# Approximation of reflected SDEs in non-smooth time-dependent domains and application to fully nonlinear PDEs with Neumann boundary condition on time-dependent domains

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# Approximation of reflected SDEs in non-smooth time-dependent domains and application to fully nonlinear PDEs with Neumann boundary condition on time-dependent domains

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We consider a class of reflected SDEs in non-smooth time-dependent domains with time sections that are increasing with time. We provide a strong approximation for this type of equations using a sequence of standard SDEs. As a consequence, we obtain an approximation scheme for the associated generalized BSDEs in this markovian setting using standard BSDEs. As a by-product, we get an approximation by a sequence of standard PDEs for the solution of a system of PDEs with nonlinear boundary conditions on time-dependent domains.

**Keywords:** Reflected SDEs, Time-dependent domains, Generalized BSDEs, Viscosity solution of PDEs, Nonlinear Neumann boundary conditions.
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## 1 Introduction

Let $d \geq 1$, $T > 0$ be fixed, and let $D'$ be a bounded open connected subset of $\mathbb{R}^{1+d}$. We will refer to $D = D' \cap \big([0,T] \times \mathbb{R}^d\big)$, as a time-dependent domain. For $t \in [0,T]$, the time sections of $D$ are defined by $D_t = \{x : (t,x) \in D\}$ and are assumed to be convex and increasing in time. We deal with the normally reflected SDE in time-dependent domain of the following form: $\forall t \leqslant T$,

$$\begin{cases} X_t = x + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dW_r + \int_0^t \vec{\mathbf{n}}(r,X_r)d|\Lambda|_r; \\ X_t \in \bar{D}_t, \quad |\Lambda|_t = \int_0^t \chi_{\{X_r \in \partial D_r\}} d|\Lambda|_r < \infty. \end{cases} \tag{1.1}$$

This class of reflected SDEs has been introduced by Costantini-Gobet-El Karoui [5] for smooth time-dependent domains. Then, in the case of SDEs with oblique reflection in non-smooth time-dependent domains, existence of weak solutions has been established by Nyström-Önskog [12]. This result has been generalized by Lundström-Önskog [9], the authors proved the existence and uniqueness of a strong solution for obliquely reflected SDEs in non-smooth time-dependent domains. In the previous papers, the results of existence and uniqueness are obtained mainly by solving Skorohod problem.
The aim of this work is to provide an approximation scheme for the reflected SDE (1.1) using standard SDEs. This problem has been considered by Lions-Menaldi-Sznitman [8] and by Menaldi [10] where an approximation is given in convex time-independent domains when $b$ and $\sigma$ are Lipschitz. The same domain was considered by Bahlali-Maticiuc-Zalinescu [1] when the coefficients are only measurable, the authors established a weak convergence result. In general time-independent domains, Ren-Wu [15] extended these results to the case where the domain may have corners.

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In this paper, we are able to extend similar results of convergence to the case of not necessarily smooth time-dependent domains whose time sections are increasing with time which constitutes the main result of this paper. As an application, we consider the following generalized backward stochastic differential equations (BSDE for short): $\forall t \leqslant s \leqslant T$,

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T \psi(r, X_r^{t,x}, Y_r^{t,x})d|\Lambda^{t,x}|_r - \int_s^T Z_r^{t,x} dW_r, \tag{1.2}$$

This class of BSDEs has been first introduced and studied by Pardoux-Zhang [14]. Moreover, the authors showed that if $(X^{t,x}, \Lambda^{t,x})$ is the solution of a normally reflected SDE in a smooth time-independent domain, then the solution of (1.2) provides a probabilistic formula for $u$ which is a solution of a system of PDEs with nonlinear Neumann boundary conditions on smooth time-independent domain. Based on this connection between BSDEs and PDEs, and the approximation by standard SDEs of reflected SDEs in regular convex time-independent domains, many authors studied the convergence in the $S$-topology of the approximation of the generalized BSDE (1.2) associated with a reflected SDE in a regular convex when the driver function $f$ does not depend on $z$. Then, they give an approximation (by standard PDEs) for the associated PDE with nonlinear Neumann boundary conditions on smooth time-independent domain (see e.g. [4, 1, 2]). A recent paper by Bahlali-Boufoussi-Mouchtabih [3] treats the case where the driver function is allowed to depend on the variable $z$. Moreover, the authors provide a strong approximation for the generalized BSDE using a sequence of standard BSDEs. Then they use it to obtain, once again, the approximation by standard PDEs for a PDE with nonlinear boundary condition on time-independent regular convex.
Using the approximation for the normally reflected diffusion (1.1) that we give in the first part of this paper, we get an approximation for the associated generalized BSDE (1.2) when randomness stems from (1.1), using a sequence of standard BSDEs. As we know that (see Jakani [6]), the solution of the generalized BSDE (1.2) provides a solution for the following PDE with nonlinear Neumann boundary condition on the time-dependent domain $\overline{D}' \cap \big([0,T] \times \mathbb{R}^d\big)$: $\forall i = 1, ..., m$,

$$\begin{cases} \partial_t u_i(t,x) + \mathscr{L} u_i(t,x) + f_i(t,x,u(t,x), \sigma^\top(t,x) D_x u_i(t,x)) = 0, \ (t,x) \in D' \cap \big([0,T) \times \mathbb{R}^d\big); \\ \dfrac{\partial u_i}{\partial \vec{\mathbf{n}}}(t,x) + \psi_i(t,x,u(t,x)) = 0, \ (t,x) \in \big(\overline{D}' \setminus D'\big) \cap \big([0,T) \times \mathbb{R}^d\big); \\ u(T,x) = h(x), \ x \in \overline{D}_T, \end{cases} \tag{1.3}$$

where the operator $\mathscr{L}$ is defined by $\mathscr{L} = \frac{1}{2} Tr(\sigma\sigma^\top) D^2_{xx}(.) + b^\top D_x(.)$ and at a point $(t,x) \in \partial D$ we set $\frac{\partial}{\partial \vec{\mathbf{n}}} = \langle \vec{\mathbf{n}}(t,x), D_x(.) \rangle$. We then obtain an approximation for the PDE with nonlinear Neumann boundary condition on time-dependent domain (1.3) by a sequence of standard PDEs defined on $[0,T] \times \mathbb{R}^d$.
The paper is organized as follows: In Section 2, we define the geometry of the domain considered in this paper and we collect some existing results on reflected SDEs in time-dependent domains. Section 3 is devoted to the results of convergence, we propose a sequence of standard SDEs for which we establish a priori estimates and we show that it converges to the solution of the reflected SDE in the time-dependent domain $D$. Then, in Section 4, we use the approximation of reflected SDEs in a smooth time-dependent domain to provide an approximation for both generalized BSDEs associated with reflected diffusions in time-dependent domains and PDEs with normal derivative and nonlinear Neumann boundary condition defined on time-dependent domains.

# 2 Preliminaries and formulation of the problem

## 2.1 Geometry of the time-dependent domain

Let $d \geq 1$, $T > 0$ be fixed, we follow the notation of [9] and we let $D'$ be a bounded open connected subset of $\mathbb{R}^{1+d}$. We will refer to $D = D' \cap \big([0,T] \times \mathbb{R}^d\big)$, as a time-dependent domain. Given $D$ and $t \in [0,T]$, we define the time sections of $D$ as $D_t = \{x : (t,x) \in D\}$. We assume that:

$$D_t \neq \varnothing, \text{ and that } D_t \text{ is open and convex for every } t \in [0,T]. \tag{2.1}$$

Our basic assumption is the following:

$$D_t \subset D_{t'}, \text{ whenever } t \leq t', \ t,t' \in [0,T]. \tag{2.2}$$

Let $t \in [0,T]$, the boundary of $D_t$ will be denoted $\partial D_t$. Then, define $N(t,x)$ the cone of unit inward normal vectors at a boundary point $x \in \partial D_t$ which is nonempty thanks to assumption (2.1). Note that under this assumption, the domain may have corners which does not rule out the possibility of several unit inward normal vectors at the same boundary point.
Now, let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^d$ and $|x| = \langle x,x \rangle^{1/2}$ be the euclidean norm of $x \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $r > 0$, let $B(x,r)$ and $S(x,r)$ denote the ball and sphere of radius $r$, centered at $x$, respectively, i.e. $B(x,r) = \left\{ y \in \mathbb{R}^d : |x-y| < r \right\}$ and $S(x,r) = \left\{ y \in \mathbb{R}^d : |x-y| = r \right\}$. We assume that there exists a radius $r_0 > 0$ such that the exterior sphere condition holds for all time-sections of $D$. This implies that for any $t \in [0,T]$, we have:

$$B(x - r_0 \vec{\mathbf{n}}(t,x), r_0) \subset D_t^c, \tag{2.3}$$

whenever $x \in \partial D_t$ and $\vec{\mathbf{n}}(t,x) \in N(t,x)$. This is equivalent to say that

$$\langle y-x, \vec{\mathbf{n}}(t,x) \rangle + \frac{1}{2r_0}|y-x|^2 \geqslant 0, \forall x \in \partial D_t, \, y \in \overline{D}_t, \tag{2.4}$$

whenever $\vec{\mathbf{n}}(t,x) \in N(t,x)$ for $t \in [0,T]$.
Unless otherwise stated, we fix $\vec{\mathbf{n}}(t,x)$ in $N(t,x)$ and we assume that:

$$\vec{\mathbf{n}} \in \mathscr{C}_b^{1,2}(\mathbb{R}^{1+d}, \{0,1\}), \tag{2.5}$$

where $\mathscr{C}_b^{1,2}(\mathbb{R}^{1+d}, \{0,1\})$ denotes the space of bounded functions that are continuously differentiable once with respect to the time variable and twice with respect to the space variable and having bounded derivatives.
Then, let us recall the temporal variation of the domain $d(t,x) := \inf\limits_{y \in D_t} |y-x|$, $\forall t \in [0,T]$, $x \in \mathbb{R}^d$ that is assumed to satisfy for some $p \in (1,\infty)$,

$$d(.,x) \in \mathscr{W}^{1,p}([0,T],[0,\infty)), \tag{2.6}$$

for all $x \in \mathbb{R}^d$, where $\mathscr{W}^{1,p}([0,T],[0,\infty))$ denotes the Sobolev space of functions whose first order weak derivatives belong to $L^p([0,T])$ with Sobolev norm uniformly bounded in space and such that the first weak derivative $\partial_t d(t,x)$ is jointly measurable in $(t,x)$.

