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# Computational real algebraic geometry and applications to robotics

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# Computational real algebraic geometry and applications to robotics

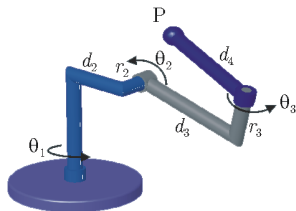
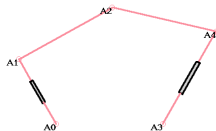
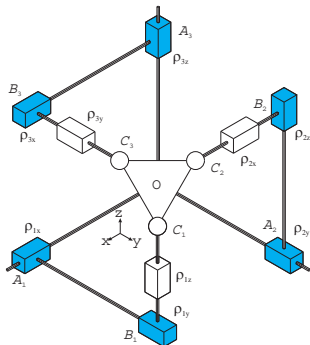
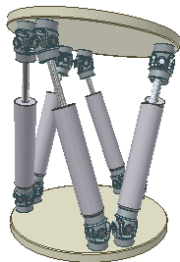
JNCF lecture, part 1

Guillaume Moroz

Inria Nancy - Grand Est

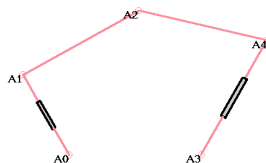
March 1<sup>st</sup>, 2021

# Mechanisms



- Joint variables
  - $r_1, r_2$
- Pose variables
  - $x, y$
- Passive variables
  - $\theta_1, \theta_2$

Parallel PR-PRR

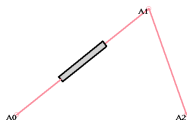


- Equations

$$(F) \begin{cases} x = \cos(\frac{2\pi}{3})r_1 + \cos(\theta_1) \\ x = 1 + \cos(\frac{\pi}{3})r_2 + \cos(\theta_2) \\ y = \sin(\frac{2\pi}{3})r_1 + \sin(\theta_1) \\ y = 1 + \sin(\frac{\pi}{3})r_2 + \sin(\theta_2) \end{cases}$$

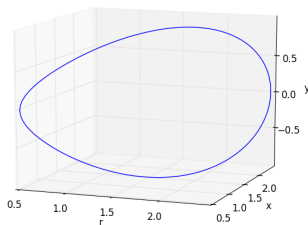
# Workspace, Joint space

- $Q$ : joint space
- $W$ : workspace



Parallel RPR-R

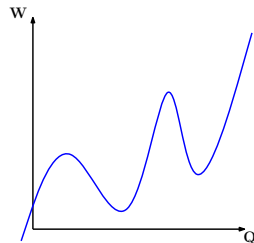
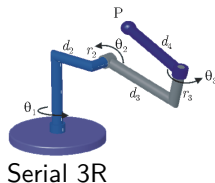
- Total space:  $Q \times W$ 
  - solutions of  $F$ :  $V(F) \subset W \times Q$



- Canonical projections:
  - $\pi_W : V(F) \rightarrow W$
  - $\pi_Q : V(F) \rightarrow Q$

# Serial robot

- Glossary:
  - P: prismatic joint
  - R: rotation joint
  - U: Cardan joint
  - S: spherical joint



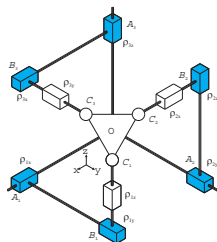
## Properties

- Inverse Kinematics (IK) hard
- Forward Kinematics (FK) easy: 1 solutions

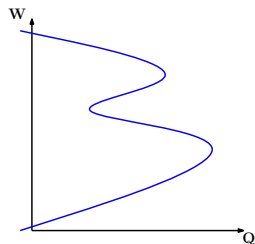
# Parallel robot

- Glossary:

- P: prismatic joint
- R: rotation joint
- U: Cardan joint
- S: spherical joint



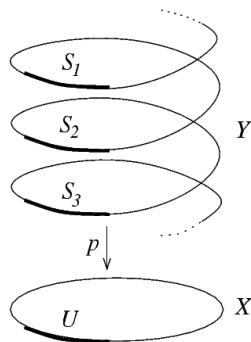
Parallel 3-PPPS



## Properties

- Inverse Kinematics (IK) easy
- Forward Kinematics (FK) hard: several solutions
- 2 solutions can cross
  - loose of control
  - break

# Covering map



## Definition

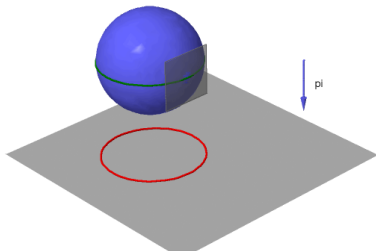
The continuous map  $f : S \rightarrow U$  is a **covering map** if:

$$f^{-1}(U) = S_1 \cup \dots \cup S_k \text{ where } \begin{cases} S_i \simeq U \\ S_i \text{ pairwise disjoint.} \end{cases}$$



# Critical points

- $V \subset \mathbb{R}^n$  smooth variety of dimension  $p$
- $\pi : V \rightarrow \mathbb{R}^p$  canonical projection



## Critical points

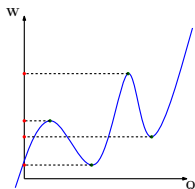
Let  $T(a)$  be the linear space tangent to  $V$  at point  $a$ .  
The **critical points**  $a$  of  $V$  for the projection  $\pi$  satisfy:

$$\dim(\pi(T(a))) < p$$

# Case of the serial robot

**Hypothesis:**  $V(F)$  smooth, bounded, equidimensional.

- FK: always 1 solution  $\Rightarrow \pi_Q : V(F) \rightarrow Q$  invertible
- IK: partition  $W$  in  $W_0, W_1, \dots, W_k$  s.t.:
  - $W_0$  are the critical values of  $\pi_W$
  - $W_1, \dots, W_k$  are the connected components of  $W \setminus W_0$
- Critical points of  $\pi_W$ : **serial singularities**



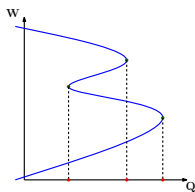
## Theorem (covering map and critical values)

For all  $1 \leq i \leq k$ , the restriction of  $\pi_W$  to  $\pi_W^{-1}(W_i)$  is a covering map above  $W_i$ .

# Case of parallel robot

**Hypothesis:**  $V(F)$  smooth, bounded, equidimensional.

- IK: always 1 solution  $\Rightarrow \pi_W : V(F) \rightarrow W$  invertible
- FK: partition  $Q$  in  $Q_0, Q_1, \dots, Q_k$  s.t.:
  - $Q_0$  are the critical values of  $\pi_Q$
  - $Q_1, \dots, Q_k$  are the connected components of  $Q \setminus Q_0$
- Critical points of  $\pi_Q$ : **parallel singularities**

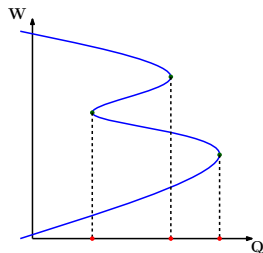


## Theorem (covering map and critical values)

For all  $1 \leq i \leq k$ , the restriction of  $\pi_Q$  to  $\pi_Q^{-1}(Q_i)$  is a covering map above  $Q_i$ .

