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Weak solutions with bounded support to an Euler-type flocking model

Debora Amadori and Cleopatra Christoforou

Abstract We give an overview of the recents results obtained in [1] on the existence and time-asymptotic flocking of weak solutions to a hydrodynamic model of flocking-type with all-to-all interaction kernel in one-space dimension. An appropriate notion of entropy weak solutions with bounded support is described, to capture the behavior of solutions to the Cauchy problem with any BV initial data that has finite total mass confined in a bounded interval and initial density uniformly positive therein. In addition, a suitable condition on the initial data is provided that allows us to show time-asymptotic flocking for such solutions.

1 Introduction

This article serves as an overview of the results obtained in [1] on *weak solutions* to the hydrodynamic model of flocking-type in one-space dimension that takes the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho \mathsf{v}) = 0, \\ \partial_t(\rho \mathsf{v}) + \partial_x\left(\rho \mathsf{v}^2 + p(\rho)\right) = \int_{\mathbb{R}} K(x,x')\rho(x,t)\rho(x',t)\left(\mathsf{v}(x',t) - \mathsf{v}(x,t)\right)\, dx' \end{cases} \tag{1}$$

where $(x,t) \in \mathbb{R}\times[0,+\infty)$. Here $\rho \geq 0$ stands for the mass variable, v for the velocity, p for the pressure and $K = K(x,x') \geq 0$ for a symmetric interaction kernel.

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Self-organization is an area that has received alot of attention in the research community and especially for systems such as flock of birds, a swarm of bacteria or a school of fish and its study gives rise to new mathematical challenges. Especially, in an effort to understand the emergence of flocking behavior, many mathematical models have been introduced arising from the inspiring work of Cucker and Smale [8], while led to many subsequent studies; cf. [6, 29] and the references therein. Most results on flocking models involve results at the particle level, the corresponding kinetic equation or its hydrodynamic formulation (cf. [17, 16, 27, 22, 13, 14, 15, 23]) and so far this subject has been investigated mainly in the context of solutions with no discontinuities.

The Euler-type flocking system (1) with pressure

$$p(\rho) = \alpha^2 \rho\,, \qquad \alpha > 0\,. \tag{2}$$

is rigorously derived as the hydrodynamic limit of the kinetic Cucker-Smale flocking model on $(x,t,\omega) \in (0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ by Karper, Mellet and Trivisa [23]. In particular, they study the singular limit corresponding to strong noise and strong local alignment with the alignment operator being the usual Cucker-Smale operator, while a diffusion term is present in the kinetic equation and show the convergence of weak solutions to the kinetic equation to strong (suitably smooth) solutions of the Euler system with pressure (2). We remark that, in the literature, hydrodynamic models for flocking are often described by a pressureless Euler system that are obtained from a microscopic description of the particles motion without a stochastic forcing. As a consequence, the system with pressure received less attention than the pressureless one, in particular for weak solutions, that is the class of solutions we aim for. A result on smooth, space-periodic solutions to this model with pressure is established in [7].

In [1], we assume that the communication rate is of *all-to-all* type, that is

$$K(x,x') \equiv 1\ . \tag{3}$$

Although this assumption, $K = 1$, simplifies the system and the nonlocal term turns into a local term, this special case still possesses many obstacles in the analysis point of view, in order to capture the existence of solutions and time-asymptotic flocking. In addition, the work on this special case indicates how the mechanism of the dissipative behavior of the solutions works and we expect this to be crucial in the extension of this analysis to general kernels.

We consider the Cauchy problem and assign initial data

$$(\rho,\mathfrak{m})(x,0) = (\rho_0(x),\mathfrak{m}_0(x)) \qquad x\in\mathbb{R}\,, \tag{4}$$

for the density $\rho = \rho(x,t)$, and momentum $\mathfrak{m} = \rho\mathrm{v}$. To capture the emergent behavior of self-organized systems, we assume that the initial total mass ρ is confined in a bounded interval, and is uniformly positive in there and the initial momentum $\mathfrak{m}_0$ taken to 0 outside that region, i.e. there exist $a_0 < b_0$ such that

$$\mathrm{supp}\{(\rho_0,\mathfrak{m}_0)\} \subset I_0 \doteq [a_0,b_0]\,, \qquad \operatorname*{ess\,inf}_{I_0} \rho_0 > 0\,, \tag{5}$$

while we use $\mathtt{v}_0 = \mathtt{m}_0/\rho_0$ only in I_0.

