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Computational real algebraic geometry and applications to robotics

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Computational real algebraic geometry and applications to robotics

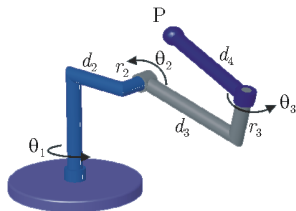
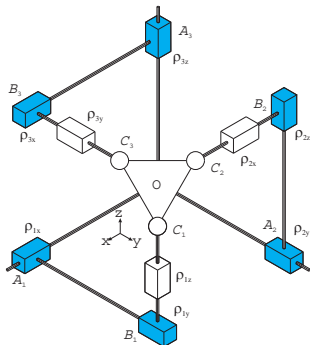
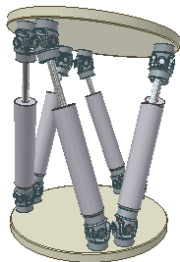
JNCF lecture, part 1

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Inria Nancy - Grand Est

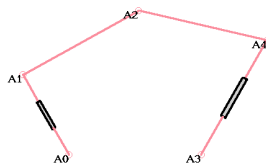
March 1st, 2021

Mechanisms



- Joint variables
 - r_1, r_2
- Pose variables
 - x, y
- Passive variables
 - θ_1, θ_2

Parallel PR-PRR

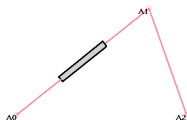


- Equations

$$(F) \begin{cases} x = \cos\left(\frac{2\pi}{3}\right)r_1 + \cos(\theta_1) \\ x = 1 + \cos\left(\frac{\pi}{3}\right)r_2 + \cos(\theta_2) \\ y = \sin\left(\frac{2\pi}{3}\right)r_1 + \sin(\theta_1) \\ y = 1 + \sin\left(\frac{\pi}{3}\right)r_2 + \sin(\theta_2) \end{cases}$$

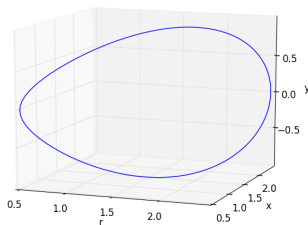
Workspace, Joint space

- Q : joint space
- W : workspace



Parallel RPR-R

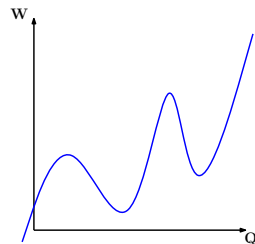
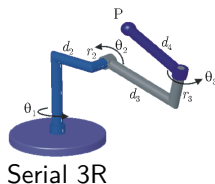
- Total space: $Q \times W$
 - solutions of F : $V(F) \subset W \times Q$



- Canonical projections:
 - $\pi_W : V(F) \rightarrow W$
 - $\pi_Q : V(F) \rightarrow Q$

Serial robot

- Glossary:
 - P: prismatic joint
 - R: rotation joint
 - U: Cardan joint
 - S: spherical joint



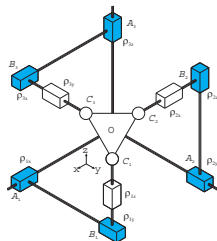
Properties

- Inverse Kinematics (IK) hard
- Forward Kinematics (FK) easy: 1 solutions

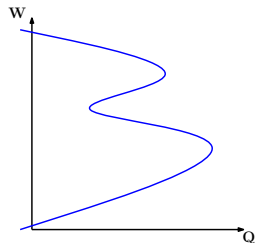
Parallel robot

- Glossary:

- P: prismatic joint
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- S: spherical joint



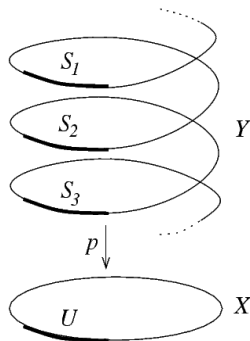
Parallel 3-PPPS



Properties

- Inverse Kinematics (IK) easy
- Forward Kinematics (FK) hard: several solutions
- 2 solutions can cross
 - loose of control
 - break

Covering map



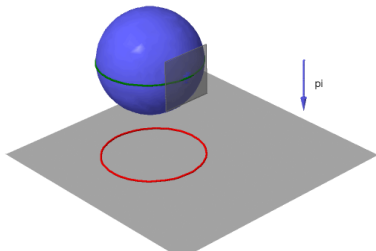
Definition

The continuous map $f : S \rightarrow U$ is a **covering map** if:

$$f^{-1}(U) = S_1 \cup \dots \cup S_k \text{ where } \begin{cases} S_i \simeq U \\ S_i \text{ pairwise disjoint.} \end{cases}$$

Critical points

- $V \subset \mathbb{R}^n$ smooth variety of dimension p
- $\pi : V \rightarrow \mathbb{R}^p$ canonical projection



Critical points

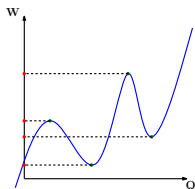
Let $T(a)$ be the linear space tangent to V at point a .
The **critical points** a of V for the projection π satisfy:

$$\dim(\pi(T(a))) < p$$

Case of the serial robot

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

- FK: always 1 solution $\Rightarrow \pi_Q : V(F) \rightarrow Q$ invertible
- IK: partition W in W_0, W_1, \dots, W_k s.t.:
 - W_0 are the critical values of π_W
 - W_1, \dots, W_k are the connected components of $W \setminus W_0$
- Critical points of π_W : **serial singularities**



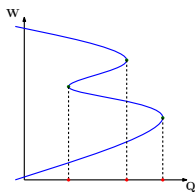
Theorem (covering map and critical values)

For all $1 \leq i \leq k$, the restriction of π_W to $\pi_W^{-1}(W_i)$ is a covering map above W_i .

Case of parallel robot

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

- IK: always 1 solution $\Rightarrow \pi_W : V(F) \rightarrow W$ invertible
- FK: partition Q in Q_0, Q_1, \dots, Q_k s.t.:
 - Q_0 are the critical values of π_Q
 - Q_1, \dots, Q_k are the connected components of $Q \setminus Q_0$
- Critical points of π_Q : **parallel singularities**

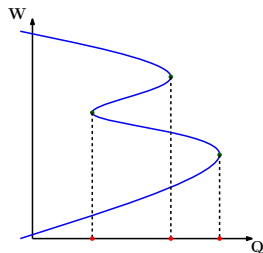


Theorem (covering map and critical values)

For all $1 \leq i \leq k$, the restriction of π_Q to $\pi_Q^{-1}(Q_i)$ is a covering map above Q_i .

Case of parallel robots

- FK: $F_q(x) = 0$, system parametrized by q
- For fixed q , finitely many solutions (0-dimensional)
- π_Q is not a covering map near q
 - \Rightarrow two sheets of solutions cross
 - $\Rightarrow F_q(x) = 0$ has singular solutions



- Remark: $V(F)$
 - not bounded: take asymptotes into account
 - not smooth: take singularities into account

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

$$\underbrace{\frac{\partial F}{\partial q}}_A dq + \underbrace{\frac{\partial F}{\partial x}}_B dx = 0$$

- Serial singularities

$$(S_s) : F = 0, \det(A) = 0$$

- W_0 , critical values of π_W : projection on x_i of solutions of (S_s)
- W_1, \dots, W_k , complement of critical values

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

$$\underbrace{\frac{\partial F}{\partial q}}_A dq + \underbrace{\frac{\partial F}{\partial x}}_B dx = 0$$

- Parallel singularities

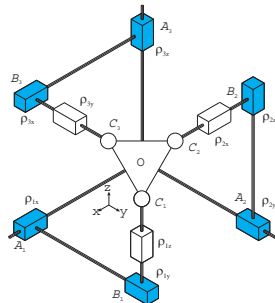
$$(S_p) : F = 0, \det(B) = 0$$

- Q_0 , critical values of π_Q : projection on q_i of solutions of (S_p)
- Q_1, \dots, Q_k , complement of critical values

