



HAL
open science

LMI-based Control of Singularly Perturbed Switched Affine Systems

Ryan P C de Souza, Zohra Kader, Stéphane Caux

► **To cite this version:**

Ryan P C de Souza, Zohra Kader, Stéphane Caux. LMI-based Control of Singularly Perturbed Switched Affine Systems. IFAC World Congress, Jul 2023, Yokohama, Japan. hal-04119047

HAL Id: hal-04119047

<https://hal.science/hal-04119047>

Submitted on 6 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LMI-based Control of Singularly Perturbed Switched Affine Systems

Ryan P. C. de Souza* Zohra Kader* Stéphane Caux*

* *LAPLACE - Université de Toulouse, 118 route de Narbonne 31062
Toulouse cedex 9, France (e-mail:
{pitanga,kader,caux}@laplace.univ-tlse.fr).*

Abstract: In this paper, we tackle the problem of stabilizing singularly perturbed switched affine systems. Even though efficient control strategies based on the solution of Linear Matrix Inequalities (LMIs) have been presented in the switched affine systems literature, the presence of the small singular perturbation parameter may introduce numerical difficulties in the solution of these LMIs due to ill conditioning. Moreover, this parameter may be uncertain or its use in the control law may not be practical. Here, we propose an LMI-based control design strategy that avoids the conditioning issues and that also provides an estimation of the domain of attraction. Furthermore, the resulting controller does not depend on the singular perturbation parameter. The proposed method is illustrated on a numerical example.

Keywords: Control of switched systems, Switched affine systems, Stabilization of hybrid systems, Singularly perturbed systems, Linear Matrix Inequalities.

1. INTRODUCTION

Switched Affine Systems (SAS) have been an interesting topic of research in the past few decades (Liberzon, 2003). Their dynamics can be described as follows: there is a finite set of subsystems (or modes) and a switching law that dictates at each time instant which one is the active mode. Moreover, the dynamics of each mode are affine on the system states. Their study is interesting from a practical viewpoint since they are suitable for modeling systems in different applications, including power converters.

Stabilization of SAS is challenging due to the fact that the goal is usually to stabilize the system at an equilibrium point which is different from the equilibrium of any individual mode. In the SAS literature, some state-dependent switching control laws have been proposed for stabilization (Bolzern and Spinelli, 2004; Hetel and Bernuau, 2015; Kader et al., 2018; Egidio et al., 2022). One important feature of these methods is that control design is carried out through the solution of Linear Matrix Inequalities (LMIs), in a process that is numerically efficient when the problem is well conditioned.

Many physical systems exhibit dynamics evolving in two time scales. The difference in the time scales is usually represented by the singular perturbation parameter ε in the system model (Kokotović et al., 1999), which is consequently ill-conditioned. For switched systems, taking into account the time-scale separation between the slow and fast dynamics is a very challenging problem. Moreover, designing LMI-based control laws in this case is not trivial due to the ill conditioning of the system matrices. Singular perturbation theory is a mature tool for stability analysis and control design for both linear and nonlinear systems (Kokotović et al., 1999; Lizarraga et al., 2005; Tognetti et al., 2020), including practical applications in

power electronics (Kimball and Krein, 2008; Ghanes et al., 2014). These techniques have been extended to the case of stabilization of singularly perturbed switched linear systems in Deaecto et al. (2012). Stability criteria have equally been proposed for singularly perturbed switched systems with arbitrary switching laws in Rejeb et al. (2018); Yang et al. (2020), and for a slow switched affine system interconnected with a fast linear time invariant system in Tang et al. (2022). Recently, in de Souza et al. (2023), state-dependent control design for a subclass of singularly perturbed SAS has been addressed. To the best of the authors' knowledge, however, the design of a state-dependent controller for general singularly perturbed SAS remains an open problem.

In this paper, we propose an LMI-based control design strategy for the stabilization of singularly perturbed SAS. Moreover, an estimation of the domain of attraction is provided when only local exponential stability is guaranteed. Since, in practice, the parameter ε may not be well-known or its use in the controller may not be desirable (Tognetti et al., 2020), the proposed controller and domain of attraction do not depend explicitly on ε . Moreover, we provide an estimation of the upper bound of ε under which the stability of the closed-loop singularly SAS is ensured. Global stabilization conditions are also proposed.

The paper is structured as follows. In Section 2, we describe the system and state the problem addressed here. In Section 3, we give a generic stabilization result valid for singularly perturbed SAS. Then, in Section 4, we present the main result of the paper using the results obtained in Section 3. The proposed technique is illustrated in Section 5 on a numerical example and then conclusions are drawn in Section 6.

Notation: If v is a vector, then $v_{(j)}$ denotes its j -th component. If M is a matrix, then M^T denotes its transpose and $\text{sym}\{M\} := M + M^T$. If M is symmetric, then $M \succ 0$ (resp. $\succeq 0$) means that M is positive definite (resp. semidefinite). The notation for negative definiteness (or semidefiniteness) is analogous. Moreover, the minimum (resp. maximum) eigenvalue of M is denoted as $\lambda_{\min}(M)$ (resp. $\lambda_{\max}(M)$). The notation $\mathcal{E}(M) := \{\eta \in \mathbb{R}^n : \eta^T M \eta \leq 1\}$ is used to denote an ellipsoid in \mathbb{R}^n , and $\mathcal{B}(r^2) := \{\eta \in \mathbb{R}^n : \eta^T \eta \leq r^2\}$ denotes a ball of radius r centered at the origin. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted as I_n . The superscript i in M^i is used as an index. The symbol 0 can be used as the scalar zero or a null matrix of appropriate dimensions, depending on the context. The symbol \star in a symmetric matrix means that the corresponding sub-matrix block can be inferred from symmetry. The convex hull of a finite set \mathcal{V} is denoted as $\text{conv}\{\mathcal{V}\}$. Given a set of real values $\{f(v) : v \in \mathcal{V}\}$, let $\arg \min_{v \in \mathcal{V}} f(v) := \{v \in \mathcal{V} : f(v) \leq f(v'), \forall v' \in \mathcal{V}\}$. The interior of a region \mathcal{R} is denoted as $\text{Int}\{\mathcal{R}\}$. Given an integer N , the unit simplex in \mathbb{R}^N is denoted as $\Delta_N := \{\alpha \in [0, 1]^N : \sum_{i=1}^N \alpha_{(i)} = 1\}$. Set \mathcal{I}_N is defined as $\mathcal{I}_N := \{1, 2, \dots, N\}$. Given a domain $\mathcal{D} \subseteq \mathbb{R}^n$ and a function $V : \mathcal{D} \rightarrow \mathbb{R}$, $\partial V / \partial \eta := [\partial V / \partial \eta_{(1)} \quad \partial V / \partial \eta_{(2)} \quad \dots \quad \partial V / \partial \eta_{(n)}]$.