To study weak solutions in the context of flocking, we need to introduce a proper notion of entropy weak solution for which the support is bounded at any given time $t > 0$ in between the extremal particle paths, that is, there exists two absolutely continuous curves $t \mapsto a(t), b(t), t \in [0, +\infty)$ with

$$a(0) = a_0\,, \quad b(0) = b_0\,; \qquad a(t) < b(t) \qquad \text{for all } t > 0 \tag{6}$$

and

$$a'(t) = \mathtt{v}(a(t)+, t)\,, \qquad b'(t) = \mathtt{v}(b(t)-, t) \qquad \text{for } a.e.\ t > 0 \tag{7}$$

such that

$$\mathrm{supp}\{(\rho, \mathtt{m})(\cdot, t)\} \subset I(t) \doteq [a(t), b(t)]\,, \qquad t > 0\,. \tag{8}$$

The following notion of *entropy weak solutions with concentration* is motivated by the *ad-hoc* boundary condition:

The vacuum region is connected with the non-vacuum one by a shock discontinuity.

This choice is made in order to allow a sharp front with finite speed to arise as expected in flocking. In this way, we exclude the case of a rarefaction connecting a vacuum region with a non-vacuum since in such a case, due to the pressure law (2), the front would not have a proper interpretation in terms of flocking because of the unbounded maximal speed. [1]

We assume that

$$\iint \{\rho\phi_t + \mathtt{m}\phi_x\}\ dxdt = 0\,, \tag{9}$$

holds true for all $\phi \in C_0^\infty(\mathbb{R} \times (0, \infty))$. Therefore, the Rankine-Hugoniot condition $[\mathtt{m}] = \dot{x}[\rho]$ must hold along discontinuities $x(t)$, in particular along the free boundaries $a(t)$ and $b(t)$, which is consistent with (7). As a consequence, conservation of mass holds:

$$M(t) = \int_{\mathbb{R}} \rho(x,t)\,dx = \int_{I(t)} \rho(x,t)\,dx = \int_{\mathbb{R}} \rho_0(x)\,dx =: \mathsf{M}\,, \qquad \forall\, t \geq 0\,. \tag{10}$$

The appropriate definition now follows:

Definition 1 Given the initial data $(\rho_0, \mathtt{m}_0) \in BV(\mathbb{R})$, together with (5), let $(\rho, \mathtt{m}) : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^2$ be a function with the following properties:

- the map $t \mapsto (\rho, \mathtt{m})(\cdot, t) \in L^1_{loc}(\mathbb{R}) \cap\ BV(\mathbb{R})$ is continuous in L^1_{loc};

[1] For instance, if a rarefaction of the 1^{st} family connects the two states $(\rho_\ell, \mathtt{v}_\ell)$ and $(\bar\rho, \mathtt{v}(\bar\rho))$, then using the pressure term (2), one has

$$\mathtt{v}(\bar\rho) = \mathtt{v}_\ell - \int_{\rho_\ell}^{\bar\rho} \frac{1}{s}\sqrt{p'(s)}\,ds = \mathtt{v}_\ell - \alpha \ln\left(\frac{\bar\rho}{\rho_\ell}\right), \qquad 0 < \bar\rho \leq \rho_\ell\,.$$

As $\bar\rho \to 0+$, then $\mathtt{v}(\bar\rho) \to \infty$.

- $\lim_{t\to 0+} (\rho, \mathfrak{m})(\cdot, t) = (\rho_0, \mathfrak{m}_0)$ in $L^1_{loc}(\mathbb{R})$;
- there exist two locally Lipschitz curves $t \mapsto a(t)$, $b(t)$, $t \in [0, +\infty)$ and a value $\rho_{inf} > 0$ such that (6), (7), (8) hold and

$$\operatorname*{ess\,inf}_{I(t)} \rho(\cdot, t) \geq \rho_{inf} > 0 \qquad \forall\, t > 0\,. \tag{11}$$