**Remark 2.1** *Thanks to assumption (2.1), the exterior cone condition assumed in [9] which is weaker than the uniform sphere condition (2.3) holds, i.e, there exists a constant $\rho \in (0,1)$ such that*

$$\bigcup_{0 \leqslant \xi \leqslant \rho} B(x - \xi \vec{\boldsymbol{n}}(t,x), \xi\rho) \subset D_t^c, \forall t \in [0,T], x \in \partial D_t. \tag{2.7}$$

*It follows, that the interior cone condition is satisfied as well:*

$$\bigcup_{0 \leqslant \xi \leqslant \rho} B(x + \xi \vec{\boldsymbol{n}}(t,x), \xi\rho) \subset \overline{D}_t, \forall t \in [0,T], x \in \partial D_t. \tag{2.8}$$

*Moreover, by Remark 2.2 in [9], there exists $\hat{\alpha} = 1 - 1/p \in (0,1)$ and $K \in (0,\infty)$ such that for all $s,t \in [0,T]$, $x \in \mathbb{R}^d$,*

$$|d(s,x) - d(t,x)| \leq K|s-t|^{\hat{\alpha}}. \tag{2.9}$$

*Recall the function $l(r) = \sup\limits_{\substack{s,t \in [0,T] \\ |s-t| \leqslant r}} \sup\limits_{x \in \bar{D}_s} \inf\limits_{y \in \bar{D}_t} |x-y|$ introduced in [5], then by Remark 2.4 in [9], the condition (2.9) is equivalent to $l(r) \leqslant K r^{\hat{\alpha}}$.*

Now, recall that for any $t \in [0,T]$, $D_t$ is convex. Then for each $t \in [0,T]$, we have for all $y \in \mathbb{R}^d \setminus \bar{D}_t$ at least one projection of $y$ onto $\partial D_t$ along $N(t,\cdot)$ denoted $\pi(t,y)$ which satisfies:

$$d(t,y) = d(y,D_t) = |y - \pi(t,y)| \text{ and } \pi(t,y) = y, \ \forall y \in \bar{D}_t.$$

Let $D$ be a time-dependent domain satisfying (2.1) and (2.2), then we have the following lemma which is an adaptation of Lemma 3.1 in [11] to our case:

**Lemma 2.1** *For any $t \in [0,T]$, and $y, y' \in \mathbb{R}^d$, we have:*

*(a)* $\langle y' - y, y - \pi(t,y)\rangle \leq 0$, $y' \in \bar{D}_t$;

*(b)* $\langle y' - y, y - \pi(t,y)\rangle \leq \langle y' - \pi(t,y'), y - \pi(t,y)\rangle$, $y' \in \bar{D}_t$;

*(c)* $|\pi(t,y) - \pi(t,y')| \leq |y - y'|$.

*Furthermore, there exists $P_0 \in D_0$ and $1 \leq \gamma < \infty$ depending on $d(P_0, \partial D_0)$ such that:*

*(d)* $\langle y - P_0, y - \pi(t,y)\rangle \geq \frac{1}{\gamma}|y - \pi(t,y)|$, *for any $y \in \mathbb{R}^d$ and $t \in [0,T]$.*

## 2.2 Reflected SDEs in time-dependent domain

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a fixed probability space on which is defined an $n$-dimensional Brownian motion $W = (W_t)_{0 \leqslant t \leqslant T}$, where $\mathscr{F} = (\mathscr{F}_t)_{0 \leq t \leq T}$ is the completed filtration of $(\sigma(W_s, 0 \leq s \leq t))_{t \leq T}$ with all $\mathbb{P}-$null sets of $\mathscr{F}$. Let $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^n$ be two continuous functions. Now, let $D$ be a time-dependent domain satisfying (2.1), (2.5) and (2.6). Then, let us recall the notion of reflected SDEs in the time-dependent domain $D$ as well as some related results. We start by recalling the definition of the solution:

**Definition 2.1 (Lundström-Önskog [9])** *A strong solution of the reflected SDE in $\bar{D}$ driven by $W$ and with coefficients $b$ and $\sigma$, direction of reflection along $\vec{\boldsymbol{n}}$ and initial condition $x \in \bar{D}_0$ is an $\mathscr{F}_t$-adapted stochastic process $X_t$ which satisfies $\mathbb{P}$-almost surely, whenever $t \in [0,T]$,*

$$\begin{cases} X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \int_0^t \vec{\boldsymbol{n}}(r, X_r)d|\Lambda|_r; \\ X_t \in \bar{D}_t, \quad |\Lambda|_t = \int_0^t \chi_{\{X_r \in \partial D_r\}} d|\Lambda|_r < \infty. \end{cases} \tag{2.10}$$

We introduce the following assumptions:

(a) The functions $b$ and $\sigma$ are Lipschitz continuous with respect to $x$, i.e., there exists a positive constant $C$ such that

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq C|x - x'|, \ \ \forall (t,x,x') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.11}$$

(b) The functions $b$ and $\sigma$ are of linear growth in $(t,x)$, i.e., there exists a positive constant $C$ such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \ \ \forall (t,x) \in [0,T] \times \mathbb{R}^d. \tag{2.12}$$

Next, we recall the following result of existence and uniqueness of the solution of the reflected SDE (2.10) under the above geometric setting:

**Theorem 2.1 (Lundström-Önskog [9])** *Under assumption (a), the reflected SDE (2.10) has a unique strong solution.*

Moreover, it has been shown in [6] that the solution satisfies the following properties:

**Proposition 2.1** *There exists a constant $C$ such that for all $x, x' \in \overline{D}_0$,*

$$\mathbb{E}[\sup_{0 \leqslant t \leqslant T} |X_t^x - X_t^{x'}|^4 + ||\Lambda^x|_t - |\Lambda^{x'}|_t|^4] \leq C|x - x'|^4. \tag{2.13}$$

*Moreover, for each $\mu > 0$, $t \in [0,T]$, there exists $C(\mu,t)$ such that for all $x \in \overline{D}_0$,*

$$\mathbb{E}[e^{\mu|\Lambda^x|_t}] \leq C(\mu,t). \tag{2.14}$$

# 3 Approximation of reflected SDEs in time-dependent domains

Let $D$ be a time-dependent domain satisfying (2.1)-(2.6). From now on we assume that the functions $b$ and $\sigma$ satisfy assumptions (a) and (b). Let $x \in \bar{D}_0$, we introduce the following penalized SDE: $\forall n \geq 1$, $\forall t \in [0,T]$,

$$X_t^n = x + \int_0^t b(r, X_r^n)dr + \int_0^t \sigma(r, X_r^n)dW_r - n\int_0^t (X_r^n - \pi(r, X_r^n))dr. \tag{3.1}$$

Note that if $X_t^n \notin \bar{D}_t$, the vector $-\frac{X_t^n - \pi(t, X_t^n)}{|X_t^n - \pi(t, X_t^n)|}$ is an element of $N(t, \pi(t, X_t^n))$. Then the penalized SDE (3.1) can be written as:

$$X_t^n = x + \int_0^t b(r, X_r^n)dr + \int_0^t \sigma(r, X_r^n)dW_r + \Lambda_t^n,$$

where $\Lambda^n$ and $|\Lambda^n|$ are given by:

$$\Lambda_t^n = \int_0^t n(X_r^n - \pi(r, X_r^n))dr,$$

and

$$|\Lambda^n|_t = \int_0^t n|X_r^n - \pi(r, X_r^n)|dr = \int_0^t n|X_r^n - \pi(r, X_r^n)|dr = \int_0^t n d(r, X_r^n)dr.$$

## 3.1 A priori estimates

**Proposition 3.1** *Under assumptions* (a) *and* (b)*, for any* $q \geq 1$, $\forall t \leq T$*, we have:*

$$\sup_{n\geq 1} \mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t^n|^{2q} + \sup_{0\leq t\leq T} |\Lambda^n|_t^q\Big] < \infty. \tag{3.2}$$

PROOF. From Itô's formula, we have: $\forall t \in [0,T]$,

$$\begin{aligned} |X_t^n - P_0|^2 + 2n\int_0^t \langle X_r^n - P_0, X_r^n - \pi(r, X_r^n)\rangle dr &= |x - P_0|^2 + \int_0^t |\sigma(r, X_r^n)|^2 dr \\ &+ 2\int_0^t \langle X_r^n - P_0, b(r, X_r^n)\rangle dr + 2\int_0^t \langle X_r^n - P_0, \sigma(r, X_r^n)\rangle dW_r. \end{aligned}$$

