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FULL DEAUTONOMISATION BY SINGULARITY CONFINEMENT AS AN INTEGRABILITY TEST

ALEXANDER STOKES^{1,†}, TAKAFUMI MASE¹, RALPH WILLOX¹, AND BASILE GRAMMATICOS²

ABSTRACT. Since its introduction, the method of full deautonomisation by singularity confinement has proved a strikingly effective way of detecting the dynamical degrees of birational mappings of the plane. This method is based on an observed link between two a priori unrelated notions: firstly the dynamical degree of the mapping and secondly the evolution of parameters required for its singularity structure to remain unchanged under a sufficiently general deautonomisation. We give a proof of this conjectured correspondence for a large class of birational mappings of the plane via the spaces of initial conditions for their deautonomised versions. We show that even for non-integrable mappings in this class, the surfaces forming these spaces have effective anticanonical divisors and one can define a kind of period map similar to that in the theory of rational surfaces associated with discrete Painlevé equations. This provides a bridge between the evolution of coefficients in the deautonomised mapping and the induced dynamics on the Picard lattice which encode the dynamical degree.

1. INTRODUCTION

The motivation for this paper stems from a number of observations of quite a remarkable phenomenon which arises in the context of the singularity analysis of birational mappings of the plane and which is related to their degree growth and, ultimately, to their integrability. This phenomenon manifests itself as the dynamical degree of a mapping being reflected in the characteristic polynomial for certain recurrences that the coefficients in the mapping have to obey for the singularity structure of the mapping to remain unchanged under a sufficiently general deautonomisation.

The first observations of this phenomenon can be traced back to so-called late-confining versions of discrete Painlevé equations [19], where singularities are still confined but not at the first opportunity. For non-autonomous mappings obtained through this kind of late confinement, as opposed to genuine discrete Painlevé equations, the dependence of the coefficients in the mapping on the independent variable ceases to be additive, multiplicative or through the arguments of elliptic functions, but rather the coefficients satisfy a recurrence relation whose characteristic polynomial has the dynamical degree of the mapping as a root. Outside the context of late confinement, it has been observed in many examples [25, 38, 15] that the minimal polynomial governing the degree growth of the equation also appears in confinement conditions on coefficients of the deautonomised mapping, though sometimes it is necessary to add to the mapping extra terms which do not affect the singularity patterns.

This relationship between dynamical degree and confinement conditions was first conjectured in [25] and, based on these and subsequent observations, the method of full deautonomisation by singularity confinement was introduced in [15, 38] as a remarkably effective integrability detector, which works even when other methods based on singularity analysis [17, 26, 37] fail. This correspondence between coefficient evolution and degree growth clearly requires explanation, and it is this problem that we address in this paper.

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Singularity confinement was proposed in [14] as a discrete counterpart to the Painlevé property of ordinary differential equations. The Painlevé property requires that if a solution develops a movable singularity, i.e. one whose location in the complex plane depends on initial conditions, then this does not induce multivaluedness which would obstruct the meaningful definition of a general solution for the equation. In the discrete case we are interested similarly in singularities which appear spontaneously when iterating the system, and require them to disappear after finitely many further iterations, i.e. the singular behaviour is confined to finitely many iterates and does not propagate. In the years after its introduction singularity confinement has been used to great effect as an integrability criterion, in particular to isolate deautonomised versions of known autonomous integrable systems. In particular, many discrete Painlevé equations were found [34, 13] by applying the method to Quispel-Roberts-Thompson (QRT) mappings [30, 31].

It is important to note, however, that singularity confinement is not a necessary criterion for integrability, in that there exist mappings with unconfined singularities which are integrable through linearisation. This is true even in the continuous case where there are linearisable differential equations which do not have the Painlevé property [36]. But most importantly, it is also not a sufficient condition since there exist examples in which all singularities are confined but the dynamics are chaotic, the most famous example being that due to Hietarinta and Viallet [20]. We shall work with a definition of integrability of birational mappings in terms of their dynamical degree, or equivalently algebraic entropy [4], following the idea that integrability in the discrete case is associated with slow growth in complexity under the dynamics, as evident in Arnol'd's notion of topological entropy [1, 2], Veselov's ideas of polynomial growth of certain characteristics [43], and related notions of entropy considered by Friedland [10] following from work with Milnor [11]. Recall that for a second-order equation defining a birational mapping $\varphi : (x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$, the limit $\lim_{k \rightarrow \infty} (\deg \varphi^k)^{1/k}$ is called the (first) dynamical degree of the mapping, in which $\deg \varphi^k$ is the degree as a rational function of initial conditions of the k -times composition of φ . The limit $\lim_{k \rightarrow \infty} \frac{1}{k} \log (\deg \varphi^k)$ is used mainly in the integrable systems community under the name 'algebraic entropy' [4]. These concepts can be similarly defined for non-autonomous equations defining families of birational mappings, and we say that a mapping, autonomous or not, is integrable if its algebraic entropy is zero or if its dynamical degree is one.

In the years since the realisation that singularity confinement is not sufficient for integrability, thankfully, a complete classification of mappings of the plane was obtained, according to their degree growth and whether they have the singularity confinement property in the sense of a space of initial conditions, both in the autonomous case by Diller and Favre [7] and in the non-autonomous case by one of the authors [24]. In particular if a non-autonomous mapping with the singularity confinement property has unbounded degree growth, then it must either fit into the framework of discrete Painlevé equations [39] or be non-integrable. Therefore for singularity confinement to function as an integrability test, it needs only be combined with something that reliably detects the dynamical degree. We will show that full deautonomisation does exactly this.

We will illustrate singularity confinement and the phenomena described above through the example

$$x_{n+1} + x_{n-1} = \frac{1}{x_n^2}, \quad (1.1)$$

which is a QRT mapping belonging to Class I in the classification of [32] (see also [35] for a derivation and [16, Appendix] for an easily consultable list). The definition of the singularity confinement property for such an equation, when recast as a mapping of the projective plane \mathbb{P}^2 , is that it is birationally conjugate to an automorphism. I.e., that one can perform a sequence of blow-ups and possibly blow-downs to obtain a rational surface on which the mapping is regularised as an automorphism. Phrased on the level of the equation itself, as it originally was in [14], a singularity is constituted by an iteration from some initial condition that results in a loss of a degree of freedom, in the sense that the iterate takes a value at which the inverse

mapping is undefined. As already mentioned above, ‘singularity confinement’ refers to the situation where finitely many further iterations lead the singularity to disappear, with the lost degree of freedom being recovered.