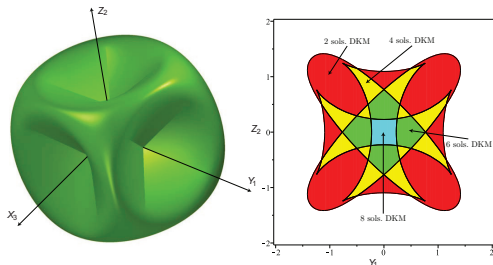
Example

- 3-PPPS:

- Parallel
- Joint variables: $x_1, y_1, y_2, z_2, x_3, z_3$
- Pose variables: $p_x, p_y, p_z, \varphi, \theta, \sigma$



- Critical values of π_Q , and partition of Q

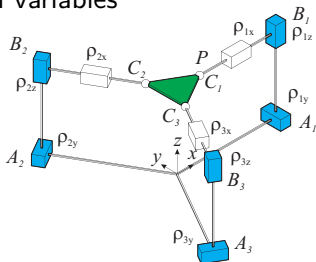
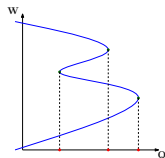


Design challenges

- $E \subset W$ given shape
 - Design a **parallel robot** without singularities in E

$$\pi_W(\text{critical points of } \pi_Q) \cap E = \emptyset$$

- Maximise the volume of E (lecture P. Lairez)
- \rightarrow design variables

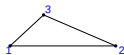


Demo

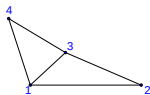
Modeling equations

- 1 Linkages
- 2 Rotations 3D
- 3 Singularities

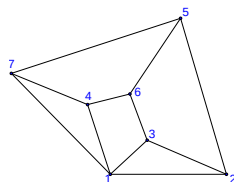
Planar Rigid Linkage: Laman Graph



3-bar



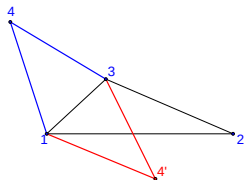
5-bar



11-bar

Constraints

- Fixed length bars: c_{ij}
- Free revolute joints
- Zero degree of freedom



- Several assembly modes
- Number depends on c_{ij}
- Max number of assembly modes?

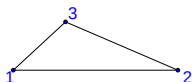
Properties of Rigid Linkages

- Construction steps



Henneberg steps: H_1 and H_2

- 3-bar rigid linkage



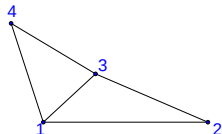
Properties of Rigid Linkages

- Construction steps



Henneberg steps: H_1 and H_2

- 5-bar rigid linkage



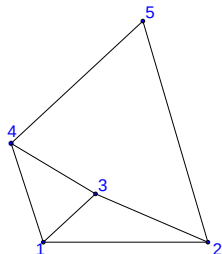
Properties of Rigid Linkages

- Construction steps



Henneberg steps: H_1 and H_2

- 7-bar rigid linkage



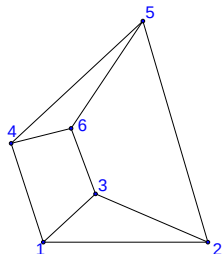
Properties of Rigid Linkages

- Construction steps



Henneberg steps: H_1 and H_2

- 9-bar rigid linkage



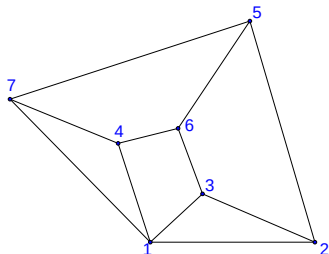
Properties of Rigid Linkages

- Construction steps



Henneberg steps: H_1 and H_2

- 11-bar rigid linkage



Theorem

A linkage is rigid \Leftrightarrow It can be constructed with H_1 and H_2

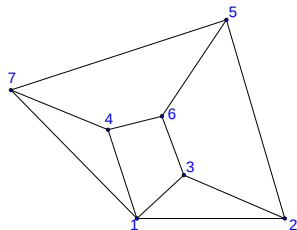
Corollary

$$\#Links = 2\#Joints - 3$$

Algebraic Modeling I

- c_{ij} : 10 parameters
- x_i, y_i : 14 variables

$$\begin{cases} x_1 = 0, y_1 = 0 \\ x_2 = 1, y_2 = 0 \end{cases}$$

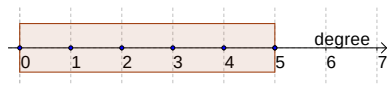


$$\begin{cases} x_3^2 + y_3^2 = c_{13} \\ (x_3 - 1)^2 + y_3^2 = c_{23} \\ (x_5 - 1)^2 + y_5^2 = c_{25} \\ (x_6 - x_3)^2 + (y_6 - y_3)^2 = c_{36} \\ x_4^2 + y_4^2 = c_{14} \end{cases} \quad \begin{cases} x_7^2 + y_7^2 = c_{17} \\ (x_6 - x_4)^2 + (y_6 - y_4)^2 = c_{46} \\ (x_5 - x_6)^2 + (y_5 - y_6)^2 = c_{56} \\ (x_7 - x_5)^2 + (y_7 - y_5)^2 = c_{57} \\ (x_4 - x_7)^2 + (y_4 - y_7)^2 = c_{47} \end{cases}$$

Number of solutions

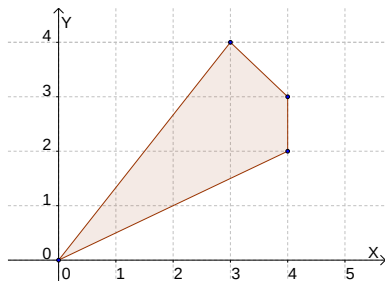
- Mixed Volume: $n! \text{Volume}(\text{Support})$ (same support)

1 variable



$$1 - X + 3X^2 - X^3 + 6X^4 - 5X^5 = 0$$

2 variables



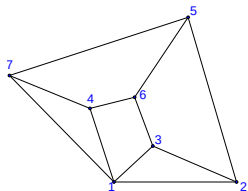
$$\begin{cases} 1 - X^4 Y^2 + 7X^4 Y^3 - 4X^3 Y^4 = 0 \\ 8 + 6X^4 Y^2 - 5X^4 Y^3 - X^3 Y^4 = 0 \end{cases}$$

- Our system: 2^{10}

Algebraic Modeling II

- Cayley-Menger matrix or distance matrix

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c_{12} & c_{13} & c_{14} & x_{15} & x_{16} & c_{17} \\ 1 & c_{12} & 0 & c_{23} & x_{24} & c_{25} & x_{26} & x_{27} \\ 1 & c_{13} & c_{23} & 0 & x_{34} & x_{35} & c_{36} & x_{37} \\ 1 & c_{14} & x_{24} & x_{34} & 0 & x_{45} & c_{46} & c_{47} \\ 1 & x_{15} & c_{25} & x_{35} & x_{45} & 0 & c_{56} & c_{57} \\ 1 & x_{16} & x_{26} & c_{36} & c_{46} & c_{56} & 0 & x_{67} \\ 1 & c_{17} & x_{27} & x_{37} & c_{47} & c_{57} & x_{67} & 0 \end{bmatrix}$$



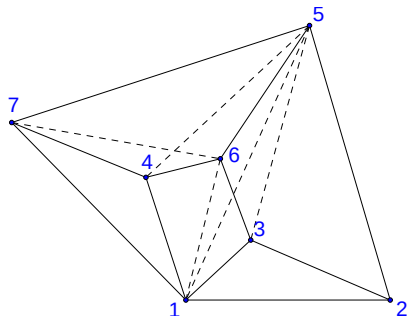
Theorem

The distance matrix has rank 4.