Then $(\rho, \mathfrak{m})$ is an entropy weak solution *with concentration* along $a(t)$ and $b(t)$ of the problem (1), (4) with (2) and (3), if

(a) the integral identities (9) and

$$\iint \left\{ \mathfrak{m}\phi_t + \left[\frac{\mathfrak{m}^2}{\rho} + p(\rho) \right] \phi_x \right\} dxdt - \iint \left[\mathfrak{m}\,\mathsf{M} - \rho\, M_1(t) \right] \phi\, dxdt$$
$$- \int_0^\infty \left[p(\rho(b(t)-, t))\phi(b(t), t) - p(\rho(a(t)+, t))\phi(a(t), t) \right] dt = 0\,, \tag{12}$$

hold true on $\mathbb{R} \times (0, \infty)$ for all test functions $\phi \in C_0^\infty(\mathbb{R} \times (0, \infty))$, with M the conserved total mass given at (10),

$$M_1(t) = \int_{\mathbb{R}} \mathfrak{m}(x, t)dx + P_b(t) - P_a(t)\,, \tag{13}$$

and

$$\begin{cases} P_b(t) := \displaystyle\int_0^t e^{-\mathsf{M}(t-s)} p(\rho(b(s)-, s))\, ds\,, \\ P_a(t) := \displaystyle\int_0^t e^{-\mathsf{M}(t-s)} p(\rho(a(s)+, s))\, ds\,; \end{cases} \tag{14}$$

(b) the solution is entropy weak for every pair of convex entropy-entropy flux (η, q) functions for the system (1) , i.e. the inequality

$$\partial_t \eta(\rho, \mathfrak{m}) + \partial_x q(\rho, \mathfrak{m}) \leq \eta_{\mathfrak{m}} \left\{ \rho M_1(t) - \mathfrak{m}\mathsf{M} \right\}$$

holds in the sense of distributions on the open set $\Omega = \{(x, t);\ t > 0,\ x \in (a(t), b(t))\} \subset \mathbb{R} \times (0, +\infty)$.

As a consequence of the boundary condition above and in conjunction with the above definition of weak solutions, we deduce a conservation of momentum property. More precisely, we introduce a quantity that we call *total momentum* $\widehat{\mathfrak{m}}$, with two delta shocks supported on the free boundaries $a(t)$ and $b(t)$, that is the distribution

$$\widehat{\mathfrak{m}}(\cdot, t) := \mathfrak{m}(\cdot, t) + \delta_{b(t)} P_b(t) - \delta_{a(t)} P_a(t)\,, \quad t > 0\,, \tag{15}$$

where δ_a denotes the Dirac delta distribution on $\mathbb{R}$ with mass at $x = a$. This new singularity of the total momentum $\widehat{\mathfrak{m}}$ along the free boundaries $a(t)$ and $b(t)$ is known as delta shock and references can be found in Dafermos [9], Chapter 9. In what follows, we use the following standard notation $< \cdot, \cdot >$:

$$< \widehat{\mathfrak{m}}(\cdot,t), \phi(\cdot,t) >:= \int_{I(t)} \mathfrak{m}(x,t)\phi(x,t)dx + P_b(t)\phi(b(t),t) - P_a(t)\phi(a(t),t), \quad t > 0$$

as the value of the functional $\widehat{\mathfrak{m}}$ over C_0^∞, for all test functions $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$, noting here that $\mathfrak{m} = 0$ for $x \notin I(t)$. Now, by definition (14), we observe that

$$P_b'(t) + \mathsf{M}P_b(t) = p(\rho(b(t)-,t)),$$

and therefore

$$\int_0^\infty < \delta_{b(t)}P_b(t), \phi_t(\cdot,t) - \mathsf{M}\phi > dt = -\int_0^\infty p(\rho(b(t)-,t))\phi(b(t),t)\,dt\,.$$

Thus, we are led to the identity:

$$\int_0^\infty < \widehat{\mathfrak{m}}(\cdot,t), \phi_t(\cdot,t) - \mathsf{M}\phi > dt = \iint_\Omega \mathfrak{m}\,(\phi_t - \mathsf{M}\phi)\;dxdt$$
$$- \int_0^\infty [p(\rho(b(t)-,t))\phi(b(t),t) - p(\rho(a(t)+,t))\phi(a(t),t)]\;dt\,.$$

