (1, 2)$  in  $\mathbb{R}$ . Then  $y_1$  and  $y_2$  defined either by (2.1) or (2.2) are well-defined and are Gevrey functions of order  $(\sigma, 1)$  in  $\mathbb{R}^2$ . They satisfy the two first equations of (1.1) and the Dirichlet boundary conditions  $y_1(t, 0) = y_2(t, 0) = 0$  for all  $t$ .*

*Proof.* We only show that  $y_1$  and  $y_2$  are well-defined and are Gevrey functions of order  $(\sigma, 1)$  in  $\mathbb{R}^2$ , the fact that they satisfy the two first equations of (1.1) and the Dirichlet boundary conditions at  $x = 0$  can be obtained by standard computations.

To show that the series in (2.1) or (2.2) are convergent and correspond to Gevrey functions, we see that they have the general form

$$y(t, x) = \sum_{k=0}^{\infty} c_k(t) \frac{x^{2k+1}}{(2k+1)!}, \quad (2.3)$$

where  $c_k$  are  $C^\infty$  functions such that for any compact  $K$  of  $\mathbb{R}$ , there exist  $M, R \in \mathbb{R}_+^*$  such that

$$\left| c_k^{(\ell)}(t) \right| \leq M \frac{(\ell+k)!^\sigma}{R^{\ell+k}} \quad (\ell, k \in \mathbb{N}, t \in K).$$

Let us consider  $r > 0$  and let us set

$$g_k(t, x) := c_k(t) \frac{x^{2k+1}}{(2k+1)!}.$$

Then,  $\partial_t^\ell \partial_x^j g_k = 0$  if  $2k+1 < j$ , and if  $2k+1 \geq j$ , we have

$$\left| \partial_t^\ell \partial_x^j g_k(t, x) \right| \leq M \frac{(\ell+k)!^\sigma}{R^{\ell+k}} \frac{r^{2k+1-j}}{(2k+1-j)!} \quad (t \in K, |x| \leq r).$$

Combining this relation with the formula  $(\ell+k)! \leq 2^{\ell+k} \ell! k!$ , leads to

$$\left| \partial_t^\ell \partial_x^j g_k(t, x) \right| \leq 2Mr \frac{2^{\sigma\ell} \ell!^\sigma j!}{R^{\ell+rj}} \left( \frac{4(2^\sigma)r^2}{R} \right)^k \frac{k!^\sigma}{(2k+1)!} \quad (t \in K, |x| \leq r).$$

Since  $\sigma < 2$ , we have

$$\sum_{k \geq 0} \left( \frac{4(2^\sigma)r^2}{R} \right)^k \frac{(k)^\sigma}{(2k+1)!} < \infty$$

and we deduce that formula (2.3) defines a  $C^\infty$  function  $y$  and that there exist  $M_1, R_1 > 0$  such that

$$\left| \partial_t^\ell \partial_x^j y(t, x) \right| \leq M_1 \frac{(\ell)^\sigma j!}{R_1^{\ell+rj}} \quad (t \in K, |x| \leq r).$$

This concludes the proof of the lemma. □

## 2.2 The boundary conditions at $x = 1$

The control  $u$  will be given by the trace of  $y_1$  at  $x = 1$ , that is

$$u(t) = \sum_{k=0}^{\infty} \frac{a^{(k)}(t)}{(2k+1)!}, \quad (2.4)$$

but it remains to deal with the Dirichlet boundary condition of  $y_2$  at  $x = 1$ :

$$y_2(t, 1) = 0 \quad (t \in (0, T)). \quad (2.5)$$

One of the main difficulties in this method based on the flatness approach for the controllability of partial differential equations is coming from this boundary conditions. The particular form of the solutions ((2.1), (2.2)) yields directly the boundary conditions at  $x = 0$  but it can be quite difficult to check any boundary condition at  $x = 1$ .

Let us start by a formal computation in the case  $d \neq 1$  (the case  $d = 1$  can be treated similarly). First, using (2.1), we write (2.5) as

$$\sum_{k=0}^{\infty} \frac{d^k}{(2k+1)!} b^{(k)}(t) = \sum_{k=1}^{\infty} d \frac{d^k - 1}{d-1} \frac{a^{(k-1)}(t)}{(2k+1)!} \quad (t \in (0, T)). \quad (2.6)$$

Let us assume that

$$a^{(k)}(0) = b^{(k)}(0) = 0 \quad (k \geq 0) \quad (2.7)$$

and let us consider the Laplace transforms of  $a$  and  $b$ :

$$\widehat{a}(s) := \int_0^{\infty} e^{-ts} a(t) dt, \quad \widehat{b}(s) := \int_0^{\infty} e^{-ts} b(t) dt \quad (s > 0).$$