Corollary

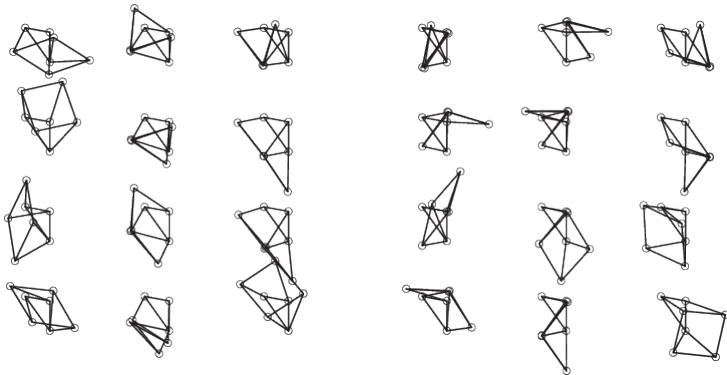
All the 5x5 minors vanish.

$$\left\{ \begin{array}{l} D(0, 4, 5, 6, 7)(c_{46}, c_{47}, c_{56}, c_{57}, x_{45}, x_{67}) = 0 \\ D(0, 1, 4, 6, 7)(c_{14}, c_{17}, c_{46}, c_{47}, x_{16}, x_{67}) = 0 \\ D(0, 1, 4, 5, 7)(c_{14}, c_{17}, c_{47}, c_{57}, x_{15}, x_{45}) = 0 \\ D(0, 1, 2, 3, 5)(c_{12}, c_{13}, c_{25}, c_{23}, x_{15}, x_{35}) = 0 \\ D(0, 1, 3, 5, 6)(c_{13}, c_{36}, c_{56}, x_{15}, x_{16}, x_{35}) = 0 \end{array} \right.$$



- Upper bound
 - Mixed volume: 56
- Lower Bound?

Sampling



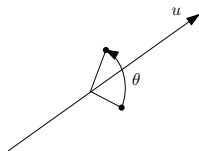
Number of assembly modes

	Maximal number of assembly modes							
bars	3	5	7	9	11	13	15	17
upper	2	4	8	24	56	136	344	880
lower	2	4	8	24	56	136	344	860

- [Bartzosa, Emiris, Legerský, Tsigaridas 2021]
- Started in 2002 with Borcea
- Bartzosa, Borcea, Emiris, Legerský, M., Streinu, Capco, Gallet, Grasegger, Koutschan, Lubbes, Schicho, Tsigaridas, ...

Rotations matrix in 3D

$$R = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$



$$R^T R = I$$

- Action of R is a rotation by θ around an axe u
- Set of rotation has dim 3

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_y(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = R_z(\psi)R_y(\varphi)R_x(\theta) \quad \text{or} \quad R = R_z(\psi)R_y(\varphi)R_z(\theta)$$

$$q_i^2 + q_j^2 + q_k^2 + q_r^2 = 1$$

$$R = \begin{pmatrix} 1 - 2(q_j^2 + q_k^2) & 2(q_i q_j - q_k q_r) & 2(q_i q_k + q_j q_r) \\ 2(q_i q_j + q_k q_r) & 1 - 2(q_i^2 + q_k^2) & 2(q_j q_k - q_i q_r) \\ 2(q_i q_k - q_j q_r) & 2(q_j q_k + q_i q_r) & 1 - 2(q_i^2 + q_j^2) \end{pmatrix}$$

Rotation of **axe:** (q_i, q_j, q_k)
angle: $2 \arccos(q_r)$

Anti-symmetric matrix - Exponential map

In 2D

$$\theta \Rightarrow \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

In 3D

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

$$R = e^A$$

Rotation of **axe:** (x, y, z)
angle: $\|(x, y, z)\|_2$

Anti-symmetric matrix - Cayley transform

In 2D

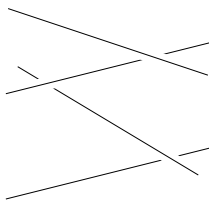
$$t = \tan\left(\frac{\theta}{2}\right) \Rightarrow \begin{pmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{pmatrix}$$

In 3D

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

$$R = (I - A)(I + A)^{-1}$$

Rotation of **axe:** (x, y, z)
angle: $2 \arctan(\|(x, y, z)\|_2)$



How many lines intersect 4 given lines?

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad P_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad M = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \\ 1 & 1 \end{pmatrix}$$

Definition

The Plücker coordinates of the line (P_1P_2) are the 6 minors of M

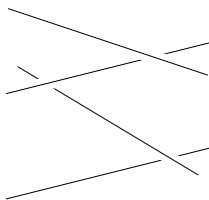
$$(d_x, d_y, d_z, m_x, m_y, m_z) = (P_2 - P_1, OP_1 \times OP_2)$$

- $(d, m) \in \mathbf{P}_5$ is on the Klein quadric

$$d \cdot m = 0$$

- Lines (d, m) and (d', m') intersect implies

$$d \cdot m' + m \cdot d' = 0$$



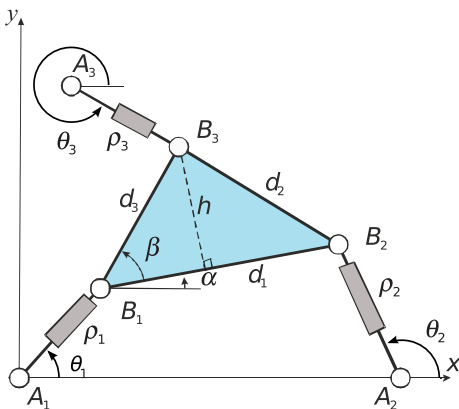
0, 1, 2 or infinitely many lines cross the 4 given lines

$$f_1(\mathbf{q}, \mathbf{x}), \dots, f_m(\mathbf{q}, \mathbf{x})$$

$$B = \begin{matrix} f_1 \\ \vdots \\ f_m \end{matrix} \left(\begin{matrix} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_m} \\ \frac{\partial f_j}{\partial x_j} \end{matrix} \right)$$

- $\det(B)$ multi-linear in its columns/rows
- $\det(B)$ of deg 1 in x_j leads to a parametrization of the singularities
- Simplifies the analysis of the singularities, as in the 3-RPR [Coste 2012]

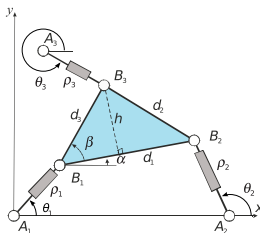
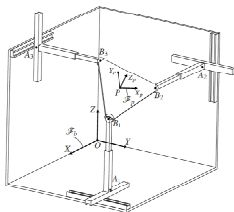
Plan parallel robot 3-RPR



- 3 degrees of freedom
- $d_1, d_2, d_3, A_1, A_2, A_3$ fixed
- Joint variables: r_1, r_2, r_3
- Pose variables: α, B_{1x}, B_{1y}

Demo

Singularity of parallel manipulator with Plücker vectors



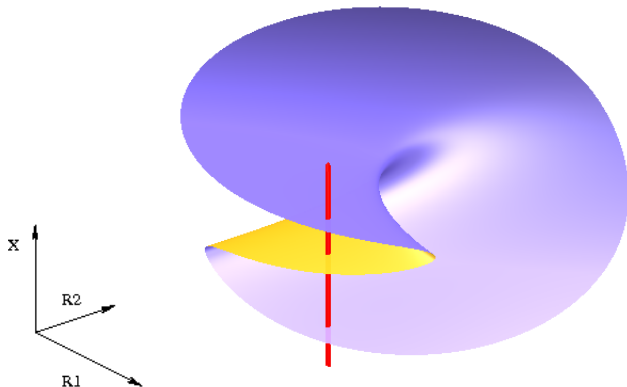
Remark

If the leg-platform joints are spherical, the rows of the inverse kinematic Jacobian matrix $A^{-1}B$ will involve the Plücker coordinates of lines associated to the legs.

