# Computational real algebraic geometry and applications to robotics 

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## Computational real algebraic geometry and applications to robotics

JNCF lecture, part 1

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March $1^{\text {st }}, 2021$

## Mechanisms



## Modeling

- Joint variables


## Parallel $\underline{P R}$ - $\underset{R}{ } R$

- $r_{1}, r_{2}$
- Pose variables
- $x, y$
- Passive variables
- $\theta_{1}, \theta_{2}$

- Equations

$$
(F)\left\{\begin{array}{l}
x=\cos \left(\frac{2 \pi}{3}\right) r_{1}+\cos \left(\theta_{1}\right) \\
x=1+\cos \left(\frac{\pi}{3}\right) r_{2}+\cos \left(\theta_{2}\right) \\
y=\sin \left(\frac{2 \pi}{3}\right) r_{1}+\sin \left(\theta_{1}\right) \\
y=1+\sin \left(\frac{\pi}{3}\right) r_{2}+\sin \left(\theta_{2}\right)
\end{array}\right.
$$

## Workspace, Joint space

- $Q$ : joint space
- W: workspace


Parallel RPR-R

- Total space: $Q \times W$
- solutions of $\mathrm{F}: V(F) \subset W \times Q$

- Canonical projections:
- $\pi_{W}: V(F) \rightarrow W$
- $\pi_{Q}: V(F) \rightarrow W$


## Serial robot

- Glossary:
- P: prismatic joint
- R: rotation joint
- U: Cardan joint
- S: spherical joint


Serial 3R


## Properties

- Inverse Kinematics (IK) hard
- Forward Kinematics (FK) easy: 1 solutions


## Parallel robot

- Glossary:
- P: prismatic joint
- R: rotation joint
- U: Cardan joint
- S: spherical joint


Parallel 3-PPPS


## Properties

- Inverse Kinemaics (IK) easy
- Forward Kinematics (FK) hard: several solutions
- 2 solutions can cross
- loose of control
- break


## Covering map



## Definition

The continuous map $f: S \rightarrow U$ is a covering map if:

$$
f^{-1}(U)=S_{1} \cup \cdots \cup S_{k} \text { where }\left\{\begin{array}{l}
S_{i} \stackrel{f}{\simeq} U \\
S_{i} \text { pairwise disjoint. }
\end{array}\right.
$$

## Critical points

- $V \subset \mathbb{R}^{n}$ smooth variety of dimension $p$
- $\pi: V \rightarrow \mathbb{R}^{p}$ canonical projection


## Critical points

Let $T(a)$ be the linear space tangent to $V$ at point $a$. The critical points a of $V$ for the projection $\pi$ satisfy:

$$
\operatorname{dim}(\pi(T(a)))<p
$$

## Case of the serial robot

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

- FK: always 1 solution $\Rightarrow \pi_{Q}: V(F) \rightarrow Q$ invertible
- IK: partition $W$ in $W_{0}, W_{1}, \ldots, W_{k}$ s.t.:
- $W_{0}$ are the critical values of $\pi_{W}$
- $W_{1}, \ldots, W_{k}$ are the connected components of $W \backslash W_{0}$
- Critical points of $\pi_{W}$ : serial singularities



## Theorem (covering map and critical values)

For all $1 \leqslant i \leqslant k$, the restriction of $\pi_{W}$ to $\pi_{W}^{-1}\left(W_{i}\right)$ is a covering map above $W_{i}$.

## Case of parallel robot

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

- IK: always 1 solution $\Rightarrow \pi_{W}: V(F) \rightarrow W$ invertible
- FK: partition $Q$ in $Q_{0}, Q_{1}, \ldots, Q_{k}$ s.t.:
- $Q_{0}$ are the critical values of $\pi_{Q}$
- $Q_{1}, \ldots, Q_{k}$ are the connected components of $Q \backslash Q_{0}$
- Critical points of $\pi_{Q}$ : parallel singularities



## Theorem (covering map and critical values)

For all $1 \leqslant i \leqslant k$, the restriction of $\pi_{Q}$ to $\pi_{Q}^{-1}\left(Q_{i}\right)$ is a covering map above $Q_{i}$.

## Properties

## Case of parallel robots

- FK: $F_{q}(x)=0$, system parametrized by $q$
- For fixed $q$, finitely many solutions (0-dimensional)
- $\pi_{Q}$ is not a covering map near $q$
$\Rightarrow$ two sheets of solutions cross
$\Rightarrow F_{q}(x)=0$ has singular solutions

- Remark: $V(F)$
- not bounded: take asymptotes into account
- not smooth: take singularities into account


## Computation

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

$$
\underbrace{\frac{\partial F}{\partial q}}_{A} d q+\underbrace{\frac{\partial F}{\partial x}}_{B} d x=0
$$

- Serial singularities

$$
\left(S_{s}\right): F=0, \operatorname{det}(A)=0
$$

- $W_{0}$, critical values of $\pi_{W}$ : projection on $x_{i}$ of solutions of $\left(S_{s}\right)$
- $W_{1}, \ldots, W_{k}$, complement of critical values


## Computation

Hypothesis: $V(F)$ smooth, bounded, equidimensional.

$$
\underbrace{\frac{\partial F}{\partial q}}_{A} d q+\underbrace{\frac{\partial F}{\partial x}}_{B} d x=0
$$

- Parallel singularities

$$
\left(S_{p}\right): F=0, \operatorname{det}(B)=0
$$

- $Q_{0}$, critical values of $\pi_{Q}$ : projection on $q_{i}$ of solutions of $\left(S_{p}\right)$
- $Q_{1}, \ldots, Q_{k}$, complement of critical values


## Example

- 3-PPPS:
- Parallel
- Joint variables: $x_{1}, y_{1}, y_{2}, z_{2}, x_{3}, z_{3}$
- Pose variables: $p_{x}, p_{y}, p_{z}, \varphi, \theta, \sigma$
- Critical values of $\pi_{Q}$, and partition of $Q$




## Design challenges

- $E \subset W$ given shape
- Design a parallel robot without singularities in $E$

$$
\pi_{W}\left(\text { critical points of } \pi_{Q}\right) \cap E=\varnothing
$$



- Maximise the volume of $E$ (lecture P. Lairez)
- $\rightarrow$ design variables



## Demo

## Demo

## Modeling equations

(1) Linkages
(2) Rotations 3D
(3) Singularities

## Planar Rigid Linkage: Laman Graph



3-bar


5-bar


11-bar

## Constraints

- Fixed length bars: $c_{i j}$
- Free revolute joints
- Zero degree of freedom

- Several assembly modes
- Number depends on $c_{i j}$
- Max number of assembly modes?