Recall Lemma 2.1 -$(d)$, then there exists $P_0 \in D_0$ such that for some $1 \leqslant \gamma < \infty$, we have:

$$\langle X_r^n - P_0, X_r^n - \pi(r, X_r^n)\rangle \geq \frac{1}{\gamma}|X_r^n - \pi(r, X_r^n)|. \tag{3.3}$$

This implies that:

$$2n\int_0^t \langle X_r^n - P_0, X_r^n - \pi(r, X_r^n)\rangle dr \geq 2n\frac{1}{\gamma}\int_0^t |X_r^n - \pi(r, X_r^n)|dr.$$

Besides, from assumption (a), we deduce that for some constant $M > 0$ depending on the Lipschitz constant of $b$ and $\sigma$, we have:

$$|b(r, X_r^n)| \leq |b(r, P_0)| + M|X_r^n - P_0|, \tag{3.4}$$

$$|\sigma(r, X_r^n)| \leq |\sigma(r, P_0)| + M|X_r^n - P_0|. \tag{3.5}$$

Therefore,

$$\begin{aligned} |X_t^n - P_0|^2 + 2n\frac{1}{\gamma}\int_0^t |X_r^n - \pi(r, X_r^n)|dr &\leq |x - P_0|^2 + (M^2 + M + 1)\int_0^t |X_r^n - P_0|^2 dr \\ &+ \int_0^t |b(r, P_0)|^2 dr + \int_0^t |\sigma(r, P_0)|^2 dr + 2\int_0^t \langle X_r^n - P_0, \sigma(r, X_r^n)\rangle dW_r. \end{aligned}$$

Then from the boundedness of $\bar{D}_0$ and thanks to assumption (b), we get:

$$|X_t^n - P_0|^2 + 2n\frac{1}{\gamma}\int_0^t |X_r^n - \pi(r, X_r^n)|dr \leq C_{D_0,\sigma,b,x} + C\int_0^t |X_r^n - P_0|^2 dr + 2\int_0^t \langle X_r^n - P_0, \sigma(r, X_r^n)\rangle dW_r,$$

and we have for $q \geq 1$,

$$|X_t^n - P_0|^{2q} + \big(2n\frac{1}{\gamma}\int_0^t |X_r^n - \pi(r, X_r^n)|dr\big)^q \leq C_{D_0,\sigma,b,x} + C\int_0^t |X_r^n - P_0|^{2q} dr + C\Big|\int_0^t \langle X_r^n - P_0, \sigma(r, X_r^n)\rangle dW_r\Big|^q. \tag{3.6}$$

First, let us examine the term $|X_t^n - P_0|^{2q}$. By taking the supremum over $[0,t]$ and the expectation, we get:

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |X_r^n - P_0|^{2q}\Big] \leq C_{D_0,\sigma,b,x} + C\mathbb{E}\Big[\int_0^r \sup_{0\leq u\leq r} |X_u^n - P_0|^{2q} du\Big] + C\mathbb{E}\Big[\sup_{0\leq r\leq t}\Big|\int_0^r \langle X_u^n - P_0, \sigma(u, X_u^n)\rangle dW_u\Big|^q\Big]. \tag{3.7}$$

From BDG inequality, it follows that:

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq r\leq t}\Big|\int_0^r \langle X_u^n - P_0, \sigma(u, X_u^n)\rangle dW_u\Big|^q\Big] &\leq C\mathbb{E}\Big[\Big|\int_0^t |X_r^n - P_0|^2 |\sigma(r, X_r^n)|^2 dr\Big|^{\frac{q}{2}}\Big] \\ &\leq C\mathbb{E}\Big[\Big|\int_0^t \big(|X_r^n - P_0|^4 + |\sigma(r, X_r^n)|^4\big) dr\Big|^{\frac{q}{2}}\Big] \\ &\leq C\mathbb{E}\Big[\int_0^t |X_r^n - P_0|^{2q} |dr + \int_0^t |\sigma(r, X_r^n)|^{2q} |dr\Big].\end{aligned}$$

Again, using (3.5), we deduce from (3.7) that:

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |X_r^n - P_0|^{2q}\Big] \leq C_{D_0,\sigma,b,x} + C\int_0^t \mathbb{E}\Big[\sup_{0\leq u\leq r} |X_u^n - P_0|^{2q}\Big] dr.$$

Finally, we apply Gronwall's lemma and we obtain: $\forall q \geq 1$,

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |X_r^n - P_0|^{2q}\Big] \leq C, \quad \forall t \leq T,$$

which implies that

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t^n|^{2q}\Big] \leq C, \forall n \geq 1.$$

Moreover, with the use of the latter estimate and taking into account (3.6), it follows that: $\forall q \geq 1$,

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |\Lambda^n|_t^q\Big] = \mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\int_0^t n|X_r^n - \pi(r, X_r^n)|dr\Big)^q\Big] \leq \frac{1}{2}\gamma C, \forall n \geq 1.$$

## 3.2 Uniform control of the distance

In this part, we are interested in the uniform control of $d(t, X_t^n)$. Note that assumptions (2.1) and (2.6) do not ensure the smoothness of the boundary. Inspired by [11], we use a smooth approximation of $D$ that allows to apply Itô's formula with a function involving the distance. More precisely, we recall the following lemma from the same work:

**Lemma 3.1** *Let $\varepsilon > 0$, there exists a $\mathscr{C}^\infty$-smooth time-dependent domain $D_\varepsilon \subset D'$ satisfying (2.1) and (2.2) such that:*

$$h(D_t, D_{\varepsilon,t}) < \varepsilon, \forall t \in [0,T], \tag{3.8}$$

*where $h$ stands for the Hausdorff distance, which is defined by:*

$$h(F,G) = \max(\sup\{d(y,F); y \in G\}, \sup\{d(y,G); y \in F\})$$

*for any two sets $F$ and $G$ of $\mathbb{R}^d$.*

Thanks to Lemma 3.1, we deduce that the cone of unit inward normal vectors at each boundary point of $D_\varepsilon$ is reduced to a unique vector that we denote $\vec{\mathbf{n}}_\varepsilon$. Then for $t \in [0,T]$ and $x \in \mathbb{R}^d \setminus \bar{D}_{\varepsilon,t}$ the projection of $x$ along $\vec{\mathbf{n}}_\varepsilon(t,x)$ will be denoted $\pi_\varepsilon(t,y)$. Note that $\pi_\varepsilon$ satisfies:

$$\pi_\varepsilon(t,y) = y, \ \forall y \in D_{\varepsilon,t}, \forall t \in [0,T],$$

and

$$d_\varepsilon(t,y) = d(y, D_{\varepsilon,t}) = |y - \pi_\varepsilon(t,y)|, \forall y \in \mathbb{R}^d, \forall t \in [0,T].$$

**Proposition 3.2** *Let $(X^n)_{n\geqslant 1}$ be the unique solution of the penalized SDE (3.1). Then for any $p > 2$, there exists $c > 0$ such that $\forall n \geqslant 1$, we have:*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} d(t, X_t^n)^p\Big] \leqslant \frac{c}{n^{\frac{p-2}{2}}}, \tag{3.9}$$

*and*

$$\mathbb{E}\Big[\int_0^T d(t, X_t^n)^p dt\Big] \leqslant \frac{c}{n^{\frac{p}{2}}}. \tag{3.10}$$

First we shall recall some properties of the projection $\pi_\varepsilon$ as stated in the following lemma which is borrowed from [11].

**Lemma 3.2** *Let $D_\varepsilon$ be a smooth approximation of $D$ satisfying (3.8). Then, there exists a constant $c \geq 0$, such that, if $\varepsilon \in (0,1)$, $y \in \mathbb{R}^d$ and $t \in [0,T]$, we have:*

*(i)* $|\pi(t,y) - \pi_\varepsilon(t,y)| \leq c \min(\sqrt{\varepsilon^2 + \varepsilon d_\varepsilon(y,t)}; \sqrt{\varepsilon}(1 + d_\varepsilon(y,t)); \sqrt{\varepsilon^2 + \varepsilon d(y,t)})$,

*(ii)* $|\pi(t,y) - \pi_\varepsilon(t,y)| \leq c\sqrt{\varepsilon}\sqrt{d_\varepsilon(t,y)}$ *whenever* $d_\varepsilon(t,y) > \varepsilon$.