## Case of parallel robots

- FK:  $F_q(x) = 0$ , system parametrized by  $q$
- For fixed  $q$ , finitely many solutions (0-dimensional)
- $\pi_Q$  is not a covering map near  $q$ 
  - $\Rightarrow$  two sheets of solutions cross
  - $\Rightarrow F_q(x) = 0$  has singular solutions



- Remark:  $V(F)$ 
  - not bounded: take asymptotes into account
  - not smooth: take singularities into account

Hypothesis:  $V(F)$  smooth, bounded, equidimensional.

$$\underbrace{\frac{\partial F}{\partial q}}_A dq + \underbrace{\frac{\partial F}{\partial x}}_B dx = 0$$

- Serial singularities

$$(S_s) : F = 0, \det(A) = 0$$

- $W_0$ , critical values of  $\pi_W$ : projection on  $x_i$  of solutions of  $(S_s)$
- $W_1, \dots, W_k$ , complement of critical values

Hypothesis:  $V(F)$  smooth, bounded, equidimensional.

$$\underbrace{\frac{\partial F}{\partial q}}_A dq + \underbrace{\frac{\partial F}{\partial x}}_B dx = 0$$

- Parallel singularities

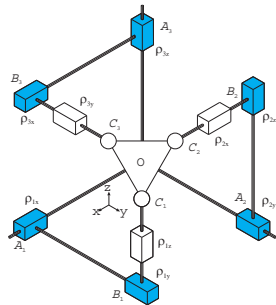
$$(S_p) : F = 0, \det(B) = 0$$

- $Q_0$ , critical values of  $\pi_Q$ : projection on  $q_i$  of solutions of  $(S_p)$
- $Q_1, \dots, Q_k$ , complement of critical values

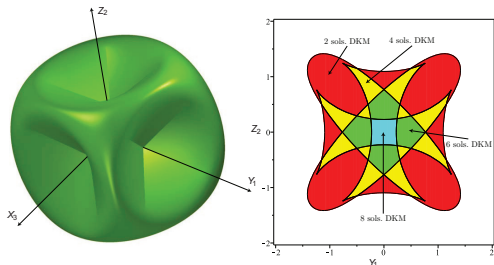
# Example

- 3-PPPS:

- Parallel
- Joint variables:  $x_1, y_1, y_2, z_2, x_3, z_3$
- Pose variables:  $p_x, p_y, p_z, \varphi, \theta, \sigma$



- Critical values of  $\pi_Q$ , and partition of  $Q$

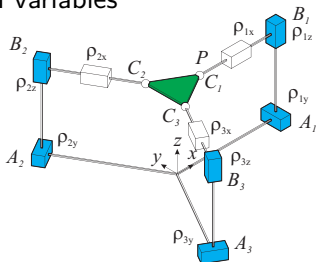
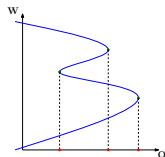


# Design challenges

- $E \subset W$  given shape
  - Design a **parallel robot** without singularities in  $E$

$$\pi_W(\text{critical points of } \pi_Q) \cap E = \emptyset$$

- Maximise the volume of  $E$  (lecture P. Lairez)
- $\rightarrow$  design variables



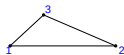


# Demo

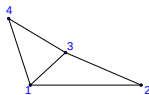
## Modeling equations

- 1 Linkages
- 2 Rotations 3D
- 3 Singularities

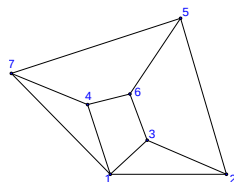
# Planar Rigid Linkage: Laman Graph



3-bar



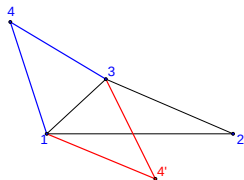
5-bar



11-bar

## Constraints

- Fixed length bars:  $c_{ij}$
- Free revolute joints
- Zero degree of freedom



- Several assembly modes
- Number depends on  $c_{ij}$
- Max number of assembly modes?

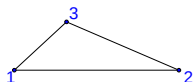
# Properties of Rigid Linkages

- Construction steps



Henneberg steps:  $H_1$  and  $H_2$

- 3-bar rigid linkage



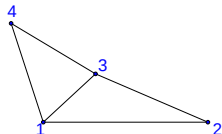
# Properties of Rigid Linkages

- Construction steps



Henneberg steps:  $H_1$  and  $H_2$

- 5-bar rigid linkage



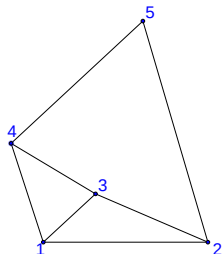
# Properties of Rigid Linkages

- Construction steps



Henneberg steps:  $H_1$  and  $H_2$

- 7-bar rigid linkage



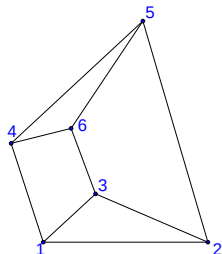
# Properties of Rigid Linkages

- Construction steps



Henneberg steps:  $H_1$  and  $H_2$

- 9-bar rigid linkage



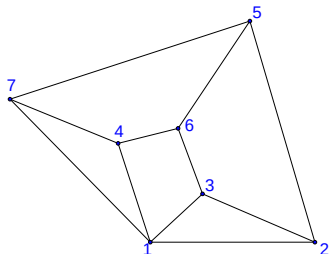
# Properties of Rigid Linkages

- Construction steps



Henneberg steps:  $H_1$  and  $H_2$

- 11-bar rigid linkage





## Theorem

A linkage is rigid  $\Leftrightarrow$  It can be constructed with  $H_1$  and  $H_2$

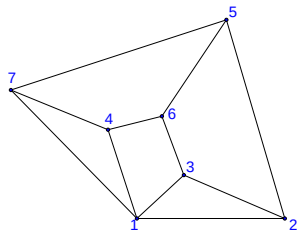
## Corollary

$$\#Links = 2\#Joints - 3$$

# Algebraic Modeling I

- $c_{ij}$ : 10 parameters
- $x_i, y_i$ : 14 variables

$$\begin{cases} x_1 = 0, y_1 = 0 \\ x_2 = 1, y_2 = 0 \end{cases}$$



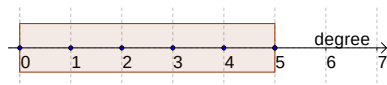
$$\begin{cases} x_3^2 + y_3^2 = c_{13} \\ (x_3 - 1)^2 + y_3^2 = c_{23} \\ (x_5 - 1)^2 + y_5^2 = c_{25} \\ (x_6 - x_3)^2 + (y_6 - y_3)^2 = c_{36} \\ x_4^2 + y_4^2 = c_{14} \end{cases}$$

$$\begin{cases} x_7^2 + y_7^2 = c_{17} \\ (x_6 - x_4)^2 + (y_6 - y_4)^2 = c_{46} \\ (x_5 - x_6)^2 + (y_5 - y_6)^2 = c_{56} \\ (x_7 - x_5)^2 + (y_7 - y_5)^2 = c_{57} \\ (x_4 - x_7)^2 + (y_4 - y_7)^2 = c_{47} \end{cases}$$

# Number of solutions

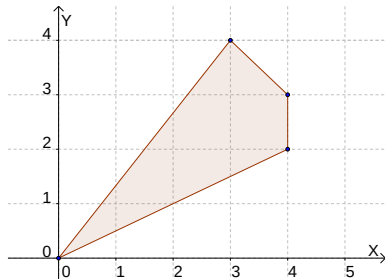
- Mixed Volume:  $n! \text{Volume}(\text{Support})$  (same support)

1 variable



$$1 - X + 3X^2 - X^3 + 6X^4 - 5X^5 = 0$$

2 variables



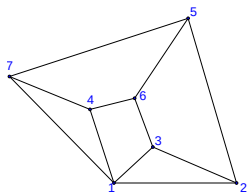
$$\begin{cases} 1 - X^4 Y^2 + 7X^4 Y^3 - 4X^3 Y^4 = 0 \\ 8 + 6X^4 Y^2 - 5X^4 Y^3 - X^3 Y^4 = 0 \end{cases}$$

- Our system:  $2^{10}$

# Algebraic Modeling II

- Cayley-Menger matrix or distance matrix

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c_{12} & c_{13} & c_{14} & x_{15} & x_{16} & c_{17} \\ 1 & c_{12} & 0 & c_{23} & x_{24} & c_{25} & x_{26} & x_{27} \\ 1 & c_{13} & c_{23} & 0 & x_{34} & x_{35} & c_{36} & x_{37} \\ 1 & c_{14} & x_{24} & x_{34} & 0 & x_{45} & c_{46} & c_{47} \\ 1 & x_{15} & c_{25} & x_{35} & x_{45} & 0 & c_{56} & c_{57} \\ 1 & x_{16} & x_{26} & c_{36} & c_{46} & c_{56} & 0 & x_{67} \\ 1 & c_{17} & x_{27} & x_{37} & c_{47} & c_{57} & x_{67} & 0 \end{bmatrix}$$



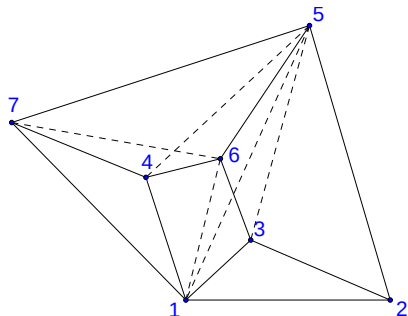
## Theorem

The distance matrix has rank 4.