## Properties of Rigid Linkages

- Construction steps

- 3-bar rigid linkage



## Properties of Rigid Linkages

- Construction steps

- 5-bar rigid linkage



## Properties of Rigid Linkages

- Construction steps

- 7-bar rigid linkage



## Properties of Rigid Linkages

- Construction steps

- 9-bar rigid linkage



## Properties of Rigid Linkages

- Construction steps

- 11-bar rigid linkage



## Known properties

## Theorem <br> A linkage is rigid $\Leftrightarrow$ It can be constructed with $H_{1}$ and $H_{2}$

## Corollary

$$
\# \text { Links }=2 \# \text { Joints }-3
$$

## Algebraic Modeling I

- $c_{i j}: 10$ parameters
- $x_{i}, y_{i}: 14$ variables

$$
\left\{\begin{array}{l}
x_{1}=0, y_{1}=0 \\
x_{2}=1, y_{2}=0
\end{array}\right.
$$



$$
\left\{\begin{array} { r l } 
{ x _ { 3 } ^ { 2 } + y _ { 3 } { } ^ { 2 } } & { = c _ { 1 3 } } \\
{ ( x _ { 3 } - 1 ) ^ { 2 } + y _ { 3 } { } ^ { 2 } } & { = c _ { 2 3 } } \\
{ ( x _ { 5 } - 1 ) ^ { 2 } + y _ { 5 } { } ^ { 2 } } & { = c _ { 2 5 } } \\
{ ( x _ { 6 } - x _ { 3 } ) ^ { 2 } + ( y _ { 6 } - y _ { 3 } ) ^ { 2 } } & { = c _ { 3 6 } } \\
{ x _ { 4 } { } ^ { 2 } + y _ { 4 } { } ^ { 2 } } & { = c _ { 1 4 } ^ { 2 } + y _ { 7 } ^ { 2 } }
\end{array} \quad \left\{\begin{array} { r l } 
{ x _ { 1 7 } }
\end{array} \quad \left\{\begin{array}{rl}
\left(x_{6}-x_{4}\right)^{2}+\left(y_{6}-y_{4}\right)^{2} & =c_{46} \\
\left(x_{5}-x_{6}\right)^{2}+\left(y_{5}-y_{6}\right)^{2} & =c_{56} \\
\left(x_{7}-x_{5}\right)^{2}+\left(y_{7}-y_{5}\right)^{2} & =c_{57} \\
\left(x_{4}-x_{7}\right)^{2}+\left(y_{4}-y_{7}\right)^{2} & =c_{47}
\end{array}\right.\right.\right.
$$

## Number of solutions

- Mixed Volume: n! Volume(Support) (same support)
1 variable


$$
1-X+3 X^{2}-X^{3}+6 X^{4}-5 X^{5}=\left\{\begin{array}{l}
1-X^{4} Y^{2}+7 X^{4} Y^{3}-4 X^{3} Y^{4}=0 \\
0 \\
8+6 X^{4} Y^{2}-5 X^{4} Y^{3}-X^{3} Y^{4}=0
\end{array}\right.
$$

- Our system: $2^{10}$


## Algebraic Modeling II

- Cayley-Menger matrix or distance matrix
$\begin{array}{lllllll}v_{1} & v_{2} & v_{3} & v_{4} & v_{5} & v_{6} & v_{7}\end{array}$
$v_{1}$
$v_{2}$
$v_{3}$
$v_{4}$
$v_{5}$
$v_{6}$
$v_{7}$$\quad\left[\begin{array}{cccccccc}0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & c_{12} & c_{13} & c_{14} & x_{15} & x_{16} & c_{17} \\ 1 & c_{12} & 0 & c_{23} & x_{24} & c_{25} & x_{26} & x_{27} \\ 1 & c_{13} & c_{23} & 0 & x_{34} & x_{35} & c_{36} & x_{37} \\ 1 & c_{14} & x_{24} & x_{34} & 0 & x_{45} & c_{46} & c_{47} \\ 1 & x_{15} & c_{25} & x_{35} & x_{45} & 0 & c_{56} & c_{57} \\ 1 & x_{16} & x_{26} & c_{36} & c_{46} & c_{56} & 0 & x_{67} \\ 1 & c_{17} & x_{27} & x_{37} & c_{47} & c_{57} & x_{67} & 0\end{array}\right]$



## Theorem

The distance matrix has rank 4.

## Corollary

All the $5 \times 5$ minors vanish.

## Algebraic Modeling II

$$
\left\{\begin{array}{l}
D(0,4,5,6,7)\left(c_{46}, c_{47}, c_{56}, c_{57}, x_{45}, x_{67}\right)=0 \\
D(0,1,4,6,7)\left(c_{14}, c_{17}, c_{46}, c_{47}, x_{16}, x_{67}\right)=0 \\
D(0,1,4,5,7)\left(c_{14}, c_{17}, c_{47}, c_{57}, x_{15}, x_{45}\right)=0 \\
D(0,1,2,3,5)\left(c_{12}, c_{13}, c_{25}, c_{23}, x_{15}, x_{35}\right)=0 \\
D(0,1,3,5,6)\left(c_{13}, c_{36}, c_{56}, x_{15}, x_{16}, x_{35}\right)=0
\end{array}\right.
$$



- Upper bound
- Mixed volume: 56
- Lower Bound?