PROOF OF PROPOSITION 3.2. We establish a uniform control of $d_\varepsilon(t, X_t^n)$. This will be done using Itô's formula with the function $\varphi_\varepsilon(t,y) := (d_\varepsilon(t,y))^p = |y - \pi_\varepsilon(t,y)|^p$, $\forall p > 2$, which is continuously differentiable with respect to $y$ and for which the derivative with respect to the time variable exists. Now, recall Lemma 3.1, then $(D_{\varepsilon,t})_{t\geqslant 0}$ is increasing in time and by definition of $\varphi_\varepsilon$, we have:

$$\partial_t \varphi_\varepsilon \leq 0.$$

Using Itô's formula and thanks to the previous remark, we get: $\forall t \in [0,T]$,

$$\begin{aligned}
\varphi_\varepsilon(t, X_t^n) &\leq \varphi_\varepsilon(0, X_0^n) + \int_0^t \partial_x \varphi_\varepsilon(s, X_s^n) dX_s^n + \frac{1}{2}\int_0^t \partial_{xx}\varphi_\varepsilon(s, X_s^n) d\langle X_i^n, X_j^n\rangle_s \\
&\leq \varphi_\varepsilon(0,x) + \int_0^t \langle \partial_x \varphi_\varepsilon(s, X_s^n), b(s, X_s^n)\rangle ds + \int_0^t \langle \partial_x \varphi_\varepsilon(s, X_s^n), \sigma(s, X_s^n) dW_s\rangle \\
&- n\int_0^t \langle \partial_x \varphi_\varepsilon(s, X_s^n), X_s^n - \pi(s, X_s^n)\rangle ds + \frac{1}{2}\int_0^t \sigma^\top(s, X_s^n) \partial_{xx}\varphi_\varepsilon(s, X_s^n)\sigma(s, X_s^n) ds.
\end{aligned}$$

First, note that Lemma 3.2-(i) gives the following boundedness from above of $\varphi_\varepsilon(0,x)$:

$$\varphi_\varepsilon(0,x) = |x - \pi_\varepsilon(0,x)|^p = |\pi(0,x) - \pi_\varepsilon(0,x)|^p \leq c^p \sqrt{\varepsilon^2 + \varepsilon d(x, D_0)}^p = (c\varepsilon)^p.$$

Next, for the term involving $\partial_x \varphi_\varepsilon$, we observe that:

$$\begin{aligned}\partial_x \varphi_\varepsilon(t,x) &= \partial_x\big((d_\varepsilon(t,x)^2)^{\frac{p}{2}}\big)\\ &= \frac{p}{2}\partial_x d_\varepsilon(t,x)^2 \times \big(d_\varepsilon(t,x)^2\big)^{\frac{p}{2}-1}\\ &= p \times (x-\pi_\varepsilon(t,x)) \times \big(d_\varepsilon(t,x)\big)^{p-2}.\end{aligned}$$

Then by taking the norm, we get:

$$|\partial_x \varphi_\varepsilon(t,X_t^n)| \leq p d_\varepsilon(t,X_t^n)^{p-1}.$$

Note that for any $c_1 > 0$, there exists $C_1$ depending on $c_1$ and $p$:

$$p d_\varepsilon(s,X_s^n)^{p-1}|b(s,X_s^n)| \leq c_1 n d_\varepsilon(s,X_s^n)^p + \frac{C_1}{n^{p-1}}|b(s,X_s^n)|^p, \tag{3.11}$$

Then we obtain the following inequality: $\forall c_1 > 0$,

$$\begin{aligned}\int_0^t \langle \partial_x \varphi_\varepsilon(s,X_s^n), b(s,X_s^n)\rangle ds &\leq p\int_0^t d_\varepsilon(s,X_s^n)^{p-1}|b(s,X_s^n)|ds\\ &\leq c_1 n\int_0^t d_\varepsilon(s,X_s^n)^p ds + \frac{C_1}{n^{p-1}}\int_0^t |b(s,X_s^n)|^p ds.\end{aligned} \tag{3.12}$$

Next, note that $-\frac{x-\pi_\varepsilon(t,x)}{|x-\pi_\varepsilon(t,x)|}$ coincides with $\vec{\mathbf{n}}_\varepsilon(t,x)$ the unit normal vector pointing toward the interior of $D_{\varepsilon,t}$ whenever $x \in \partial D_{\varepsilon,t}$ for each $t \in [0,T]$ and null elsewhere. Hence, we can see that,

$$\begin{aligned}\partial_{xx}\varphi_\varepsilon(t,x) &= \partial_x\big(-p\vec{\mathbf{n}}_\varepsilon(t,x)d_\varepsilon(t,x)^{p-1}\big)\\ &= -p\vec{\mathbf{n}}_\varepsilon(t,x)\partial_x\big((d_\varepsilon(t,x)^2)^{\frac{p-1}{2}}\big) - p(d_\varepsilon(t,x))^{p-1}\partial_x\vec{\mathbf{n}}_\varepsilon(t,x).\end{aligned}$$

Taking into account the smoothness of $D_\varepsilon$ and the boundedness of $D'$, the derivative $\partial_x\vec{\mathbf{n}}_\varepsilon$ is bounded. Thus, by taking the norm there exists a constant $c > 0$ independent of $\varepsilon$ such that:

$$\frac{1}{2}\int_0^t (\sigma^\top \partial_{xx}\varphi_\varepsilon \sigma)(s,X_s^n)ds \leq \int_0^t \big[cpd_\varepsilon(s,X_s^n)^{p-1} + cp(p-1)d_\varepsilon(s,X_s^n)^{p-2}\big]|\sigma(s,X_s^n)|^2 ds.$$

Similarly, we can see that for any $c_2, c_3 > 0$ there exist $C_2, C_3 > 0$ depending on $c_2$ and $c_3$ respectively and $p$ such that,

$$cpd_\varepsilon(s,X_s^n)^{p-1} \leq c_2 n d_\varepsilon(s,X_s^n)^p + \frac{C_2}{n^{p-1}}|\sigma(s,X_s^n)|^{2p},$$

and

$$cp(p-1)d_\varepsilon(s,X_s^n)^{p-2}|\sigma(s,X_s^n)|^2 \leq c_3 n d_\varepsilon(s,X_s^n)^p ds + \frac{C_3}{n^{\frac{p-2}{2}}}|\sigma(s,X_s^n)|^p.$$

Therefore, the following upper bound holds: $\forall c_2, c_3 > 0$,

$$\begin{aligned}\frac{1}{2}\int_0^t (\sigma^\top \partial_{xx}\varphi_\varepsilon \sigma)(s,X_s^n)ds \leq (c_2+c_3)n\int_0^t d_\varepsilon(s,X_s^n)^p ds + \frac{C_2}{n^{p-1}}\int_0^t |\sigma(s,X_s^n)|^{2p}ds\\ + \frac{C_3}{n^{\frac{p-2}{2}}}\int_0^t |\sigma(s,X_s^n)|^p ds.\end{aligned}$$

Now, let us examine the term involving the penalization term,

$$\begin{aligned}-n\int_0^t <\partial_x\varphi_\varepsilon(s,X_s^n), X_s^n - \pi(s,X_s^n)> ds &= -np\int_0^t d_\varepsilon(s,X_s^n)^p ds\\ &- np\int_0^t d_\varepsilon(s,X_s^n)^{p-2}\langle X_s^n - \pi_\varepsilon(s,X_s^n), \pi_\varepsilon(s,X_s^n) - \pi(s,X_s^n)\rangle \chi_{\{d_\varepsilon(s,X_s^n)>\varepsilon\}}(s,X_s^n)ds\\ &- np\int_0^t d_\varepsilon(s,X_s^n)^{p-2}\langle X_s^n - \pi_\varepsilon(s,X_s^n), \pi_\varepsilon(s,X_s^n) - \pi(s,X_s^n)\rangle \chi_{\{d_\varepsilon(s,X_s^n)\leq\varepsilon\}}(s,X_s^n)ds.\end{aligned}$$

On the one hand, Lemma 3.2-$(ii)$ yields:

$$\begin{aligned}|-np\int_0^t d_\varepsilon(s,X_s^n)^{p-2}\langle X_s^n-\pi_\varepsilon(s,X_s^n),\pi_\varepsilon(s,X_s^n)-\pi(s,X_s^n)\rangle\chi_{\{d_\varepsilon(s,X_s^n)>\varepsilon\}}(s,X_s^n)ds| \\ \leq npc\sqrt{\varepsilon}\int_0^t d_\varepsilon(s,X_s^n)^{p-2}|X_s^n-\pi_\varepsilon(s,X_s^n)|d_\varepsilon(s,X_s^n)^{\frac{1}{2}}\chi_{\{d_\varepsilon(s,X_s^n)>\varepsilon\}}(s,X_s^n)ds \\ \leq npc\sqrt{\varepsilon}\int_0^t d_\varepsilon(s,X_s^n)^{\frac{2p-1}{2}}ds.\end{aligned}$$

This implies that, for any $c_4>0$ there exists $C_4>0$ depending on $c_4$ and $p$ such that:

$$\begin{aligned}|-np\int_0^t d_\varepsilon(s,X_s^n)^{p-2}\langle X_s^n-\pi(s,X_s^n),\pi_\varepsilon(s,X_s^n)-\pi(s,X_s^n)\rangle\chi_{\{d_\varepsilon(s,X_s^n)>\varepsilon\}}(s,X_s^n)ds| \\ \leq c_4n\int_0^t d_\varepsilon(s,X_s^n)^p ds+C_4n^{\frac{1}{2p}}\varepsilon^p.\end{aligned}$$

On the other hand, the second term can be dominated as follows:

$$\begin{aligned}np\int_0^t d_\varepsilon(s,X_s^n)^{p-2}|X_s^n-\pi_\varepsilon(s,X_s^n)||\pi_\varepsilon(s,X_s^n)-\pi(s,X_s^n)|(1-\chi_{\{d_\varepsilon(s,X_s^n)\leq\varepsilon\}}(s,X_s^n))ds \\ \leq np\sqrt{\varepsilon}\int_0^t d_\varepsilon(s,X_s^n)^{p-1}(1+d_\varepsilon(s,X_s^n))ds \\ \leq cn\varepsilon^{p-\frac{1}{2}}.\end{aligned}$$