## Corollary

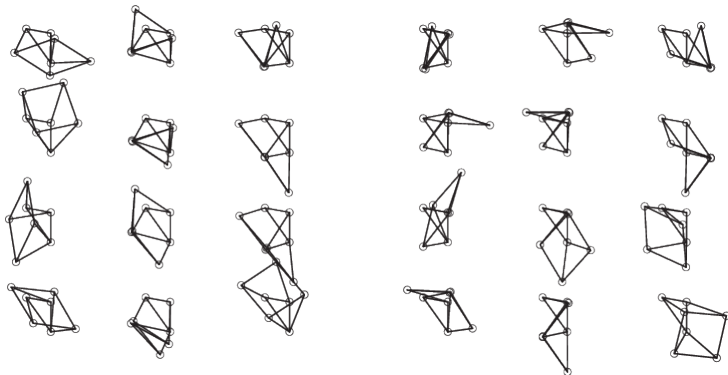
All the 5x5 minors vanish.

$$\left\{ \begin{array}{l} D(0, 4, 5, 6, 7)(c_{46}, c_{47}, c_{56}, c_{57}, x_{45}, x_{67}) = 0 \\ D(0, 1, 4, 6, 7)(c_{14}, c_{17}, c_{46}, c_{47}, x_{16}, x_{67}) = 0 \\ D(0, 1, 4, 5, 7)(c_{14}, c_{17}, c_{47}, c_{57}, x_{15}, x_{45}) = 0 \\ D(0, 1, 2, 3, 5)(c_{12}, c_{13}, c_{25}, c_{23}, x_{15}, x_{35}) = 0 \\ D(0, 1, 3, 5, 6)(c_{13}, c_{36}, c_{56}, x_{15}, x_{16}, x_{35}) = 0 \end{array} \right.$$



- Upper bound
  - Mixed volume: 56
- Lower Bound?

# Sampling



# Number of assembly modes

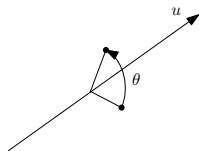
Maximal number of assembly modes

bars	3	5	7	9	11	13	15	17
upper	2	4	8	24	56	136	344	880
lower	2	4	8	24	56	136	344	860

- [Bartzosa, Emiris, Legerský, Tsigaridas 2021]
- Started in 2002 with Borcea
- Bartzosa, Borcea, Emiris, Legerský, M., Streinu, Capco, Gallet, Grasegger, Koutschan, Lubbes, Schicho, Tsigaridas, ...

# Rotations matrix in 3D

$$R = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$



$$R^T R = I$$

- Action of  $R$  is a rotation by  $\theta$  around an axe  $u$
- Set of rotation has dim 3



$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$R_y(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = R_z(\psi)R_y(\varphi)R_x(\theta) \quad \text{or} \quad R = R_z(\psi)R_y(\varphi)R_z(\theta)$$

$$q_i^2 + q_j^2 + q_k^2 + q_r^2 = 1$$

$$R = \begin{pmatrix} 1 - 2(q_j^2 + q_k^2) & 2(q_i q_j - q_k q_r) & 2(q_i q_k + q_j q_r) \\ 2(q_i q_j + q_k q_r) & 1 - 2(q_i^2 + q_k^2) & 2(q_j q_k - q_i q_r) \\ 2(q_i q_k - q_j q_r) & 2(q_j q_k + q_i q_r) & 1 - 2(q_i^2 + q_j^2) \end{pmatrix}$$

Rotation of **axe:**  $(q_i, q_j, q_k)$   
**angle:**  $2 \arccos(q_r)$

# Anti-symmetric matrix - Exponential map

In 2D

$$\theta \Rightarrow \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

In 3D

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

$$R = e^A$$

Rotation of **axe:**  $(x, y, z)$   
**angle:**  $\|(x, y, z)\|_2$

# Anti-symmetric matrix - Cayley transform

In 2D

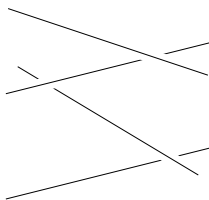
$$t = \tan\left(\frac{\theta}{2}\right) \Rightarrow \begin{pmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{pmatrix}$$

In 3D

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

$$R = (I - A)(I + A)^{-1}$$

Rotation of **axe:**  $(x, y, z)$   
**angle:**  $2 \arctan(\|(x, y, z)\|_2)$



How many lines intersect 4 given lines?

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad P_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad M = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \\ 1 & 1 \end{pmatrix}$$

## Definition

The Plücker coordinates of the line  $(P_1P_2)$  are the 6 minors of  $M$

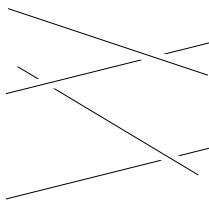
$$(d_x, d_y, d_z, m_x, m_y, m_z) = (P_2 - P_1, OP_1 \times OP_2)$$

- $(d, m) \in \mathbf{P}_5$  is on the Klein quadric

$$d \cdot m = 0$$

- Lines  $(d, m)$  and  $(d', m')$  intersect implies

$$d \cdot m' + m \cdot d' = 0$$



0, 1, 2 or infinitely many lines cross the 4 given lines

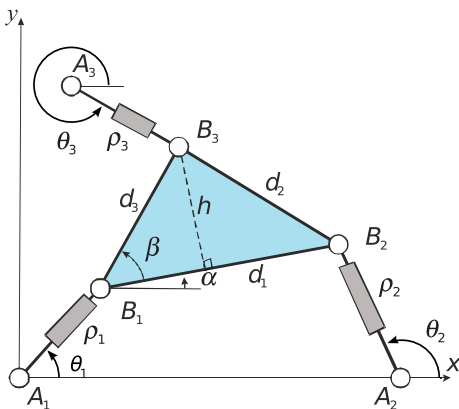
$$f_1(\mathbf{q}, \mathbf{x}), \dots, f_m(\mathbf{q}, \mathbf{x})$$

$$B = \begin{matrix} f_1 \\ \vdots \\ f_m \end{matrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_m} \\ \\ \frac{\partial f_j}{\partial x_j} \end{pmatrix}$$

- $\det(B)$  multi-linear in its columns/rows
- $\det(B)$  of deg 1 in  $x_j$  leads to a parametrization of the singularities
- Simplifies the analysis of the singularities, as in the 3-RPR [Coste 2012]



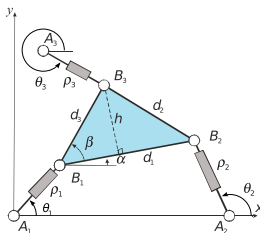
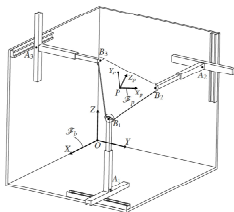
# Plan parallel robot 3-RPR



- 3 degrees of freedom
- $d_1, d_2, d_3, A_1, A_2, A_3$  fixed
- Joint variables:  $r_1, r_2, r_3$
- Pose variables:  $\alpha, B_{1x}, B_{1y}$

# Demo

# Singularity of parallel manipulator with Plücker vectors



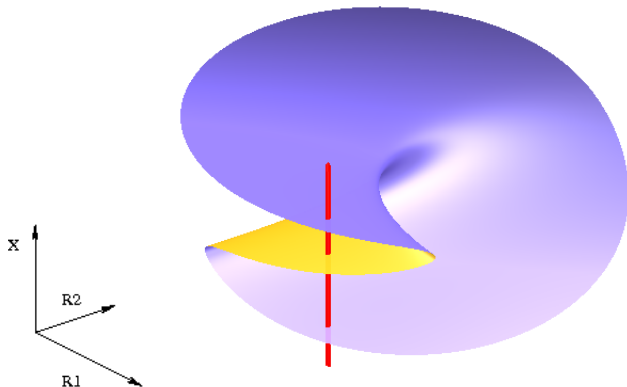
## Remark

If the leg-platform joints are spherical, the rows of the inverse kinematic Jacobian matrix  $A^{-1}B$  will involve the Plücker coordinates of lines associated to the legs.