## Sampling




## Number of assembly modes

Maximal number of assembly modes

| bars | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| upper | 2 | 4 | 8 | 24 | 56 | 136 | 344 | 880 |
| lower | 2 | 4 | 8 | 24 | 56 | 136 | 344 | 860 |

- [Bartzosa, Emiris, Legerský, Tsigaridas 2021]
- Started in 2002 with Borcea
- Bartzosa, Borcea, Emiris, Legerský, M., Streinu, Capco, Gallet, Grasegger, Koutschan, Lubbes, Schicho, Tsigaridas, ...


## Rotations matrix in 3D

$$
R=\left(\begin{array}{lll}
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3} \\
z_{1} & z_{2} & z_{3}
\end{array}\right)
$$

$$
R^{T} R=I
$$

- Action of $R$ is a rotation by $\theta$ around an axe $u$
- Set of rotation has dim 3


## Euler matrix

$$
\begin{gathered}
R_{x}(\theta)=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos (\theta) & \sin (\theta) \\
0 & -\sin (\theta) & \cos (\theta)
\end{array}\right) \\
R_{y}(\varphi)=\left(\begin{array}{ccc}
\cos (\varphi) & 0 & \sin (\varphi) \\
0 & 1 & 0 \\
-\sin (\varphi) & 0 & \cos (\varphi)
\end{array}\right) \\
R_{z}(\psi)=\left(\begin{array}{ccc}
\cos (\psi) & \sin (\psi) & 0 \\
-\sin (\psi) & \cos (\psi) & 0 \\
0 & 0 & 1
\end{array}\right) \\
R=R_{z}(\psi) R_{y}(\varphi) R_{x}(\theta) \text { or } R=R_{z}(\psi) R_{y}(\varphi) R_{z}(\theta)
\end{gathered}
$$

## Quaternions matrix

$$
\begin{gathered}
q_{i}^{2}+q_{j}^{2}+q_{k}^{2}+q_{r}^{2}=1 \\
R=\left(\begin{array}{cll}
1-2\left(q_{j}^{2}+q_{k}^{2}\right) & 2\left(q_{i} q_{j}-q_{k} q_{r}\right) & 2\left(q_{i} q_{k}+q_{j} q_{r}\right) \\
2\left(q_{i} q_{j}+q_{k} q_{r}\right) & 1-2\left(q_{i}^{2}+q_{k}^{2}\right) & 2\left(q_{j} q_{k}-q_{i} q_{r}\right) \\
2\left(q_{i} q_{k}-q_{j} q_{r}\right) & 2\left(q_{j} q_{k}+q_{i} q_{r}\right) & 1-2\left(q_{i}^{2}+q_{j}^{2}\right)
\end{array}\right) \\
\text { Rotation of } \\
\text { axe: } \quad\left(q_{i}, q_{j}, q_{k}\right) \\
\text { angle: }
\end{gathered}
$$

## Anti-symmetric matrix - Exponential map

In 2D

$$
\theta \Rightarrow\left(\begin{array}{cc}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{array}\right)
$$

In 3D

$$
\begin{gathered}
A=\left(\begin{array}{ccc}
0 & z & -y \\
-z & 0 & x \\
y & -x & 0
\end{array}\right) \\
R=e^{A}
\end{gathered}
$$

Rotation of axe: $\quad(x, y, z)$ angle: $\|(x, y, z)\|_{2}$

## Anti-symmetric matrix - Cayley transform

In 2D

$$
t=\tan \left(\frac{\theta}{2}\right) \Rightarrow\left(\begin{array}{ll}
\frac{1-t^{2}}{1+t^{2}} & \frac{-2 t}{1+t^{2}} \\
\frac{2 t}{1+t^{2}} & \frac{1-t^{2}}{1+t^{2}}
\end{array}\right)
$$

In 3D

$$
\begin{aligned}
& A=\left(\begin{array}{ccc}
0 & z & -y \\
-z & 0 & x \\
y & -x & 0
\end{array}\right) \\
& R=(I-A)(I+A)^{-1}
\end{aligned}
$$

Rotation of axe: $(x, y, z)$ angle: $\quad 2 \arctan \left(\|(x, y, z)\|_{2}\right)$

## Plücker coordinates



How many lines intersect 4 given lines?

## Plücker coordinates

$$
P_{1}=\left(\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right) \quad P_{2}=\left(\begin{array}{l}
x_{2} \\
y_{2} \\
z_{2}
\end{array}\right) \quad M=\left(\begin{array}{cc}
x_{1} & x_{2} \\
y_{1} & y_{2} \\
z_{1} & z_{2} \\
1 & 1
\end{array}\right)
$$

## Definition

The Plücker coordinates of the line $\left(P_{1} P_{2}\right)$ are the 6 minors of $M$

$$
\left(d_{x}, d_{y}, d_{z}, m_{x}, m_{y}, m_{z}\right)=\left(P_{2}-P_{1}, O P_{1} \times O P_{2}\right)
$$

- $(d, m) \in \mathbf{P}_{5}$ is on the Klein quadric

$$
d \cdot m=0
$$

- Lines $(d, m)$ and $\left(d^{\prime}, m^{\prime}\right)$ intersect implies

$$
d \cdot m^{\prime}+m \cdot d^{\prime}=0
$$

## Plücker coordinates


$0,1,2$ or infinitely many lines cross the 4 given lines

## Singularity modeling

$$
\begin{array}{r}
f_{1}(q, x), \ldots, f_{m}(q, x) \\
B=\begin{array}{c}
\frac{\partial}{\partial x_{1}} \cdots \frac{\partial}{\partial x_{m}} \\
\vdots \\
f_{m}
\end{array}\left(\begin{array}{c}
\frac{\partial f_{i}}{\partial x_{j}}
\end{array}\right)
\end{array}
$$