2. PROBLEM STATEMENT

Consider a singularly perturbed switched affine system written as follows:

$$\begin{cases} \dot{x} = A_{11}(u)x + A_{12}(u)z + B_1 u & (1a) \\ \varepsilon \dot{z} = A_{21}(u)x + A_{22}(u)z + B_2 u, & (1b) \end{cases}$$

where $x \in \mathbb{R}^{n_x}$ is the slow state, $z \in \mathbb{R}^{n_z}$ is the fast state, $\varepsilon > 0$ is a small parameter, and $u \in \mathbb{R}^m$ is a control variable that can only belong to a finite set $\mathcal{V} := \{v_1, v_2, \dots, v_N\}$. Matrices $A_{11}(u)$, $A_{12}(u)$, $A_{21}(u)$ and $A_{22}(u)$ are functions of u . The following assumption regarding \mathcal{V} is made.

Assumption 1. $\text{conv}\{\mathcal{V}\}$ has nonempty interior and $0 \in \text{Int}\{\text{conv}\{\mathcal{V}\}\}$. \square

Remark 1. System (1) can be rewritten in the more familiar form:

$$\begin{cases} \dot{x} = \tilde{A}_{11}^\sigma x + \tilde{A}_{12}^\sigma z + \tilde{b}_1^\sigma & (2a) \\ \varepsilon \dot{z} = \tilde{A}_{21}^\sigma x + \tilde{A}_{22}^\sigma z + \tilde{b}_2^\sigma, & (2b) \end{cases}$$

where $\sigma \in \mathcal{I}_N$ denotes the switching function, $\tilde{A}_{11}^i = A_{11}(v_i)$, $\tilde{A}_{12}^i = A_{12}(v_i)$, $\tilde{A}_{21}^i = A_{21}(v_i)$, $\tilde{A}_{22}^i = A_{22}(v_i)$, $\tilde{b}_1^i = B_1 v_i$ and $\tilde{b}_2^i = B_2 v_i$, $\forall i \in \mathcal{I}_N$. Moreover, under Assumption 1, it can be shown using (Hetel and Bernuau, 2015, Prop. 1) that one can always pass from the representation in (2) to the one in (1). \square

Define the full state $\eta := [x^T \quad z^T]^T \in \mathbb{R}^n$, where $n = n_x + n_z$. Let us also define $A_\varepsilon(u)$ and B_ε as:

$$A_\varepsilon(u) := \begin{bmatrix} A_{11}(u) & A_{12}(u) \\ \varepsilon^{-1} A_{21}(u) & \varepsilon^{-1} A_{22}(u) \end{bmatrix}, \quad (3)$$

$$B_\varepsilon := \begin{bmatrix} B_1 \\ \varepsilon^{-1} B_2 \end{bmatrix}. \quad (4)$$

We address in this paper the use of state-dependent switching laws of the form:

$$u(\eta) \in \arg \min_{v \in \mathcal{V}} \eta^T H v, \quad (5)$$

with H a parameter to be determined. The closed-loop system (1), (5), can be equivalently described as:

$$\dot{\eta} = A_\varepsilon(u(\eta))\eta + B_\varepsilon u(\eta). \quad (6)$$

The closed-loop system (6) is a discontinuous system (Cortes, 2008) and its solutions are considered here in the Filippov sense (Filippov, 1988). Thus, to the system (1), (5) we associate the differential inclusion

$$\dot{\eta}(t) \in \mathcal{F}(\eta(t)), \quad (7)$$

where $\mathcal{F}(\eta)$ is the set-valued map defined by

$$\mathcal{F}(\eta) = \text{conv} \left\{ A_\varepsilon(\tilde{v})\eta + B_\varepsilon \tilde{v} : \tilde{v} \in \arg \min_{v \in \mathcal{V}} \eta^T H v \right\}. \quad (8)$$

Definition 1. The trajectory $\eta(t)$ is a solution of (6) if it is absolutely continuous in the interval $[0, t_1] \subset \mathbb{R}$ for any $t_1 > 0$ and satisfies the associated differential inclusion (7), (8) for almost all $t \in [0, t_1]$. \square

Note that in the case of piecewise constant control inputs, the existence of at least one solution of (7), (8) is guaranteed. This is because the obtained set-valued map (8) is Lebesgue measurable, upper-semicontinuous, compact, non-empty and convex (see Bacciotti and Rosier (2005)).

Definition 2. The origin is an equilibrium point of (1), (5), if $0 \in \mathcal{F}(0)$. \square

The statement of Definition 2 implies that $\eta = 0$ is an equilibrium of the closed-loop system (6) if $0 \in \text{conv}\{B_\varepsilon v : v \in \mathcal{V}\}$. Note that this is implied by Assumption 1.