Hence, the integral identity (12) reduces to:

$$\int_0^\infty < \widehat{\mathfrak{m}}(\cdot,t), \phi_t(\cdot,t) > dt + \iint \left[\frac{\mathfrak{m}^2}{\rho} + p(\rho)\right]\phi_x dxdt$$
$$+ \iint \rho\, M_1(t)\phi\, dxdt - \mathsf{M}\int_0^\infty < \widehat{\mathfrak{m}}(\cdot,t), \phi > dt = 0\,.$$

If we choose the test function $\phi(x,t) = \phi_1(x)\psi(t)$, with $\phi_1(x) = 1$ for all $x \in \cup_{t\in[T_1,T_2]}I(t)$ and $\psi(t) = 0$ for $t \notin [T_1,T_2]$, with $0 < T_1 < T_2$, and notice that $M_1(t) =< \widehat{\mathfrak{m}}(\cdot,t), \phi_1 >$, we get $M_1(T_2) - M_1(T_1) = 0$. Since this holds true for arbitrary times, $0 < T_1 < T_2$, we deduce the conservation of (extended) momentum $M_1(t)$. By the time continuity of $\int \mathfrak{m}(\cdot,t)\,dx$ and the definition (14), we conclude that $M_1(t) = M_1(0+)$ that is

$$\int_{I(t)} \mathfrak{m}(x,t)\,dx + P_b(t) - P_a(t) = \int_{\mathbb{R}} \mathfrak{m}_0(x)\,dx =: \mathsf{M}_1 \qquad \forall\, t \geq 0\,. \tag{16}$$

2 Theorems

In this section, we state the main results of [1]. More precisely:

- Theorem 1 in Section 2.1 concerns the global in time existence of solutions obeying the notion introduced in Definition 1 without any restriction on the size of the total variation of the initial data, that conserve mass and momentum and at the

same time, the non-vacuum region Ω is separated by the vacuum ones by the two free boundaries alongside delta shocks, which are present in the total momentum;

- Theorem 2 in Section 2.2 states that, under an appropriate condition on the initial data, the solution admits time-asymptotic flocking.

2.1 Global Existence

First, we state the result on global existence in time of entropy weak solutions with concentration to (1) with bounded support.

Theorem 1 *Assume that the initial data* $(\rho_0, \mathtt{m}_0) \in BV(\mathbb{R})$ *and satisfy* (5) *with pressure* (2)*. Then the Cauchy problem* (1)*,* (4) *with* (3)*, admits an entropy weak solution with concentration* $(\rho, \mathtt{m})$ *in the sense of Definition 1. Moreover, conservation of mass* (10) *and of momentum* (16) *hold true.*

From the definition of solution $(\rho, \mathtt{m})$, we clarify that we use variable $\mathtt{v}$ only in the support $I(t)$ for all $t > 0$ satisfying $\mathtt{v} = \frac{\mathtt{m}}{\rho}$, where it is well defined, while only the variables ρ and $\mathtt{m}$ are used in the complement of $I(t)$. Now, by conservation of mass and momentum, we have the values M and M_1 in (10) and (16) respectively, and the average velocity $\bar{\mathtt{v}}$ defined by

$$\bar{\mathtt{v}} \doteq \mathsf{M}_1/\mathsf{M}\,. \tag{17}$$

By means of (10) and (16), system (1) rewrites as

$$\begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0, \\ \partial_t(\rho \mathtt{v}) + \partial_x\left(\rho \mathtt{v}^2 + p(\rho)\right) = -\mathsf{M}\rho\,(\mathtt{v} - \bar{\mathtt{v}})\ . \end{cases} \tag{18}$$

Thus, to prove Theorem 1, it is equivalent to establish an entropy weak solution with concentration to (18) for the initial data assumed in the theorem that conserves mass and momentum. Now, system (18) belongs to the class of system of balance laws and the Cauchy problem for strictly hyperbolic systems of balance has been studied in [11, 10] under appropriate conditions. However, the previous results do not apply to system (18) because of large data and the loss of strict hyperbolicity due to the vacuum present in $\mathbb{R} \setminus [a_0, b_0]$. For the notion of the physical vacuum boundary for (18), we refer to [24, 25]. About the asymptotic behavior of the solutions it is conjectured to obey the porous media equation and for the case of the pressure (2), we refer to [19], but when the initial density ρ_0 tends to a positive value as $x \to \pm\infty$, which is not our case. Other works [18, 20] study the system with the pressure $p(\rho) = \rho^\gamma$ with $\gamma > 1$ and therefore, their analysis cannot be applied again here.