⇒ singularities can be interpreted geometrically

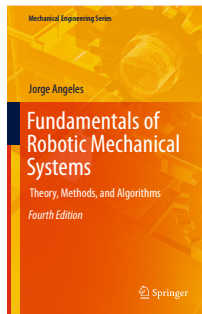
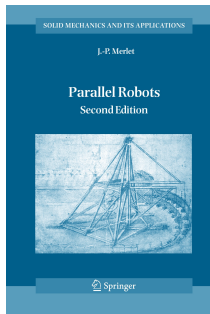
Cuspidal configuration

- **Cuspidal point:** point of order ≥ 3
- **Characterization:** A cuspidal robot has at least one cuspidal point



Demo

Further reading



À suivre

Solving systems

- 1 With initial point
 - Newton
- 2 Without initial point
 - Symbolic approaches
 - Numerical approaches

Computational real algebraic geometry and applications to robotics

JNCF lecture, part 2

Guillaume Moroz

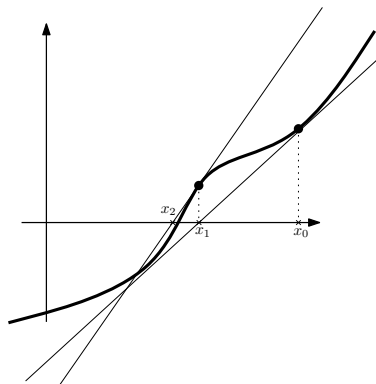
Inria Nancy - Grand Est

March 4th, 2021

Solving systems

- ① With initial point
 - Newton
- ② Without initial point
 - Symbolic approaches
 - Numerical approaches

$$f(x) = 0$$

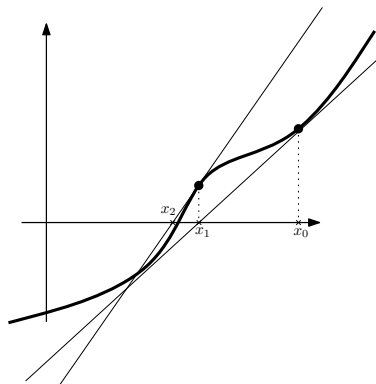


x_0 = initial point

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Good for path tracking

$$f(x) = 0$$



x_0 = initial point

$$x_{n+1} = x_n - Df(x_n)^{-1}f(x_n)$$

Good for path tracking

$$K(f, x_0) = \sup_x \|Df(x_0)^{-1}D^2f(x)\|$$
$$\beta(f, x_0) = \|Df(x_0)^{-1}f(x_0)\|$$

Theorem (Kantorovich)

If $\beta(f, x_0)K(f, x_0) \leq 1/2$,
then f has a unique solution in $B(x_0, 2\beta(f, x_0))$.

$$\gamma(f, x_0) = \sup_{k \geq 2} \left\| Df(x_0)^{-1} \frac{D^k f(x_0)}{k!} \right\|^{\frac{1}{k-1}}$$
$$\beta(f, x_0) = \|Df(x_0)^{-1} f(x_0)\|$$

Theorem (Smale)

If $\beta(f, x_0)\gamma(f, x_0) \leq 3 - 2\sqrt{2}$,
then f has a unique solution in $B(x_0, \frac{1-\sqrt{2}/2}{\gamma(f, x_0)})$.

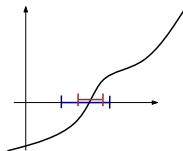
Reliable interval Newton

B a box interval containing p

1 variable

$$N(x) = p - \left(\frac{f(x) - f(p)}{x - p} \right)^{-1} f(p)$$

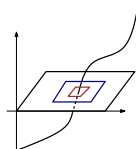
$N(B) \subset B \Rightarrow N$ has a fix point in B
 $\Rightarrow f$ has a solution in B



2 variables or more

$$N(B) = p - \square J(B)^{-1} \begin{pmatrix} f(p) \\ g(p) \end{pmatrix}$$

$N(B) \subset B \Rightarrow f = g = 0$ has a solution in B

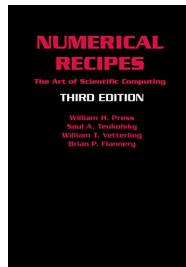
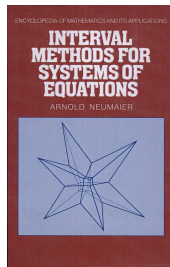
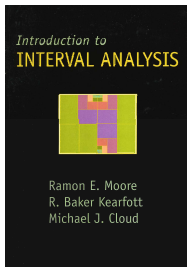
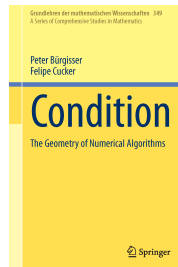
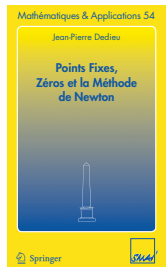
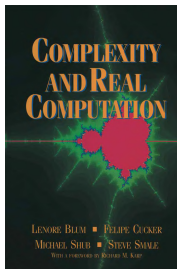


Interval operations

$$[a, b] \oplus [c, d] = [a + c, b + d]$$

$$[a, b] \otimes [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

Further reading



Computer Algebra Systems

- Maple
- Mathematica
- Mathmagix (mmx)
- Matlab
- SageMath
- Xcas
- ...

Numerical Newton

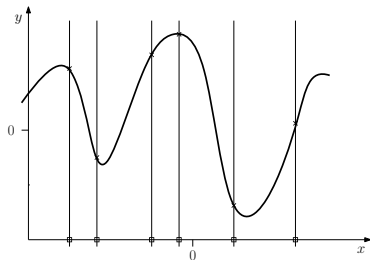
- Boost (C++)
- MPSolve (C) [Bini, Fiorentino, Robol]
- GNU Scientific Library (C)
- Roots, Optim, NLSolve (Julia)
- Scipy (Python)
- ...

Interval and ball arithmetic Libraries

- Arb (C, Python) [Johansson]
- GAOL (C++) [Goualard]
- Intlab (Octave/Matlab) [Rump]
- mpfi (C) [Revol]
- numerix (mmx, C++) [van der Hoeven]
- IntervalArithmetic (Julia) [Benet, Sanders]
- ...

Univariate Representation

Reduce problem to univariate polynomial



$$p(x) = 0$$
$$y = q(x)$$

Resultant definition

Given two polynomials in $\mathbb{C}[y]$:

- $P = p_0 y^d + \dots + p_d$ with roots $\sigma_1, \dots, \sigma_d$
- $Q = q_0 y^d + \dots + q_d$ with roots τ_1, \dots, τ_d

Definition: Resultant

$$\begin{aligned} \text{Res}(P, Q) &= p_0^d q_0^d \prod_{i,j} (\sigma_i - \tau_j) \\ &= p_0^d \boxed{Q}(\sigma_1) \cdots \boxed{Q}(\sigma_d) \\ &= (-1)^{d^2} q_0^d \boxed{P}(\tau_1) \cdots \boxed{P}(\tau_d) \end{aligned}$$

Bivariate case

Given two polynomials in $\mathbb{C}[x, y]$ of degree d in x and y :

- $P = p_0(x)y^d + \dots + p_d(x)$

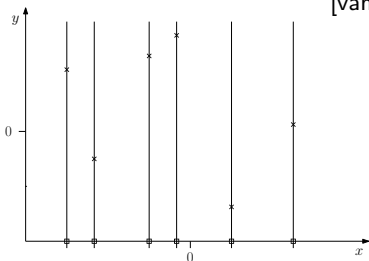
- $Q = q_0(x)y^d + \dots + q_d(x)$

- $\text{Res}(P, Q)$ is a polynomial in x

- $O(n^{3-1/\omega+\varepsilon})$ arithmetic operations

[Villard 2018]

- $O(n^{2+\varepsilon})$ randomized in a finite field
[van der Hoeven, Lecerf 2019]



$$\text{Res}_d(P(x, y), Q(x, y))(\alpha) = \text{Res}_d(P(\alpha, y), Q(\alpha, y))$$

First subresultant definition

Given two polynomials in $\mathbb{C}[y]$:

- $P = p_0 y^d + \dots + p_d$ with roots $\sigma_1, \dots, \sigma_d$
- $Q = q_0 y^d + \dots + q_d$ with roots τ_1, \dots, τ_d