Definition 3. The origin of the closed-loop system (1), (5) (or equivalently the associated differential inclusion (7), (8)) is said to be *locally exponentially stable* if there exist $\mathcal{D} \subseteq \mathbb{R}^n$ with $0 \in \text{Int}\{\mathcal{D}\}$, $\kappa_1 > 0$ and $\kappa_2 > 0$, such that for all Filippov solutions $\eta(t)$ of (7), (8), with $\eta(0) \in \mathcal{D}$, $\|\eta(t)\| \leq \kappa_1 e^{-\kappa_2 t} \|\eta(0)\|$, $\forall t \geq 0$. If $\mathcal{D} = \mathbb{R}^n$, then the origin is said to be *globally exponentially stable*. \square

Sufficient conditions for the local exponential stability of systems modeled by a differential inclusion (7), (8), are given by the existence of a strict Lyapunov function $V : \mathcal{D} \rightarrow \mathbb{R}$, with $0 \in \text{Int}\{\mathcal{D}\}$, $V(0) = 0$, $V(\eta) > 0$, $\forall \eta \in \mathcal{D} \setminus \{0\}$, such that

$$\sup_{\varsigma \in \mathcal{F}(\eta)} \frac{\partial V}{\partial \eta} \varsigma < -2\chi V(\eta), \quad \forall \eta \in \mathcal{D} \setminus \{0\}, \quad (9)$$

where $\chi > 0$ is a desired decay rate. A sufficient condition for global stability is obtained if (9) holds with $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ radially unbounded.

The difficulty in designing a controller for the stabilization of (1) lies in the numerical issues induced by the small parameter ε . This may pose a challenge in the LMI-based techniques which are usually adopted in the SAS literature (such as those used, e.g., in Bolzern and Spinelli (2004)). Additionally, ε may not be well-known, or it may not be practical to use it in the control law. In this case, it is desirable that the controller does not depend on this parameter, and as a result the control design strategy must be carried out accordingly. The problem addressed in this paper is summarized as follows.

Problem. Design an LMI-based ε -independent switching law of the form (5) that ensures exponential stabilization of the origin of the singularly perturbed switched affine

system (1) (or, equivalently, (2)), while alleviating the numerical issues due to the presence of parameter ε .

3. SYSTEM STABILIZATION

In this section, we present some generic results concerning the stabilization of the closed-loop system (1), (5), as well as the estimation of the domain of attraction.

Firstly, note that, thanks to Assumption 1, we can write $\text{conv}\{\mathcal{V}\}$ as a region in \mathbb{R}^m delimited by n_h hyperplanes in the following manner:

$$\text{conv}\{\mathcal{V}\} = \{v \in \mathbb{R}^m : h_q^T v \leq 1, q = 1, \dots, n_h\}, \quad (10)$$

where $h_q \in \mathbb{R}^m, \forall q \in \mathcal{I}_{n_h}$. This representation of $\text{conv}\{\mathcal{V}\}$ will be useful throughout this paper for estimating the domain of attraction.

Consider the next lemma, which provides an important result that will be useful later on. It gives ε -dependent conditions for the stabilization of the origin of (1). This result is similar to others reported in the literature on switched systems, such as Hetel et al. (2015) and Delpoux et al. (2015). However, these works do not consider general switched affine systems, differently from the current paper.

Lemma 1. Given a parameter $\chi > 0$, assume that there exist $\rho > 0$ and matrices $Q = Q^T \succ 0$ and $Y \in \mathbb{R}^{m \times n}$ such that:

$$\text{sym}\{A_\varepsilon(v_i)Q + B_\varepsilon Y\} + 2\chi Q \prec 0, \quad \forall i \in \mathcal{I}_N, \quad (11)$$

$$\begin{bmatrix} 1 & h_q^T Y \\ \star & Q \end{bmatrix} \succ 0, \quad \forall q \in \mathcal{I}_{n_h}, \quad (12)$$

$$\begin{bmatrix} \rho I_n & I_n \\ \star & Q \end{bmatrix} \succ 0. \quad (13)$$

Then, the origin of the closed-loop system (1), (5), is locally exponentially stable by choosing $H = Q^{-1}B_\varepsilon$ in (5). In addition, $\Omega = \mathcal{E}(Q^{-1})$ is an estimation of the domain of attraction that contains the ball $\mathcal{B}(0, 1/\rho)$. \square

Proof. Using (11) and considering $P = Q^{-1}$, then:

$$2\eta^T P A_\varepsilon(v_i)\eta + 2\eta^T P B_\varepsilon K\eta + 2\chi\eta^T P\eta < 0, \forall i \in \mathcal{I}_N, \quad (14)$$

for all $\eta \in \mathbb{R}^n \setminus \{0\}$, with $K = YP$.

Define the set $\mathcal{C}_\mathcal{V}(K) := \{\eta \in \mathbb{R}^n : K\eta \in \text{conv}\{\mathcal{V}\}\}$. Using Assumption 1 and continuity arguments, it can be seen that $\mathcal{C}_\mathcal{V}(K)$ has nonempty interior. On the other hand, using the Schur complement, (12) is equivalent to $1 - h_q^T Y Q^{-1} Y^T h_q > 0, \forall q \in \mathcal{I}_{n_h}$. It can be shown (Boyd et al., 1994, Sec. 5.2.2) that this implies $\mathcal{E}(P) \subset \mathcal{C}_\mathcal{V}(K)$. Thus, there exists $\alpha(\eta) \in \Delta_N$ such that $K\eta = \sum_{j=1}^N \alpha_{(j)}(\eta)v_j$, for all $\eta \in \mathcal{E}(P)$. Substituting in (14) and taking into account the fact that $\sum_{j=1}^N \alpha_{(j)}(\eta) = 1$:

$$\sum_{j=1}^N \alpha_{(j)}(\eta) (\eta^T P A_\varepsilon(v_j)\eta + \eta^T P B_\varepsilon v_j + \chi\eta^T P\eta) < 0, \quad (15)$$

for every $i \in \mathcal{I}_N$ and for all $\eta \in \mathcal{E}(P) \setminus \{0\}$. Since $\alpha_{(j)}(\eta) \in [0, 1], \forall j \in \mathcal{I}_N$, then for each η there must be at least one minimizer v_j^* for which $\eta^T P A_\varepsilon(v_i)\eta + \eta^T P B_\varepsilon v_j^* + \chi\eta^T P\eta$ is negative. Choosing $j = j^*$ such that $v_{j^*} = \arg \min_{v \in \mathcal{V}} \eta^T P B_\varepsilon v$ then guarantees that:

$$\eta^T P A_\varepsilon(v_i)\eta + \eta^T P B_\varepsilon v_{j^*} + \chi\eta^T P\eta < 0, \quad (16)$$

for every $i \in \mathcal{I}_N$ and for all $\eta \in \mathcal{E}(P) \setminus \{0\}$. In particular, (16) is valid for $i = j^*$.