For example, while iterating the equation above, if some iterate x_n takes the value 0 while $x_{n-1} = u$ is some finite constant, then the value of the subsequent iterate is $x_{n+1} = \infty$ irrespective of the precise value of u . This constitutes a loss of degree of freedom in the sense that the one-parameter family of pairs $(x_{n-1}, x_n) = (u, 0)$ all lead to the same value $(x_n, x_{n+1}) = (0, \infty)$. In terms of the iteration mapping $(x_{n-1}, x_n) \mapsto (x_n, x_{n+1})$, this is nothing but the curve given by $x_n = 0$ being contracted to a point, i.e. a curve being blown down by the mapping.

Computing further we find that $x_{n+2} = 0$ but then, crucially, see that the value of the next iterate $x_{n+3} = \frac{1}{x_{n+2}^2} - x_{n+1}$ is not defined, with the point $(x_{n+1}, x_{n+2}) = (\infty, 0)$ being an indeterminacy of the mapping. The way in which singularity confinement is verified, is to take $x_n = \varepsilon$ for some small parameter ε which leads to

$$x_{n-1} = u, \quad x_n = \varepsilon, \quad x_{n+1} = \frac{1}{\varepsilon^2} - u, \quad x_{n+2} = -\varepsilon + O(\varepsilon^2), \quad x_{n+3} = u + O(\varepsilon). \quad (1.2)$$

Then, defining the values of iterates as the limits of the above as $\varepsilon \rightarrow 0$, we have

$$x_{n-1} = u, \quad x_n = 0, \quad x_{n+1} = \infty^2, \quad x_{n+2} = 0, \quad x_{n+3} = u, \quad (1.3)$$

in which $x_{n+1} = \infty^2$ means the iterate is of order $1/\varepsilon^2$ as above. In singularity confinement parlance we say that the lost degree of freedom is recovered through the reappearance of u in the value of x_{n+3} , and that the singularity is confined. The results of the above calculation are then summarized by saying that equation (1.1) admits the (confined) singularity pattern $\{0, \infty^2, 0\}$.

To demonstrate the use of singularity confinement in the deautonomisation procedure, let us consider the following generalisation of mapping (1.1)

$$x_{n+1} + x_{n-1} = \frac{a_n + b_n x_n}{x_n^2}, \quad (1.4)$$

where the n -dependence of the functions a_n, b_n is to be determined by requiring that the singularity structure of (1.4) is the same as in the autonomous case. As above, some iterate x_n taking the value 0 constitutes a singularity of the mapping (1.4), so we let $x_n = \varepsilon$ and $x_{n-1} = u$, and compute

$$\begin{aligned} x_n &= \varepsilon, \\ x_{n+1} &= \frac{a_n}{\varepsilon^2} + \frac{b_n}{\varepsilon} - u, \\ x_{n+2} &= -\varepsilon + \frac{b_{n+1}}{a_n} \varepsilon^2 + O(\varepsilon^3), \\ x_{n+3} &= \frac{a_{n+2} - a_n}{\varepsilon^2} - \left(b_{n+2} - \frac{2a_{n+2}b_{n+1}}{a_n} + b_n \right) \frac{1}{\varepsilon} + O(1). \end{aligned} \quad (1.5)$$

In order for the singularity to be confined in the same way as above, the singular part of the expansion of x_{n+3} in ε must vanish, so we must have $a_{n+2} = a_n$ for all n , as well as

$$b_{n+2} - 2b_{n+1} + b_n = 0. \quad (1.6)$$

Solving these ‘confinement’ conditions we find

$$x_{n+1} + x_{n-1} = \frac{\alpha + \beta n}{x_n} + \frac{(-1)^n \gamma + \delta}{x_n^2}, \quad (1.7)$$

for some constants $\alpha, \beta, \gamma, \delta$, which reduces to a known example of a discrete Painlevé equation [33]. More relevant to the present paper is the fact that the characteristic polynomial of the linear recurrence (1.6) is $t^2 - 2t + 1$, which has no roots larger than one.

In the deautonomisation which led to the mapping (1.7), we required that the evolution of coefficients be such that the deautonomised mapping admits the exact same confined singularity pattern $\{0, \infty^2, 0\}$. However it is possible to choose a_n, b_n such that the singularity associated

with the value 0 is confined but after more iterations rather than at the earliest opportunity, a notion called late confinement [15]. In particular, calculating further along the lines above but assuming that the previous confinement conditions do not hold, we find

$$\begin{aligned}
x_n &= \varepsilon, \\
x_{n+1} &= \frac{a_n}{\varepsilon^2} + \frac{b_n}{\varepsilon} - u, \\
x_{n+2} &= -\varepsilon + \frac{b_{n+1}}{a_n} \varepsilon^2 + O(\varepsilon^3), \\
x_{n+3} &= \frac{a_{n+2} - a_n}{\varepsilon^2} - \left(b_{n+2} - \frac{2a_{n+2}b_{n+1}}{a_n} + b_n \right) \frac{1}{\varepsilon} + O(1), \\
x_{n+4} &= \varepsilon + \frac{a_n b_{n+3} - a_{n+2} b_{n+1} + a_n b_{n+1}}{a_n(a_{n+2} - a_n)} \varepsilon^2 + O(\varepsilon^3), \\
x_{n+5} &= \frac{A_n}{\varepsilon^2} - \left(B_n + 2A_n \left(\frac{b_{n+1}}{a_n} + \frac{b_{n+3}}{a_n - a_{n+2}} \right) \right) \frac{1}{\varepsilon} + O(1),
\end{aligned} \tag{1.8}$$

where $A_n = a_{n+4} - a_{n+2} + a_n$ and $B_n = b_{n+4} - 2b_{n+3} + b_{n+2} - 2b_{n+1} + b_n$. Therefore we have an opportunity to confine the singularity at a later stage, by ensuring that the singular powers of ε in x_{n+5} vanish, such that we have singularity confinement with the singularity pattern $\{0, \infty^2, 0, \infty^2, 0\}$. In this case the confinement conditions are given by the pair of recurrences

$$a_{n+4} - a_{n+2} + a_n = 0, \quad b_{n+4} - 2b_{n+3} + b_{n+2} - 2b_{n+1} + b_n = 0, \tag{1.9}$$

whose characteristic polynomial

$$(t^4 - t^2 + 1)(t^4 - 2t^3 + t^2 - 2t + 1), \tag{1.10}$$

has six roots of modulus one, as well as a pair λ, λ^{-1} where

$$\lambda = \frac{1 + \sqrt{2} + \sqrt{-1 + 2\sqrt{2}}}{2} \cong 1.8832. \tag{1.11}$$

The mapping (1.4) with coefficients satisfying (1.9) but otherwise generic is indeed non-integrable with dynamical degree given by this largest root λ of the characteristic polynomial, which can be proved by a number of methods, e.g. [41, 17, 37].