Then, the relation (2.6) for  $T = \infty$  yields that

$$\sum_{k=0}^{\infty} \frac{d^k s^k}{(2k+1)!} \widehat{b}(s) = \sum_{k=1}^{\infty} d \frac{d^k - 1}{d-1} \frac{s^{k-1}}{(2k+1)!} \widehat{a}(s) \quad (s > 0).$$

One possible solution of the above equation is

$$\widehat{a}(s) = \sum_{k=0}^{\infty} \frac{d^k s^k}{(2k+1)!} \widehat{f}(s), \quad \widehat{b}(s) = \sum_{k=1}^{\infty} d \frac{d^k - 1}{d-1} \frac{s^{k-1}}{(2k+1)!} \widehat{f}(s) \quad (s > 0),$$

for some function  $f$ . If we assume that  $f$  satisfies the same property (2.7) as  $a$  and  $b$ , this leads us to consider the following form for  $a$  and  $b$  (if  $d \neq 1$ ):

$$a(t) = \sum_{m=0}^{\infty} \frac{d^m}{(2m+1)!} f^{(m)}(t), \quad b(t) = \sum_{m=1}^{\infty} d \frac{d^m - 1}{d-1} \frac{f^{(m-1)}(t)}{(2m+1)!}. \quad (2.8)$$

In the case  $d = 1$ , a similar formal computation leads to the following formulae

$$a(t) = \sum_{m=0}^{\infty} \frac{f^{(m)}(t)}{(2m+1)!}, \quad b(t) = \sum_{m=1}^{\infty} m \frac{f^{(m-1)}(t)}{(2m+1)!}. \quad (2.9)$$

*Remark 2.3.* Without condition (2.7), the relation (2.6) for  $T = \infty$  yields a formula of the form

$$\sum_{k=0}^{\infty} \frac{d^k p^k}{(2k+1)!} \widehat{b}(p) - \sum_{k=1}^{\infty} d \frac{d^k - 1}{d-1} \frac{p^{k-1}}{(2k+1)!} \widehat{a}(p) = R(p) \quad (p > 0),$$

for a function  $R$  that can be expressed as a series of  $s$ . Such a Bézout identity may be difficult to solve in the general case and one can quote the recent article [21] where the author presents a practical algorithm for computing this kind of Bézout relations. A similar difficulty appears for the controllability by a flatness approach of the beam equation where there are four boundary conditions: the authors in [13] are using Mikusiński's operators (see [20]) to overcome this problem.

The following result gives a rigorous framework to the above formal computations:

**Lemma 2.4.** *Assume  $f$  is a Gevrey function of order  $\sigma \in (1, 2)$ . Then formulae (2.8) (respectively (2.9)) define Gevrey functions  $a$  and  $b$  of order  $\sigma$ . Moreover, if  $(y_1, y_2)$  is defined by (2.2) (respectively (2.1)), then  $y_2$  satisfies (2.5).*

*Proof.* Assume  $K$  is a compact of  $\mathbb{R}$  and consider  $M, R$  such that

$$\left| f^{(m)}(t) \right| \leq M \frac{m!^\sigma}{R^m} \quad (m \in \mathbb{N}, t \in K).$$

Then

$$\left| \frac{d^\ell}{dt^\ell} \left( \frac{d^m}{(2m+1)!} f^{(m)} \right) (t) \right| \leq M \frac{2^{\sigma\ell} \ell!^\sigma}{R^\ell} \left( \frac{2^\sigma d}{R} \right)^m \frac{m!^\sigma}{(2m+1)!} \quad (\ell, m \in \mathbb{N}, t \in K)$$

and we deduce that  $a$  defined by (2.8) or (2.9) is a  $C^\infty$  function satisfying

$$\left| a^{(\ell)}(t) \right| \leq M_1 \frac{2^{\sigma\ell} \ell!^\sigma}{R^\ell} \quad (\ell \in \mathbb{N}, t \in K)$$

for some  $M_1 > 0$ . This shows that  $a$  is a Gevrey function of order  $\sigma$ . A similar estimate holds for  $b$ , leading to the fact that  $b$  is a Gevrey function of order  $\sigma$ .

Moreover, a similar computation shows that we can apply the Fubini theorem in the following computations. If  $d \neq 1$ , using (2.1), deduce that

$$\begin{aligned} y_2(t, 1) &= \sum_{k=0}^{\infty} d^k \frac{b^{(k)}(t)}{(2k+1)!} - \sum_{k=1}^{\infty} d \frac{d^k - 1}{d-1} \frac{a^{(k-1)}(t)}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} d^{k+1} \frac{d^m - 1}{d-1} \frac{f^{(m+k-1)}(t)}{(2k+1)!(2m+1)!} - \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} d^{m+1} \frac{d^k - 1}{d-1} \frac{f^{(m+k-1)}(t)}{(2m+1)!(2k+1)!} \\ &= 0. \end{aligned}$$