Consider the Lyapunov function $V(\eta) = \eta^T P\eta$. Since $\partial V/\partial \eta = 2\eta^T P$ and taking $i = j^*$, we obtain from (16):

$$\frac{\partial V}{\partial \eta} (A_\varepsilon(v_{j^*})\eta + B_\varepsilon v_{j^*}) < -2\chi\eta^T P\eta, \quad (17)$$

for all $\eta \in \mathcal{E}(P) \setminus \{0\}$. By considering $H = PB_\varepsilon$ and noting that $\mathcal{F}(\eta)$ is compact and convex:

$$\sup_{\varsigma \in \mathcal{F}(\eta)} \frac{\partial V}{\partial \eta} \varsigma < -2\chi V(\eta), \quad \forall \eta \in \mathcal{E}(P) \setminus \{0\}, \quad (18)$$

and therefore the origin is locally exponentially stable with decay rate χ . Since $\Omega = \mathcal{E}(P)$ is a level set of $V(\eta)$, then it can be taken as an estimation of the domain of attraction.

Finally, LMI (13) implies that $\rho\eta^T \eta - \eta^T Q^{-1}\eta > 0, \forall \eta \in \mathbb{R}^n$. Thus, $\eta \in \mathcal{B}(1/\rho) \iff \rho\eta^T \eta \leq 1 \implies \eta^T Q^{-1}\eta < 1 \iff \eta \in \Omega$. Therefore, $\mathcal{B}(1/\rho) \subseteq \Omega$. \blacksquare

Remark 2. From (11), one can see that stabilizability of the pairs $(A_\varepsilon(v_i), B_\varepsilon), i = 1, \dots, N$, is a necessary condition for applying the results of Lemma 1. \square

Variable ρ has been included in order to offer a way to maximize the size of the domain of attraction. Indeed, this can be accomplished by minimizing ρ subject to the constraints (11)-(13), $\rho > 0$, and $Q = Q^T \succ 0$, which is a standard convex optimization problem. Alternatively, one could also maximize the size of Ω by minimizing $\log \det Q^{-1}$, as in (Boyd et al., 1994, Section 5.2.2).

There are two drawbacks in directly using Lemma 1 for the design of the controller: (i) since ε can be very small and cause ill conditioning in the system matrices, the solution of (11) may suffer from numerical problems; and (ii) if ε is not well-known, Lemma 1 cannot be applied, since both LMI (11) and the controller require the knowledge of this parameter. In the next section, an ε -independent stabilizing state-dependent switching law that circumvents both of the aforementioned drawbacks is proposed.

4. MAIN RESULT

Consider the following lemma, which will be useful later in proving the main result of the paper.

Lemma 2. Consider symmetric matrices $\Psi^i, \Theta^i, \Sigma^i \in \mathbb{R}^{n \times n}, i = 1, \dots, N$, of same dimensions, having the following form¹:

$$\Psi^i = \begin{bmatrix} \Psi_1^i & \Psi_{12}^i \\ \star & \Psi_2^i \end{bmatrix}, \Theta^i = \begin{bmatrix} 0 & \Theta_{12}^i \\ \star & \Theta_2^i \end{bmatrix}, \Sigma^i = \begin{bmatrix} 0 & 0 \\ \star & \Sigma_2^i \end{bmatrix}, \quad (19)$$

where $\Sigma_2^i \prec 0, \forall i \in \mathcal{I}_N$. There exists $\mu > 0$ such that

$$\Psi^i + \varepsilon^{-1}\Theta^i + \varepsilon^{-2}\Sigma^i \prec 0, \quad \forall i \in \mathcal{I}_N, \quad (20)$$

holds for all $\varepsilon \in (0, \mu)$ if and only if $\Psi_1^i \prec 0$ and $\Theta_{12}^i = 0, \forall i \in \mathcal{I}_N$. In addition, assuming $\Psi_2^i - \Psi_{12}^i{}^T \Psi_1^i{}^{-1} \Psi_{12}^i \preceq 0, \forall i \in \mathcal{I}_N$, and defining:

$$\mu_i = \begin{cases} +\infty, & \text{if } \Theta_2^i \preceq 0, \\ \lambda_{\max}^{-1} \left((-\Sigma_2^i)^{-1/2} \Theta_2^i (-\Sigma_2^i)^{-1/2} \right), & \text{otherwise.} \end{cases} \quad (21)$$

for every $i \in \mathcal{I}_N$, then the bound μ can be estimated as $\mu = \min_{i \in \mathcal{I}_N} \mu_i$. \square

Proof. See Appendix A. \blacksquare

¹ $\Psi_1^i \in \mathbb{R}^{n_x \times n_x}, \Psi_{12}^i, \Theta_{12}^i \in \mathbb{R}^{n_x \times n_z}, \Psi_2^i, \Theta_2^i, \Sigma_2^i \in \mathbb{R}^{n_z \times n_z}$.

The next theorem presents the main result of the paper. It provides an LMI-based approach for the design of stabilizing ε -independent switching laws for the singularly perturbed switched affine system (1). The singular perturbation parameter ε does not appear in the LMIs and local exponential stabilization is shown for all $\varepsilon \in (0, \varepsilon^*)$ while both the upper bound ε^* and an estimation of the domain of attraction are provided.