More generally, consider the mapping

$$x_{n+1} + x_{n-1} = \frac{1}{x_n^{2m}}, \tag{1.12}$$

for some integer $m \geq 1$, which is known [15, 37] to have dynamical degree $m + \sqrt{m^2 - 1}$. Similar analysis as above shows that this mapping admits the confined singularity pattern $\{0, \infty^{2m}, 0\}$. If we proceed to deautonomise, by taking

$$x_{n+1} + x_{n-1} = \frac{a_n}{x_n^{2m}}, \tag{1.13}$$

we see by similar calculations to those for the $m = 1$ case above, that for the pattern $\{0, \infty^{2m}, 0\}$ to persist we must require the confinement condition $a_{n+2} = a_n$. Note that this is the same condition as for the $m = 1$ case which, most importantly, does not give any information on the dynamical degree of the mapping. However, in performing these calculations with $x_n = \varepsilon$ one notices that adding terms $\frac{1}{x^j}$, $j = 1, \dots, 2m - 1$ to the right-hand side of (1.12) does not affect the leading behaviours $x_{n+1} \sim \frac{1}{\varepsilon^{2m}}$, $x_{n+2} \sim \varepsilon$, and hence that there still is an opportunity to confine at the next iterate. Doing so leads to the ‘full deautonomisation’ of the mapping (1.12) given by

$$x_{n+1} + x_{n-1} = \frac{a_n^{(2m)}}{x_n^{2m}} + \sum_{i=2}^{2m-1} \frac{a_n^{(i)}}{x_n^i} + \frac{b_n}{x_n}. \tag{1.14}$$

The confinement conditions on the coefficients $a_n^{(i)}$ and b_n required for this mapping to admit the singularity pattern $\{0, \infty^{2m}, 0\}$ are

$$a_{n+2}^{(i)} = (-1)^i a_n^{(i)}, \quad i = 2, \dots, 2m, \quad b_{n+2} - 2mb_{n+1} + b_n = 0. \quad (1.15)$$

The characteristic polynomial for the recurrence satisfied by b_n is

$$t^2 - 2mt + 1, \quad (1.16)$$

whose largest root $\lambda = m + \sqrt{m^2 - 1}$ coincides with the dynamical degree of the mapping. It is worth emphasizing that this is also the dynamical degree of the non-autonomous mapping with coefficients that obey (1.15).

The remarkable connection that is observed here between degree growth and evolution of coefficients is what we aim to explain in this paper. The key to this explanation is offered by an extension of the notion of ‘period mapping’, originally introduced for integrable mappings of the plane. In the non-autonomous case, the singularity confinement property of a family φ_n of birational mappings of the plane is equivalent to the existence of a space of initial conditions: a family of rational surfaces to which the mappings lift to isomorphisms (cf. Definition 2.4). In the theory of rational surfaces associated with discrete Painlevé equations due to Sakai [39], there is a known bridge between parameter evolution and degree growth, namely the interaction between dynamics on the Picard lattice of the surfaces and a kind of period map, whose construction goes back to the work of Looijenga [23]. This bridge relies on, amongst other things, the fact that the surfaces providing the space of initial conditions for the mapping have an effective anticanonical divisor, i.e. the fact that they admit a zero-free rational two-form .

In fact the question of the existence of effective anticanonical and, more generally, antipluricanonical divisors (here meaning a divisor representing some negative multiple of the canonical class) appears frequently in the study of automorphisms of algebraic surfaces and their dynamics [3, 6, 8, 9, 12, 18, 27, 45]. In particular, a conjecture of Gizatullin (communicated to Harbourne by Dolgachev and Looijenga [18]) stated that if a rational surface X has an automorphism φ of infinite order, then X must have an effective antipluricanonical divisor. In all counterexamples to this conjecture presented by Harbourne [18], φ has zero entropy and the pair (X, φ) is non-minimal in the sense that there exists a birational morphism $\pi : X \rightarrow X'$ such that $\pi \circ \varphi \circ \pi^{-1}$ is an automorphism of X' but π is not an isomorphism. McMullen then posed in [27, Section 12] a refined version of the conjecture, by asking whether for minimal (X, φ) , φ being of infinite order is sufficient to guarantee that X has an effective antipluricanonical divisor. For certain classes of φ coming from group actions the conjecture is true [45], but these also have zero entropy, i.e. also correspond to integrable mappings. A negative answer to McMullen’s question was given by Bedford and Kim [3] who constructed a family of mappings which give automorphisms of infinite order with nonzero entropy, on rational surfaces which admit no effective antipluricanonical divisor.

In this paper we show that in a large class of non-autonomous mappings, namely those that preserve rational two-forms (hence including all mappings of QRT-type), even in non-integrable cases the space of initial conditions can be chosen via minimisation so that the surfaces are anticanonical, i.e. have effective anticanonical divisors. Then a period map exists and can be used to show that the evolution of coefficients under deautonomisation must correspond to the dynamics on the Picard lattice and thus encodes the dynamical degree. The main results of the paper can be summarised in the following theorem, which provides a justification of the method of full deautonomisation by singularity confinement as an integrability test.

Theorem 1.1. *If an autonomous mapping has a space of initial conditions and unbounded degree growth, and preserves a rational two-form, the period map parametrisation of the coefficients in a sufficient deautonomisation gives a linear system of difference equations for their evolution, whose characteristic polynomial has the dynamical degree of the mapping as a root.*

There are some subtle technical points relating to the applicability of the method of full deautonomisation as an integrability detector that we should point out here. The first is that,

when applying the method to an autonomous mapping which preserves a rational two-form, so whose space of initial conditions is within the scope of Theorem 2.23 on effectiveness of the anticanonical divisor, one must ascertain the extent to which the mapping must be generalised via the addition of extra terms in order for the confinement conditions to reliably detect the dynamical degree. While it is possible to define a sufficiently general deautonomisation in terms of families of surfaces forming the space of initial conditions (cf. Definition 2.30), in order to make it possible to address this question directly on the level of the equations themselves, we provide the reader with sufficiently general deautonomisations for QRT-type classes I-VI in Appendix B.