If  $d = 1$ , using (2.2), we find similarly

$$\begin{aligned} y_2(t, 1) &= \sum_{k=0}^{\infty} \frac{b^{(k)}(t)}{(2k+1)!} - \sum_{k=1}^{\infty} k \frac{a^{(k-1)}(t)}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} m \frac{f^{(m+k-1)}(t)}{(2k+1)!(2m+1)!} - \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} k \frac{f^{(m+k-1)}(t)}{(2m+1)!(2k+1)!} \\ &= 0. \end{aligned}$$

□

*Remark 2.5.* We notice that in Lemma 2.4, we do not need the condition (2.7) for  $a, b$  or  $f$ . In particular, by combining Lemmas 2.2 and 2.4, we have obtained that for any Gevrey function  $f$  of order  $\sigma \in (1, 2)$ , the formulas (2.1) and (2.8) (for  $d \neq 1$ ) and (2.2) and (2.9) (for  $d = 1$ ) yield Gevrey functions  $y_1$  and  $y_2$  satisfying the first four equations of (1.1) with the control  $u$  defined by (2.4).

It remains to determine  $f$  so that the corresponding solutions satisfy the desired initial and final conditions. This is the aim of the next sections.

### 3 Proof of Theorem 1.1

With the above section, we can construct solutions of (1.1) by using the formulas (2.1) and (2.8) (for  $d \neq 1$ ) or (2.2) and (2.9) (for  $d = 1$ ). In this construction, we need to determine a suitable function  $f$ . To this end, we first consider the operator associated with (1.1)

$$\mathcal{D}(A) := [H^2(0, 1) \cap H_0^1(0, 1)]^2, \quad A : \mathcal{D}(A) \rightarrow L^2(0, 1)^2, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} -\partial_x^2 y_1 \\ -\frac{1}{d} \partial_x^2 y_2 - y_1 \end{bmatrix}. \quad (3.1)$$

The operator  $-A$  is the infinitesimal generator of an analytic semigroup and if there is no control (that is if  $u = 0$ ), the system (1.1) writes

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} (0) = \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix}. \quad (3.2)$$

One can compute the eigenvalues and eigenvectors of  $A$  in classical way, and we have the following standard result (see, for instance, [4]).

**Proposition 3.1.** *Assume  $d \neq 1$ . The spectrum of  $A$  is composed of eigenvalues*

$$\sigma(A) = \{\lambda_{1,n}, \lambda_{2,n}\}_{n \geq 1}, \quad \lambda_{1,n} = \frac{(\pi n)^2}{d}, \quad \lambda_{2,n} = (\pi n)^2$$

and a corresponding family of eigenfunctions is

$$\psi_{1,n}(x) = \begin{bmatrix} 0 \\ \sin(n\pi x) \end{bmatrix}, \quad \psi_{2,n}(x) = \begin{bmatrix} \sin(n\pi x) \\ \frac{d}{(1-d)n^2\pi^2} \sin(n\pi x) \end{bmatrix} \quad (x \in [0, 1]).$$

The eigenvalues of  $A$  are simple if and only if  $\sqrt{d} \notin \mathbb{Q}$ .

In particular, if  $(y_1^0, y_2^0) \in L^2(0, 1)^2$ , then the solution of (1.1) with  $u = 0$  (or equivalently of (3.2)) has the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} (t) = \sum_{n \geq 1} \alpha_{1,n} \psi_{1,n} e^{-\lambda_{1,n} t} + \sum_{n \geq 1} \alpha_{2,n} \psi_{2,n} e^{-\lambda_{2,n} t}, \quad (3.3)$$

with  $(\alpha_{1,n})_n, (\alpha_{2,n})_n \in \ell^2(\mathbb{N}^*)$ . This shows in particular, that  $(y_1, y_2)$  is analytic at any  $(t, x) \in (0, T] \times [0, 1]$  and we have

$$y_1(T, x) = \sum_{n \geq 1} \alpha_{2,n} \sin(n\pi x) e^{-\lambda_{2,n} T}, \quad (3.4)$$

$$y_2(T, x) = \sum_{n \geq 1} \alpha_{1,n} \sin(n\pi x) e^{-\lambda_{1,n} T} + \sum_{n \geq 1} \alpha_{2,n} \frac{d}{(1-d)n^2\pi^2} \sin(n\pi x) e^{-\lambda_{2,n} T}. \quad (3.5)$$

In order to prove Theorem 1.1, we use the linearity of our system, and we construct our solution as the difference of the solution (3.3) associated with the initial conditions  $(y_1^0, y_2^0) \in L^2(0, 1)^2$  and with  $u = 0$  and a solution with initial conditions

$$y_1(0, \cdot) = y_2(0, \cdot) = 0 \quad \text{in } (0, 1) \quad (3.6)$$

and with the final conditions given by (3.4) and (3.5). As explained in Section 2, to build such solutions, we only need to define the Gevrey function  $f$  of order  $\sigma \in (1, 2)$  such that the corresponding solution  $(y_1, y_2)$  satisfies the above initial and final conditions.