- $\operatorname{det}(B)$ multi-linear in its columns/rows
- $\operatorname{det}(B)$ of deg 1 in $x_{j}$ leads to a parametrization of the singularities
- Simplifies the analysis of the singularities, as in the 3-RPR [Coste 2012]


## Plan parallel robot 3-RPR



- 3 degrees of freedom
- $d_{1}, d_{2}, d_{3}, A_{1}, A_{2}, A_{3}$ fixed
- Joint variables: $r_{1}, r_{2}, r_{3}$
- Pose variables: $\alpha, B_{1 x}, B_{1 y}$


## Demo

## Demo

## Singularity of parallel manipulator with Plücker vectors



## Remark

If the leg-platorm joints are spherical, the rows of the inverse kinematic Jacobian matrix $A^{-1} B$ will involve the Plücker coordinates of lines associated to the legs.
$\Rightarrow$ singularities can be interpreted geometrically

## Cuspidal configuration

- Cuspidal point: point of order $\geqslant 3$
- Characterization: A cuspidal robot has at least one cuspidal point



## Demo

## Demo

## Further reading



## À suivre

## Solving systems

(1) With initial point

- Newton
(2) Without initial point
- Symbolic approaches
- Numerical approaches


## Computational real algebraic geometry and applications to robotics

JNCF lecture, part 2

Guillaume Moroz<br>Inria Nancy - Grand Est

March $4^{\text {th }}, 2021$

## Solving systems

(1) With initial point

- Newton
(2) Without initial point
- Symbolic approaches
- Numerical approaches


## Newton

$$
f(x)=0
$$



$$
\begin{aligned}
x_{0} & =\text { initial point } \\
x_{n+1} & =x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}
\end{aligned}
$$

## Good for path tracking

## Newton

$$
f(x)=0
$$



$$
\begin{aligned}
x_{0} & =\text { initial point } \\
x_{n+1} & =x_{n}-\operatorname{Df}\left(x_{n}\right)^{-1} f\left(x_{n}\right)
\end{aligned}
$$

## Good for path tracking

## Reliable Newton with Kantorovich

$$
\begin{aligned}
& K\left(f, x_{0}\right)=\sup _{x}\left\|D f\left(x_{0}\right)^{-1} D^{2} f(x)\right\| \\
& \beta\left(f, x_{0}\right)=\left\|D f\left(x_{0}\right)^{-1} f\left(x_{0}\right)\right\|
\end{aligned}
$$

Theorem (Kantorovich)

$$
\text { If } \beta\left(f, x_{0}\right) K\left(f, x_{0}\right) \leqslant 1 / 2
$$

then $f$ has a unique solution in $B\left(x_{0}, 2 \beta\left(f, x_{0}\right)\right)$.

## Reliable Newton with Smale

$$
\begin{aligned}
& \gamma\left(f, x_{0}\right)=\sup _{k \geqslant 2}\left\|D f\left(x_{0}\right)^{-1} \frac{D^{k} f\left(x_{0}\right)}{k!}\right\|^{\frac{1}{k-1}} \\
& \beta\left(f, x_{0}\right)=\left\|D f\left(x_{0}\right)^{-1} f\left(x_{0}\right)\right\|
\end{aligned}
$$

## Theorem (Smale)

$$
\begin{gathered}
\text { If } \beta\left(f, x_{0}\right) \gamma\left(f, x_{0}\right) \leqslant 3-2 \sqrt{2} \\
\text { then } f \text { has a unique solution in } B\left(x_{0}, \frac{1-\sqrt{2} / 2}{\gamma\left(f, x_{0}\right)}\right) .
\end{gathered}
$$

## Reliable interval Newton

## $B$ a box interval containing $p$

1 variable

$$
N(x)=p-\left(\frac{f(x)-f(p)}{x-p}\right)^{-1} f(p)
$$

$N(B) \subset B \Rightarrow N$ has a fix point in $B$ $\Rightarrow f$ has a solution in $B$

2 variables or more

$$
N(B)=p-\square J(B)^{-1}\binom{f(p)}{g(p)}
$$

$N(B) \subset B \Rightarrow f=g=0$ has a solution in $B$
Interval operations

$$
\begin{aligned}
& {[a, b] \oplus[c, d]=[a+c, b+d]} \\
& {[a, b] \otimes[c, d]=[\min (a c, a d, b c, b d), \max (a c, a d, b c, b d)]}
\end{aligned}
$$

## Further reading



Ramon E. Moore R. Baker Kearfott

Michael J. Cloud


Grunclehrea der mathematischea Wissenschatten 349 Gunolehies der matematischea Wissenscortt

Peter Bürgisser
Felipe Cucker

## Condition

The Geometry of Numerical Algorithms

## NUMERICAL

 RECIPESThe Art of Scientific Emmputing THIRD EDITIUN

Willorn H. Press
Saut A. Teuknisky
Williom T. Vetterling
Brtan P. Flannery

## Further computing

## Computer Algebra Systems

- Maple
- Mathematica
- Mathemagix (mmx)
- Matlab
- SageMath
- Xcas
- ...
- Boost (C++)
- MPSolve (C) [Bini, Fiorentino, Robol]
- GNU Scientific Library (C)
- Roots, Optim, NLsolve (Julia)
- Scipy (Python)

Interval and ball arithmetic Libraries

- Arb (C, Python) [Johansson]
- GAOL (C++) [Goualard]
- Intlab (Octave/Matlab) [Rump]
- mpfi (C) [Revol]
- numerix (mmx, $\mathrm{C}++$ ) [van der Hoeven]
- IntervalArithmetic (Julia) [Benet,Sanders]