A second subtle point, which also has implications for the implementation of the method, arises on the level of geometry. The family of examples constructed by Bedford and Kim [3] show that a period map construction, as we will present here, might not be possible for just any, general, birational mapping with a space of initial conditions because the existence of an effective anticanonical divisor is not guaranteed in the general setting. Therefore, in such examples, any hope of reading off the dynamical degree for the mapping from confinement conditions on coefficients in the mapping must be given up. The problem being that finding a linearisation of the conditions on the coefficients that might lead to a characteristic polynomial that encodes the dynamical degree is pretty much impossible, as is to be expected given the absence of a period map parametrisation of the coefficients in the mapping. This is similar to cases where the anticanonical divisor is effective, but for which the geometry is such that the parametrisation of its coefficients by the period map is highly non-trivial. This is the case in, for example, elliptic discrete Painlevé equations or QRT-type mappings of Class VIII, though for these mappings there are ways to overcome these difficulties and obtain an appropriate parametrisation of coefficients and therefore a linearisation of the confinement conditions such that full deautonomisation by singularity confinement still works [16].

1.1. Outline of the paper. Readers interested in the utilisation of the method as an integrability test can proceed first to Sections 3-6, and additionally see Appendix B for details of sufficiently general deautonomisations for QRT classes I-V, before referring to Section 2 for the geometric theory behind it. Readers interested in the geometry of the space of initial conditions may proceed to Section 2 for the setup and proofs of the main results relating to anticanonical divisors and the period map and to Appendix A for a local classification of the anticanonical divisors of rational surfaces associated with non-integrable mappings. They can then consult Sections 3-6 for concrete examples.

2. GEOMETRY

In this Section we give a rigorous formulation and justification for the full deautonomisation procedure as a detector of the dynamical degree of a mapping that possesses a space of initial conditions. We first recall the definition of the singularity confinement property, for a second-order mapping, in the sense of the existence of a space of initial conditions for the mapping, in both the autonomous and non-autonomous cases. We then review the known classifications of autonomous and non-autonomous mappings according to their degree growth and according to whether they have a space of initial conditions, and deduce which cases full deautonomisation should be shown to discern between. Next we outline the period map defined for complex rational surfaces with effective anticanonical divisor, which will be the main tool in our formulation. For this tool to be applicable in the cases we need, we must prove Theorem 2.23, which states that in a wide class of mappings – including deautonomised versions of all those in the classification of QRT-type mappings in [32] (or [35]) – if a mapping with a space of initial conditions is non-integrable, then the surfaces forming this space still have effective anticanonical divisors. We then show that coefficients in a non-autonomous mapping can always be parametrised by appropriate computation of the period map, in a way that linearises their evolution and such that the characteristic polynomial of the linearised recurrences will have the dynamical degree of the mapping as a root.

Most of the setup for this Section is given in [24, Appendix A], but we recall the relevant parts here in order to make the present paper self-contained. In what follows we work over \mathbb{C} and use the following notation and conventions:

- X : a smooth projective rational surface.
- $\text{Div}(X)$: the group of divisors on X .
- \sim : linear equivalence of divisors.
- $\text{Pic}(X)$: the Picard group of X , which is isomorphic to $\text{Div}(X)/\sim$. We often use the same symbol for an element of $\text{Pic}(X)$ as for its corresponding linear equivalence class of divisors and we write the group operation on $\text{Pic}(X)$ additively.
- $[D]$: the linear equivalence class of $D \in \text{Div}(X)$. We will generally use calligraphic script to indicate classes of divisors, i.e. $\mathcal{D} = [D]$.
- $|\mathcal{F}|$: the linear system of $\mathcal{F} \in \text{Pic}(X)$.
- $\text{Pic}^+(X) = \{\mathcal{F} \in \text{Pic}(X) \mid \mathcal{F} = [F] \text{ where } F \in \text{Div}(X) \text{ is effective}\}$: the set of effective classes.
- $D_1 \cdot D_2$: the intersection number of the divisors D_1 and D_2 . This is well-defined on linear equivalence classes and we use the same notation for the intersection number $\mathcal{D}_1 \cdot \mathcal{D}_2$ of $\mathcal{D}_1, \mathcal{D}_2 \in \text{Pic}(X)$.
- $(D)^2 = (\mathcal{D})^2$: the self-intersection number of $D \in \text{Div}(X)$ or $\mathcal{D} = [D] \in \text{Pic}(X)$.
- $\mathcal{K}_X \in \text{Pic}(X)$: the canonical bundle or canonical divisor class of X .
- $\text{div}(\omega) \in \text{Div}(X)$: the divisor of a rational two-form ω on X , so $[\text{div}(\omega)] = \mathcal{K}_X$.
- $\mathcal{O}_{\mathbb{P}^n}(1) \in \text{Pic}(\mathbb{P}^n)$: the twisting sheaf corresponding to the class of a hyperplane in \mathbb{P}^n .
- $\text{Pic}_{\mathbb{Q}}(X) = \text{Pic}(X) \otimes \mathbb{Q}$, $\text{Pic}_{\mathbb{R}}(X) = \text{Pic}(X) \otimes \mathbb{R}$, $\text{Pic}_{\mathbb{C}}(X) = \text{Pic}(X) \otimes \mathbb{C}$.
- $\rho(X)$: the Picard number of X , which is equal to $\text{rank Pic}(X)$ since X is rational.
- $H^i(X, D)$: the i -th cohomology group of the divisor D on X .
- $h^i(D) = h^i(X, D) = \dim H^i(X, D)$.
- $b_1 = \text{rank } H^1(X; \mathbb{Z})$: the first Betti number.
- $h^{p,q} = \dim H^q(X, \Omega^p)$: the Hodge numbers.
- $g_a(C) = \dim H^1(C, \mathcal{O}_C)$: the arithmetic genus of an irreducible curve C .

2.1. Spaces of initial conditions for birational mappings of the plane. The singularity confinement property of a second-order system of difference equations defining a birational mapping of the plane is equivalent to the existence of a space of initial conditions. The terminology ‘space of initial conditions’ or ‘space of initial values’ originates in Okamoto’s work [29] using blow-ups to construct augmented phase spaces on which Hamiltonian forms of the Painlevé differential equations are regularised. Solutions of these equations can be globally defined by analytic continuation from any point in the space, hence the term space of initial conditions. The same terminology continues to be used in the discrete case to describe rational surfaces on which birational mappings are regularised in an analogous way. In the autonomous case this means that the birational mapping becomes an automorphism of the surface. In the non-autonomous case the n -dependence in the coefficients means that the equation becomes a family of mappings, for which a space of initial conditions is a family of rational surfaces between which the mappings become isomorphisms.

2.1.1. Autonomous case. Consider an autonomous second-order discrete equation

$$(x_{n+1}, y_{n+1}) = (f(x_n, y_n), g(x_n, y_n)), \quad (2.1)$$

where f, g are rational functions of their arguments with coefficients independent of n , such that the mapping $(x_n, y_n) \mapsto (f(x_n, y_n), g(x_n, y_n))$ is birational. By taking (x_n, y_n) as an affine chart we can extend this to a birational mapping

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (2.2)$$

where here and from this point onwards we use dashed arrows to denote birational mappings between surfaces, with solid arrows being morphisms. Note that scalar three-point equations

$x_{n+1} = F(x_n, x_{n-1})$ with F homographic in x_{n-1} , such as the examples considered in the previous section, can be brought to this form by setting $y_n = x_{n-1}$.