The last line follows from Lemma 3.2-$(i)$. As a conclusion, we have:

$$\begin{aligned}\varphi_\varepsilon(t,X_t^n)+(p-c_1-c_2-c_3-c_4)n\int_0^t\varphi_\varepsilon(s,X_s^n)ds\leq(c^p+C_4n^{\frac{1}{2p}})\varepsilon^p+cn\varepsilon^{p-\frac{1}{2}} \\ +\frac{C_1}{n^{p-1}}\int_0^t|b(s,X_s^n)|^p ds+\int_0^t\Big(\frac{C_2}{n^{p-1}}|\sigma(s,X_s^n)|^{2p}+\frac{C_3}{n^{\frac{p-2}{2}}}|\sigma(s,X_s^n)|^p\Big)ds \\ +\int_0^t\langle\partial_x\varphi_\varepsilon(s,X_s^n),\sigma(s,X_s^n)dW_s\rangle. \qquad (3.13)\end{aligned}$$

It suffices to choose $c_1,c_2,c_3$ and $c_4$ positive such that $p-c_1-c_2-c_3-c_4>1$. Then, by taking the expectation, there exists $C>0$ depending on $c_1,c_2,c_3,c_4$ and $p$ such that:

$$\begin{aligned}\mathbb{E}\Big[d_\varepsilon(t,X_t^n)^p+n\int_0^t d_\varepsilon(s,X_s^n)^p ds\Big]\leq(c^p+C_4n^{\frac{1}{2p}})\varepsilon^p+cn\varepsilon^{p-\frac{1}{2}} \\ +\frac{C}{n^{\frac{p-2}{2}}}\mathbb{E}\Big[\int_0^t\big(|b(s,X_s^n)|^p+|\sigma(s,X_s^n)|^p+|\sigma(s,X_s^n)|^{2p}\big)ds\Big]. \qquad (3.14)\end{aligned}$$

By taking the supremum over $[0,T]$ and by recalling (3.2) and assumption (b), we conclude that:

$$\sup_{0\leq t\leq T}\mathbb{E}\big[d_\varepsilon(t,X_t^n)^p\big]\leqslant(c^p+C_4n^{\frac{1}{2p}})\varepsilon^p+cn\varepsilon^{p-\frac{1}{2}}+\frac{C}{n^{\frac{p-2}{2}}},\forall p>2.$$

Next, using B-D-G inequality there exists $c>0$ such that:

$$\begin{aligned}\mathbb{E}\big[\sup_{0\leq t\leq T}|\int_0^t\langle\partial_x\varphi_\varepsilon(s,X_s^n),\sigma(s,X_s^n)dW_s\rangle|\big]\leq c\mathbb{E}\big[\int_0^T d_\varepsilon(s,X_s^n)^{p-1}|\sigma(s,X_s^n)|ds\big] \\ \leq c_5\mathbb{E}\big[\int_0^T nd_\varepsilon(s,X_s^n)^p ds\big]+\frac{C_5}{n^{p-1}}\mathbb{E}\big[\int_0^T|\sigma(s,X_s^n)|^p ds\big],\forall c_5>0,\end{aligned}$$

where $C_5>0$ is independent of $n$. Hence, from (3.13) and (3.14), we conclude that there exists $c>0$ such that

$$\mathbb{E}\big[\sup_{0\leq t\leq T}d_\varepsilon(t,X_t^n)^p\big]\leqslant(c^p+C_4n^{\frac{1}{2p}})\varepsilon^p+cn\varepsilon^{p-\frac{1}{2}}+\frac{C}{n^{\frac{p-2}{2}}},\forall p>2.$$

Finally, since $h(D_t,D_{\varepsilon,t})<\varepsilon$, we have $d(s,X^n)\leq d_\varepsilon(s,X_s^n)+\varepsilon$. Then, it suffices to take the limit as $\varepsilon\to 0$.

### 3.3 Convergence of the Penalized SDE

First, we show that $(X^n)_{n\geqslant 1}$ is a Cauchy sequence as stated in the following proposition:

**Proposition 3.3** *Let $(X^n)_{n\geqslant 1}$ be the solution of the penalized SDE (3.1). Then, for any $p \geqslant 1$,*

$$\lim_{m,n\to\infty} \mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^n - X_t^m|^p\big] = 0. \tag{3.15}$$

PROOF. Let $m,n \geqslant 1$ and $0 \leqslant t \leqslant T$, by applying Itô's formula to $|X_s^n - X_s^m|^2$, we get:

$$\begin{aligned}
|X_t^n - X_t^m|^2 &= 2\int_0^t \langle X_s^n - X_s^m, b(s,X_s^n) - b(s,X_s^m)\rangle ds + \frac{1}{2}\int_0^t |\sigma(s,X_s^n) - \sigma(s,X_s^m)|^2 ds \\
&\quad - 2n\int_0^t \langle X_s^n - X_s^m, X_s^n - \pi(s,X_s^n)\rangle ds + 2m\int_0^t \langle X_s^n - X_s^m, X_s^m - \pi(s,X_s^m)\rangle ds \\
&\quad + 2\int_0^t \langle X_s^n - X_s^m, (\sigma(s,X_s^n) - \sigma(s,X_s^m))dW_s\rangle.
\end{aligned}$$

Using the Lipschitz continuity of $b$ and $\sigma$, we deduce that:

$$\begin{aligned}
|X_t^n - X_t^m|^2 &\leq c_{b,\sigma}\int_0^t |X_s^n - X_s^m|^2 ds - 2n\int_0^t \langle X_s^n - X_s^m, X_s^n - \pi(s,X_s^n)\rangle ds \\
&\quad + 2m\int_0^t \langle X_s^n - X_s^m, X_s^m - \pi(s,X_s^m)\rangle ds + 2\int_0^t \langle X_s^n - X_s^m, (\sigma(s,X_s^n) - \sigma(s,X_s^m))dW_s\rangle.
\end{aligned} \tag{3.16}$$

Note that,

$$\begin{aligned}
-\langle X_s^n - X_s^m, X_s^n - \pi(s,X_s^n)\rangle &= -|X_s^n - \pi(s,X_s^n)|^2 \\
&\quad + \Big\langle \pi(s,X_s^n) - \pi(s,X_s^m), -\frac{X_s^n - \pi(s,X_s^n)}{|X_s^n - \pi(s,X_s^n)|}\Big\rangle d(s,X_s^n) - \Big\langle \pi(s,X_s^m) - X_s^m, X_s^n - \pi(s,X_s^n)\Big\rangle.
\end{aligned} \tag{3.17}$$

By recalling the exterior sphere property (2.4) and using the Lipschitz continuity of $\pi$ w.r.t. $y$, we deduce that:

$$\begin{aligned}
-2n\int_0^t \langle X_s^n - X_s^m, X_s^n - \pi(s,X_s^n)\rangle ds &\leq cn\int_0^t |\pi(s,X_s^n) - \pi(s,X_s^m)|^2 d(s,X_s^n) ds \\
&\quad + 2n\int_0^t d(s,X_s^n) d(s,X_s^m) ds \\
&\leq cn\int_0^t d(s,X_s^n)|X_s^n - X_s^m|^2 ds + 2n\int_0^t d(s,X_s^n) d(s,X_s^m) ds.
\end{aligned}$$

We do likewise with the third term in the right hand side of (3.16). Then, we obtain the following inequality:

$$|X_t^n - X_t^m|^2 \leq M_t^{m,n} + H_t^{m,n} + \int_0^t \Psi_s^{m,n}|X_s^n - X_s^m|^2 ds,$$

where $M^{m,n}$ is a local martingale and the processes $H^{m,n}$ and $\Psi^{m,n}$ are defined by:

$$\begin{aligned}
H_t^{m,n} &:= 2(m+n)\int_0^t d(s,X_s^n) d(s,X_s^m) ds \\
\Psi_s^{m,n} &:= c_{b,\sigma,r}(1 + nd(s,X_s^n) + md(s,X_s^m)).
\end{aligned}$$

Then thanks to Stochastic Gronwall's inequality (see e.g. Lemma 2.3 in Ren-Wu [15], Theorem 4 in Scheutzow [16]), we get:

$$\begin{aligned}
&\mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^n - X_t^m| e^{-\frac{c_{b,\sigma,r}}{2}\int_0^T (1+nd(s,X_s^n)+md(s,X_s^m))ds}\big] \\
&\leq 3c\mathbb{E}\big[(n+m)\int_0^T d(s,X_s^n) d(s,X_s^m) ds\big]^{\frac{1}{2}} \\
&\leq c\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^m)\int_0^T nd(s,X_s^n) ds + \sup_{0\leq t\leq T} d(t,X_t^n)\int_0^T md(s,X_s^m) ds\big]^{\frac{1}{2}} \\
&\leq c\big(\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^m) + \sup_{0\leq t\leq T} d(t,X_t^n)\big]\big)^{\frac{1}{2}} \times \big(\mathbb{E}\big[n\int_0^T d(s,X_s^n) ds + m\int_0^T d(s,X_s^m) ds\big]\big)^{\frac{1}{2}}.
\end{aligned}$$