*Remark 3.2.* Finding a suitable function  $f$  can be a difficult task in general. In order to do this, we can start by searching a function  $\tilde{f}$  such that the solutions (3.3) can be written formally under the form (2.1) and (2.8) where in the last relation  $f$  is replaced by  $\tilde{f}$ .



Relation (3.3) implies

$$y_1(t, x) = \sum_{k \geq 0} \sum_{n \geq 1} (-1)^k \alpha_{2,n} (n\pi)^{2k+1} e^{-\lambda_{2,n} t} \frac{x^{2k+1}}{(2k+1)!},$$

$$y_2(t, x) = \sum_{k \geq 0} \sum_{n \geq 1} (-1)^k (n\pi)^{2k+1} \left[ \alpha_{1,n} e^{-\lambda_{1,n} t} + \frac{d}{(1-d)n^2\pi^2} \alpha_{2,n} e^{-\lambda_{2,n} t} \right] \frac{x^{2k+1}}{(2k+1)!}.$$

By identifying with (2.1), we obtain that

$$a^{(k)}(t) = \sum_{n \geq 1} (-1)^k \alpha_{2,n} (n\pi)^{2k+1} e^{-\lambda_{2,n} t},$$

$$d^k b^{(k)}(t) - d \frac{d^k - 1}{d - 1} a^{(k-1)}(t) = \sum_{n \geq 1} (-1)^k (n\pi)^{2k+1} \left[ \alpha_{1,n} e^{-\lambda_{1,n} t} + \frac{d}{(1-d)n^2\pi^2} \alpha_{2,n} e^{-\lambda_{2,n} t} \right].$$

Combining these two equations, we deduce

$$b^{(k)}(t) = \sum_{n \geq 1} \left[ (n\pi) (-\lambda_{1,n})^k \alpha_{1,n} e^{-\lambda_{1,n} t} + \frac{d}{(1-d)n\pi} \alpha_{2,n} (-\lambda_{2,n})^k e^{-\lambda_{2,n} t} \right].$$

We thus need to find a function  $\tilde{f}$  such that (see (2.8)):

$$\sum_{m=0}^{\infty} \frac{d^m}{(2m+1)!} \tilde{f}^{(m+k)}(t) = \sum_{n \geq 1} (n\pi) \alpha_{2,n} (-\lambda_{2,n})^k e^{-\lambda_{2,n} t}, \quad (3.7)$$

$$\sum_{m=1}^{\infty} d \frac{d^m - 1}{d - 1} \frac{\tilde{f}^{(m+k-1)}(t)}{(2m+1)!} = \sum_{n \geq 1} \left[ (n\pi) (-\lambda_{1,n})^k \alpha_{1,n} e^{-\lambda_{1,n} t} + \frac{d}{(1-d)n\pi} \alpha_{2,n} (-\lambda_{2,n})^k e^{-\lambda_{2,n} t} \right]. \quad (3.8)$$

This suggests to search  $\tilde{f}$  under the form

$$\tilde{f}(t) := \sum_{n \geq 1} \beta_{1,n} e^{-\lambda_{1,n} t} + \sum_{n \geq 1} \beta_{2,n} e^{-\lambda_{2,n} t}.$$

We see below that one can indeed find  $(\beta_{1,n})$  and  $(\beta_{2,n})$  so that  $\tilde{f}$  satisfies (3.7) and (3.8). Finally, to recover (3.6), we use a Gevrey function  $\eta$  equal to 0 for  $t \leq 0$  and equal to 1 for  $t \geq T$ , and we then set  $f := \tilde{f}\eta$ . In what follows, we apply this strategy to prove Theorem 1.1.

Let us consider  $T_0 > c(\Lambda_d)$ ,  $\sigma \in (1, 2)$  and let us define

$$\eta(t) := \theta_\sigma \left( \frac{t - T_0}{T - T_0} \right),$$

where  $\theta_\sigma$  is given by (1.3). The function  $\eta$  is Gevrey of order  $\sigma$  and satisfies  $\eta \equiv 0$  in  $(-\infty, T_0]$ ,  $\eta(T) = 1$  and  $\eta^{(k)}(T) = 0$  for all  $k \geq 1$ . We set

$$\beta_{1,n} := \alpha_{1,n} \frac{\pi^4 n^4 (d-1)}{d^{5/2} \sin\left(\frac{\pi n}{\sqrt{d}}\right)}, \quad \beta_{2,n} := \alpha_{2,n} \frac{\pi^2 n^2 \sqrt{d}}{\sin(\pi n \sqrt{d})}, \quad (3.9)$$

and we define

$$\tilde{f}(t) := \sum_{n \geq 1} \beta_{1,n} e^{-\lambda_{1,n} t} + \sum_{n \geq 1} \beta_{2,n} e^{-\lambda_{2,n} t} \quad \text{and} \quad f := \tilde{f}\eta.$$