⇒ singularities can be interpreted geometrically

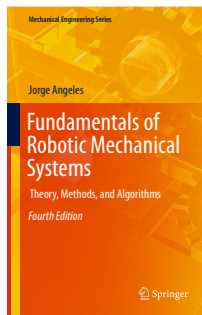
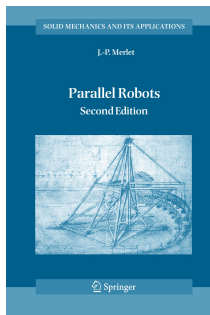
# Cuspidal configuration

- **Cuspidal point:** point of order  $\geq 3$
- **Characterization:** A cuspidal robot has at least one cuspidal point



# Demo

# Further reading



# À suivre

## Solving systems

- 1 With initial point
  - Newton
- 2 Without initial point
  - Symbolic approaches
  - Numerical approaches

# Computational real algebraic geometry and applications to robotics

JNCF lecture, part 2

Guillaume Moroz

Inria Nancy - Grand Est

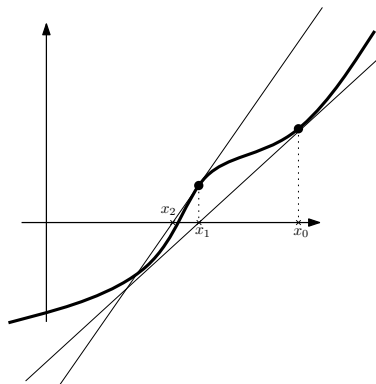
March 4<sup>th</sup>, 2021



## Solving systems

- 1 With initial point
  - Newton
- 2 Without initial point
  - Symbolic approaches
  - Numerical approaches

$$f(x) = 0$$

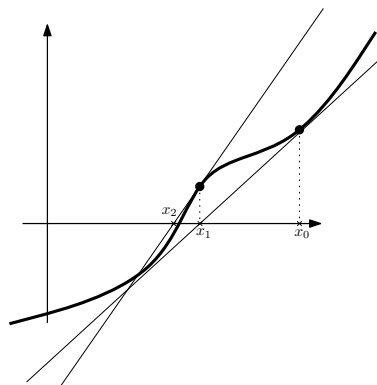


$x_0$  = initial point

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Good for path tracking

$$f(x) = 0$$



$x_0$  = initial point

$$x_{n+1} = x_n - Df(x_n)^{-1}f(x_n)$$

Good for path tracking

$$K(f, x_0) = \sup_x \|Df(x_0)^{-1}D^2f(x)\|$$
$$\beta(f, x_0) = \|Df(x_0)^{-1}f(x_0)\|$$

## Theorem (Kantorovich)

If  $\beta(f, x_0)K(f, x_0) \leq 1/2$ ,  
then  $f$  has a unique solution in  $B(x_0, 2\beta(f, x_0))$ .

$$\gamma(f, x_0) = \sup_{k \geq 2} \left\| Df(x_0)^{-1} \frac{D^k f(x_0)}{k!} \right\|^{\frac{1}{k-1}}$$
$$\beta(f, x_0) = \|Df(x_0)^{-1} f(x_0)\|$$

## Theorem (Smale)

If  $\beta(f, x_0)\gamma(f, x_0) \leq 3 - 2\sqrt{2}$ ,  
then  $f$  has a unique solution in  $B(x_0, \frac{1-\sqrt{2}/2}{\gamma(f, x_0)})$ .

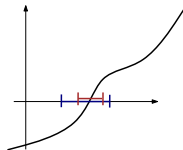
# Reliable interval Newton

$B$  a box interval containing  $p$

## 1 variable

$$N(x) = p - \left( \frac{f(x) - f(p)}{x - p} \right)^{-1} f(p)$$

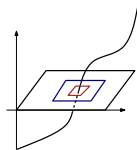
$N(B) \subset B \Rightarrow N$  has a fix point in  $B$   
 $\Rightarrow f$  has a solution in  $B$



## 2 variables or more

$$N(B) = p - \square J(B)^{-1} \begin{pmatrix} f(p) \\ g(p) \end{pmatrix}$$

$N(B) \subset B \Rightarrow f = g = 0$  has a solution in  $B$

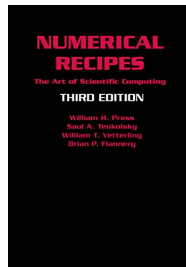
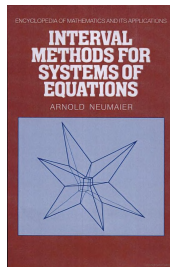
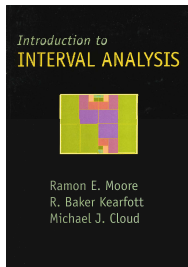
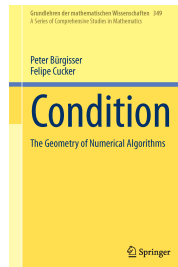
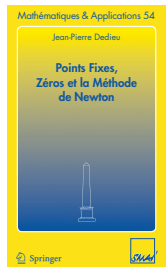
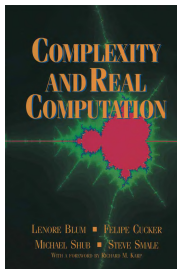


## Interval operations

$$[a, b] \oplus [c, d] = [a + c, b + d]$$

$$[a, b] \otimes [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

# Further reading



## Computer Algebra Systems

- Maple
- Mathematica
- Mathmagix (mmx)
- Matlab
- SageMath
- Xcas
- ...

## Numerical Newton

- Boost (C++)
- MPSolve (C) [Bini, Fiorentino, Robol]
- GNU Scientific Library (C)
- Roots, Optim, NLSolve (Julia)
- Scipy (Python)
- ...

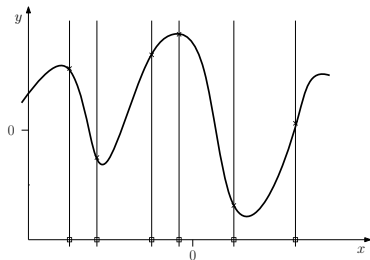
## Interval and ball arithmetic Libraries

- Arb (C, Python) [Johansson]
- GAOL (C++) [Goualard]
- Intlab (Octave/Matlab) [Rump]
- mpfi (C) [Revol]
- numerix (mmx, C++) [van der Hoeven]
- IntervalArithmetic (Julia) [Benet, Sanders]
- ...



# Univariate Representation

Reduce problem to univariate polynomial



$$p(x) = 0$$
$$y = q(x)$$

# Resultant definition

Given two polynomials in  $\mathbb{C}[y]$  :

- $P = p_0 y^d + \dots + p_d$  with roots  $\sigma_1, \dots, \sigma_d$
- $Q = q_0 y^d + \dots + q_d$  with roots  $\tau_1, \dots, \tau_d$

## Definition: Resultant

$$\begin{aligned} \text{Res}(P, Q) &= p_0^d q_0^d \prod_{i,j} (\sigma_i - \tau_j) \\ &= p_0^d \boxed{Q}(\sigma_1) \cdots \boxed{Q}(\sigma_d) \\ &= (-1)^{d^2} q_0^d \boxed{P}(\tau_1) \cdots \boxed{P}(\tau_d) \end{aligned}$$

# Resultant definition

## Bezout

$$\varphi: \mathbb{C}[y]_{d-1} \times \mathbb{C}[y]_{d-1} \rightarrow \mathbb{C}[y]_{2d-1}$$
$$U \quad , \quad V \quad \mapsto UP + VQ$$

$$\varphi(U, V) = 1 \Leftrightarrow \gcd(P, Q) = 1$$
$$\Leftrightarrow \varphi \text{ invertible}$$

- **Resultant**  $r$  is the determinant of the **Sylvester Matrix**
- $r \in \langle P, Q \rangle = I$

## Definition: Sylvester matrix

$$\begin{pmatrix} 1 & 2 & \dots & & d+1 & d+2 & d+3 & \dots \\ p_0 & & & & q_0 & & & \\ p_1 & p_0 & & & q_1 & q_0 & & \\ p_2 & p_1 & p_0 & & q_2 & q_1 & q_0 & \\ p_3 & p_2 & p_1 & p_0 & q_3 & q_2 & q_1 & q_0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# Bivariate case

Given two polynomials in  $\mathbb{C}[x, y]$  of degree  $d$  in  $x$  and  $y$ :