## Univariate Representation

Reduce problem to univariate polynomial


$$
\begin{aligned}
p(x) & =0 \\
y & =q(x)
\end{aligned}
$$

## Resultant definition

Given two polynomials in $\mathbb{C}[y]$ :

- $P=p_{0} y^{d}+\cdots+p_{d}$ with roots $\sigma_{1}, \ldots, \sigma_{d}$
- $Q=q_{0} y^{d}+\cdots+q_{d}$ with roots $\tau_{1}, \ldots, \tau_{d}$


## Definition: Resultant

$$
\begin{aligned}
\operatorname{Res}(P, Q) & =p_{0}^{d} q_{0}^{d} \prod_{i, j}\left(\sigma_{i}-\tau_{j}\right) \\
& =p_{0}^{d} \frac{Q}{} \quad\left(\sigma_{1}\right) \cdots \square\left(\sigma_{d}\right) \\
& \left.=(-1)^{d^{2}} q_{0}^{d} \frac{P}{} \frac{P}{}\left(\tau_{1}\right) \cdots \square\right)
\end{aligned}
$$

## Resultant definition

## Bezout

$$
\begin{array}{cccl}
\varphi: \mathbb{C}[y]_{d-1} & \times & \mathbb{C}[y]_{d-1} & \rightarrow \mathbb{C}[y]_{2 d-1} \\
U & , & \mapsto U P+V Q \\
\varphi(U, V)=1 & \Leftrightarrow \operatorname{gcd}(P, Q)=1 \\
& \Leftrightarrow \varphi \text { invertible }
\end{array}
$$

- Resultant $r$ is the determinant of the Sylvester Matrix
- $r \in\langle P, Q\rangle=I$

Definition: Sylvester matrix

$$
\begin{array}{cccccccccc}
1 & 2 & \cdots & & & d+1 & d+2 & d+3 & \cdots & \\
\left(\begin{array}{cccccccc}
p_{0} & & & & & q_{0} & & \\
p_{1} & p_{0} & & & & q_{1} & q_{0} & \\
p_{2} & p_{1} & p_{0} & & & q_{2} & q_{1} & q_{0} \\
p_{3} & p_{2} & p_{1} & p_{0} & & q_{3} & q_{2} & q_{1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & q_{0} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
\end{array}
$$

## Bivariate case

Given two polynomials in $\mathbb{C}[x, y]$ of degree $d$ in $x$ and $y$ :

- $P=p_{0}(x) y^{d}+\cdots+p_{d}(x)$
- $Q=q_{0}(x) y^{d}+\cdots+q_{d}(x)$
- $\operatorname{Res}(P, Q)$ is a polynomial in $x$
- $O\left(n^{3-1 / \omega+\varepsilon}\right)$ arithmetic operations
[Villard 2018]
- $O\left(n^{2+\varepsilon}\right)$ randomized in a finite field [van der Hoeven, Lecerf 2019]


$$
\operatorname{Res}_{d}(P(x, y), Q(x, y))(\alpha)=\operatorname{Res}_{d}(P(\alpha, y), Q(\alpha, y))
$$

## First subresultant definition

Given two polynomials in $\mathbb{C}[y]$ :

- $P=p_{0} y^{d}+\cdots+p_{d}$ with roots $\sigma_{1}, \ldots, \sigma_{d}$
- $Q=q_{0} y^{d}+\cdots+q_{d}$ with roots $\tau_{1}, \ldots, \tau_{d}$


## Properties

- Sres $_{1}=s_{1} y+s_{0}$ has degree at most 1
- Sres $_{1} \in\langle P, Q\rangle=I$

Definition: first subresultant

$$
\begin{aligned}
\operatorname{Sres}_{1}(P, Q)=p_{0}^{d-1} \quad & \left(y-\sigma_{1}\right) \prod_{j \neq 1} \xlongequal[\sigma_{j}-\sigma_{1}]{ }\left(\sigma_{j}\right) \\
& +\cdots+ \\
& \left(y-\sigma_{d}\right) \prod_{j \neq d} \frac{Q}{\sigma_{j}-\sigma_{d}}\left(\sigma_{j}\right) \\
&
\end{aligned}
$$

## First subresultant definition

$$
\begin{array}{rlrl}
\varphi: \mathbb{C}[y]_{d-2} & \times \mathbb{C}[y]_{d-2} & \rightarrow \mathbb{C}[y]_{2 d-2} \\
U & V, \quad \mapsto & \mapsto P+V Q
\end{array}
$$

The $s_{0}$ and $s_{1}$ are the determinants of minors of the Sylvester Matrix
Sylvester matrix

$$
\begin{aligned}
& 12 \cdots \quad d+1 \quad d+2 \quad d+3 \quad \cdots \\
& \left(\begin{array}{ccccc}
p_{0} & & & & \\
p_{1} & p_{0} & & & \\
p_{2} & p_{1} & p_{0} & & \\
p_{3} & p_{2} & p_{1} & p_{0} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right.
\end{aligned}
$$

## Parametrization of $y$



$$
\begin{array}{r}
s_{1}(x) y+s_{0}(x)=0 \\
r(x)=0
\end{array}
$$

## Three variables and more

$$
f_{1}\left(x_{1}, \cdots, x_{n}\right)=\cdots=f_{n}\left(x_{1}, \cdots, x_{n}\right)=0
$$

## Problem

$$
\text { Find } p\left(x_{1}\right)=q_{1} f_{1}+\cdots+q_{n} f_{n}
$$

Matrix of $1, x_{1}, \ldots, x_{1}^{D}$ modulo $\left\langle f_{1}, \ldots, f_{n}\right\rangle$

$$
\begin{aligned}
& 1 \quad \cdots \quad x_{1}^{D} \\
& M=\begin{array}{c}
m_{1} \\
\vdots \\
m_{k}
\end{array}(\square) \\
& \Rightarrow \quad v \in \operatorname{Ker}(M) \text { iff } v_{0}+\cdots+v_{D} X^{D} \in\left\langle f_{1}, \ldots, f_{n}\right\rangle
\end{aligned}
$$