From (1.2), for any  $T_0 > c(\Lambda_d)$ , there exists a constant  $C > 0$  such that for all  $n \geq 1$ ,

$$\left| \alpha_{1,n} \frac{\pi^4 n^4 (d-1)}{d^{5/2} \sin\left(\frac{\pi n}{\sqrt{d}}\right)} \right| \leq C |\alpha_{1,n}| e^{T_0 \lambda_{1,n}}, \quad \left| \alpha_{2,n} \frac{\pi^2 n^2 \sqrt{d}}{\sin(\pi n \sqrt{d})} \right| \leq C |\alpha_{2,n}| e^{T_0 \lambda_{2,n}}$$

In particular,  $\tilde{f}$  is an analytic function in  $(T_0, \infty)$  and  $f$  is a Gevrey function of order  $\sigma$  in  $\mathbb{R}$ . Moreover, we have

$$f^{(m)} = \sum_{j=0}^m \binom{m}{j} \tilde{f}^{(j)} \eta^{(m-j)}$$

so that

$$f^{(m)}(0) = 0, \quad f^{(m)}(T) = \tilde{f}^{(m)}(T) = \sum_{n \geq 1} \beta_{1,n} (-\lambda_{1,n})^m e^{-\lambda_{1,n} T} + \sum_{n \geq 1} \beta_{2,n} (-\lambda_{2,n})^m e^{-\lambda_{2,n} T}.$$

In particular, from (2.8), for every  $k \geq 0$  we have

$$a^{(k)}(0) = b^{(k)}(0) = 0, \tag{3.10}$$

$$a^{(k)}(T) = \sum_{m=0}^{\infty} \frac{d^m}{(2m+1)!} \left( \sum_{n \geq 1} \beta_{1,n} (-\lambda_{1,n})^{m+k} e^{-\lambda_{1,n} T} + \sum_{n \geq 1} \beta_{2,n} (-\lambda_{2,n})^{m+k} e^{-\lambda_{2,n} T} \right), \tag{3.11}$$

$$b^{(k)}(T) = \sum_{m=0}^{\infty} \frac{d^m - 1}{d-1} \frac{d}{(2m+1)!} \left( \sum_{n \geq 1} \beta_{1,n} (-\lambda_{1,n})^{m+k-1} e^{-\lambda_{1,n} T} + \sum_{n \geq 1} \beta_{2,n} (-\lambda_{2,n})^{m+k-1} e^{-\lambda_{2,n} T} \right). \tag{3.12}$$

We also have

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{d^m}{(2m+1)!} (-\lambda_{1,n})^m &= 0, & \sum_{m=0}^{\infty} \frac{d^m}{(2m+1)!} (-\lambda_{2,n})^m &= \frac{\sin(\pi n \sqrt{d})}{\pi n \sqrt{d}}, \\ \sum_{m=0}^{\infty} \frac{(-\lambda_{1,n})^m}{(2m+1)!} &= \sin\left(\frac{\pi n}{\sqrt{d}}\right) \frac{\sqrt{d}}{\pi n}, & \sum_{m=0}^{\infty} \frac{(-\lambda_{2,n})^m}{(2m+1)!} &= 0. \end{aligned}$$

Using the above formula into (3.11) and (3.12) and applying the Fubini theorem, we deduce

$$\begin{aligned} a^{(k)}(T) &= \sum_{n \geq 1} \beta_{2,n} (-\lambda_{2,n})^k \frac{\sin(\pi n \sqrt{d})}{\pi n \sqrt{d}} e^{-\lambda_{2,n} T}, \\ b^{(k)}(T) &= \sum_{n \geq 1} \frac{-d}{d-1} \beta_{1,n} \sin\left(\frac{\pi n}{\sqrt{d}}\right) \frac{\sqrt{d}}{\pi n} (-\lambda_{1,n})^{k-1} e^{-\lambda_{1,n} T} + \sum_{n \geq 1} \frac{d}{d-1} \beta_{2,n} \frac{\sin(\pi n \sqrt{d})}{\pi n \sqrt{d}} (-\lambda_{2,n})^{k-1} e^{-\lambda_{2,n} T}. \end{aligned}$$