- $P = p_0(x)y^d + \dots + p_d(x)$

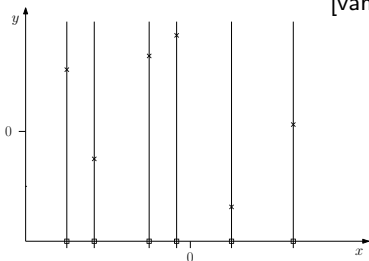
- $Q = q_0(x)y^d + \dots + q_d(x)$

- $\text{Res}(P, Q)$  is a polynomial in  $x$

- $O(n^{3-1/\omega+\varepsilon})$  arithmetic operations

[Villard 2018]

- $O(n^{2+\varepsilon})$  randomized in a finite field  
[van der Hoeven, Lecerf 2019]



$$\text{Res}_d(P(x, y), Q(x, y))(\alpha) = \text{Res}_d(P(\alpha, y), Q(\alpha, y))$$

# First subresultant definition

Given two polynomials in  $\mathbb{C}[y]$  :

- $P = p_0 y^d + \dots + p_d$  with roots  $\sigma_1, \dots, \sigma_d$
- $Q = q_0 y^d + \dots + q_d$  with roots  $\tau_1, \dots, \tau_d$

## Properties

- $Sres_1 = s_1 y + s_0$  has degree at most 1
- $Sres_1 \in \langle P, Q \rangle = I$

## Definition: first subresultant

$$Sres_1(P, Q) = p_0^{d-1} \left( (y - \sigma_1) \prod_{j \neq 1} \frac{Q(\sigma_j)}{\sigma_j - \sigma_1} + \dots + (y - \sigma_d) \prod_{j \neq d} \frac{Q(\sigma_j)}{\sigma_j - \sigma_d} \right)$$

# First subresultant definition

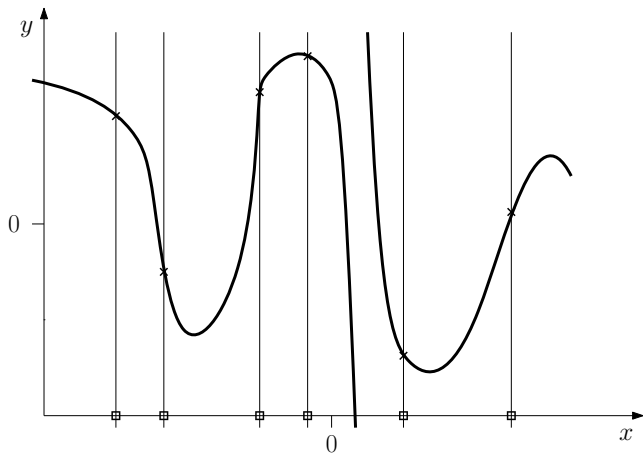
$$\varphi : \mathbb{C}[y]_{d-2} \times \mathbb{C}[y]_{d-2} \rightarrow \mathbb{C}[y]_{2d-2}$$
$$U \quad , \quad V \quad \mapsto UP + VQ$$

The  $s_0$  and  $s_1$  are the determinants of minors of the Sylvester Matrix

## Sylvester matrix

$$\begin{pmatrix} 1 & 2 & \dots & & d+1 & d+2 & d+3 & \dots \\ p_0 & & & & q_0 & & & \\ p_1 & p_0 & & & q_1 & q_0 & & \\ p_2 & p_1 & p_0 & & q_2 & q_1 & q_0 & \\ p_3 & p_2 & p_1 & p_0 & q_3 & q_2 & q_1 & q_0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# Parametrization of $y$



$$s_1(x)y + s_0(x) = 0$$

$$r(x) = 0$$

# Three variables and more

$$f_1(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

## Problem

$$\text{Find } p(x_1) = q_1 f_1 + \dots + q_n f_n$$

Matrix of  $1, x_1, \dots, x_1^D$  modulo  $\langle f_1, \dots, f_n \rangle$

$$M = \begin{matrix} & 1 & \dots & x_1^D \\ m_1 & \left( \right. & & \\ \vdots & & & \\ m_k & \left. \right) & & \end{matrix}$$

$$\Rightarrow v \in \text{Ker}(M) \text{ iff } v_0 + \dots + v_D x^D \in \langle f_1, \dots, f_n \rangle$$



# Three variables and more

$$f_1(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

## Problem

$$\text{Find } p(x_1) = q_1 f_1 + \dots + q_n f_n$$

Matrix of multiplication by  $x_1$  modulo  $\langle f_1, \dots, f_n \rangle$

$$M = \begin{matrix} & x_1 m_1 & \dots & x_1 m_k \\ \begin{matrix} m_1 \\ \vdots \\ m_k \end{matrix} & \left( \begin{array}{ccc} & & \\ & & \\ & & \end{array} \right) \end{matrix}$$

$$\Rightarrow \chi_M(x_1) \in \langle f_1, \dots, f_n \rangle$$

# Three variables and more

$$f_1(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

## Problem

$$\text{Find } p(x_1) = q_1 f_1 + \dots + q_n f_n$$

Matrix of multiplication by  $x_1$  modulo  $f_1(x_1)$

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{d-1} \end{pmatrix}$$

$$\Rightarrow \chi_M(x_1) = f_1(x_1) \in \langle f_1 \rangle$$

# Normal form and univariate representation

## Normal form

$$x_1 m_i \pmod{\langle f_1, \dots, f_n \rangle}$$

- Euclidean division by Gröbner basis

## Groebner Bases computation

- $n > 2$  in  $\tilde{O}((nd)^{(\omega+1)n})$  operations [Bardet, Faugère, Salvy 2015]
- $n = 2$  in  $\tilde{O}(d^2)$  with terse representation [van der Hoeven, Larrieu 2018]

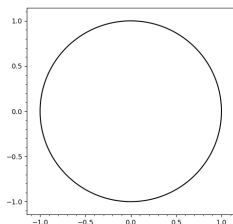
## Univariate Representation

- Multivariate subresultant
- Rational Univariate Representation [Kronecker 1882, Rouillier 1999]
  - u-resultant and its derivatives at point  $(t, a_1, \dots, a_n)$

$$\text{u-resultant} = C \prod_{\zeta | F(\zeta)=0} (u_0 + u_1 \zeta_n + \dots + u_n \zeta_n)$$

- Lexicographical Gröbner bases
  - Polynomial parametrization generically, bigger size [Dahan, Schost 2004]

# Univariate Representation with Geometric Resolution



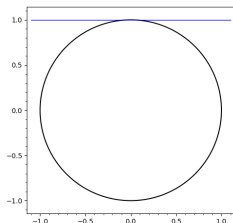
$$f(x, y) = 0$$

## Theorem (Hensel lifting)

$$\begin{cases} y_0 & = \text{root of } f(0, y) = 0 \\ y_{n+1}(x) & = y_n(x) - \frac{f(x, y_n(x))}{\frac{\partial f}{\partial y}(x, y_n(x))} \pmod{x^{2^{n+1}}} \end{cases}$$

Then  $y_n(x)$  is a root of  $f(x, y) = 0 \pmod{x^{2^n}}$ .

# Univariate Representation with Geometric Resolution



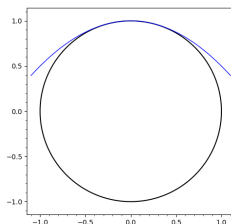
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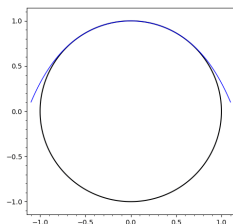
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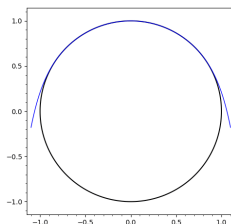
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# Univariate Representation with Geometric Resolution



$$f(x, y) = 0$$

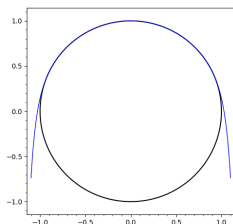
## Theorem (Hensel lifting)

$$\begin{cases} y_0 & = \text{root of } f(0, y) = 0 \\ y_{n+1}(x) & = y_n(x) - \frac{f(x, y_n(x))}{\frac{\partial f}{\partial y}(x, y_n(x))} \pmod{x^{2^{n+1}}} \end{cases}$$

Then  $y_n(x)$  is a root of  $f(x, y) = 0 \pmod{x^{2^n}}$ .