Definition 2.1 (space of initial conditions for an autonomous mapping). We say that an autonomous equation defining a mapping φ as above has a space of initial conditions if there exists a rational surface X and a birational map $\pi : X \dashrightarrow \mathbb{P}^2$ such that $\tilde{\varphi} := \pi^{-1} \circ \varphi \circ \pi$ is an automorphism of X :

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\varphi}} & X \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^2 & \dashrightarrow^{\varphi} & \mathbb{P}^2. \end{array}$$

The automorphism $\tilde{\varphi}$ then induces, by pullback and pushforward, linear transformations of $\text{Pic}(X)$ which we denote by $\tilde{\varphi}^*$ and $\tilde{\varphi}_* = (\tilde{\varphi}^*)^{-1}$ respectively. These maps are lattice automorphisms of $\text{Pic}(X)$, i.e. \mathbb{Z} -module automorphisms which preserve the symmetric bilinear form given by the intersection product, they fix \mathcal{K}_X and preserve effectiveness of divisor classes.

2.1.2. *Non-autonomous case.* In the non-autonomous case, consider a second-order discrete system $(x_{n+1}, y_{n+1}) = (f_n(x_n, y_n), g_n(x_n, y_n))$, defining a family of mappings

$$\varphi_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2. \quad (2.3)$$

In addition to a family of rational surfaces X_n on which φ_n conjugates to an isomorphism $X_n \rightarrow X_{n+1}$, for each value of n , in the non-autonomous case it is necessary to include in the definition of a space of initial conditions some extra conditions in order to formulate a general theory. This is because allowing for arbitrary n -dependence in the mappings and surfaces leads to many pathological examples and prevents us from making any meaningful statements in general; see [24] for a detailed discussion of this. In order to state the definition we will work with for non-autonomous mappings we require the following.

Definition 2.2 (basic rational surface). A basic rational surface is a rational surface that admits \mathbb{P}^2 as a minimal model, i.e. there exists a birational morphism from X to \mathbb{P}^2 . Such a surface can be obtained from \mathbb{P}^2 by a finite number of blow-ups, without the need for blow-downs.

Definition 2.3 (geometric basis). Let X be a basic rational surface and $(e^{(0)}, \dots, e^{(r)})$ a \mathbb{Z} -basis for $\text{Pic}(X)$. We call $(e^{(0)}, \dots, e^{(r)})$ a geometric basis if there exists a composition of blow-ups $\pi = \pi^{(1)} \circ \dots \circ \pi^{(r)} : X \rightarrow \mathbb{P}^2$ such that $e^{(0)} = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$, where $e^{(i)}$ is the class of the total transform (under $\pi^{(i+1)} \circ \dots \circ \pi^{(r)}$) of the exceptional curve of $\pi^{(i)}$ for $i = 1, \dots, r$. In this case we say that $(e^{(0)}, \dots, e^{(r)})$ is the geometric basis of $\text{Pic}(X)$ corresponding to π .

Definition 2.4 (space of initial conditions for a non-autonomous mapping; without blow-downs). A space of initial conditions for a non-autonomous mapping $\varphi_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ consists of sequences $(X_n)_n$ and $(\pi_n)_n$, where each X_n is a rational surface and π_n is a birational morphism $\pi_n : X_n \rightarrow \mathbb{P}^2$ written as a sequence of blow-ups $\pi_n = \pi_n^{(1)} \circ \dots \circ \pi_n^{(r)}$, such that the following conditions hold:

- The mappings φ_n become isomorphisms $\tilde{\varphi}_n := \pi_{n+1}^{-1} \circ \varphi_n \circ \pi_n$ as in Figure 1.
- Let $e_n = (e_n^{(0)}, \dots, e_n^{(r)})$ be the geometric basis for $\text{Pic}(X_n)$ corresponding to π_n . Then the matrices of $\tilde{\varphi}_n^* : \text{Pic}(X_{n+1}) \rightarrow \text{Pic}(X_n)$ with respect to these bases do not depend on n .
- The set of effective classes in $\text{Pic}(X_n)$ in terms of the basis $(e_n^{(0)}, \dots, e_n^{(r)})$ does not depend on n , i.e. if $\sum_i a^{(i)} e_n^{(i)} \in \text{Pic}(X_n)$ is effective, then $\sum_i a^{(i)} e_k^{(i)} \in \text{Pic}(X_k)$ is effective for any k .

Using the geometric bases e_n we may identify all $\text{Pic}(X_n)$ with a single \mathbb{Z} -module

$$\text{Pic}(\mathcal{X}) = \sum_{i=0}^r \mathbb{Z} e^{(i)}, \quad (2.4)$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X_{n-1} & \xrightarrow{\tilde{\varphi}_{n-1}} & X_n & \xrightarrow{\tilde{\varphi}_n} & X_{n+1} & \longrightarrow & \cdots \\
& & \downarrow \pi_{n-1} & & \downarrow \pi_n & & \downarrow \pi_{n+1} & & \\
\cdots & \dashrightarrow & \mathbb{P}^2 & \dashrightarrow^{\varphi_{n-1}} & \mathbb{P}^2 & \dashrightarrow^{\varphi_n} & \mathbb{P}^2 & \dashrightarrow & \cdots,
\end{array}$$

FIGURE 1. Space of initial conditions for a non-autonomous mapping; without blow-downs

via

$$\begin{aligned}
\iota_n : \text{Pic}(\mathcal{X}) &\longrightarrow \text{Pic}(X_n), \\
\iota_n(e^{(i)}) &= e_n^{(i)},
\end{aligned} \tag{2.5}$$

where (for reasons that will become clear when we discuss deautonomisation) we use \mathcal{X} to distinguish this module from the Picard group of a single surface X . Denote the map induced by φ_n^* , for any n , by $\Phi : \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{X})$, which is well-defined because of the conditions in Definition 2.4:

$$\begin{array}{ccc}
\text{Pic}(\mathcal{X}) & \xleftarrow{\Phi} & \text{Pic}(\mathcal{X}) \\
\downarrow \iota_n & & \downarrow \iota_{n+1} \\
\text{Pic}(X_n) & \xleftarrow{\tilde{\varphi}_n^*} & \text{Pic}(X_{n+1}).
\end{array}$$