Theorem 1. Consider system (1). For a given parameter $\chi > 0$, assume that there exist $\rho > 0$, $Q_1 = Q_1^T \succ 0$, $Q_2 = Q_2^T \succ 0$, $Y_1 \in \mathbb{R}^{m \times n_x}$, $\Pi_1^i = \Pi_1^{i,T} \succ 0$ and $\Pi_2^i = \Pi_2^{i,T}$, $i = 1, \dots, N$, such that the following LMI conditions are satisfied:

$$\begin{bmatrix} 1 & h_q^T Y_1 \\ \star & Q_1 \end{bmatrix} \succ 0, \quad \forall q \in \mathcal{I}_{n_h}, \quad (22)$$

$$\begin{bmatrix} \rho I_{n_x} & 0 & I_{n_x} \\ \star & \rho I_{n_z} & 0 \\ \star & \star & Q_1 \end{bmatrix} \succ 0, \quad (23)$$

and

$$\text{sym} \{A_{11}(v_i)Q_1 + B_1 Y_1\} + 2\chi Q_1 + \Pi_1^i \prec 0, \quad (24)$$

$$\text{sym} \{A_{22}(v_i)Q_2\} + \Pi_2^i \prec 0, \quad (25)$$

$$\begin{bmatrix} \Pi_1^i & A_{12}(v_i)Q_2 + Q_1 A_{21}^T(v_i) + Y_1^T B_2^T \\ \star & \Pi_2^i \end{bmatrix} \succeq 0, \quad (26)$$

for every $i \in \mathcal{I}_N$. Then, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, the origin of the closed-loop system (1), (5), is locally exponentially stable with decay rate χ by choosing H as:

$$H = \begin{bmatrix} Q_1^{-1} B_1 \\ Q_2^{-1} B_2 \end{bmatrix}. \quad (27)$$

In addition, $\Omega^* = \mathcal{E}(Q_{\varepsilon^*}^{-1})$ is an estimation of the domain of attraction, where:

$$Q_{\varepsilon^*}^{-1} = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & \varepsilon^* Q_2^{-1} \end{bmatrix}, \quad (28)$$

and $\mathcal{B}(1/\rho) \subseteq \Omega^*$. \square

Proof. Consider the following inequality:

$$\text{sym} \{A_\varepsilon(v_i)Q_\varepsilon + B_\varepsilon Y\} + 2\chi Q_\varepsilon + \Pi_\varepsilon^i \prec 0, \quad \forall i \in \mathcal{I}_N, \quad (29)$$

with $Q_\varepsilon = Q_\varepsilon^T$ and $\Pi_\varepsilon^i = \Pi_\varepsilon^{i,T}$ given by

$$Q_\varepsilon = \begin{bmatrix} Q_1 & 0 \\ 0 & \varepsilon^{-1} Q_2 \end{bmatrix}, \quad \Pi_\varepsilon^i = \begin{bmatrix} \Pi_1^i & \varepsilon^{-1} \Pi_{12}^i \\ \star & \varepsilon^{-2} \Pi_2^i \end{bmatrix}, \quad (30)$$

for every $i \in \mathcal{I}_N$, and $Y = [Y_1 \ 0]$. Moreover, note that $A_\varepsilon(v_i)$ and B_ε can be written as:

$$A_\varepsilon(v_i) = \begin{bmatrix} A_{11}(v_i) & A_{12}(v_i) \\ 0 & 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} 0 & 0 \\ A_{21}(v_i) & A_{22}(v_i) \end{bmatrix}, \quad (31)$$

$$B_\varepsilon = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \quad (32)$$

Using (31)-(32) and developing, (29) can be written in the form of expression (20) with Ψ^i , Θ^i and Σ^i having the structure shown in (19), where:

$$\begin{cases} \Psi_1^i = \text{sym} \{A_{11}(v_i)Q_1 + B_1 Y_1\} + 2\chi Q_1 + \Pi_1^i \prec 0; & (33a) \\ \Psi_{12}^i = 0; & (33b) \\ \Psi_2^i = 0; & (33c) \\ \Theta_{12}^i = A_{12}(v_i)Q_2 + Q_1 A_{21}^T(v_i) + Y_1^T B_2^T + \Pi_{12}^i; & (33d) \\ \Theta_2^i = 2\chi Q_2; & (33e) \\ \Sigma_2^i = \text{sym} \{A_{22}(v_i)Q_2\} + \Pi_2^i. & (33f) \end{cases}$$

The feasibility of (25) implies directly that $\Sigma_2^i \prec 0$ for all $i \in \mathcal{I}_N$. Thus, we can apply Lemma 2 to show that there exists $\mu > 0$ such that (29) holds for all $\varepsilon \in (0, \mu)$ iff $\Psi_1^i \prec 0$ and $\Theta_{12}^i = 0$, $\forall i \in \mathcal{I}_N$. It can be readily seen that the feasibility of (24) implies $\Psi_1^i \prec 0$. The condition $\Theta_{12}^i = 0$ is enforced by making $\Pi_{12}^i = -(A_{12}(v_i)Q_2 + Q_1 A_{21}^T(v_i) + Y_1^T B_2^T)$. Therefore, LMI (26) implies that:

$$\begin{bmatrix} \Pi_1^i & -\Pi_{12}^i \\ \star & \Pi_2^i \end{bmatrix} \succeq 0, \quad \forall i \in \mathcal{I}_N. \quad (34)$$

Applying the Schur complement on the first row and column of (34), we have (since $\Pi_1^i \succ 0$) that (34) \iff $\Pi_2^i - \Pi_{12}^{i,T} \Pi_1^{i-1} \Pi_{12}^i \succeq 0 \iff \varepsilon^{-2} \Pi_2^i - \varepsilon^{-2} \Pi_{12}^{i,T} \Pi_1^{i-1} \Pi_{12}^i \succeq 0$, which, by using once again the Schur complement, is equivalent to $\Pi_\varepsilon^i \succeq 0$ for any $i \in \mathcal{I}_N$.