# Univariate Representation with Geometric Resolution



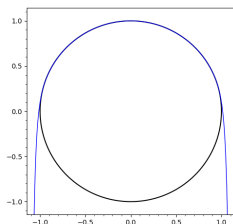
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# Univariate Representation with Geometric Resolution



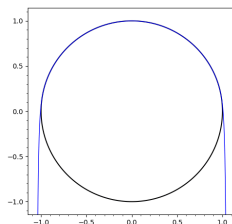
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# Univariate Representation with Geometric Resolution



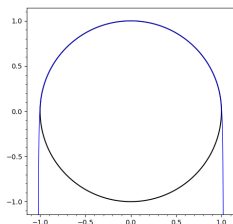
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# Univariate Representation with Geometric Resolution



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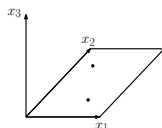
Then  $y_n(x)$  is a root of  $f(x, y) = 0 \pmod{x^{2^n}}$ .

# Univariate Representation with Geometric Resolution

$$f_1(x_1, x_2, x_3) = \dots = f_3(x_1, x_2, x_3) = 0$$

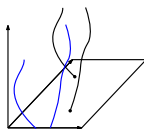
- RUR of  $f_1(x_1, x_2, 0)$  and  $f_2(x_1, x_2, 0)$

$$p_1(x_1) = 0 \quad x_2 = q_{12}(x_1)$$



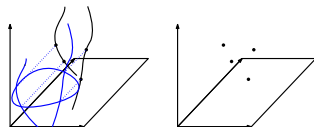
- Lift  $x_3$  in  $f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3) = 0$

$$\tilde{p}_1(x_1, x_3) = 0 \quad x_2 = \tilde{q}_{12}(x_1, x_3)$$



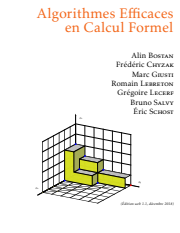
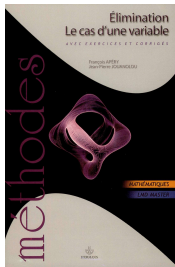
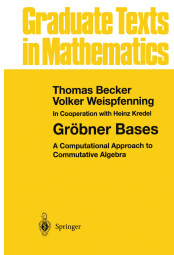
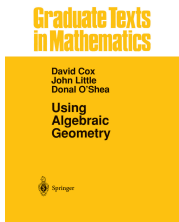
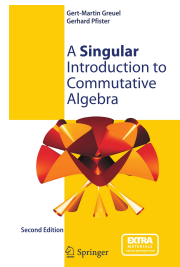
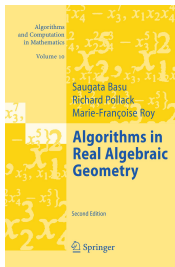
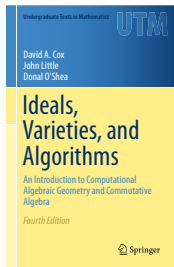
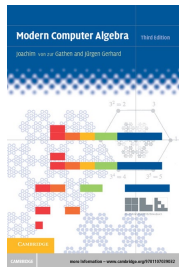
- RUR of  $\tilde{p}_{11}(x_1, x_3)$  and  $f_3(x_1, \tilde{q}_{12}(x_1, x_3), x_3)$

$$p_2(x_1) = 0 \quad x_2 = q_{22}(x_1) \quad x_3 = q_{23}(x_1)$$



$\tilde{O}(d^{3n})$  operations [Giusti, Lecerf, Salvy 2001]

# Further reading

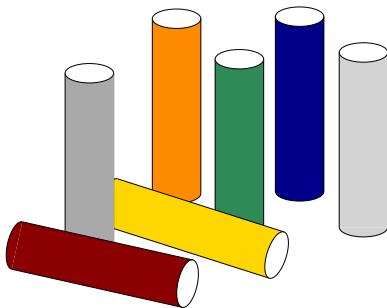


## Computer Algebra systems

- Giac/xcas
- Maple
- Magma
- Mathmagix
- Mathematica
- Mcaulay2
- MuPAD
- SageMath
- Singular
- ...

## Special purpose libraries/software

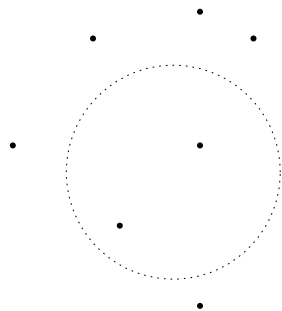
- FGb (C) [Faugère]
- Flint (C) [Hart]
- msolve [Berthomieux, Eder, Safey El Din]
- RS/RS3 (C) [Rouillier]
- borderbasis (mmx, C++) [Trebuchet, Mourrain]
- algebramix (mmx, C++) [van der Hoeven, Lecerf]
- geomsolvex (mmx, C++) [Lecerf]
- larrix (mmx, C++) [Larrieu]
- ...



Is it possible to arrange 7 infinite cylinders of unit radius such that they are mutually touching?



# Path continuation



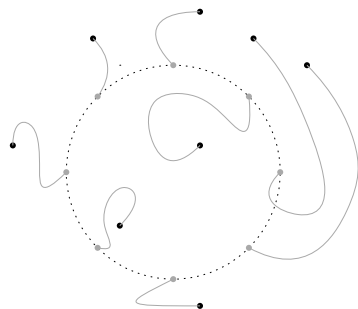
- $f(x) = 0$
- $\deg d$

- We know the solutions of  $x^d - 1 = 0$

$$(1 - t)(x^d - 1) + tf(x) = 0$$

- Multivariate case: compute the mixed volume first
- Find 1 solution by following only one path  
Polynomial time in  $\binom{n+d}{d}$  average [Beltran, Pardo 2009], [Lairez 2017]
- Roots distribution of random polynomials [Edelman, Kostlan 1995]

# Path continuation



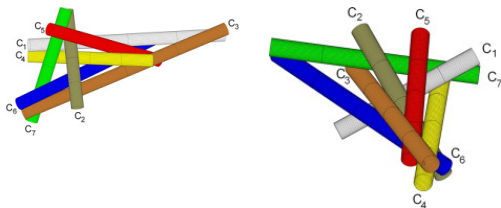
- $f(x) = 0$
- $\deg d$

- We know the solutions of  $x^d - 1 = 0$

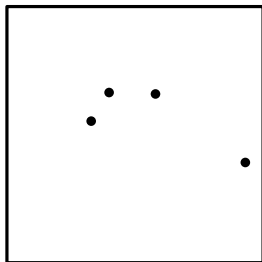
$$(1 - t)(x^d - 1) + tf(x) = 0$$

- Multivariate case: compute the mixed volume first
- Find 1 solution by following only one path  
Polynomial time in  $\binom{n+d}{d}$  average [Beltran, Pardo 2009], [Lairez 2017]
- Roots distribution of random polynomials [Edelman, Kostlan 1995]

2 arrangements of 7 cylinders mutually touching [Bozóki et al. 2015]



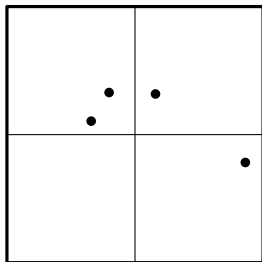
- Modeled with 20 equations in 20 variables
- Mixed volume 121 098 993 664
- First real solution after 80 000 000 paths
- Solution certified with Smale theorem



Test each cell

- 1 solution, guaranteed  $\Rightarrow$  keep
- 0 solution, guaranteed  $\Rightarrow$  remove
- don't know  $\Rightarrow$  subdivide

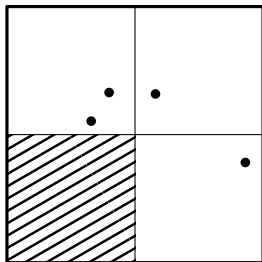
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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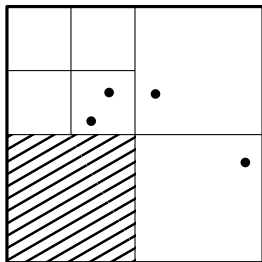
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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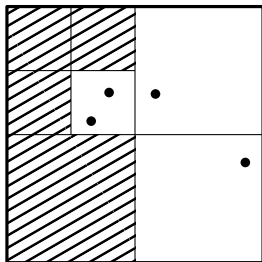
Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]