The canonical class in each $\text{Pic}(X_n)$ is identified with the element

$$\mathcal{K}_{\mathcal{X}} := \iota_n^{-1}(\mathcal{K}_{X_n}) = -3e^{(0)} + e^{(1)} + \cdots + e^{(r)}, \tag{2.6}$$

and the intersection form on $\text{Pic}(X_n)$ can be used to equip $\text{Pic}(\mathcal{X})$ with the symmetric bilinear form defined by

$$e^{(i)} \cdot e^{(j)} = \begin{cases} +1 & (i = j = 0) \\ -1 & (i = j \neq 0) \\ 0 & (i \neq j). \end{cases} \tag{2.7}$$

We will refer to $\text{Pic}(\mathcal{X})$ equipped with this bilinear form as the Picard lattice of the space of initial conditions X_n . Then Φ is a lattice automorphism of $\text{Pic}(\mathcal{X})$ that fixes the element $\mathcal{K}_{\mathcal{X}}$. It also preserves effectiveness in the sense that if $\mathcal{F} \in \text{Pic}(\mathcal{X})$ is such that $\iota_{n+1}(\mathcal{F})$ is an effective class on X_{n+1} , then $\iota_n \circ \Phi(\mathcal{F})$ is an effective class on X_n .

While all the examples we will study in Sections 3-6 will have spaces of initial conditions as defined above, our main general results will rely on the notion of minimality of a space of initial conditions, which requires us to give a more general definition allowing blow-downs in the construction of X_n from \mathbb{P}^2 .

Definition 2.5 (space of initial conditions for a non-autonomous mapping; allowing blow-downs). A space of initial conditions for a non-autonomous mapping φ_n , allowing blowdowns, consists of sequences $(X_n)_n$, $(Y_n)_n$, $(\pi_n)_n$, $(\rho_n)_n$, and $(\sigma_n)_n$, with

- X_n a rational surface with birational morphism $\pi_n : X_n \rightarrow \mathbb{P}^2$, written as a sequence of blow-ups as $\pi_n = \pi_n^{(r)} \circ \cdots \circ \pi_n^{(1)}$,
- Y_n a rational surface with birational morphism $\rho_n : Y_n \rightarrow \mathbb{P}^2$, written as a sequence of blow-ups as $\rho_n = \rho_n^{(s)} \circ \cdots \circ \rho_n^{(1)}$,
- $\sigma_n : Y_n \rightarrow X_n$ a birational morphism written as sequences of blow-ups as $\sigma_n = \sigma_n^{(s-r)} \circ \cdots \circ \sigma_n^{(1)}$,

such that the following conditions hold:

- The mappings $\tilde{\varphi}_n := \sigma_{n+1} \circ \rho_{n+1}^{-1} \circ \varphi_n \circ \rho_n \circ \sigma_n^{-1} : X_n \rightarrow X_{n+1}$ are isomorphisms as in Figure 2.

- Let $f_n = (f_n^{(0)}, \dots, f_n^{(s)})$ be the geometric basis for $\text{Pic}(Y_n)$ corresponding to ρ_n . Then the set of effective classes in $\text{Pic}(Y_n)$ in terms of the basis f_n does not depend on n .
- Let $e_n = (e_n^{(0)}, \dots, e_n^{(r)})$ be the geometric basis for $\text{Pic}(X_n)$ corresponding to π_n . Then the set of effective classes in $\text{Pic}(X_n)$ in terms of the basis e_n does not depend on n .
- Let $F_n^{(j)} \in \text{Pic}(Y_n)$ be the class of (the total transform of) the exceptional curve contracted by $\sigma_n^{(j)}$, for $j = 1, \dots, s - r$. Then the expression for $F_n^{(j)}$ in terms of the basis f_n does not depend on n .
- The expression for $\sigma_n^*(e_n^{(i)})$ in the basis f_n for $\text{Pic}(Y_n)$ does not depend on n .
- The matrices of $\tilde{\varphi}_n^* : \text{Pic}(X_{n+1}) \rightarrow \text{Pic}(X_n)$ with respect to the bases e_{n+1} and e_n do not depend on n .

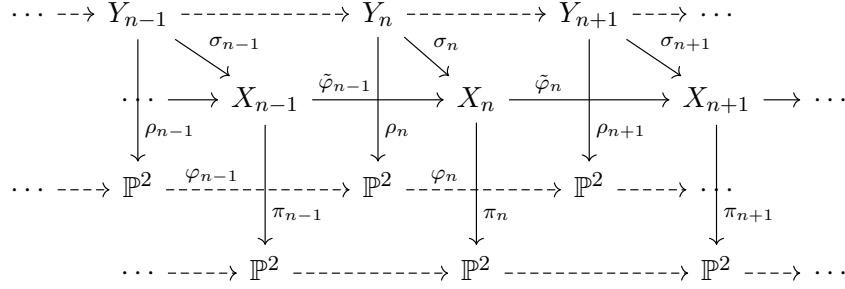


FIGURE 2. Space of initial conditions for a non-autonomous mapping; allowing blow-downs

If a mapping has a space of initial conditions as above then we may still define the Picard lattice $\text{Pic}(\mathcal{X})$ by identification of all $\text{Pic}(X_n)$ via the geometric bases corresponding to π_n , with the lattice automorphism Φ of $\text{Pic}(\mathcal{X})$ defined as above.

Remark 2.6. The key feature of the more general Definition 2.5 of a space of initial conditions is that σ_n may contract curves on Y_n which ρ_n does not, in which case the mapping $\pi_{n+1} \circ \tilde{\varphi}_n \circ \pi_n^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ will not coincide with φ_n , but rather be conjugate to it by a birational coordinate change on \mathbb{P}^2 .

2.1.3. *Minimisation of a space of initial conditions.* Allowing for blow-downs in the definition of a space of initial conditions means we can introduce the notion of a minimal space of initial conditions for a given non-autonomous mapping.

Definition 2.7 (minimisation of a space of initial conditions). Let $\varphi_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a non-autonomous mapping with a space of initial conditions allowing blow-downs as in Definition 2.5. A minimisation of the space of initial conditions consists of sequences $(X'_n)_n$ and $(\mu_n)_n$ with

- X'_n a rational surface
- a birational morphism $\mu_n : X_n \rightarrow X'_n$ which is a sequence of blow-ups $\mu_n = \mu_n^{(1)} \circ \dots \circ \mu_n^{(r')}$

such that the following conditions hold:

- For every n , the map $\tilde{\varphi}'_n := \sigma_{n+1}^{-1} \circ \tilde{\varphi}_n \circ \sigma_n : X'_n \rightarrow X'_{n+1}$ is an isomorphism.
- Let $e_n = (e_n^{(0)}, \dots, e_n^{(r)})$ be the geometric basis for $\text{Pic}(X_n)$ corresponding to π_n , and let $E_n^{(k)} \in \text{Pic}(X_n)$ be the class of (the total transform of) the exceptional divisor contracted by $\mu_n^{(k)}$ for $k = 1, \dots, r'$. Then the expression for $E_n^{(k)}$ in terms of the basis e_n is independent of n .
- There exist birational morphisms $\pi'_n : X'_n \rightarrow \mathbb{P}^2$ such that the geometric bases corresponding to π'_n satisfy the requirements for X'_n to form a space of initial conditions for φ_n in the sense of Definition 2.5.