Properties

- $Sres_1 = s_1 y + s_0$ has degree at most 1
- $Sres_1 \in \langle P, Q \rangle = I$

Definition: first subresultant

$$Sres_1(P, Q) = p_0^{d-1} \left((y - \sigma_1) \prod_{j \neq 1} \frac{Q(\sigma_j)}{\sigma_j - \sigma_1} + \dots + (y - \sigma_d) \prod_{j \neq d} \frac{Q(\sigma_j)}{\sigma_j - \sigma_d} \right)$$

First subresultant definition

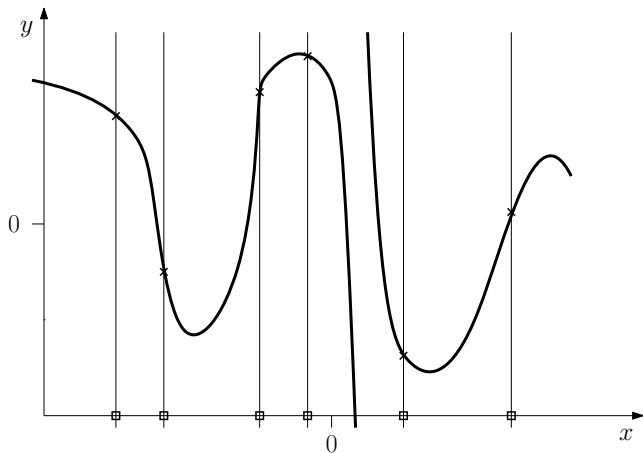
$$\varphi : \mathbb{C}[y]_{d-2} \times \mathbb{C}[y]_{d-2} \rightarrow \mathbb{C}[y]_{2d-2}$$
$$U \quad , \quad V \quad \mapsto UP + VQ$$

The s_0 and s_1 are the determinants of minors of the Sylvester Matrix

Sylvester matrix

$$\begin{array}{cccccccc} 1 & 2 & \cdots & & d+1 & d+2 & d+3 & \cdots \\ \left(\begin{array}{cccccccc} p_0 & & & & q_0 & & & \\ p_1 & p_0 & & & q_1 & q_0 & & \\ p_2 & p_1 & p_0 & & q_2 & q_1 & q_0 & \\ p_3 & p_2 & p_1 & p_0 & q_3 & q_2 & q_1 & q_0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{array} \right) \end{array}$$

Parametrization of y



$$s_1(x)y + s_0(x) = 0$$

$$r(x) = 0$$

Three variables and more

$$f_1(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

Problem

$$\text{Find } p(x_1) = q_1 f_1 + \dots + q_n f_n$$

Matrix of $1, x_1, \dots, x_1^D$ modulo $\langle f_1, \dots, f_n \rangle$

$$M = \begin{matrix} m_1 \\ \vdots \\ m_k \end{matrix} \begin{pmatrix} 1 & \dots & x_1^D \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \end{pmatrix}$$

$$\Rightarrow v \in \text{Ker}(M) \text{ iff } v_0 + \dots + v_D x^D \in \langle f_1, \dots, f_n \rangle$$

Three variables and more

$$f_1(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

Problem

$$\text{Find } p(x_1) = q_1 f_1 + \dots + q_n f_n$$

Matrix of multiplication by x_1 modulo $\langle f_1, \dots, f_n \rangle$

$$M = \begin{matrix} & x_1 m_1 & \dots & x_1 m_k \\ \begin{matrix} m_1 \\ \vdots \\ m_k \end{matrix} & \left(\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right) \end{matrix}$$

$$\Rightarrow \chi_M(x_1) \in \langle f_1, \dots, f_n \rangle$$

Three variables and more

$$f_1(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

Problem

$$\text{Find } p(x_1) = q_1 f_1 + \dots + q_n f_n$$

Matrix of multiplication by x_1 modulo $f_1(x_1)$

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{d-1} \end{pmatrix}$$

$$\Rightarrow \chi_M(x_1) = f_1(x_1) \in \langle f_1 \rangle$$

Normal form and univariate representation

Normal form

$$x_1 m_i \pmod{\langle f_1, \dots, f_n \rangle}$$

- Euclidean division by Gröbner basis

Groebner Bases computation

- $n > 2$ in $\tilde{O}((nd)^{(\omega+1)n})$ operations [Bardet, Faugère, Salvy 2015]
- $n = 2$ in $\tilde{O}(d^2)$ with terse representation [van der Hoeven, Larrieu 2018]

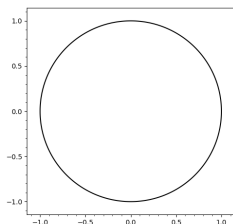
Univariate Representation

- Multivariate subresultant
- Rational Univariate Representation [Kronecker 1882, Rouillier 1999]
 - u-resultant and its derivatives at point (t, a_1, \dots, a_n)

$$\text{u-resultant} = C \prod_{\zeta | F(\zeta)=0} (u_0 + u_1 \zeta_n + \dots + u_n \zeta_n)$$

- Lexicographical Gröbner bases
 - Polynomial parametrization generically, bigger size [Dahan, Schost 2004]

Univariate Representation with Geometric Resolution



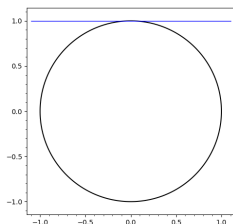
$$f(x, y) = 0$$

Theorem (Hensel lifting)

$$\begin{cases} y_0 & = \text{root of } f(0, y) = 0 \\ y_{n+1}(x) & = y_n(x) - \frac{f(x, y_n(x))}{\frac{\partial f}{\partial y}(x, y_n(x))} \pmod{x^{2^{n+1}}} \end{cases}$$

Then $y_n(x)$ is a root of $f(x, y) = 0 \pmod{x^{2^n}}$.

Univariate Representation with Geometric Resolution



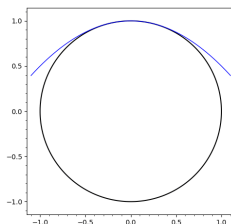
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Univariate Representation with Geometric Resolution



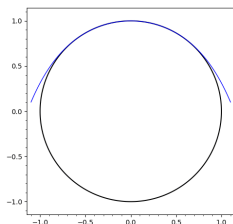
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Univariate Representation with Geometric Resolution



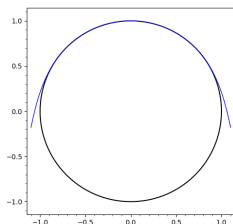
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Univariate Representation with Geometric Resolution



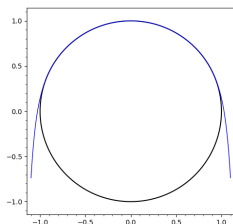
$$f(x, y) = 0$$

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Univariate Representation with Geometric Resolution



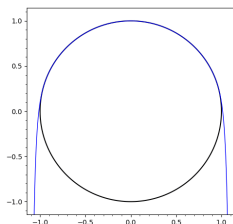
$$f(x, y) = 0$$

Theorem (Hensel lifting)

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Univariate Representation with Geometric Resolution



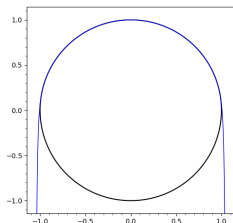
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Univariate Representation with Geometric Resolution



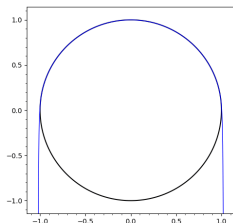
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Univariate Representation with Geometric Resolution



$$f(x, y) = 0$$

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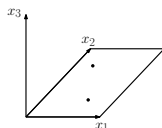
Then $y_n(x)$ is a root of $f(x, y) = 0 \pmod{x^{2^n}}$.