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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

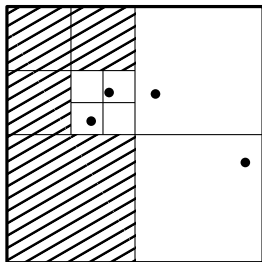


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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

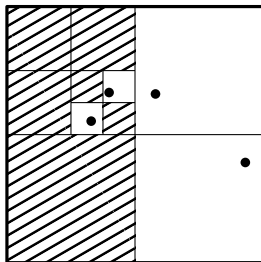




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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

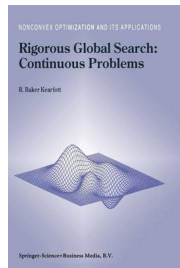
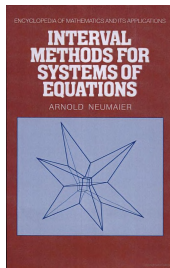
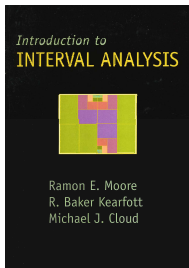
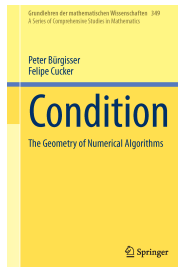
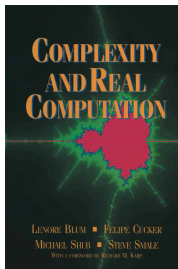


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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

# Further reading



## Path continuation

- Bertini (C++, Python) [Bates, Amethyst, Hauenstein, Sommese, Wampler]
- Hom4PS (C++) [Tien-Yien Li, Tianran Chen, Tsung-Lin Lee]
- PHCpack (Ada) [Vershelde]
- analyziz (mmx, C++) [van der Hoeven]
- ...

## Subdivision multivariate

- ibex (C++) [Chabert]
- Realpaver (C) [Granvilliers]
- realroot (mmx, C++) [Mourrain]
- subdivision\_solver (C++, Python) [Imbach]
- voxelize (C++) [M.]
- ...

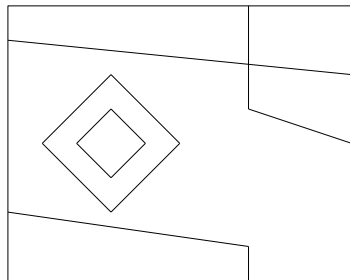
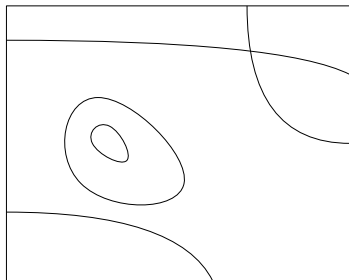
## Subdivision univariate

- ANewDsc (C) [Kobel, Rouillier, Sagraloff]
- Clenshaw (C, Python) [M.]
- Ccluster (C, Julia) [Imbach, Pan, Yap]
- real\_roots (sage) [Witty]
- RS (C) [Rouillier]
- SLV (C) [Tsigaridas]
- ...

## Drawing reliably the projection of the solution

- 1 Isotopy
- 2 Projection
- 3 Drawing reliably

# Drawing hypersurfaces



## Definition (isotopy)

A triangulation  $T$  is isotope to  $V \subset \mathbb{R}^n$  if there exists  $\varphi : [0, 1] \times T \rightarrow \mathbb{R}^n$  such that:

- $\varphi$  is continuous
- $\varphi(0, T) = T$  and  $\varphi(1, T) = V$
- $\varphi_{t_0} : T \rightarrow \varphi(t_0, T)$  is an homeomorphism

# Size of the triangulation [Kerber and Sagraloff 2011]

$f(x_1, \dots, x_n)$  of degree  $d$

2 variables

Theorem

In the worst case  $\Omega(d^2)$  vertices and  $O(d^2)$  segments

3 variables

Theorem

In the worst case  $\Omega(d^3)$  vertices and  $O(d^5)$  triangles

$n$  variables

Theorem

In the worst case  $\Omega(d^n)$  vertices and  $O(d^{3/4 \cdot 2^n - 1})$  simplices

# Size of the triangulation [Kerber and Sagraloff 2011]

$f$  of degree  $d$  and vertices on the surface

2 variables

Theorem

In the worst case  $\Omega(d^{\cancel{2}^3})$  vertices and  $O(d^{\cancel{2}^3})$  segments

3 variables

Theorem

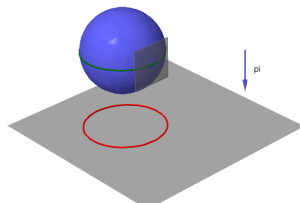
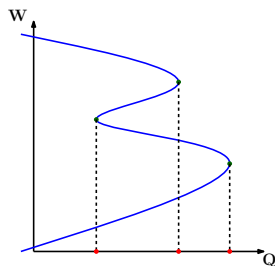
In the worst case  $\Omega(d^{\cancel{3}^4})$  vertices and  $O(d^{\cancel{3}^7})$  triangles

$n$  variables

Theorem

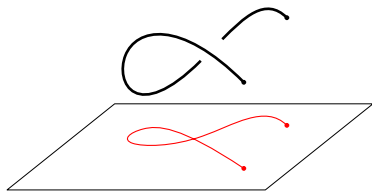
In the worst case  $\Omega(d^{n+1})$  vertices and  $O(d^{\cancel{3}^4} 2^{n-1})$  simplices





$V$  solutions of  $f_1 = \dots = f_k = \det\left(\frac{\partial f_i}{\partial x_j}\right) = 0$  in  $\mathbb{R}^n \times \mathbb{R}^k$

Goal: draw the projection of  $V$  in  $Q$

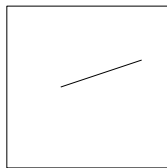
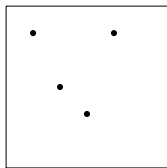
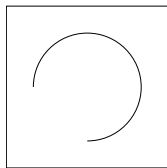


## Definition (Semi-algebraic)

A **semi-algebraic** set is a set solution of a system of equalities and inequalities.

## Theorem (Tarski-Seidenberg)

The projection of a semi-algebraic set is semi-algebraic.



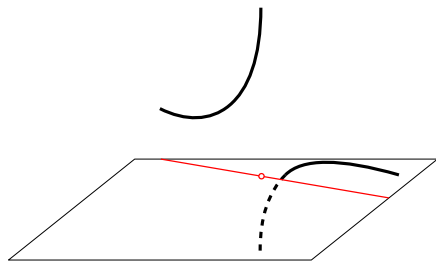
## Definition (Zariski closure)

The **Zariski closure** of  $E$  is the minimal set  $\bar{E}$  containing  $E$  that is solution of a system of equalities.

## Theorem (elimination)

- $V$  solution of  $p_1(q, x) = \dots = p_{k+1}(q, x) = 0$
- $G$  Gröbner basis of  $F$  with respect to the lexdeg ordering with  $x > q$
- $G_q$  the polynomials in  $\underline{G} \cap \mathbb{Q}[q]$   
 $\Rightarrow \pi_{\mathbb{Q}}(V) = \text{solutions of } G_q$

# Projective elimination



$$p_1(q, x) = \cdots = p_{k+1}(q, x) = 0$$

## Theorem

If  $V_p$  is the projective closure of  $V$  in  $\mathbb{C}^n \times \mathbf{P}_k$ , then

$$\overline{\pi_Q(V)} = \pi_Q(V_p)$$



# Geometry of the projection

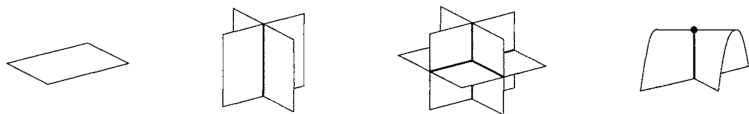


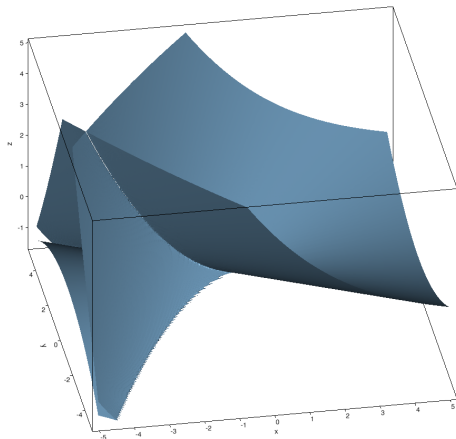
Figure 1: *Local singularities of images of generic maps of surfaces into 3-space*

[Guryonov 1997]

There exists a hypersurface  $\Delta \subset C^\infty(M, \mathbb{R}^3)$  s.t. if  $f \in C^\infty(M, \mathbb{R}^3) \setminus \Delta$ , then the neighborhood of any point of  $f(M)$  is one of the 4 above.