We give a diagram showing the birational mappings involved in minimisation of a space of initial conditions in Figure 3.

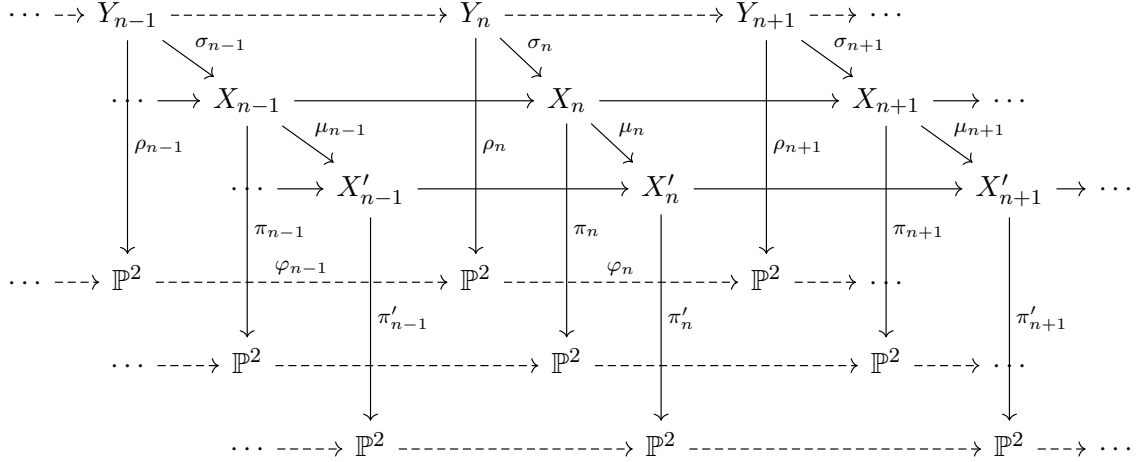


FIGURE 3. Minimisation of a space of initial conditions

Definition 2.8 (minimal space of initial conditions). For a space of initial conditions X_n , we say that a minimisation is trivial if the maps μ_n are isomorphisms. If there does not exist any nontrivial minimisation, then we say the space of initial conditions is minimal.

Whether a space of initial conditions can be minimised or not is characterised by the following.

Lemma 2.9 ([24]). *If there exist a finite set of elements of $\text{Pic}(\mathcal{X})$ corresponding to mutually disjoint exceptional curves of the first kind which are permuted by Φ , then blowing down these curves provides a nontrivial minimisation of the space of initial conditions.*

2.2. Degree growth and entropy for birational mappings of the plane. Hereafter, the definition of integrability of a birational mapping of the plane we will work with is that its algebraic entropy vanishes, or equivalently that its dynamical degree is one. For a birational map f from \mathbb{P}^2 to itself, written in homogeneous coordinates as

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (2.8)$$

$$(x_0 : x_1 : x_2) \mapsto (f_0(x_0, x_1, x_2) : f_1(x_0, x_1, x_2) : f_2(x_0, x_1, x_2)),$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree with no common factors, its degree $\deg f$ is defined as the common degree of the polynomials f_i .

Definition 2.10 (dynamical degree, algebraic entropy [4]). For a mapping $\varphi_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, the numbers

$$d = \lim_{k \rightarrow \infty} \left(\deg \varphi^{(k)} \right)^{1/k}, \quad \text{and} \quad \varepsilon = \lim_{k \rightarrow \infty} \frac{1}{k} \log \left(\deg \varphi^{(k)} \right), \quad (2.9)$$

where $\varphi^{(k)} = \varphi_{n+k-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n$, are called the dynamical degree and algebraic entropy [4] respectively. Note that in the autonomous case $\varphi_n = \varphi$ and $\varphi^{(k)} = \varphi^k$, and the definition of d coincides with that of the (first) dynamical degree of a dominant rational self-map of \mathbb{P}^2 .

Remark 2.11. If instead of \mathbb{P}^2 we consider equations as birational mappings of $\mathbb{P}^1 \times \mathbb{P}^1$ the degrees of iterates are in general different but their rate of growth is the same. Therefore the dynamical degree and algebraic entropy of a mapping $(x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$ does not depend on the choice of compactification of \mathbb{C}^2 . In fact, it is often more convenient to use $\mathbb{P}^1 \times \mathbb{P}^1$ for calculation in particular examples as we will do in Sections 3-6, but we will use \mathbb{P}^2 in this Section for the general theory.

The kinds of mappings with a space of initial conditions between which full deautonomisation should be shown to distinguish are provided by the following classifications.

2.2.1. *Autonomous case.* In [7], Diller and Favre classified bimeromorphic self-maps of compact surfaces according to the rate of growth of their spectral radii under iteration. Restating some of their results in the language of birational maps, as studied in discrete integrable systems, yields the following.

Proposition 2.12 (Diller-Favre [7]). *An autonomous birational mapping φ of the plane belongs to one of the following types:*

type (1): *The degree of φ^n is bounded.*

This type of mapping has a space of initial conditions.

type (2): *The degree of φ^n grows linearly.*

This type of mapping does not have a space of initial conditions.

type (3): *The degree of φ^n grows quadratically.*

This type of mapping has a space of initial conditions, which is a rational elliptic surface.

type (4): *The degree of φ^n grows exponentially and φ has a space of initial conditions.*

The surface X providing the space of initial conditions has $\text{rank Pic}(X) > 10$.

type (5): *The degree of φ^n grows exponentially and φ does not have a space of initial conditions.*

2.2.2. *Non-autonomous case.* If a mapping, autonomous or not, has a space of initial conditions then the dynamical degree is encoded in the induced dynamics on its Picard lattice.

Lemma 2.13 (Takenawa [41]). *If a non-autonomous mapping φ_n has a space of initial conditions, the degree of $\varphi^{(k)}$ is given by*

$$\deg \varphi^{(k)} = \Phi^k e^{(0)} \cdot e^{(0)}, \quad (2.10)$$

so the dynamical degree of the mapping is given by the largest eigenvalue of Φ , and the algebraic entropy by its logarithm. Similarly if an autonomous mapping φ has a space of initial conditions then the dynamical degree is given by the largest eigenvalue of φ^ .*

Using the above fact and the work of Diller-Favre [7], one of the authors obtained the following classification of non-autonomous mappings with a space of initial conditions.