Finally, using these formulae into (2.1) leads to

$$\begin{aligned} y_1(T, x) &= \sum_{n \geq 1} \beta_{2,n} \frac{\sin(\pi n \sqrt{d})}{\pi n \sqrt{d}} e^{-\lambda_{2,n} T} \sum_{k=0}^{\infty} (-\lambda_{2,n})^k \frac{x^{2k+1}}{(2k+1)!}, \\ y_2(T, x) &= \sum_{k=0}^{\infty} d^k \frac{x^{2k+1}}{(2k+1)!} \sum_{n \geq 1} \frac{-d}{d-1} \beta_{1,n} \sin\left(\frac{\pi n}{\sqrt{d}}\right) \frac{\sqrt{d}}{\pi n} (-\lambda_{1,n})^{k-1} e^{-\lambda_{1,n} T} \\ &\quad + \sum_{k=0}^{\infty} d^k \frac{x^{2k+1}}{(2k+1)!} \sum_{n \geq 1} \frac{d}{d-1} \beta_{2,n} \frac{\sin(\pi n \sqrt{d})}{\pi n \sqrt{d}} (-\lambda_{2,n})^{k-1} e^{-\lambda_{2,n} T} \\ &\quad - \sum_{k=0}^{\infty} d \frac{d^k - 1}{d-1} \sum_{n \geq 1} \beta_{2,n} (-\lambda_{2,n})^{k-1} \frac{\sin(\pi n \sqrt{d})}{\pi n \sqrt{d}} e^{-\lambda_{2,n} T} \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

The above relation can be written as

$$y_1(T, x) = \sum_{n \geq 1} \beta_{2,n} \frac{\sin(\pi n \sqrt{d})}{\pi^2 n^2 \sqrt{d}} \sin(n\pi x) e^{-\lambda_{2,n} T}, \quad (3.13)$$

$$y_2(T, x) = \sum_{n \geq 1} \frac{d^{5/2}}{\pi^4 n^4 (d-1)} \beta_{1,n} \sin\left(\frac{\pi n}{\sqrt{d}}\right) \sin(n\pi x) e^{-\lambda_{1,n} T} + \sum_{n \geq 1} \frac{d}{n^2 \pi^2 (1-d)} \beta_{2,n} \frac{\sin(\pi n \sqrt{d})}{\pi^2 n^2 \sqrt{d}} \sin(n\pi x) e^{-\lambda_{2,n} T}. \quad (3.14)$$

Using (3.9) into (3.13) and (3.14) yields (3.4) and (3.5). This concludes the proof of Theorem 1.1.

## 4 Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1. We start by considering the spectral analysis of the operator  $A$  defined by (3.1) (with  $d = 1$ ).

**Proposition 4.1.** *Assume  $d = 1$ . The spectrum of  $A$  is composed of simple eigenvalues*

$$\sigma(A) = \{\lambda_n\}_{n \geq 1}, \quad \lambda_n = (\pi n)^2,$$

*with algebraic multiplicity equal to 2. A corresponding family of generalized eigenfunctions is*

$$\psi_{1,n}(x) = \begin{bmatrix} 0 \\ \sin(n\pi x) \end{bmatrix}, \quad \psi_{2,n}(x) = \begin{bmatrix} \sin(n\pi x) \\ 0 \end{bmatrix} \quad (x \in [0, 1]).$$

In particular, if  $(y_1^0, y_2^0) \in L^2(0, 1)^2$ , then the solution of (1.1) with  $u = 0$  (or equivalently of (3.2)) has the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} (t) = \sum_{n \geq 1} \alpha_{1,n} \psi_{1,n} e^{-\lambda_n t} + \sum_{n \geq 1} \alpha_{2,n} \psi_{2,n} e^{-\lambda_n t} + \sum_{n \geq 1} \alpha_{2,n} \psi_{1,n} t e^{-\lambda_n t},$$

with  $(\alpha_{1,n})_n, (\alpha_{2,n})_n \in \ell^2(\mathbb{N}^*)$ . This shows that  $(y_1, y_2)$  is analytic at any  $(t, x) \in (0, T] \times [0, 1]$  and we have

$$y_1(T, x) = \sum_{n \geq 1} \alpha_{2,n} \sin(n\pi x) e^{-\lambda_n T}, \quad y_2(T, x) = \sum_{n \geq 1} \alpha_{1,n} \sin(n\pi x) e^{-\lambda_n T} + \sum_{n \geq 1} \alpha_{2,n} \sin(n\pi x) T e^{-\lambda_n T}. \quad (4.1)$$

In order to prove Theorem 1.2, we are going to construct a solution of (1.1) with initial conditions

$$y_1(0, \cdot) = y_2(0, \cdot) = 0 \quad \text{in } (0, 1)$$

and with the final conditions given by (4.1). As explained in Section 2, we build such solutions in the form (2.1), where  $a$  and  $b$  are given by (2.8). It remains to define the Gevrey function  $f$  of order  $\sigma \in (1, 2)$  such that the corresponding solution  $(y_1, y_2)$  satisfies the above initial and final conditions.