One of the main challenges to construct a convergent approximate sequence to system (18), (4) is the loss of strict hyperbolicity. To overcome this, we transform the problem into Lagrangian coordinates in the spirit of Wagner [30]. Recasting system (18) from Eulerian $(\rho(x,t), \mathtt{v}(x,t))$ into the Lagrangian variables $(u(y,t), v(y,t))$,

we get the equations

$$\begin{cases} \partial_\tau u - \partial_y v = 0, \\ \partial_\tau v + \partial_y(\alpha^2/u) = -\mathsf{M}(v - \bar{\mathsf{v}}) \end{cases} \tag{19}$$

with the domain $\{(y,t);\ t \geq 0,\ y \in (0,\mathsf{M})\}$. However, the equivalence of weak solutions between the Eulerian and Lagrangian solutions does not apply from [30] because of the finite mass condition (5) in our case. Actually, because of this, the problem in Lagrangian coordinates is not Cauchy any more but boundary value problem and further difficulties arise. Now, system (19) has been studied but mostly in the context of the Cauchy problem or with different boundary conditions, cf. [28, 11, 26, 2, 12].

Our strategy to prove Theorem 1 is: First, using the Riemann solution to (18) around the vacuum that is admissible for the flocking model and the definition of *weak solution with concentration*, we recast the problem (18), (4) in Lagrangian variables to (19) with *non-reflecting* boundary conditions at $y = 0, \mathsf{M}$. Actually, the boundary conditions are expressed by the fact that, when a wave-front reaches the boundary, there is no resulting emitted wave. This is the natural counterpart to the behavior of the free boundaries that, in Eulerian variables, delimit the non-vacuum region. Next, we construct approximate solutions (u^ν, v^ν) to (18), (4) using the front tracking algorithm (cf. Bressan [5] and Holden–Risebro [21]), define appropriate Lyapunov functionals and show that the total variation in space and time of the approximate sequence remains bounded. Combining these results, we prove the convergence to an entropy weak solution of the system in Lagrangian coordinates. The last step is to transform this analysis to problem (18) in Eulerian. We work, at the level of the approximate solutions, to show the equivalence between Eulerian and Lagrangian variables in the spirit of [30] and within the domain $I(t) = [a(t), b(t)]$ where no vacuum is present, see Fig. 1. In this way, we construct the approximate

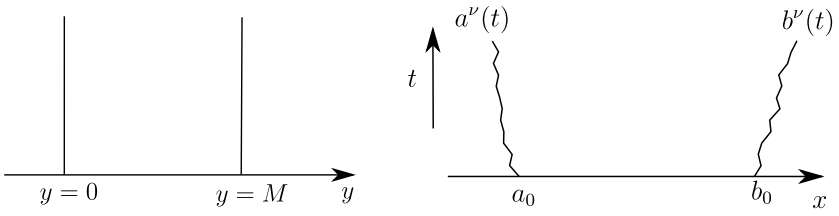

Fig. 1 On the left, the domain in Lagrangian variables for (u^ν, v^ν) and on the right, the domain in Eulerian variables for $(\rho^\nu, \mathsf{v}^\nu)$

solutions $(\rho^\nu, \mathsf{v}^\nu)$ to (18) that inherits the convergence property from the change of coordinates and it is appropriately extended to the half-plane. Also, this coordinate transformation together with the approximation scheme allow us to pass from the non-reflecting boundary conditions at $y = 0, \mathsf{M}$ to the free boundaries $a(t) < b(t)$ in the Eulerian coordinates. It is shown finally that, in the limit, $(\rho^\nu, \mathsf{v}^\nu)$ converges

to an *entropy weak solution with concentration* that conserves mass and (extended) momentum.