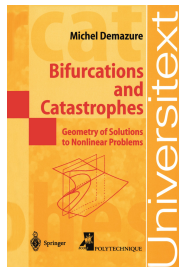
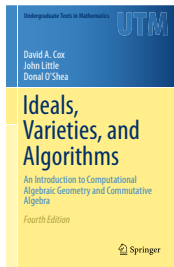
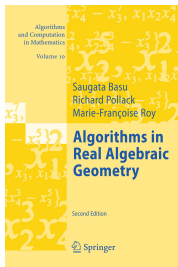
- Classification of generic singularities started with Whitney
- Thom introduced the transversality theorem
- Arnold and others provided numerous classifications

# Geometry of the projection



For the projection of critical points, we also have swallowtail singularities

# Further reading





## Quantifier elimination

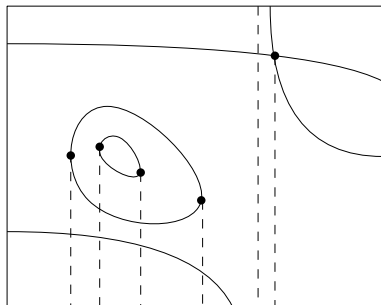
- Redlog (Reduce)
- RegularChains (Maple)
- Resolve (Mathematica)
- QEPCAD (C)
- Tarski (C++)
- ...

[Dolzmann, Sturm]  
[Moreno Maza et al]  
[Strzebonski]  
[Brown]  
[Brown]

## Satisfiability of formula

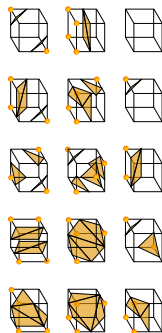
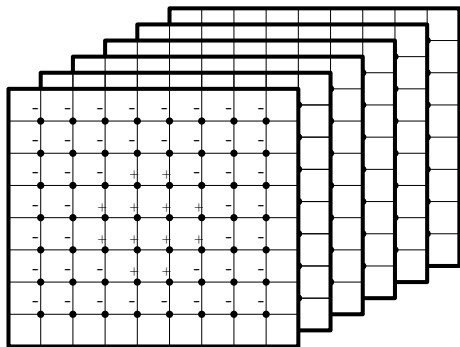
- RAGLib (Maple)
- RealCertify (Maple)
- RSolver (OCaml)
- TSSOS (Julia)
- ...

[Safey el Din]  
[Magron, Safey el Din]  
[Ratschan]  
[Lassere, Magron, Wang]



- Topology of curves in  $\tilde{O}(d^6 + \tau d^5)$  operations  
[Kobel, Sagraloff, 2015], [Niang Diatta, Diatta, Rouillier, Roy, Sagraloff, submitted 2018]
- Topology of surfaces with  $O(d^5)$  simplices  
[Berberich, Kerber, Sagraloff 2009]

# Marching cubes [Lorensen and Cline 1987]



- 1 Evaluate  $f(x, y, z)$  of deg  $d$  on a grid of size  $N = n \times n \times n$
- 2 For each cube, compute a triangle of the surface to plot

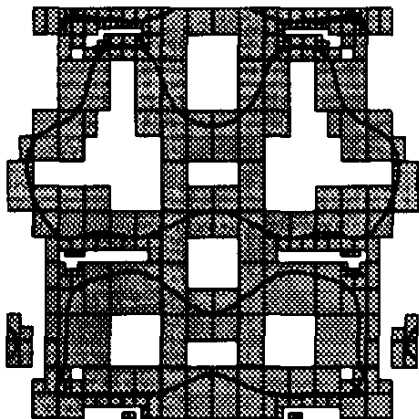
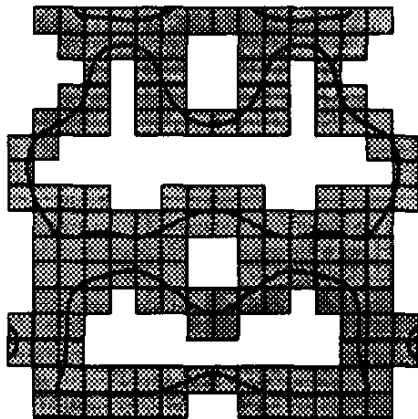
**Complexity:** Cost of evaluating  $f$  on 1 point with Hörner:  $O(d^3)$

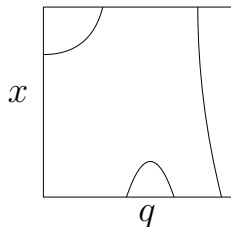
Total naive:  $O(d^3 n^3)$

Reuse computation by coordinates:  $O(dn^3)$

# Subdivision pruning

Interval arithmetic to remove boxes [Snyder 1992, Plantinga-Vegter 2006]

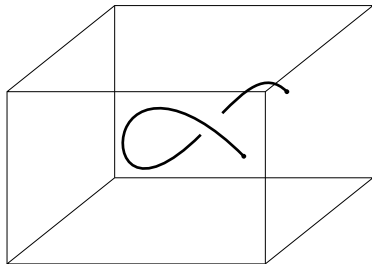




$$0 \notin \square \det(J_x(B)) = \square \det \left( \begin{array}{c} \frac{\partial f_i}{\partial x_j}(B) \end{array} \right)$$

$\Rightarrow V$  is globally parametrized by  $q$  in  $B$

# Surface tracking



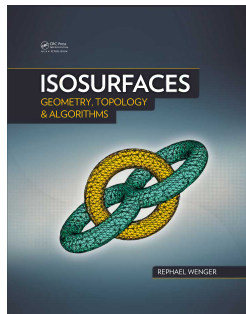
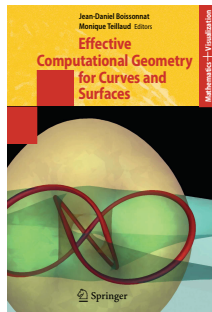
## Regular case

- First marching segments [Dobkin et al. 1990]
- First marching triangles [Hilton et al. 1996]
- Recent result on marching simplices, topology guaranteed [Boissonnat, Kachanovich, Wintraecken, 2020]

## Singular surfaces

- Use isosingular deflation near singularities [Bates, Brake, Hauenstein, Sommese, Wampler, 2014]

# Further reading



# Further computing

## Drawing with symbolic approach

- `algsolve` (Maple)
- `isotop` (C, Maple)
- `EXACUS` (C++)
- ...

[Deconink, Patterson, van Hoesj]  
[Peñeranda, Pouget, Lazard, Rouillier]  
[Melhorn et al]

## Drawing with subdivision

- `axl` (C++, `mmx`)
- `ibex` (C++)
- `Realpaver` (C)
- `voxelize` (C++)
- ...

[Christoflorou, Mantzaflaris, Mourrain, Wintz]  
[Chabert]  
[Granvilliers]  
[M.]

## Drawing with marching cube

- `JuliaGeometry` (Julia)
- `scikit-image` (C, python)
- `MathMod` (C++)
- ...

[Kelly]  
[Lewiner]  
[Taha]

## Drawing with continuation

- `bertini_real`
- `CGAL` (C++)
- `GUDHI` (C++)
- ...

[Brake et al]  
[Rineau, Yvinec]  
[Kachanovich]



Merci !

- Complexity of gröbner bases in dimension 3 with terse representation?
- What are the actual bounds on the number of simplices in drawings in 3D, in  $nD$ ?
- What if we use polynomial pieces of degree  $k$  instead of linear pieces?
- Reliable drawing of singular surfaces with prescribed singularities?