Let us define

$$\eta(t) := \theta_\sigma \left( \frac{t}{T} \right),$$

where  $\theta_\sigma$  is given by (1.3). The function  $\eta$  is Gevrey of order  $\sigma$  and satisfies  $\eta \equiv 0$  in  $(-\infty, 0]$ ,  $\eta(T) = 1$  and  $\eta^{(k)}(T) = 0$  for all  $k \geq 1$ . We now set

$$\beta_{1,n} := 2(-1)^{n+1} n^2 \pi^2 \left( \alpha_{1,n} - \frac{3}{2n\pi} \alpha_{2,n} \right), \quad \beta_{2,n} := 2(-1)^{n+1} n^2 \pi^2 \alpha_{2,n} \quad (4.2)$$

and we define

$$\tilde{f}(t) := \sum_{n \geq 1} \beta_{1,n} e^{-\lambda_n t} + \sum_{n \geq 1} \beta_{2,n} t e^{-\lambda_n t}, \quad f := \tilde{f} \eta.$$

In particular,  $\tilde{f}$  is an analytic function in  $(0, \infty)$  and  $f$  is a Gevrey function of order  $\sigma$  in  $\mathbb{R}$ . Moreover, we have

$$f^{(m)} = \sum_{j=0}^m \binom{m}{j} \tilde{f}^{(j)} \eta^{(m-j)}$$

so that

$$\begin{aligned} f^{(m)}(0) &= 0, \\ f^{(m)}(T) &= \tilde{f}^{(m)}(T) = \sum_{n \geq 1} \beta_{1,n} (-\lambda_n)^m e^{-\lambda_n T} + \sum_{n \geq 1} \beta_{2,n} \left( (-\lambda_n)^m T + m (-\lambda_n)^{m-1} \right) e^{-\lambda_n T}. \end{aligned}$$

In particular, from (2.9), for every  $k \geq 0$  we have

$$a^{(k)}(0) = b^{(k)}(0) = 0, \tag{4.3}$$

$$\begin{aligned} a^{(k)}(T) &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \left( \sum_{n \geq 1} \beta_{1,n} (-\lambda_n)^{m+k} e^{-\lambda_n T} \right. \\ &\quad \left. + \sum_{n \geq 1} \beta_{2,n} \left( (-\lambda_n)^{m+k} T + (m+k) (-\lambda_n)^{m+k-1} \right) e^{-\lambda_n T} \right), \end{aligned} \tag{4.4}$$

$$\begin{aligned} b^{(k)}(T) &= \sum_{m=0}^{\infty} \frac{m}{(2m+1)!} \left( \sum_{n \geq 1} \beta_{1,n} (-\lambda_n)^{m+k-1} e^{-\lambda_n T} \right. \\ &\quad \left. + \sum_{n \geq 1} \beta_{2,n} \left( (-\lambda_n)^{m+k-1} T + (m+k-1) (-\lambda_n)^{m+k-2} \right) e^{-\lambda_n T} \right). \end{aligned} \tag{4.5}$$

We also have

$$\sum_{m=0}^{\infty} \frac{(-\lambda_n)^{m-1}}{(2m+1)!} = 0, \quad \sum_{m=0}^{\infty} \frac{m (-\lambda_n)^{m-1}}{(2m+1)!} = \frac{1}{2} \frac{(-1)^{n+1}}{n\pi}, \quad \sum_{m=0}^{\infty} \frac{m(m-1) (-\lambda_n)^{m-2}}{(2m+1)!} = \frac{3}{4} \frac{(-1)^{n+1}}{n^2 \pi^2}.$$

Using the above formula into (4.4) and (4.5), we deduce

$$\begin{aligned} a^{(k)}(T) &= \sum_{n \geq 1} \beta_{2,n} (-\lambda_n)^k e^{-\lambda_n T} \frac{1}{2} \frac{(-1)^{n+1}}{n\pi} \\ b^{(k)}(T) &= \sum_{n \geq 1} \beta_{1,n} (-\lambda_n)^k \frac{1}{2} \frac{(-1)^{n+1}}{n\pi} e^{-\lambda_n T} \\ &\quad + \sum_{n \geq 1} \beta_{2,n} \left( \frac{1}{2} \frac{(-1)^{n+1}}{n\pi} (-\lambda_n)^k T + k (-\lambda_n)^{k-1} \frac{1}{2} \frac{(-1)^{n+1}}{n\pi} + (-\lambda_n)^k \frac{3}{4} \frac{(-1)^{n+1}}{n^2 \pi^2} \right) e^{-\lambda_n T}. \end{aligned}$$

Finally, using these formulae into (2.2) leads to

$$y_1(T, x) = \sum_{n \geq 1} \beta_{2,n} \frac{(-1)^{n+1}}{2n^2 \pi^2} \sin(n\pi x) e^{-\lambda_n T},$$

$$y_2(T, x) = \sum_{n \geq 1} \beta_{1,n} \frac{(-1)^{n+1}}{2n^2\pi^2} \sin(n\pi x) e^{-\lambda_n T} + \sum_{n \geq 1} \beta_{2,n} \frac{(-1)^{n+1}}{2n^2\pi^2} \sin(n\pi x) T e^{-\lambda_n T} \\ + \sum_{n \geq 1} \beta_{2,n} \frac{3}{4} \frac{(-1)^{n+1}}{n^3\pi^3} \sin(n\pi x) e^{-\lambda_n T}$$

and in particular,

$$y_2(T, x) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{2n^2\pi^2} \left( \beta_{1,n} + \frac{3}{2} \frac{\beta_{2,n}}{n\pi} \right) \sin(n\pi x) e^{-\lambda_n T} + \sum_{n \geq 1} \beta_{2,n} \frac{(-1)^{n+1}}{2n^2\pi^2} \sin(n\pi x) T e^{-\lambda_n T}.$$

By using the formula (4.2) into the above relations we deduce (4.1) and this ends the proof of Theorem 1.2.