2.2 Time-Asymptotic Flocking

Another important issue we address, except of the global in time existence, is the long-time behavior of the entropy weak solution with concentration to (1), (4) with (2), which is interesting in the context of self-organization. The terminology “flocking" corresponds to the phenomenon in which self-organized individuals, using only limited environmental information and simple rules, get organized into an ordered motion. In the spirit of [17, 16], this behavior is captured in the following definition:

Definition 2 We say that the *entropy weak solution with concentration* $(\rho, \mathtt{m})(x,t)$ to system (1), admits time-asymptotic flocking if:

1. the support $I(t)$ of the solution remains bounded for all times, i.e.

$$\sup_{0\le t<\infty}\{b(t)-a(t)\}<\infty\,. \tag{20}$$

2. the velocity satisfies

$$\lim_{t\to\infty}\operatorname*{ess\,sup}_{x_1,x_2\in I(t)}|\mathtt{v}(x_1,t)-\mathtt{v}(x_2,t)|=0. \tag{21}$$

Indeed, condition (20) assures that the support of the solution is uniformly bounded, thus defining the “flock", while condition (21) yields that alignment occurs, i.e. the diameter of the set of velocity states within the support $I(t)$ goes to zero time-asymptotically.

We observe that solutions obtained in Theorem 1 satisfy condition (20) immediately by combining (8), (11) and (10). Therefore, to establish the *time-asymptotic flocking* property, it suffices to show (21), or equivalently that

$$\lim_{t\to\infty}\operatorname*{ess\,sup}_{x_1,x_2\in I(t)}|\mathtt{v}(x_1,t)-\bar{\mathtt{v}}|=0$$

where $\bar{\mathtt{v}}$ is defined at (17). Assuming that the initial data satisfy

$$e^{2q}\mathsf{M}^2<\alpha\max\{\rho_0(a_0+),\rho_0(b_0-)\}\,, \tag{22}$$

where q stands for the initial bulk:

$$q:=\frac{1}{2}\mathrm{TV}\{\ln(\rho_0)\}+\frac{1}{2\alpha}\mathrm{TV}\{\mathtt{v}_0\} \tag{23}$$

and TV the total variation in the support I_0, we show that flocking occurs. In fact, if initial bulk q is controlled by the initial density at the endpoints a_0 and b_0 according to (22), then velocity decays to $\bar{\mathtt{v}}$ at an exponential in time rate. The result is:

Theorem 2 *Let* $(\rho, \mathtt{m})$ *be the entropy weak solution with concentration to* (1), (2), (4) *with* (3), *the initial data* $(\rho_0, \mathtt{m}_0) \in BV(\mathbb{R})$ *satisfying* (5) *and with* $q > 0$ *given in* (23) *as obtained in Theorem 1. Suppose that* (22) *holds true, then the solution* $(\rho, \mathtt{m})$ *admits* time-asymptotic flocking. *More precisely, the oscillation of the velocity decays exponentially fast, i.e. there exists* $t_0 > 0$ *such that*

$$\operatorname*{ess\,sup}_{x_1, x_2 \in I(t)} |\mathtt{v}(x_1, t) - \mathtt{v}(x_2, t)| \leq C_2' e^{-C_1' t}, \qquad \forall\, t \geq t_0 \tag{24}$$

for some positive constants C_1', C_2'.

Our strategy to prove Theorem 2 involves careful analysis of the long time behavior of the approximate solutions constructed in the existence part using the results about the *wave strength dissipation* for the system of isothermal flow that is, the homogeneous version of (1)–(2) as obtained in [2, 3, 4]. These properties are used to provide uniform bounds on the vertical traces of approximate solutions and for the time-asymptotic analysis. In addition, we introduce a new functional, called *the total variation that is weighted by generation order*

$$V(t) = \sum_{k \geq 1} \xi^k F_k(t)$$

for $\xi \geq 1$ taking values in an appropriate interval and $F_k(t)$ being the total variation functional of generation k. This allows us to detect a *geometric decay property* in terms of the number of wave reflections for the homogeneous system and it turns out that we can control under assumption (22), the possible increase of the functional which is due to the reflections produced by the damping term leading to the time-exponential decay established in (24).

In a work in progress, we are going to relax the sufficient condition (22) to allow for time-asymptotic flocking with *any* $(\rho_0, \mathtt{m}_0) \in BV(\mathbb{R})$ that satisfies (5).

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