Univariate Representation with Geometric Resolution

$$f_1(x_1, x_2, x_3) = \dots = f_3(x_1, x_2, x_3) = 0$$

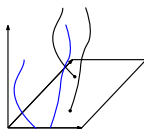
- RUR of $f_1(x_1, x_2, 0)$ and $f_2(x_1, x_2, 0)$

$$p_1(x_1) = 0 \quad x_2 = q_{12}(x_1)$$



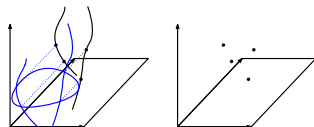
- Lift x_3 in $f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) = 0$

$$\tilde{p}_1(x_1, x_3) = 0 \quad x_2 = \tilde{q}_{12}(x_1, x_3)$$



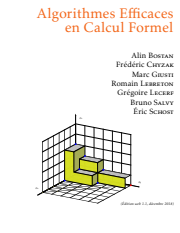
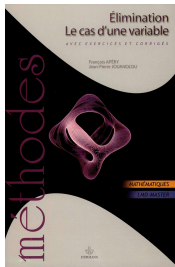
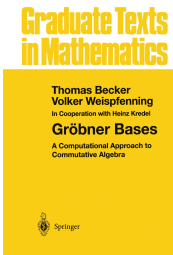
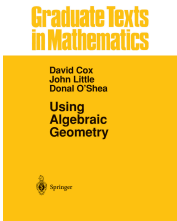
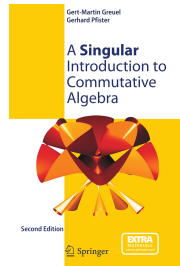
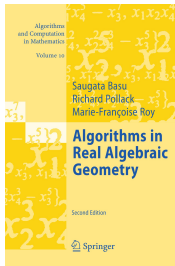
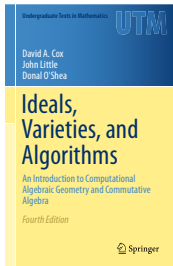
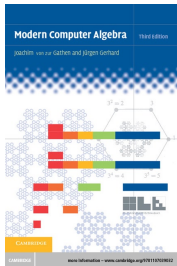
- RUR of $\tilde{p}_{11}(x_1, x_3)$ and $f_3(x_1, \tilde{q}_{12}(x_1, x_3), x_3)$

$$p_2(x_1) = 0 \quad x_2 = q_{22}(x_1) \quad x_3 = q_{23}(x_1)$$



$\tilde{O}(d^{3n})$ operations [Giusti, Lecerf, Salvy 2001]

Further reading

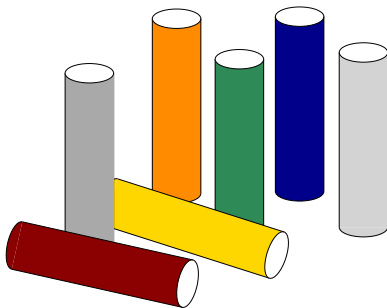


Computer Algebra systems

- Giac/xcas
- Maple
- Magma
- Mathmagix
- Mathematica
- Mcaulay2
- MuPAD
- SageMath
- Singular
- ...

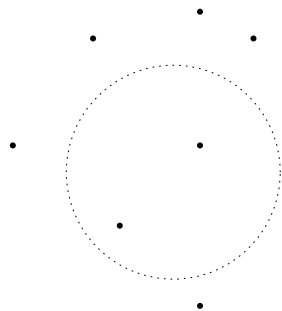
Special purpose libraries/software

- FGb (C) [Faugère]
- Flint (C) [Hart]
- msolve [Berthomieux, Eder, Safey El Din]
- RS/RS3 (C) [Rouillier]
- borderbasis (mmx, C++) [Trebuchet, Mourrain]
- algebramix (mmx, C++) [van der Hoeven, Lecerf]
- geomsolvex (mmx, C++) [Lecerf]
- larrix (mmx, C++) [Larrieu]
- ...



Is it possible to arrange 7 infinite cylinders of unit radius such that they are mutually touching?

Path continuation



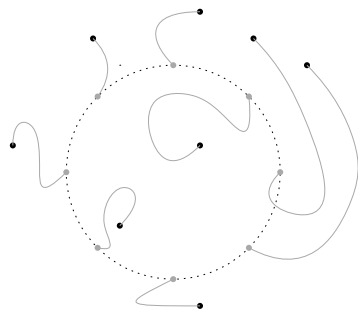
- $f(x) = 0$
- $\deg d$

- We know the solutions of $x^d - 1 = 0$

$$(1 - t)(x^d - 1) + tf(x) = 0$$

- Multivariate case: compute the mixed volume first
- Find 1 solution by following only one path
Polynomial time in $\binom{n+d}{d}$ average [Beltran, Pardo 2009], [Lairez 2017]
- Roots distribution of random polynomials [Edelman, Kostlan 1995]

Path continuation



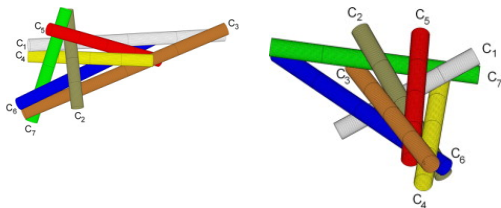
- $f(x) = 0$
- $\deg d$

- We know the solutions of $x^d - 1 = 0$

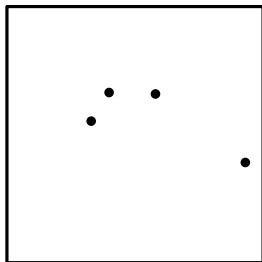
$$(1 - t)(x^d - 1) + tf(x) = 0$$

- Multivariate case: compute the mixed volume first
- Find 1 solution by following only one path
Polynomial time in $\binom{n+d}{d}$ average [Beltran, Pardo 2009], [Lairez 2017]
- Roots distribution of random polynomials [Edelman, Kostlan 1995]

2 arrangements of 7 cylinders mutually touching [Bozóki et al. 2015]



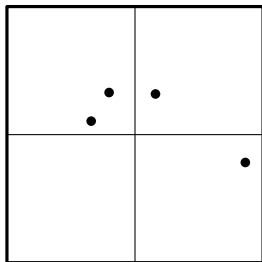
- Modeled with 20 equations in 20 variables
- Mixed volume 121 098 993 664
- First real solution after 80 000 000 paths
- Solution certified with Smale theorem



Test each cell

- 1 solution, guaranteed \Rightarrow keep
- 0 solution, guaranteed \Rightarrow remove
- don't know \Rightarrow subdivide

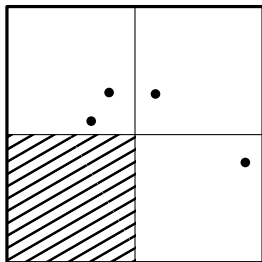
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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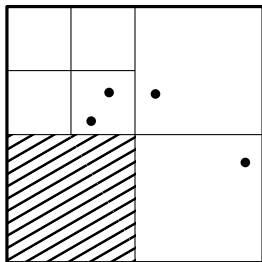
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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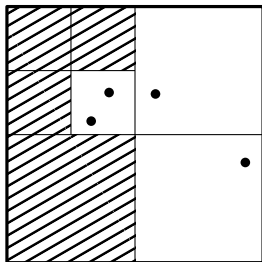
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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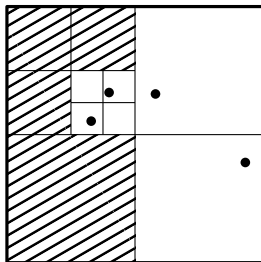
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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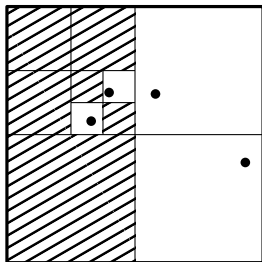
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

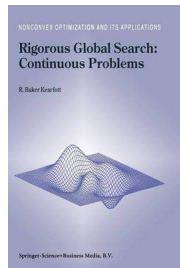
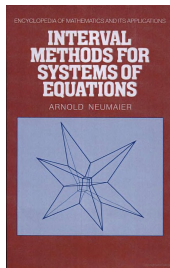
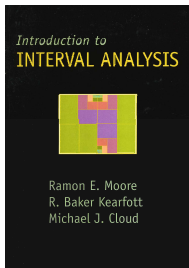
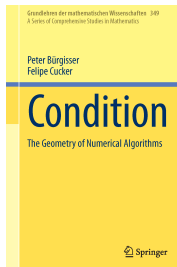
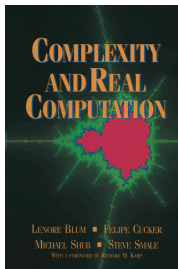


Test each cell

- 1 solution, guaranteed \Rightarrow keep
- 0 solution, guaranteed \Rightarrow remove
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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

Further reading



Path continuation

- Bertini (C++, Python) [Bates, Amethyst, Hauenstein, Sommese, Wampler]
- Hom4PS (C++) [Tien-Yien Li, Tianran Chen, Tsung-Lin Lee]
- PHCpack (Ada) [Vershelde]
- analyziz (mmx, C++) [van der Hoeven]
- ...