## 5 Proof of Theorem 1.4

In order to prove Theorem 1.4, we follow again the strategy of Section 2 and build solutions in the form (2.1) or (2.2), where  $a$  and  $b$  are given by (2.8) or (2.9). It remains to define the Gevrey function  $f$  of order  $\sigma \in (1, 2)$  such that the corresponding solution  $(y_1, y_2)$  satisfies the above initial and final conditions.

Let us consider two stationary states  $(\bar{y}_1^{(1)}, \bar{y}_2^{(1)})$  and  $(\bar{y}_1^{(2)}, \bar{y}_2^{(2)})$ . They are given by the formula (1.4) for some  $\bar{u}^{(1)}, \bar{u}^{(2)} \in \mathbb{R}$ .

Let us define

$$f(t) := \left( \bar{u}^{(2)} - \bar{u}^{(1)} \right) \theta_\sigma \left( \frac{t}{T} \right) + \bar{u}^{(1)},$$

where  $\theta_\sigma$  is given by (1.3). The function  $f$  is Gevrey of order  $\sigma$  and satisfies

$$f(0) = \bar{u}^{(1)}, \quad f(T) = \bar{u}^{(2)}, \quad f^{(m)}(0) = f^{(m)}(T) = 0 \quad (m \geq 1).$$

In particular, since  $a$  and  $b$  are given by (2.8) or (2.9), we have

$$a(0) = \bar{u}^{(1)}, \quad a(T) = \bar{u}^{(2)}, \quad a^{(k)}(0) = a^{(k)}(T) = 0 \quad (k \geq 1), \\ b(0) = \frac{d}{6} \bar{u}^{(1)}, \quad b(T) = \frac{d}{6} \bar{u}^{(2)}, \quad b^{(k)}(0) = b^{(k)}(T) = 0 \quad (k \geq 1).$$

In particular,  $y_1$  and  $y_2$  given by (2.1) or (2.2) satisfy

$$y_1(0, x) = \bar{u}^{(1)} x, \quad y_2(0, x) = \frac{d}{6} \bar{u}^{(1)} (x - x^3), \\ y_1(T, x) = \bar{u}^{(2)} x, \quad y_2(T, x) = \frac{d}{6} \bar{u}^{(2)} (x - x^3).$$

This ends the proof of Theorem 1.4.

## 6 Conclusion

We have obtained the null-controllability of a coupled system of heat equations in dimension 1 in space by using the flatness approach. The system is in ‘‘cascade’’, with a control acting only on one boundary of the first equation. This is a particular example of the general case treated in [4]

$$\left\{ \begin{array}{l} \partial_t y - (D\partial_x^2 + A)y = 0 \quad \text{in } (0, T) \times (0, 1), \\ y(t, 0) = 0 \quad \text{in } (0, T), \\ y(t, 1) = Bu(t) \quad \text{in } (0, T), \\ y(0, \cdot) = y^0 \quad \text{in } (0, 1), \end{array} \right. \quad (6.1)$$

where  $y = (y_1, \dots, y_n)$ ,  $A \in \mathcal{M}_n(\mathbb{R})$ ,  $D = \text{diag}\left(\frac{1}{d_1}, \dots, \frac{1}{d_n}\right)$ ,  $u = (u_1, \dots, u_m)$  and  $B \in \mathcal{M}_{n,m}(\mathbb{R})$ . We have treated the case  $n = 2$  and  $m = 1$ , with

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For this particular example, we have managed to recover the controllability results in [4] with the flatness method. Note that we avoid in particular the use of any bi-orthogonal basis as it is standard for the study of the controllability of such systems.

In the future, we would like to generalize our result and in particular to deal with general systems of the form (6.1). Another interesting problem would be to consider, as in [6], the case where in the equation of  $y_2$  in (1.1) we replace the right-hand side by  $cy_1$  with a function  $c$  depending on  $x$ . Finally, one could wonder if this flatness method can be applied to systems of heat equations coupled from the boundary: see, for instance [7], where the coupling is done with Robin boundary condition, or [8], with Kirchhoff boundary conditions.

## Acknowledgments

The authors were partially supported by the ANR Grant ‘‘TRECOS’’ (ANR-20-CE40-0009).

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