Recall Proposition 3.1, then from (3.2) we deduce that $\mathbb{E}\big[n\int_0^T d(s,X_s^n)ds\big] = \mathbb{E}\big[|\Lambda^n|_T\big] \leq c$. This implies that,

$$\mathbb{E}\big[n\int_0^T d(s,X_s^n)ds + m\int_0^T d(s,X_s^m)ds\big] \leq c$$

On the other hand, from (3.9) it follows that for $p>2$, there exists $c>0$ that may change from line to line such that:

$$\begin{aligned}\Big(\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^m) + \sup_{0\leq t\leq T} d(t,X_t^n)\big]\Big)^{\frac{1}{2}} &\leq \Big(\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^m) + \sup_{0\leq t\leq T} d(t,X_t^n)\big]^p\Big)^{\frac{1}{2p}} \\ &\leq c\Big(\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^m)\big]^p + \big[\sup_{0\leq t\leq T} d(t,X_t^n)\big]^p\Big)^{\frac{1}{2p}} \\ &\leq c\Big(\frac{1}{m^{\frac{p-2}{2}}} + \frac{1}{n^{\frac{p-2}{2}}}\Big)^{\frac{1}{2p}}.\end{aligned}$$

Therefore, the following holds: $\forall p>2$,

$$\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|e^{-\frac{c_{b,\sigma,r}}{2}(1+|\Lambda^n|_T+|\Lambda^m|_T)}\big] \leq c\Big(\frac{1}{m^{\frac{p-2}{2}}} + \frac{1}{n^{\frac{p-2}{2}}}\Big)^{\frac{1}{2p}} \underset{m,n\to\infty}{\longrightarrow} 0.$$

First, observe that: $\forall \kappa>0$,

$$\begin{aligned}\sup_{0\leq t\leq T}|X_t^n - X_t^m| &= \sup_{0\leq t\leq T}|X_t^n - X_t^m|\Big(\chi_{\left\{e^{-\frac{c_{b,\sigma,r}}{2}(1+|\Lambda^n|_T+|\Lambda^m|_T)}\leqslant\kappa\right\}} + \chi_{\left\{e^{-\frac{c_{b,\sigma,r}}{2}(1+|\Lambda^n|_T+|\Lambda^m|_T)}>\kappa\right\}}\Big) \\ &\leq \sup_{0\leq t\leq T}|X_t^n - X_t^m|\chi_{\left\{1+|\Lambda^n|_T+|\Lambda^m|_T\geqslant\ln(\frac{1}{\kappa})\right\}} + \frac{1}{\kappa}\sup_{0\leq t\leq T}|X_t^n - X_t^m|e^{-\frac{c_{b,\sigma,r}}{2}(1+|\Lambda^n|_T+|\Lambda^m|_T)}.\end{aligned}$$

By taking the expectation, we obtain: $\forall \kappa>0$,

$$\begin{aligned}\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|\big] &\leq \big(\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|\big]\big)^{\frac{1}{2}}\big(\mathbb{P}\big[1+|\Lambda^n|_T+|\Lambda^m|_T \geqslant \ln\big(\frac{1}{\kappa}\big)\big]\big)^{\frac{1}{2}} \\ &\quad + \frac{1}{\kappa}\Big(\frac{1}{m^{\frac{p-2}{2}}} + \frac{1}{n^{\frac{p-2}{2}}}\Big)^{\frac{1}{2p}} \\ &\leq \big(\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|\big]\big)^{\frac{1}{2}}\Big(\frac{\mathbb{E}\big[1+|\Lambda^n|_T+|\Lambda^m|_T\big]}{\ln\big(\frac{1}{\kappa}\big)}\Big)^{\frac{1}{2}} + \frac{1}{\kappa}\Big(\frac{1}{m^{\frac{p-2}{2}}} + \frac{1}{n^{\frac{p-2}{2}}}\Big)^{\frac{1}{2p}}.\end{aligned}$$

Thanks to the estimate (3.2), there exists a constant $c>0$ independent of $n$ and $m$ such that $\forall \kappa>0$,

$$\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|\big] \leq c\Big(\frac{1}{\ln\big(\frac{1}{\kappa}\big)}\Big)^{\frac{1}{2}} + \frac{1}{\kappa}\Big(\frac{1}{m^{\frac{p-2}{2}}} + \frac{1}{n^{\frac{p-2}{2}}}\Big)^{\frac{1}{2p}}.$$

It follows that:

$$\limsup_{m,n\to\infty}\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|\big] = 0.$$

Again from the estimate (3.2), we can see that for any $p\geqslant 1$ the family $(|X_t^n - X_t^m|^p)_{m,n}$ is uniformly integrable. Therefore, the following convergence holds: $\forall p\geqslant 1$,

$$\mathbb{E}\big[\sup_{0\leq t\leq T}|X_t^n - X_t^m|^p\big] \underset{m,n\to\infty}{\longrightarrow} 0.$$

It follows that $(X^n)_{n\geqslant 1}$ is a Cauchy sequence in the space $\mathscr{S}^p$ defined by:

$$\mathscr{S}^2 = \{(\psi_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-progressively measurable such that } \mathbb{E}[\sup_{0\leq t\leq T}\ |\ \psi_t\ |^p] < \infty\},$$

then $(X^n)_{n\geqslant 1}$ converges.

To conclude this section, we show that the limit of $(X^n,\Lambda^n)_{n\geqslant 1}$ is the solution of the reflected SDE (2.10).

**Proposition 3.4** *For any $p \geqslant 1$, we have:*

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^n - X_t|^p\big] = 0. \tag{3.18}$$

PROOF. Let $n \geqslant 1$, by applying Itô's formula to $|X_t^n - X_t|^2$, we get:

$$\begin{aligned}|X_t^n - X_t|^2 &= 2\int_0^t \langle X_s^n - X_s, b(s,X_s^n) - b(s,X_s)\rangle ds + \frac{1}{2}\int_0^t |\sigma(s,X_s^n) - \sigma(s,X_s)|^2 ds \\ &+ 2\int_0^t \langle X_s^n - X_s, (\sigma(s,X_s^n) - \sigma(s,X_s))dW_s\rangle + 2\int_0^t (X_s^n - X_s)d\Lambda_s^n - 2\int_0^t (X_s^n - X_s)d\Lambda_s.\end{aligned}$$

The Lipschitz continuity of $b$ and $\sigma$, implies that;

$$\begin{aligned}|X_t^n - X_t|^2 &\leq c_{b,\sigma}\int_0^t |X_s^n - X_s|^2 ds + 2\int_0^t \langle X_s^n - X_s, (\sigma(s,X_s^n) - \sigma(s,X_s))dW_s\rangle \\ &- 2n\int_0^t \langle X_s^n - X_s, X_s^n - \pi(s,X_s^n)\rangle ds + 2\int_0^t (X_s^n - X_s)d\Lambda_s.\end{aligned} \tag{3.19}$$

Then, by repeating the same calculus as in the previous proof, we obtain:

$$\begin{aligned}-2n\int_0^t \langle X_s^n - X_s, X_s^n - \pi(s,X_s^n)\rangle ds &\leq cn\int_0^t |\pi(s,X_s^n) - \pi(s,X_s)|^2 d(s,X_s^n) ds \\ &+ 2n\int_0^t d(s,X_s^n) d(s,X_s) ds \\ &\leq cn\int_0^t d(s,X_s^n)|X_s^n - X_s|^2 ds + 2n\int_0^t d(s,X_s^n) d(s,X_s) ds.\end{aligned}$$

Since $(X_t)_{0\leqslant t\leqslant T}$ is the solution of the reflected SDE in the time-dependent domain (2.10), then $X_t \in D_t$. This implies that $d(t,X_t) = 0$ for any $0 \leqslant t \leqslant T$. Therefore, the inequality (3.19) becomes:

$$|X_t^n - X_t|^2 \leq M_t^n + c\int_0^t (1 + nd(s,X_s^n))|X_s^n - X_s|^2 ds + 2\int_0^t (X_s^n - X_s)d\Lambda_s,$$

where $M^m$ is a local martingale. Besides, note that

$$|X_s^n - X_s| \leq |X_s^n - \pi(s,X_s^n) + \pi(s,X_s^n) - X_s| \leq d(s,X_s^n) + c,$$

since $\pi(s,X_s^n)$ and $X_s$ belong to the bounded set $D_T$.
It follows that,

$$|X_t^n - X_t|^2 \leq M_t^n + c\int_0^t (1 + nd(s,X_s^n))|X_s^n - X_s|^2 ds + c\int_0^t (1 + d(s,X_s^n))d|\Lambda_s|.$$

Again thanks to Stochastic Gronwall's inequality (see e.g. Lemma 2.3 in Ren-Wu [15], Theorem 4 in Scheutzow [16]), we get:

$$\begin{aligned}\mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^n - X_t^{t,x}| e^{-\frac{c_{b,\sigma,r}}{2}\int_0^T nd(s,X_s^n)ds}\big] &\leq 3c\mathbb{E}\Big[\int_0^T d(s,X_s^n)d|\Lambda|_s\Big]^{\frac{1}{2}} \\ &\leq c\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^n)|\Lambda|_T\big]^{\frac{1}{2}} \\ &\leq c\Big\{\big(\mathbb{E}\big[\sup_{0\leq t\leq T} d(t,X_t^n)^p\big]\big)^{\frac{1}{p}} \times \big(\mathbb{E}\big[|\Lambda|_T{}^q\big]\big)^{\frac{1}{q}}\Big\}^{\frac{1}{2}},\ p>2,\ q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

Thanks to (2.14) and (3.2), we deduce that the right hand side tends to zero as $n \to \infty$. The remaining of the proof is similar to the proof of Proposition 3.3.