## Three variables and more

$$
f_{1}\left(x_{1}, \cdots, x_{n}\right)=\cdots=f_{n}\left(x_{1}, \cdots, x_{n}\right)=0
$$

## Problem

$$
\text { Find } p\left(x_{1}\right)=q_{1} f_{1}+\cdots+q_{n} f_{n}
$$

Matrix of multiplication by $x_{1}$ modulo $\left\langle f_{1}, \ldots, f_{n}\right\rangle$

$$
\begin{aligned}
M= & \begin{array}{ccc}
m_{1} \\
\vdots \\
m_{k}
\end{array}\left(\begin{array}{lll}
x_{1} m_{1} & \cdots & x_{1} m_{k} \\
& & \\
& \Rightarrow & \\
& \\
& \\
& \\
& \\
\left.x_{1}\right) \in\left\langle f_{1}, \ldots, f_{n}\right\rangle
\end{array}\right.
\end{aligned}
$$

## Three variables and more

$$
f_{1}\left(x_{1}, \cdots, x_{n}\right)=\cdots=f_{n}\left(x_{1}, \cdots, x_{n}\right)=0
$$

## Problem

Find $p\left(x_{1}\right)=q_{1} f_{1}+\cdots+q_{n} f_{n}$

Matrix of multiplication by $x_{1}$ modulo $f_{1}\left(x_{1}\right)$

$$
\begin{aligned}
M & =\left(\begin{array}{ccccc}
0 & 0 & \ldots & 0 & -c_{0} \\
1 & 0 & \ldots & 0 & -c_{1} \\
0 & 1 & \ldots & 0 & -c_{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -c_{d-1}
\end{array}\right) \\
& \Rightarrow \chi_{M}\left(x_{1}\right)=f_{1}\left(x_{1}\right) \in\left\langle f_{1}\right\rangle
\end{aligned}
$$

## Normal form and univariate representation

Normal form

$$
x_{1} m_{i} \bmod \left\langle f_{1}, \ldots, f_{n}\right\rangle
$$

- Euclidean division by Gröbner basis


## Groebner Bases computation

- $n>2$ in $\widetilde{O}\left((n d)^{(\omega+1) n}\right)$ operations
[Bardet, Faugère, Salvy 2015]
- $n=2$ in $\widetilde{O}\left(d^{2}\right)$ with terse representation [van der Hoeven, Larrieu 2018]

Univariate Representation

- Multivariate subresultant
- Rational Univariate Representation [Kronecker 1882, Rouillier 1999]
- u-resultant and its derivatives at point $\left(t, a_{1}, \ldots, a_{n}\right)$

$$
\text { u-resultant }=C \prod_{\zeta \mid F(\zeta)=0}\left(u_{0}+u_{1} \zeta_{n}+\cdots+u_{n} \zeta_{n}\right)
$$

- Lexicographical Gröbner bases
- Polynomial parametrization generically, bigger size [Dahan, Schost 2004]


## Univariate Representation with Geometric Resolution



## Theorem (Hensel lifting)

$$
\left\{\begin{array}{ll}
y_{0} & =\text { root of } f(0, y)=0 \\
y_{n+1}(x) & =y_{n}(x)-\frac{f\left(x, y_{n}(x)\right)}{\frac{\partial f}{\partial y}\left(x, y_{n}(x)\right)}
\end{array} \bmod x^{2^{n+1}}\right.
$$

Then $y_{n}(x)$ is a root of $f(x, y)=0 \bmod x^{2^{n}}$.

## Univariate Representation with Geometric Resolution



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\end{array} \bmod x^{2^{n+1}}\right.
$$

Then $y_{n}(x)$ is a root of $f(x, y)=0 \bmod x^{2^{n}}$.

## Univariate Representation with Geometric Resolution

$$
f_{1}\left(x_{1}, x_{2}, x_{3}\right)=\cdots=f_{3}\left(x_{1}, x_{2}, x_{3}\right)=0
$$

- RUR of $f_{1}\left(x_{1}, x_{2}, 0\right)$ and $f_{2}\left(x_{1}, x_{2}, 0\right)$

$$
p_{1}\left(x_{1}\right)=0 \quad x_{2}=q_{12}\left(x_{1}\right)
$$



- Lift $x_{3}$ in $f_{1}\left(x_{1}, x_{2}, x_{3}\right)=f_{2}\left(x_{1}, x_{2}, x_{3}\right)=0$

$$
\widetilde{p}_{1}\left(x_{1}, x_{3}\right)=0 \quad x_{2}=\widetilde{q}_{12}\left(x_{1}, x_{3}\right)
$$



- RUR of $\tilde{p}_{11}\left(x_{1}, x_{3}\right)$ and

$$
f_{3}\left(x_{1}, \tilde{q}_{12}\left(x_{1}, x_{3}\right), x_{3}\right)
$$

$$
p_{2}\left(x_{1}\right)=0 \quad x_{2}=q_{22}\left(x_{1}\right) \quad x_{3}=q_{23}\left(x_{1}\right)
$$


$\widetilde{O}\left(d^{3 n}\right)$ operations [Giusti, Lecerf, Salvy 2001]

## Further reading



David Cox
John Little
Donal O'Shea
Using
Algebraic
Geometry



Thomas Becker Volker Weispfenning
in Cooperation with Heinaz Krodel
Gröbner Bases
A Computational Approach to
Commutative Algebra

[^0]


Algorithmes Efficaces en Calcul Formel

Alin Bostan
Frédéric Chyzaz Marc GiustI
Romain Lebreto Grégoire Lecerf Bruno Saivy


## Further computing

Computer Algebra systems

- Giac/xcas
- Maple
- Magma
- Mathemagix
- Mathematica
- Mcaulay2
- MuPAD
- SageMath
- Singular
- ...