Subdivision multivariate

- ibex (C++) [Chabert]
- Realpaver (C) [Granvilliers]
- realroot (mmx, C++) [Mourrain]
- subdivision_solver (C++, Python) [Imbach]
- voxelize (C++) [M.]
- ...

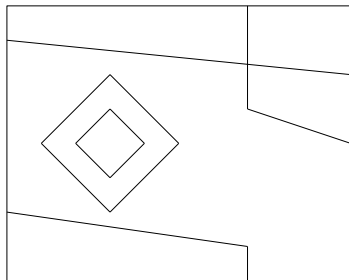
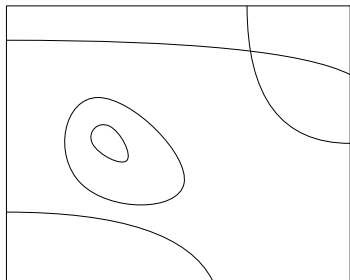
Subdivision univariate

- ANewDsc (C) [Kobel, Rouillier, Sagraloff]
- Clenshaw (C, Python) [M.]
- Ccluster (C, Julia) [Imbach, Pan, Yap]
- real_roots (sage) [Witty]
- RS (C) [Rouillier]
- SLV (C) [Tsigaridas]
- ...

Drawing reliably the projection of the solution

- 1 Isotopy
- 2 Projection
- 3 Drawing reliably

Drawing hypersurfaces



Definition (isotopy)

A triangulation T is isotope to $V \subset \mathbb{R}^n$ if there exists $\varphi : [0, 1] \times T \rightarrow \mathbb{R}^n$ such that:

- φ is continuous
- $\varphi(0, T) = T$ and $\varphi(1, T) = V$
- $\varphi_{t_0} : T \rightarrow \varphi(t_0, T)$ is an homeomorphism

Size of the triangulation [Kerber and Sagraloff 2011]

$f(x_1, \dots, x_n)$ of degree d

2 variables

Theorem

In the worst case $\Omega(d^2)$ vertices and $O(d^2)$ segments

3 variables

Theorem

In the worst case $\Omega(d^3)$ vertices and $O(d^5)$ triangles

n variables

Theorem

In the worst case $\Omega(d^n)$ vertices and $O(d^{3/4 \cdot 2^n - 1})$ simplices

Size of the triangulation [Kerber and Sagraloff 2011]

f of degree d and vertices on the surface

2 variables

Theorem

In the worst case $\Omega(d^{\cancel{2}^3})$ vertices and $O(d^{\cancel{2}^3})$ segments

3 variables

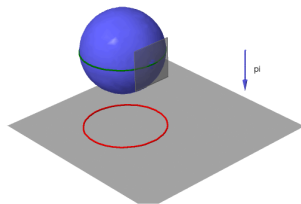
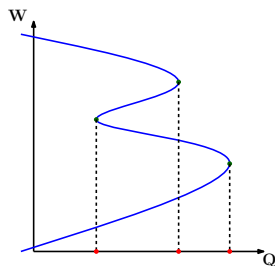
Theorem

In the worst case $\Omega(d^{\cancel{3}^4})$ vertices and $O(d^{\cancel{3}^7})$ triangles

n variables

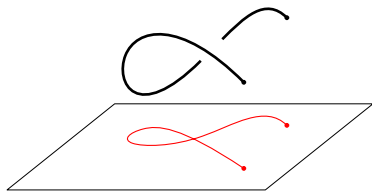
Theorem

In the worst case $\Omega(d^{n+1})$ vertices and $O(d^{\cancel{3}^4} 2^{n-1})$ simplices



V solutions of $f_1 = \dots = f_k = \det\left(\frac{\partial f_i}{\partial x_j}\right) = 0$ in $\mathbb{R}^n \times \mathbb{R}^k$

Goal: draw the projection of V in Q

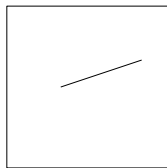
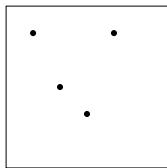
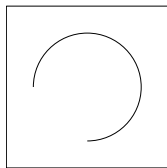


Definition (Semi-algebraic)

A **semi-algebraic** set is a set solution of a system of equalities and inequalities.

Theorem (Tarski-Seidenberg)

The projection of a semi-algebraic set is semi-algebraic.



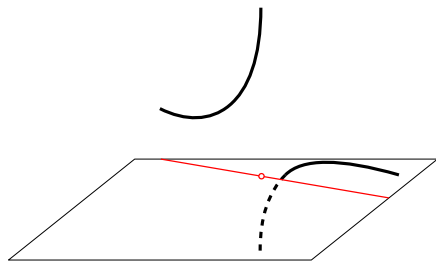
Definition (Zariski closure)

The **Zariski closure** of E is the minimal set \bar{E} containing E that is solution of a system of equalities.

Theorem (elimination)

- V solution of $p_1(q, x) = \dots = p_{k+1}(q, x) = 0$
- G Gröbner basis of F with respect to the lexdeg ordering with $x > q$
- G_q the polynomials in $\underline{G \cap \mathbb{Q}[q]}$
 $\Rightarrow \pi_Q(V) = \text{solutions of } G_q$

Projective elimination



$$p_1(q, x) = \cdots = p_{k+1}(q, x) = 0$$

Theorem

If V_p is the projective closure of V in $\mathbb{C}^n \times \mathbf{P}^k$, then

$$\overline{\pi_Q(V)} = \pi_Q(V_p)$$

Geometry of the elimination

$$x, y \in \mathbb{R}, z = a + ib \in \mathbb{C}$$

$$\begin{cases} 0 = f_1(x, y, a + ib) = f_1(x, y, a) - \frac{b^2}{2} f_1''(x, y, a) + \dots \\ \quad + ib \left(f_1'(x, y, a) - \frac{b^2}{6} f_1'''(x, y, a) + \dots \right) \\ 0 = f_2(x, y, a + ib) = f_2(x, y, a) - \frac{b^2}{2} f_2''(x, y, a) + \dots \\ \quad + ib \left(f_2'(x, y, a) - \frac{b^2}{6} f_2'''(x, y, a) + \dots \right) \end{cases}$$

Case $b = 0$: 2 equation in x, y, a

Case $b \neq 0$: 4 equation in x, y, a, b

\Rightarrow possibly isolated points with $x, y \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathbb{R}$

In general: $\overline{\pi_Q(V)}$ can have stable real components of dim $n - 2$

Geometry of the projection

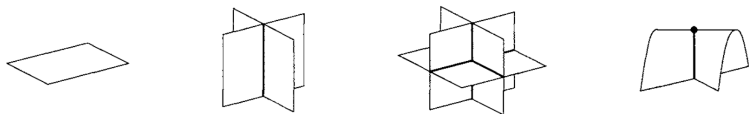


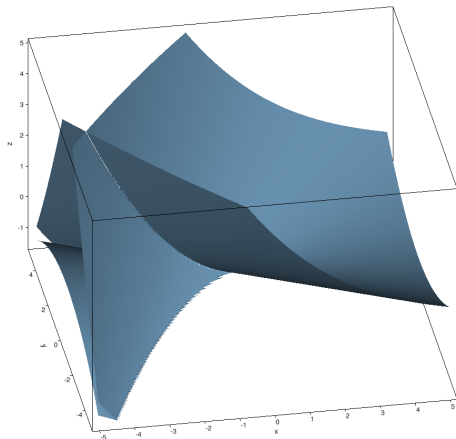
Figure 1: *Local singularities of images of generic maps of surfaces into 3-space*

[Guryonov 1997]

There exists a hypersurface $\Delta \subset C^\infty(M, \mathbb{R}^3)$ s.t. if $f \in C^\infty(M, \mathbb{R}^3) \setminus \Delta$, then the neighborhood of any point of $f(M)$ is one of the 4 above.