Note that the above convergence implies the convergence of $\Lambda^n$ to $\Lambda$ as stated in the following corollary:

**Corollary 3.1** *For any $p \geqslant 1$, we have:*

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{0\leq t\leq T} |\Lambda_t^n - \Lambda_t|^p\big] = 0. \tag{3.20}$$

PROOF. Recall that

$$\Lambda_t^n - \Lambda_t = X_t^n - X_t + \int_0^t (b(r,X_r) - b(r,X_r^n))dr + \int_0^t (\sigma(r,X_r) - \sigma(r,X_r^n))dW_r.$$

Then using the Lipschitz continuity of $b$ and $\sigma$, we get: $\forall p \geqslant 1$,

$$|\Lambda_t^n - \Lambda_t|^p \leq |X_t^n - X_t|^p + \big(\int_0^t (b(r,X_r) - b(r,X_r^n))dr\big)^p + \big|\int_0^t (\sigma(r,X_r) - \sigma(r,X_r^n))dW_r\big|^p.$$

By taking the expectation, together with the use of B-D-G inequality and the convergence (3.18), we obtain the result.

**Remark 3.1** *It should be pointed out that the smoothness of the direction of reflection (2.5) and the assumption on the boundary of the domain (2.6) are not required to obtain the convergence of the approximation (3.1). However, in case these conditions are not satisfied, the existence and uniqueness of a solution of the reflected SDE (2.10) as well as its properties are not ensured within the geometric setting of Lundström-Önskog [9]. In another framework, when the SDE (2.10) is reflected in $D \subset \mathbb{R}^{d+1}$ which is assumed to be at least $\mathscr{H}_2$-time-dependent domain, the cone of unit inward normal vectors is reduced to a unique element and the time sections of $D$ satisfy the uniform exterior and interior sphere condition. Therefore, the existence and uniqueness of the solution of the reflected SDE (2.10) within this geometric setting as well as the estimate (2.14) follow without the need for the assumptions (2.5) and (2.6) (see Theorem 3.2, Theorem 3.4 and Proposition 3.5 in Costantini-Gobet-El Karoui [5]). Moreover, with this type of regularity, the control of the distance $d(t,X_t^n)$ is obtained without resorting to the approximation of the domain (3.1).*

**Remark 3.2** *In the case of time-independent domains, the estimates (3.2) coincide with the estimates obtained for the approximation of reflected SDEs in regular convex domain given in Bahlali-Maticiuc-Zalinescu [1] and Menaldi [10]. We also note that the convergence results (3.18) and (3.20) are similar to the results of convergence obtained in time-independent domains: for convex domain (Menaldi [10] and Slominski [17]) and for non-convex domains (see e.g. Lions-Sznitmann [7] and Ren-Wu [15]).*

# 4 Application to generalized BSDEs and PDEs with boundary condition on non-smooth time-dependent domains

Let $D$ be a time-dependent domain satisfying (2.1)-(2.6). In this section, we give an application of the results obtained in the previous section by providing an approximation of generalized BSDEs when the underlying process is a reflected diffusion in a time-dependent domain. This approximation consists of a sequence of standard BSDEs associated with standard diffusions. Therefore, by considering the associated PDEs we also get an approximation of PDEs with boundary conditions on a time-dependent domain using a sequence of standard PDEs defined on $\mathbb{R}^d$. However, this requires an additional smoothness condition on the domain. From now on, we assume that the cone of unit inward normal vectors at each boundary point is reduced to a unique element $\vec{\mathbf{n}}$ that is assumed to satisfy (2.5).

**Remark 4.1** *By assuming that $\vec{\mathbf{n}}$ is unique at each boundary point, we note that Lemma 2.2 in Boufoussi-Casteren [4] can be generalized to the case $\phi \in \mathscr{C}_b^{1,2}(\mathbb{R}^{1+d})$. The result is based essentially on the convergences (3.18) and (3.20). In fact we have: $\forall p \geqslant 1$,*

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{0\leq t\leq T} \big|\int_0^t \langle \phi(r,X_r), d\Lambda_r\rangle - \int_0^t \langle \phi(r,X_r^n), d\Lambda_r^n\rangle\big|^p\big] = 0.$$

*Therefore, by taking $\phi := \vec{n}(.,.)$, we get:*

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t\langle\vec{n}(r,X_r),d\Lambda_r\rangle-\int_0^t\langle\vec{n}(r,X_r^n),d\Lambda_r^n\rangle\Big|^p\Big]=\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\leq t\leq T}\big||\Lambda^n|_t-|\Lambda|_t\big|^p\Big]=0.$$

Now, let $(t,x)\in\overline{D}$ be fixed, we define $(X_s^{t,x},\Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ as the unique solution of: $\forall t\leqslant s\leqslant T$,

$$\begin{cases}X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dW_r+\int_t^s\vec{\mathbf{n}}(r,X_r^{t,x})d|\Lambda^{t,x}|_r;\\ X_s^{t,x}\in\bar{D}_s,\ |\Lambda^{t,x}|_s=\int_0^s\chi_{\{X_r^{t,x}\in\partial D_r\}}d|\Lambda^{t,x}|_r.\end{cases}\tag{4.1}$$

Then, let $m\geqslant 1$ and let us introduce the following functions:

$$\begin{aligned}&f:(t,x,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\longmapsto f(t,x,\vec{y},z)=(f_i(t,x,\vec{y},z_i))_{i=1,\dots,m}\in\mathbb{R}^m,\\ &\psi:(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^m\longmapsto\psi(t,x,\vec{y})=(\psi_i(t,x,\vec{y}))_{i=1,\dots,m}\in\mathbb{R}^m,\\ &h:x\in\mathbb{R}^d\longmapsto h(x)=(h_i(x))_{i=1,\dots,m}\in\mathbb{R}^m.\end{aligned}$$

From now on, we make the following assumptions:

- $(H_0)$ The function $h$ is continuous and is of polynomial growth.
- $(H_1)$
  - $(i)$ $(t,x)\longmapsto f(t,x,\vec{y},z)$ and $(t,x)\longmapsto\psi(t,x,\vec{y})$ are uniformly continuous with respect to $(\vec{y},z)$ and $\vec{y}$ respectively.
  - $(ii)$ $(t,x)\longmapsto f(t,x,0,0)$ and $(t,x)\longmapsto\psi(t,x,0)$ are of polynomial growth.
  - $(iii)$ $f$ and $\psi$ are Lipschitz continuous with respect to $(\vec{y},z)$ and $\vec{y}$ respectively.
  - $(iii)$ $\exists\beta<0$ such that $\langle y-\bar{y},\psi(t,x,y)-\psi(t,x,\bar{y})\rangle\leq\beta\mid y-\bar{y}\mid^2$.

## 4.1 Associated BSDEs

Now, let $(X_s^{t,x},\Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ be the unique solution of the reflected SDE (4.1), then we introduce the associated multidimensional generalized BSDE: $\forall t\leqslant s\leqslant T$,

$$Y_s^{t,x}=h(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^T\psi(r,X_r^{t,x},Y_r^{t,x})d|\Lambda^{t,x}|_r-\int_s^T Z_r^{t,x}dW_r,\tag{4.2}$$

which has a unique solution thanks to Theorem 1.6 in Pardoux-Zhang [14] that we denote $(Y^{t,x},Z^{t,x})$. Moreover, the following estimates hold:

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant T}\mid Y_s^{t,x}\mid^2+\int_t^T\mid Y_r^{t,x}\mid^2 d|\Lambda^{t,x}|_r+\int_t^T||Z_r^{t,x}||^2dr\right]<\infty.\tag{4.3}$$

Next, let $n\geqslant 1$ and let $X^{t,x,n}$ be the unique solution of the penalized SDE: $\forall t\leqslant s\leqslant T$,

$$\begin{cases}X_s^{t,x,n}=x+\int_t^s b(r,X_r^{t,x,n})dr+\int_t^s\sigma(r,X_r^{t,x,n})dW_r+\Lambda_s^{t,x,n};\\ \Lambda_r^{t,x,n}=-n\int_t^s(X_r^{t,x,n}-\pi(r,X_r^{t,x,n}))dr,\\ |\Lambda^{t,x,n}|_s=\int_0^s n|X_s^{t,x,n}-\pi(s,X_s^{t,x,n})|dr.\end{cases}\tag{4.4}$$

Then, consider the associated sequence of standard BSDEs: $\forall n\geqslant 1$, $\forall t\leqslant s\leqslant T$,

$$\begin{aligned}Y_s^{t,x,n}=h(X_T^{t,x,n})+\int_s^T f(r,X_r^{t,x,n},Y_r^{t,x,n},Z_r^{t,x,n})dr+\int_s^T\psi(r,X_r^{t,x,n},Y_r^{t,x,n})d|\Lambda^{t,x,n}|_r\\ -\int_s^T Z_r^{t,x,n}dW_r.\end{aligned}\tag{4.5}$$

We first give the estimates for the penalized BSDE:

**Proposition 4.1** *Let $n \geqslant 1$ and $(Y^{t,x,n}, Z^{t,x,n})$ be the solution of the the penalized BSDE (4.5). Then, we have the following estimates: $\forall n \geqslant 1$,*

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant T} |\, Y_s^{t,x,n}\,|^2 + \int_t^T |\, Y_r^{t,x,n}\,|^2\, d|\Lambda^{t,x,n}|_r + \int_t^T ||Z_r^{t,x,n}||^2 dr\right] < c, \tag{4.6}$$