Special purpose libraries/software

- FGb (C)
[Faugère]
- Flint (C) [Hart]
- msolve [Berthomieux, Eder, Safey El Din]
- RS/RS3 (C) [Rouillier]
- borderbasix (mmx, C++) [Trebuchet, Mourrain]
- algebramix (mmx, C++) [van der Hoeven, Lecerf]
- geomsolvex (mmx, C++)
- larrix (mmx, C++)
- ...
[Lecerf]
[Larrieu]


## Path continuation



Is it possible to arrange 7 infinite cylinders of unit radius such that they are mutually touching?

## Path continuation

> - $f(x)=0$
> - $\operatorname{deg} d$

- We know the solutions of $x^{d}-1=0$

$$
(1-t)\left(x^{d}-1\right)+t f(x)=0
$$

- Multivariate case: compute the mixed volume first
- Find 1 solution by following only one path

Polynomial time in $\binom{n+d}{d}$ average [Beltran, Pardo 2009], [Lairez 2017]

- Roots distribution of random polynomials [Edelman, Kostlan 1995]


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## Path continuation

2 arrangements of 7 cylinders mutually touching [Bozóki et al. 2015]


- Modeled with 20 equations in 20 variables
- Mixed volume 121098993664
- First real solution after 80000000 paths
- Solution certified with Smale theorem


## Subdivision



## Test each cell

- 1 solution, guaranteed $\quad \Rightarrow$ keep
- 0 solution, guaranteed $\quad \Rightarrow$ remove
- don't know $\quad \Rightarrow$ subdivide

Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

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Complexity analysis: continuous amortization [Burr, Krahmer, Yap, 2009]

## Further reading



[^1]Peter Bürgisser
Felipe Cucker

## Condition

The Geometry of Numerical Algorithms


## Further computing

## Path continuation

- Bertini (C++, Python)
- Hom4PS (C++)
- PHCpack (Ada)
- analyziz (mmx, $\mathrm{C}++$ )
[Bates, Amethyst, Hauenstein, Sommese, Wampler] [Tien-Yien Li, Tianran Chen, Tsung-Lin Lee] [Verschelde] [van der Hoeven]


## Subdivision multivariate

- ibex $(\mathrm{C}++)$
[Chabert]
Subdivision univariate
- Realpaver (C)
[Granvilliers]
- ANewDsc (C) [Kobel,Rouillier,Sagraloff]
- realroot (mmx, $\mathrm{C}++$ )
[Mourrain]
- subdivision_solver ( $\mathrm{C}++$, Python)
[Imbach]
[M.]
- voxelize $(\mathrm{C}++)$
- Clenshaw (C, Python)
[M.]
- ...
- Ccluster (C, Julia) [Imbach,Pan, Yap]
- real_roots (sage)
[Witty]
- RS (C)
[Rouillier]
- SLV (C)
[Tsigaridas]


# Drawing reliably the projection of the solution 

(1) Isotopy
(2) Projection
(3) Drawing reliably

## Drawing hypersurfaces



## Definition (isotopy)

A triangulation $T$ is isotope to $V \subset \mathbb{R}^{n}$ if there exists $\varphi:[0,1] \times T \rightarrow \mathbb{R}^{n}$ such that:

- $\varphi$ is continuous
- $\varphi(0, T)=T$ and $\varphi(1, T)=V$
- $\varphi_{t_{0}}: T \rightarrow \varphi\left(t_{0}, T\right)$ is an homeomorphism


## Size of the triangulation [Kerber and Sagraloff 2011]

$$
f\left(x_{1}, \ldots, x_{n}\right) \text { of degree } d
$$

2 variables

## Theorem

In the worst case $\Omega\left(d^{2}\right)$ vertices and $O\left(d^{2}\right)$ segments

3 variables

## Theorem

In the worst case $\Omega\left(d^{3}\right)$ vertices and $O\left(d^{5}\right)$ triangles
n variables

## Theorem

In the worst case $\Omega\left(d^{n}\right)$ vertices and $O\left(d^{3 / 4 \cdot 2^{n}-1}\right)$ simplices

## Size of the triangulation [Kerber and Sagraloff 2011]

## $f$ of degree $d$ and vertices on the surface

2 variables
Theorem
In the worst case $\Omega\left(d^{\not \chi^{2} 3}\right)$ vertices and $O\left(d^{\not \mathfrak{Z}^{3}}\right)$ segments

3 variables

## Theorem

In the worst case $\Omega\left(d^{\not \beta^{4}}\right)$ vertices and $O\left(d^{57}\right)$ triangles
n variables

## Theorem

In the worst case $\Omega\left(d^{n+1}\right)$ vertices and $O\left(d^{3 / 42^{n}-1}\right)$ simplices

## In Robotic


$V$ solutions of $f_{1}=\cdots=f_{k}=\operatorname{det}\left(\frac{\partial f_{i}}{\partial x_{j}}\right)=0$ in $\mathbb{R}^{n} \times \mathbb{R}^{k}$
Goal: draw the projection of $V$ in $Q$

## Projection



## Definition (Semi-algebraic)

A semi-algebraic set is a set solution of a system of equalities and inequalities.

## Theorem (Tarski-Seidenberg)

The projection of a semi-algebraic set is semi-algebraic.

## Zariski closure



## Definition (Zariski closure)

The Zariski closure of $E$ is the minimal set $\bar{E}$ containing $E$ that is solution of a system of equalities.