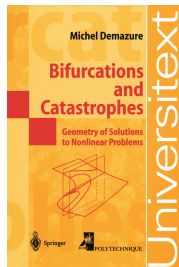
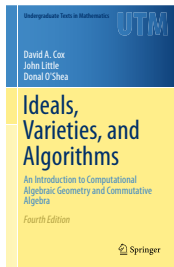
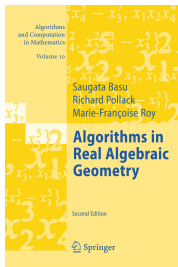
- Classification of generic singularities started with Whitney
- Thom introduced the transversality theorem
- Arnold and others provided numerous classifications

Geometry of the projection



For the projection of critical points, we also have swallowtail singularities

Further reading



Quantifier elimination

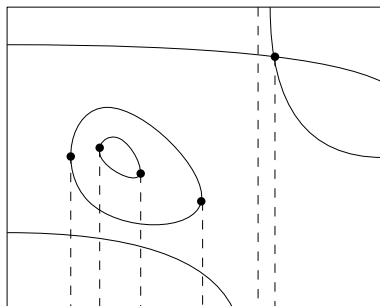
- Redlog (Reduce)
- RegularChains (Maple)
- Resolve (Mathematica)
- QEPCAD (C)
- Tarski (C++)
- ...

[Dolzmann, Sturm]
[Moreno Maza et al]
[Strzebonski]
[Brown]
[Brown]

Satisfiability of formula

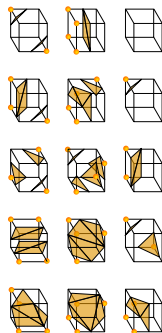
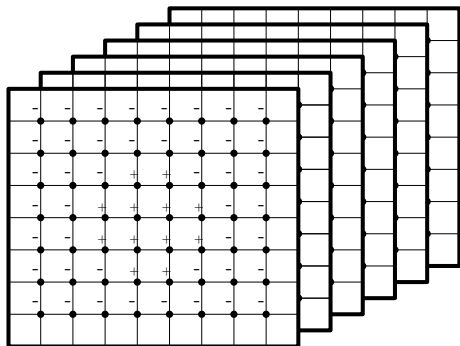
- RAGLib (Maple)
- RealCertify (Maple)
- RSolver (OCaml)
- TSSOS (Julia)
- ...

[Safey el Din]
[Magron, Safey el Din]
[Ratschan]
[Lassere, Magron, Wang]



- Topology of curves in $\tilde{O}(d^6 + \tau d^5)$ operations
[Kobel, Sagraloff, 2015], [Niang Diatta, Diatta, Rouillier, Roy, Sagraloff, submitted 2018]
- Topology of surfaces with $O(d^5)$ simplices
[Berberich, Kerber, Sagraloff 2009]

Marching cubes [Lorensen and Cline 1987]



- 1 Evaluate $f(x, y, z)$ of deg d on a grid of size $N = n \times n \times n$
- 2 For each cube, compute a triangle of the surface to plot

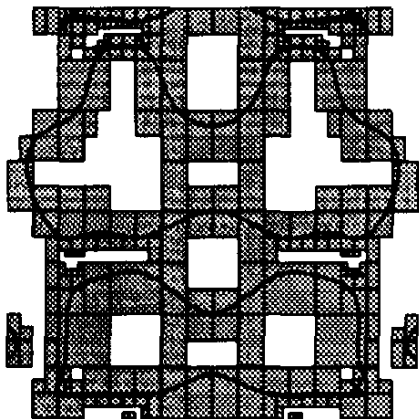
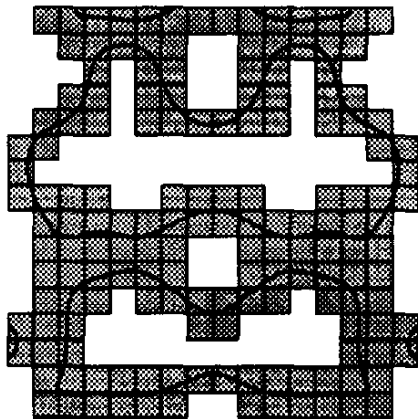
Complexity: Cost of evaluating f on 1 point with Hörner: $O(d^3)$

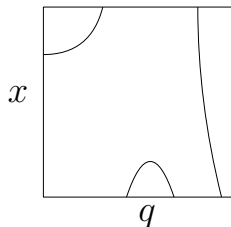
Total naive: $O(d^3 n^3)$

Reuse computation by coordinates: $O(dn^3)$

Subdivision pruning

Interval arithmetic to remove boxes [Snyder 1992, Plantinga-Vegter 2006]

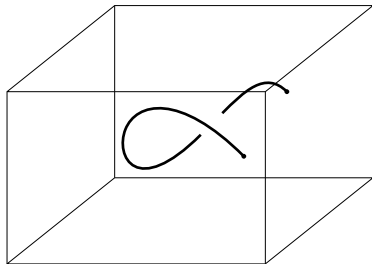




$$0 \notin \square \det(J_x(B)) = \square \det \left(\begin{array}{c} \frac{\partial f_i}{\partial x_j}(B) \end{array} \right)$$

$\Rightarrow V$ is globally parametrized by q in B

Surface tracking



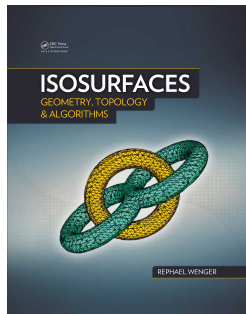
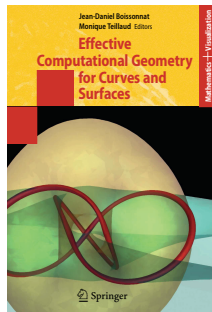
Regular case

- First marching segments [Dobkin et al. 1990]
- First marching triangles [Hilton et al. 1996]
- Recent result on marching simplices, topology guaranteed [Boissonnat, Kachanovich, Wintraecken, 2020]

Singular surfaces

- Use isosingular deflation near singularities [Bates, Brake, Hauenstein, Sommese, Wampler, 2014]

Further reading



Further computing

Drawing with symbolic approach

- algcurve (Maple)
- isotop (C, Maple)
- EXACUS (C++)
- ...

[Deconink, Patterson, van Hoesj]
[Peñeranda, Pouget, Lazard, Rouillier]
[Melhorn et al]

Drawing with subdivision

- axl (C++, mmx)
- ibex (C++)
- Realpaver (C)
- voxelize (C++)
- ...

[Christoflorou, Mantzaflaris, Mourrain, Wintz]
[Chabert]
[Granvilliers]
[M.]

Drawing with marching cube

- JuliaGeometry (Julia)
- scikit-image (C, python)
- MathMod (C++)
- ...

[Kelly]
[Lewiner]
[Taha]

Drawing with continuation

- bertini_real
- CGAL (C++)
- GUDHI (C++)
- ...

[Brake et al]
[Rineau, Yvinec]
[Kachanovich]

Merci !

- Complexity of gröbner bases in dimension 3 with terse representation?
- What are the actual bounds on the number of simplices in drawings in 3D, in nD ?
- What if we use polynomial pieces of degree k instead of linear pieces?
- Reliable drawing of singular surfaces with prescribed singularities?