Therefore, we have shown that (24)-(26) are sufficient conditions for (29) and for $\Pi_\varepsilon^i \succeq 0$, for all $i \in \mathcal{I}_N$ and for all $\varepsilon \in (0, \mu)$, for some $\mu > 0$. Since $\Pi_\varepsilon^i \succeq 0$, (29) implies that (11) holds with $Q = Q_\varepsilon^{-1}$ and $Y = [Y_1 \ 0]$. Because of this, using the result from Lemma 1, one can show that the origin of the closed-loop system (1), (5) is locally exponentially stable with decay rate χ by selecting $H = Q_\varepsilon^{-1} B_\varepsilon$, whose expression is given in (27). More specifically, (18) holds with $V(\eta) = V_\varepsilon(\eta) := \eta^T Q_\varepsilon^{-1} \eta$ and $K = Y Q_\varepsilon^{-1}$.

Moreover, exactly as in the proof of Lemma 1, $\Omega_\varepsilon := \mathcal{E}(Q_\varepsilon^{-1})$ is an estimation of the domain of attraction if $1 - h_q^T Y Q_\varepsilon^{-1} Y^T h_q > 0$, $\forall q \in \mathcal{I}_{n_h}$. Since $Y = [Y_1 \ 0]$, then this can be rewritten as $1 - h_q^T Y_1 Q_1^{-1} Y_1^T h_q > 0$, $\forall q \in \mathcal{I}_{n_h}$. Note that, using the Schur complement, this is implied by (22).

We can follow the same arguments as in the proof of Lemma 1 to show that, taking $Q = Q_\varepsilon$, then (13) implies $\mathcal{B}(1/\rho) \subset \Omega_\varepsilon$. Note that, here, LMI (13) can be written as:

$$\begin{bmatrix} \rho I_{n_x} & 0 & I_{n_x} & 0 \\ \star & \rho I_{n_z} & 0 & I_{n_z} \\ \star & \star & Q_1 & 0 \\ \star & \star & \star & \varepsilon^{-1} Q_2 \end{bmatrix} \succ 0. \quad (35)$$

Since (23) holds, then using the Schur complement, it can be shown that (35) $\iff \varepsilon^{-1} Q_2 - (1/\rho) I_{n_z} \succ 0$. This is valid for all $\varepsilon \in (0, \nu)$, with $\nu = \rho \lambda_{\min}(Q_2)$. Thus, for all $\varepsilon \in (0, \nu)$, we have that $\mathcal{B}(1/\rho) \subset \Omega_\varepsilon$.

Let $\varepsilon^* = \min\{\mu, \nu\}$, so that all results proven so far are valid for all $\varepsilon \in (0, \varepsilon^*)$. Note that $\eta \in \Omega^* \iff x^T Q_1^{-1} x + \varepsilon^* z^T Q_2^{-1} z \leq 1 \iff x^T Q_1^{-1} x + \varepsilon z^T Q_2^{-1} z \leq 1 \iff \eta \in \Omega_\varepsilon$, $\forall \varepsilon \in (0, \varepsilon^*)$, since $Q_2^{-1} \succ 0$. This implies that $\Omega^* \subseteq \bigcap_{\varepsilon \in (0, \varepsilon^*)} \Omega_\varepsilon$, and therefore Ω^* is contained in the domain of attraction. Since $\varepsilon^* \leq \nu$ and $\Omega^* = \Omega_{\varepsilon^*}$, then $\mathcal{B}(1/\rho) \subseteq \Omega^*$. This concludes the proof. \blacksquare

The bound ε^* for which Theorem 1 is valid can be estimated by simply applying Lemma 2 with the expressions for the sub-block matrices given in (33).

Remark 3. From (26) and the fact that $\Pi_1^i \succ 0$, we have that $\Pi_2^i \succeq 0$. Then, from (25), we conclude that $A_{22}(v_i)$ must be Hurwitz, for all $i \in \mathcal{I}_N$. Otherwise, Theorem 1 cannot be applied since (25) would not be feasible. \square

Remark 4. If a bound $\bar{\varepsilon}$ is known such that $\varepsilon \leq \bar{\varepsilon} < \varepsilon^*$, then a less conservative estimation of the domain of attraction can be obtained as $\bar{\Omega} = \mathcal{E}(Q_{\bar{\varepsilon}}^{-1})$. \square

The next result presents conditions under which global stabilization can be ensured.

Corollary 1. Assume that there exist matrices $Q_1 = Q_1^T \succ 0$, $Q_2 = Q_2^T \succ 0$, $\Pi_1^i = \Pi_1^{i,T} \succ 0$ and $\Pi_2^i = \Pi_2^{i,T}$, $i = 1, \dots, N$, such that LMI conditions (24)-(26) are satisfied with $Y_1 = 0$. Then, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, the origin of the closed-loop system (1), (5), (27), is globally exponentially stable with decay rate χ . \square

Proof. It has been shown in the proof of Theorem 1 that LMIs (24)-(26) imply that (18) holds with $K = [Y_1 \ 0]Q_{\varepsilon}^{-1}$. If these LMIs are satisfied with $Y_1 = 0$, then (18) holds with $K = 0$. As a result of Assumption 1, $\mathcal{C}_{\mathcal{V}}(K = 0) = \{\eta \in \mathbb{R}^n : 0 \in \text{conv}\{\mathcal{V}\}\} = \mathbb{R}^n$. Therefore, (18) holds for all $\eta \in \mathbb{R}^n \setminus \{0\}$, meaning that the stability result obtained in Theorem 1 holds globally. \blacksquare

Note that Corollary 1 presents stricter conditions with respect to Theorem 1. Indeed, it can be seen that now both $A_{11}(v_i)$ and $A_{22}(v_i)$ must be Hurwitz for every $i \in \mathcal{I}_N$ in order for the corresponding LMIs to be feasible. However, global stability of the closed-loop system is ensured.

The results presented in Theorem 1 and Corollary 1 allow us to design a controller that does not depend on the value of ε (though it must be bounded by ε^*). It can be seen in the proofs that this is accomplished by considering a Lyapunov function $V_{\varepsilon}(\eta) = \eta^T Q_{\varepsilon}^{-1} \eta$ that is in fact parameterized by ε . This means that, for any $\varepsilon \in (0, \varepsilon^*)$, there is a Lyapunov function that can be used to prove the stability results. Even if the value of ε is unknown, we know that a corresponding Lyapunov function exists as long as $\varepsilon < \varepsilon^*$.