## Theorem (elimination)

- $V$ solution of $p_{1}(q, x)=\cdots=p_{k+1}(q, x)=0$
- $G$ Gröbner basis of $F$ with respect to the lexdeg ordering with $x>q$
- $G_{q}$ the polynomials in $G \cap \mathbb{Q}[q]$

$$
\Rightarrow \overline{\pi_{Q}(V)}=\text { solutions of } G_{q}
$$

## Projective elimination



## Theorem

If $V_{p}$ is the projective closure of $V$ in $\mathbb{C}^{n} \times \mathbf{P}_{k}$, then

$$
\overline{\pi_{Q}(V)}=\pi_{Q}\left(V_{p}\right)
$$

## Geometry of the elimination

$$
x, y \in \mathbb{R}, z=a+i b \in \mathbb{C}
$$

$$
\left\{\begin{aligned}
0=f_{1}(x, y, a+i b)= & f_{1}(x, y, a)-\frac{b^{2}}{2} f_{1}^{\prime \prime}(x, y, a)+\cdots \\
& +i b\left(f_{1}^{\prime}(x, y, a)-\frac{b^{2}}{6} f_{1}^{\prime \prime \prime}(x, y, a)+\cdots\right) \\
0=f_{2}(x, y, a+i b)= & f_{2}(x, y, a)-\frac{b^{2}}{2} f_{2}^{\prime \prime}(x, y, a)+\cdots \\
& +i b\left(f_{2}^{\prime}(x, y, a)-\frac{b^{2}}{6} f_{2}^{\prime \prime \prime}(x, y, a)+\cdots\right)
\end{aligned}\right.
$$

Case $b=0: 2$ equation in $x, y, a$
Case $b \neq 0$ : 4 equation in $x, y, a, b$
$\Rightarrow$ possibly isolated points with $x, y \in \mathbb{R}$ and $z \in \mathbb{C} \backslash \mathbb{R}$
In general: $\overline{\pi_{Q}(V)}$ can have stable real components of $\operatorname{dim} n-2$

## Geometry of the projection



Figure 1: Local singularities of images of generic maps of surfaces into 3-space
[Guryonov 1997]

There exists a hypersurface $\Delta \subset C^{\infty}\left(M, \mathbb{R}^{3}\right)$ s.t. if $f \in C^{\infty}\left(M, \mathbb{R}^{3}\right) \backslash \Delta$, then the neighborhood of any point of $f(M)$ is one of the 4 above.

- Classification of generic singularities started with Whitney
- Thom introduced the transversality theorem
- Arnold and others provided numerous classifications


## Geometry of the projection



For the projection of critical points, we also have swallowtail singularities

## Further reading



## Further computing

## Quantifier elimination

- Redlog (Reduce)
- RegularChains (Maple)
- Resolve (Mathematica)
- QEPCAD (C)
- Tarski (C++)
- ...

Satisfiability of formula

- RAGLib (Maple)
- RealCertify (Maple)
- RSolver (OCaml)
- TSSOS (Julia)
[Dolzmann, Sturm]
[Moreno Maza et al] [Strzebonski]
[Brown]
[Brown]
- ...
[Safey el Din] [Magron, Safey el Din] [Ratschan] [Lassere, Magron, Wang]


## Drawing with symbolic approach



- Topology of curves in $\widetilde{O}\left(d^{6}+\tau d^{5}\right)$ operations [Kobel, Sagraloff, 2015], [Niang Diatta, Diatta, Rouillier, Roy, Sagraloff, submitted 2018]
- Topology of surfaces with $O\left(d^{5}\right)$ simplices
[Berberich, Kerber, Sagraloff 2009]


## Marching cubes [Lorensen and Cline 1987]


(1) Evaluate $f(x, y, z)$ of deg $d$ on a grid of size $N=n \times n \times n$
(2) For each cube, compute a triangle of the surface to plot

Complexity: Cost of evaluating $f$ on 1 point with Hörner: $O\left(d^{3}\right)$ Total naive: $O\left(d^{3} n^{3}\right)$
Reuse computation by coordinates: $O\left(d n^{3}\right)$

## Subdivision pruning

Interval arithmetic to remove boxes [Snyder 1992, Plantinga-Vegter 2006]


## Jacobian



$$
\begin{gathered}
0 \notin \square \operatorname{det}\left(J_{x}(B)\right)=\square \operatorname{det}\left(\begin{array}{c}
\frac{\partial f_{i}}{\partial x_{j}}(B)
\end{array}\right) \\
\Rightarrow \mathrm{V} \text { is globally parametrized by } q \text { in } B
\end{gathered}
$$

## Surface tracking



## Regular case

- First marching segments
[Dobkin et al. 1990]
- First marching triangles
[Hilton et al. 1996]
- Recent result on marching simplices, topology guaranteed
[Boissonnat, Kachanovich, Wintraecken, 2020]
Singular surfaces
- Use isosingular deflation near singularities
[Bates, Brake, Hauenstein, Sommese, Wampler, 2014]


## Further reading



## Further computing

Drawing with symbolic approach

- algcurve (Maple)
- isotop (C, Maple)
- EXACUS (C++)

Drawing with subdivision

- axl (C++, mmx)
- ibex (C++)
- Realpaver (C)
- voxelize (C++)
- ...

Drawing with marching cube

- JuliaGeometry (Julia)
- scikit-image (C, python)
- MathMod (C++)
- ...
[Christoflorou, Mantzaflaris, Mourrain, Wintz]
[Chabert]

Drawing with continuation
[Kelly]
[Lewiner]
[Taha]

- bertini_real
- CGAL (C++)
- GUDHI (C ++ )
- ...
[Brake et al]
[Rineau, Yvinec]
[Kachanovich]
[Deconink, Patterson, van Hoeij] [Peñeranda, Pouget, Lazard, Rouillier] [Melhorn et al]

Merci!

## Open problems

- Complexity of gröbner bases in dimension 3 with terse representation?
- What are the actual bounds on the number of simplices in drawings in 3D, in nD?
- What if we use polynomial pieces of degree $k$ instead of linear pieces?
- Reliable drawing of singular surfaces with prescribed singularities?


[^0]:    (6) Springer

[^1]:    Gundelehrea der mathematischea Wissenschatten 349
    1 Selles of Comperhessire Suctics is Matiencmatics