*where $c$ is a constant independent of $n$.*

PROOF. From Itô's formula, we have:

$$\begin{aligned}|Y_s^{t,x,n}|^2 + \int_s^T ||Z_r^{t,x,n}||^2 dr = |h(X_T^{t,x,n})|^2 + 2\int_s^T \langle Y_r^{t,x,n}, f(r, X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n})\rangle dr \\ +2\int_s^T \langle Y_r^{t,x,n}, \psi(r, X_r^{t,x,n}, Y_r^{t,x,n})\rangle d|\Lambda^{t,x,n}|_r - 2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r\rangle.\end{aligned}$$

Besides, using assumptions $(H_1)(ii)-(iv)$, we have:

$$\begin{aligned}\langle Y_r^{t,x,n}, f(r, X_r^{t,x,n}, Y_r^{t,x,n})\rangle &\leq |Y_r^{t,x,n}||f(r, X_r^{t,x,n}, Y_r^{t,x,n})| \\ &\leq |Y_r^{t,x,n}|\{c(1+|X_r^{t,x,n}|+|Y_r^{t,x,n}|+|Z_r^{t,x,n}|)\} \\ &\leq \frac{c}{c_1}\{(1+|X_r^{t,x,n}|^2+|Y_r^{t,x,n}|^2\} + c_1||Z_r^{t,x,n}||^2\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\langle Y_r^{t,x,n}, \psi(r, X_r^{t,x,n}, Y_r^{t,x,n})\rangle &\leq \langle Y_r^{t,x,n}, \psi(r, X_r^{t,x,n}, Y_r^{t,x,n}) - \psi(r, X_r^{t,x,n}, 0)\rangle + \langle Y_r^{t,x,n}, \psi(r, X_r^{t,x,n}, 0)\rangle \\ &\leq \beta|Y_r^{t,x,n}|^2 + c_2|Y_r^{t,x,n}|^2 + \frac{c}{c_1}\{1+|X_r^{t,x,n}|^2)\}.\end{aligned}$$

Note that the constants $c_1$ and $c_2$ can be chosen such that $(-\beta - c_2) > 0$ and $(1-c_1) > 0$. Therefore, the equation (4.7) yields,

$$\begin{aligned}|Y_s^{t,x,n}|^2 + (-\beta - c_2)\int_s^T |Y_r^{t,x,n}|^2 d|\Lambda^{t,x,n}|_r + (1-c_1)\int_s^T ||Z_r^{t,x,n}||^2 dr \leq c\{1+|X_T^{t,x,n}|^2 \\ + \int_s^T |X_r^{t,x,n}|^2 dr + \int_s^T |X_r^{t,x,n}|^2 d|\Lambda^{t,x,n}|_r\} + c\int_s^T |Y_r^{t,x,n}|^2 dr - 2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r\rangle.\end{aligned}$$

Next, using the estimates (3.2)and by taking the expectation, we get:

$$\begin{aligned}\mathbb{E}\big[|Y_s^{t,x,n}|^2 + (-\beta - c_2)\int_s^T |Y_r^{t,x,n}|^2 d|\Lambda^{t,x,n}|_r + (1-c_1)\int_s^T ||Z_r^{t,x,n}||^2 dr\big] \leq c\{1 \\ + \mathbb{E}\big[\sup_{t\leq r\leq T} |X_r^{t,x,n}|^2 + |\Lambda^n|^q_{[t,s]} + \int_s^T |Y_r^{t,x,n}|^2 dr\big]\}.\end{aligned}$$

Then, thanks to Gronwall's lemma, we obtain:

$$\sup_{t\leq s\leq T} \mathbb{E}\big[|Y_s^{t,x,n}|^2 + \int_s^T |Y_r^{t,x,n}|^2 d|\Lambda^{t,x,n}|_r + \int_s^T ||Z_r^{t,x,n}||^2 dr\big] \leq c.$$

Finally, it suffices to apply BDG inequality to conclude.

The convergence of $(Y^{t,x,n}, Z^{t,x,n})_{n\geqslant 1}$ is stated in the following proposition:

**Proposition 4.2** *Let $(Y^{t,x,n}, Z^{t,x,n})_{n\geqslant 1}$ and $(Y^{t,x}, Z^{t,x})$ be the unique solutions of the sequence of BSDEs (4.5) and the generalized BSDE (4.2) respectively. Then, the following convergences hold true:*

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{t\leq s\leq T} |Y_s^{t,x,n} - Y_s^{t,x}|^2 + \int_t^T ||Z_r^{t,x,n} - Z_r^{t,x}||^2 dr\big] = 0. \tag{4.8}$$

**Remark 4.2** *The convergence (4.8) was established in Bahlali-Boufoussi-Mouchtabih [3] when $(X_s^{t,x},\Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ is the solution of the reflected SDE (4.1) in a regular convex time-independent domain. The proof is very technical and relies essentially on the results of convergence and properties of the penalized SDE (4.4) established in [1, 4, 17] and the properties of the reflected SDE in a regular convex time-independent domain that can be found in [7, 18, 10]. The properties of the solution $(X_s^{t,x},\Lambda_s^{t,x})_{t\leqslant s\leqslant T}$ and the results of convergence obtained in Section 2 and Section 3 are the generalizations of these results within our geometric setting. Therefore, the proof of Proposition 4.2 can be obtained by mimicking the proof of Theorem 6 in Bahlali-Boufoussi-Mouchtabih [3], we omit any further details.*

## 4.2 Associated PDEs

Now, we are in a position to get an approximation of PDEs with nonlinear Neumann boundary condition on time-dependent domains. Let us set $\tilde{D}=\overline{D'}\cap\big([0,T]\times\mathbb{R}^d\big)$. Then, following the notation of Lundström-Önskog [9], let us recall the spaces:

$$D^{\mathrm{o}}=D'\cap\big([0,T)\times\mathbb{R}^d\big),$$
$$\partial D=\big(\overline{D'}\setminus D'\big)\cap\big([0,T)\times\mathbb{R}^d\big).$$

Next, let us consider the following system of PDEs: $\forall i=1,...,m$,

$$\begin{cases}\partial_t u_i(t,x)+\mathscr{L}u_i(t,x)+f_i(t,x,u(t,x),\sigma^\top(t,x)D_xu_i(t,x))=0,\ (t,x)\in D^{\mathrm{o}};\\ \dfrac{\partial u_i}{\partial\vec{\mathbf{n}}}(t,x)+\psi_i(t,x,u(t,x))=0,\ (t,x)\in\partial D;\\ u(T,x)=h(x),\ x\in\overline{D}_T,\end{cases}\tag{4.9}$$

where the operator $\mathscr{L}$ is defined by $\mathscr{L}=\frac{1}{2}Tr(\sigma\sigma^\top)D^2_{xx}(.)+b^\top D_x(.)$ and at a point $(t,x)\in\partial D$ we set $\frac{\partial}{\partial\vec{\mathbf{n}}}=\langle\vec{\mathbf{n}}(t,x),D_x(.)\rangle$. Let $u:\tilde{D}\mapsto\mathbb{R}^m$ be the deterministic function defined by $Y^{t,x}$ the solution of the multidimensional generalized BSDE (4.2) as follows:

$$u_i(t,x):=Y_t^{t,x,i},\ \forall i=1,\dots,m.\tag{4.10}$$

It follows from Theorem 2 in Jakani [6], that $u$ is a viscosity solution of the system of PDEs with boundary condition of Neumann type on time-dependent domain (4.9) in the sense of Definition 2.2 in Jakani [6].
Finally, let $(Y^{t,x,n},Z^{t,x,n})_{n\geqslant1}$ be the unique solution of (4.5), it is well known that the sequence of deterministic functions $(u^n)_{n\geqslant1}$ given by $u^n(t,x)=Y_t^{t,x,n}$ for any $n\geqslant1$ is a viscosity solution of the following PDEs system: $\forall x\in\mathbb{R}^d$, $\forall 0\leqslant t<T$, $\forall i=1,\dots,m$,

$$\begin{cases}\partial_t u_i^n(t,x)+\mathscr{L}u_i^n(t,x)+f_i(t,x,u^n(t,x),\sigma^\top(t,x)D_xu_i^n(t,x))\\ \qquad\qquad -n\psi_i(t,x,u^n(t,x))\langle\vec{\mathbf{n}}(t,x),x-\pi(t,x)\rangle=0,\\ u^n(T,x)=h(x).\end{cases}\tag{4.11}$$

As an application of the approximation provided for generalized BSDEs (4.5), we obtain an approximation for the system of PDEs (4.9):

**Proposition 4.3** *The following convergence holds: $\forall(t,x)\in\tilde{D}$, $\forall i=1,\dots,m$,*

$$\lim_{n\to\infty}u_i^n(t,x)=u_i(t,x).\tag{4.12}$$

PROOF. Let $(t,x)\in\tilde{D}$, then from the definition of $(u_i^n)_{i=1,\dots,m}$ and $(u_i)_{i=1,\dots,m}$, it follows that: $\forall i=1,\dots,m$,

$$\lim_{n\to\infty}|u_i^n(t,x)-u_i(t,x)|^2=\lim_{n\to\infty}|Y_t^{t,x,n,i}-Y_t^{t,x,i}|^2\leq\lim_{n\to\infty}\mathbb{E}\big[\sup_{t\leq s\leq T}|Y_s^{t,x,n,i}-Y_s^{t,x,i}|^2\big]=0.$$

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