5. NUMERICAL EXAMPLE

In this section, we illustrate our approach by means of a numerical example. Consider system (6) with:

$$A_{\varepsilon}(v_1) = \begin{bmatrix} -1 & 0 \\ 0 & -4000 \end{bmatrix}, \quad (36)$$

$$A_{\varepsilon}(v_2) = \begin{bmatrix} -1 & 1 \\ -20000 & -4000 \end{bmatrix}, \quad (37)$$

$$B_{\varepsilon} = \begin{bmatrix} -2.5 \\ 30000 \end{bmatrix}, \quad (38)$$

$$v_1 = 0.61, v_2 = -0.39, \quad (39)$$

and $u(\eta)$ given by (5).

By taking $\varepsilon = 5 \times 10^{-5}$, the system dynamics can be re-expressed as (1), with: $A_{11}(v_1) = A_{11}(v_2) = A_{21}(v_2) = -1$, $A_{12}(v_1) = A_{21}(v_1) = 0$, $A_{12}(v_2) = 1$, $A_{22}(v_i) = -0.2$, $i = 1, 2$, $B_1 = -2.5$ and $B_2 = 1.5$.

We can now apply the results of Theorem 1. By solving LMIs (23)-(26) choosing $\chi = 0.3$, we obtain: $Q_1 = 25.8$, $Q_2 = 28.2$, $Y_1 = 0.76$, $\Pi_1^1 = \Pi_1^2 = 20.7$, and $\Pi_2^1 = \Pi_2^2 = 5.63$. The bound ε^* has been estimated as $\varepsilon^* = 0.3$. Since $\varepsilon < \varepsilon^*$, we can use the controller proposed in Theorem 1.

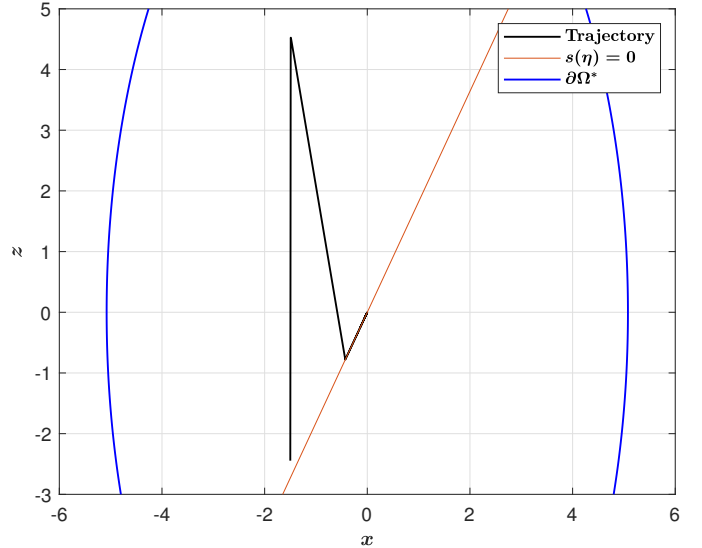


Fig. 1. Trajectory of the system in the example.

Fig. 1 shows the simulation results. In Fig. 1, the border of the estimation of the domain of attraction ($\partial\Omega^*$) is also shown. It can be seen in Fig. 1 that the trajectory indeed stabilizes at the desired equilibrium point. The switching surface $s(\eta) := \eta^T H(v_1 - v_2) = 0$ is also shown in Fig. 1. The expression for $s(\eta)$ is obtained from (5), with H determined using (27).

It is worth noting in Fig. 1 that, at first, the trajectory behaves as a singularly perturbed system: the variation of the fast variable z is large in comparison with that of the slow variable x . However, the trajectory eventually reaches the surface $s(\eta) = 0$. Then, the trajectory slides on this surface until converging to the origin. While switching occurs on this surface, the separation between the slow and fast dynamics is not observed. However, before attaining the switching surface, the difference between the two time scales is evident. It is important to remark that the same controller (and hence the same switching surface) can be used to stabilize the system for any $\varepsilon \in (0, \varepsilon^*)$, which ensures some robustness with respect to this parameter.

6. CONCLUSION

In this paper, we have proposed an ε -independent switching controller which ensures the stabilization of a singularly perturbed SAS with a given imposed decay rate. Moreover, an estimation of the domain of attraction is also provided and conditions are presented for global stabilization.

This work can be viewed as a preliminary step in the stabilization of switched affine systems from a singular perturbation perspective. The end goal is to apply to the systems addressed here the very useful tools in classical singular perturbations theory, most notably model reduction, in which the control signal is based only on the slow states. Conditions for achieving this are the object of further investigation.

We also hope in future works to relax some of the LMI conditions stated here in order to enlarge the possible applications and also obtain less conservative results. Furthermore, the maximization of the upper bound ε^* may

also be another interesting result to pursue. A way of dealing with the growth in the number of LMIs when many switching modes are present is also an important point worth looking into.

REFERENCES

- Bacciotti, A. and Rosier, L. (2005). *Liapunov functions and stability in control theory*. Springer Science & Business Media.
- Bolzern, P. and Spinelli, W. (2004). Quadratic stabilization of a switched affine system about a nonequilibrium point. In *Proceedings of the 2004 American Control Conference*, 3890–3895. Boston, MA, USA.
- Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*. SIAM.
- Cortes, J. (2008). Discontinuous dynamical systems. *IEEE Control Systems Magazine*, 28(3), 36–73.
- de Souza, R.P.C., Kader, Z., and Caux, S. (2023). Stabilization of a class of singularly perturbed switched systems. In *2023 American Control Conference (ACC)*. San Diego, CA, USA. Accepted for publication.
- Deaecto, G.S., Daafouz, J., and Geromel, J.C. (2012). \mathcal{H}_2 and \mathcal{H}_∞ performance optimization of singularly perturbed switched systems. *SIAM Journal on Control and Optimization*, 50(3), 1597–1615.
- Delpoux, R., Hetel, L., and Kruszewski, A. (2015). Parameter-dependent relay control: Application to PMSM. *IEEE Transactions on Control Systems Technology*, 23(4), 1628–1637.
- Egidio, L.N., Deaecto, G.S., and Jungers, R.M. (2022). Stabilization of rank-deficient continuous-time switched affine systems. *Automatica*, 143, 110426.
- Filippov, A.F. (1988). *Differential equations with discontinuous righthand sides*. Springer Science & Business Media.
- Ghanes, M., Bethoux, O., Hilaire, M., and Barbot, J.P. (2014). Fuel cell system control under converter losses with experimental results. *IFAC Proceedings Volumes*, 47(3), 8588–8593.
- Hetel, L. and Bernuau, E. (2015). Local stabilization of switched affine systems. *IEEE Transactions on Automatic Control*, 60(4), 1158–1163.
- Hetel, L., Fridman, E., and Floquet, T. (2015). Variable structure control with generalized relays: A simple convex optimization approach. *IEEE Transactions on Automatic Control*, 60(2), 497–502.
- Kader, Z., Fiter, C., Hetel, L., and Belkoura, L. (2018). Stabilization of switched affine systems with disturbed state-dependent switching laws. *International Journal of Robust and Nonlinear Control*, 28(2), 582–595.
- Kimball, J.W. and Krein, P.T. (2008). Singular perturbation theory for DC–DC converters and application to PFC converters. *IEEE Transactions on Power Electronics*, 23(6), 2970–2981.
- Kokotović, P., Khalil, H.K., and O’Reilly, J. (1999). *Singular perturbation methods in control: analysis and design*. SIAM.
- Liberzon, D. (2003). *Switching in systems and control*. Springer Science & Business Media.
- Lizarraga, I., Tarbouriech, S., and Garcia, G. (2005). Control of singularly perturbed systems under actuator saturation. *IFAC Proceedings Volumes*, 38(1), 243–248.
- Rejeb, J.B., Morărescu, I.C., Girard, A., and Daafouz, J. (2018). Stability analysis of a general class of singularly perturbed linear hybrid systems. *Automatica*, 90, 98–108.
- Tang, Y., Fiter, C., and Hetel, L. (2022). About switched affine system interconnected with fast LTI dynamics. *IFAC-PapersOnLine*, 55(25), 67–72.
- Tognetti, E.S., Calliero, T.R., Morărescu, I.C., and Daafouz, J. (2020). LMI-based output feedback control of singularly perturbed systems with guaranteed cost. In *2020 59th IEEE Conference on Decision and Control (CDC)*, 4566–4571. Jeju, South Korea.
- Yang, W., Wang, Y.W., Wen, C., and Daafouz, J. (2020). Exponential stability of singularly perturbed switched systems with all modes being unstable. *Automatica*, 113, 108800.

Appendix A. PROOF OF LEMMA 2

From (20), we have that, for each $i \in \mathcal{I}_N$:

$$\Psi^i + \varepsilon^{-1} \begin{bmatrix} 0 & \Theta_{12}^i \\ \star & \varepsilon^{-1}\Sigma_2^i + \Theta_2^i \end{bmatrix} \prec 0. \quad (\text{A.1})$$

Since $\Sigma_2^i \prec 0$, we know that there exists $\mu_{i,1} > 0$ such that $\varepsilon^{-1}\Sigma_2^i + \Theta_2^i \prec 0$ holds for all $\varepsilon \in (0, \mu_{i,1})$. Indeed, if $\Theta_2^i \preceq 0$, then $\mu_{i,1} = +\infty$. Otherwise, $\mu_{i,1} = 1/\lambda_{\max}((-\Sigma_2^i)^{-1/2}\Theta_2^i(-\Sigma_2^i)^{-1/2})$.

According to (Deaecto et al., 2012, Lem. 1), there exists $\mu_{i,2} > 0$ such that (A.1) holds for all $\varepsilon \in (0, \mu_{i,2})$ iff $\Psi_1^i \prec 0$, $\Theta_{12}^i = 0$ and $\varepsilon^{-1}\Sigma_2^i + \Theta_2^i \prec 0$, the latter being valid for all $\varepsilon \in (0, \mu_{i,1})$. Therefore, (A.1) holds for all $\varepsilon \in (0, \mu_i)$ with $\mu_i = \min\{\mu_{i,1}, \mu_{i,2}\}$.

The estimation of $\mu_{i,2}$ is carried out as follows. First, rewrite (A.1) as:

$$\begin{bmatrix} \Psi_1^i & \Psi_{12}^i \\ \star & \Psi_2^i + \varepsilon^{-2}\Sigma_2^i + \varepsilon^{-1}\Theta_2^i \end{bmatrix} \prec 0, \quad (\text{A.2})$$

which, using the Schur complement (given that Ψ_1^i is nonsingular), is equivalent to $\Psi_2^i + \varepsilon^{-2}\Sigma_2^i + \varepsilon^{-1}\Theta_2^i - \Psi_{12}^i{}^T \Psi_1^{i-1} \Psi_{12}^i \prec 0$ or, equivalently:

$$-\varepsilon^{-1}(\varepsilon^{-1}\Sigma_2^i + \Theta_2^i) \succ \Psi_2^i - \Psi_{12}^i{}^T \Psi_1^{i-1} \Psi_{12}^i. \quad (\text{A.3})$$

Since $\Psi_2^i - \Psi_{12}^i{}^T \Psi_1^{i-1} \Psi_{12}^i \preceq 0$, then clearly $\mu_{i,2} = +\infty$. Therefore, $\mu_i = \min\{\mu_{i,1}, \mu_{i,2}\} = \mu_{i,1}$, $\forall i \in \mathcal{I}_N$, and thus μ_i can be written as in (21). The bound μ can be taken as the most conservative one among the bounds for each $i \in \mathcal{I}_N$. Thus, $\mu = \min_{i \in \mathcal{I}_N} \mu_i$.