Proposition 2.14 ([24]). *A non-autonomous mapping of the plane with a space of initial conditions belongs to one of the following three types according to the Jordan normal form of Φ :*

type (a):

$$\Phi \sim \begin{pmatrix} \mu_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \mu_{r+1} \end{pmatrix},$$

where μ_i are all roots of unity. In particular $\Phi^l = \text{id}$ for some $l > 0$ so the degree growth of the mapping is bounded.

type (b):

$$\Phi \sim \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & \mu_1 & \\ & & & & \ddots \\ & & & & & \mu_{r-2} \end{pmatrix},$$

where μ_i are all roots of unity. In this case the degree grows quadratically.

type (c):

$$\Phi \sim \begin{pmatrix} \lambda & & & & \\ & \frac{1}{\lambda} & & & \\ & & \mu_1 & & \\ & & & \ddots & \\ & & & & \mu_{r-1} \end{pmatrix},$$

where $\lambda > 1$ is a reciprocal quadratic integer or a Salem number, and $|\mu_i| = 1$. In this case the degree grows exponentially and the dynamical degree is λ .

We can partially distinguish between the different types above according to $\text{rank Pic}(\mathcal{X})$ as follows

Proposition 2.15 ([24]). *If $\text{rank Pic}(\mathcal{X}) \leq 10$, then the mapping belongs to either type (a) or type (b) of Proposition 2.14.*

For type (b), when the degree growth is quadratic, one of the authors also showed [24] that a minimal space of initial conditions is formed of generalised Halphen surfaces as defined by Sakai as follows.

Definition 2.16 (generalised Halphen surface [39], Sakai surface). A rational surface X is called a generalised Halphen surface if it has an effective anticanonical divisor $D \in |-\mathcal{K}_X|$ of canonical type, i.e if $D = \sum_i m_i D_i$, $m_i > 0$, is its decomposition into irreducible components then $[D_i] \cdot \mathcal{K}_X = 0$ for all i . A generalised Halphen surface has $\dim |-\mathcal{K}_X|$ being equal to either 1 or 0. In the first case X is a rational elliptic surface and in the second case X has a unique anticanonical divisor. The latter is the type of surface which forms the spaces of initial conditions for discrete Painlevé equations, and we call such X a Sakai surface.

From Proposition 2.14 we see that if a non-autonomous mapping has a space of initial conditions and unbounded degree growth, then it must either be a discrete Painlevé equation or non-integrable. So to put the full deautonomisation method on a rigorous footing we need to show that confinement conditions on a sufficiently general deautonomisation distinguish between these two cases.

Remark 2.17. The restriction to basic rational surfaces in the Definitions 2.4 and 2.5 of a space of initial conditions is justified for surfaces associated with mappings of type (b) or (c), since in both cases the surfaces will have infinitely many exceptional curves of the first kind, so by a theorem of Nagata [28] they must be basic rational surfaces.

2.3. Deautonomisation. We now define what it means for a non-autonomous mapping to be a deautonomisation of an autonomous one with reference to the space of initial conditions and dynamics on the Picard lattice.

Definition 2.18 (deautonomisation). Consider a family of mappings $\varphi_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with a space of initial conditions as above. Consider also an autonomous mapping $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with a space of initial conditions in the sense of Definition 2.1 being a basic rational surface Y with $\text{rank Pic}(Y) = \text{rank Pic}(\mathcal{X})$. Consider a birational morphism (given by a sequence of blow-ups) and denote the geometric basis corresponding to that morphism by $(f^{(0)}, \dots, f^{(r)})$. We say that φ_n is a deautonomisation of ψ if under the identification

$$\begin{aligned} \kappa : \text{Pic}(\mathcal{X}) &\rightarrow \text{Pic}(Y), \\ e^{(i)} &\mapsto f^{(i)}, \end{aligned} \tag{2.11}$$

the following two conditions hold:

- If $\mathcal{F} \in \text{Pic}(\mathcal{X})$ represents an effective class $\iota_n(\mathcal{F})$ in $\text{Pic}(X_n)$, then $\kappa(\mathcal{F}) \in \text{Pic}(Y)$ is also effective.
- The linear maps Φ and ψ^* coincide under the identification, i.e. the following diagram commutes:

$$\begin{array}{ccc} \text{Pic}(X_n) & \xleftarrow{\varphi_n^*} & \text{Pic}(X_{n+1}) \\ \uparrow \iota_n & & \iota_{n+1} \uparrow \\ \text{Pic}(\mathcal{X}) & \xleftarrow{\Phi} & \text{Pic}(\mathcal{X}) \\ \downarrow \kappa & & \kappa \downarrow \\ \text{Pic}(Y) & \xleftarrow{\psi^*} & \text{Pic}(Y). \end{array}$$

Remark 2.19. This definition is consistent with that of [5] formulated in the context of deautonomising QRT mappings to obtain discrete Painlevé equations. The notion of deautonomisation there involves a choice of fibre from the elliptic fibration of the space of initial conditions for a QRT mapping, which then becomes the unique anticanonical divisor of the surface after allowing locations of centres of blow-ups to move. In particular, a different choice of singular fibre will lead to a different choice of which classes stay effective after deautonomisation, and therefore corresponds to a different way of injecting $\text{Pic}^+(X_n)$ into $\text{Pic}^+(Y)$ via $\kappa \circ \iota_n^{-1}$, but this does not change Φ nor the degree growth of the deautonomised mappings.

The crucial fact that we will use for the remainder of the paper is the following

Lemma 2.20. *If φ_n is a deautonomisation of an autonomous mapping in the sense of Definition 2.18, then it has exactly the same degree growth, dynamical degree and algebraic entropy.*

2.4. Period map for rational surfaces with effective anticanonical divisor. In this Subsection, let X be a rational surface with an effective anticanonical divisor $D = \sum_i m_i D_i \in |-\mathcal{K}_X|$. Denote the sublattice of $\text{Pic}(X)$ spanned by the classes $\mathcal{D}_i = [D_i]$ of the irreducible components of D by

$$Q = \sum_i \mathbb{Z} \mathcal{D}_i. \quad (2.12)$$

Its orthogonal complement is another sublattice, which we denote

$$Q^\perp = \{ \mathcal{F} \in \text{Pic}(X) \mid \mathcal{F} \cdot \mathcal{D}_i = 0 \ \forall i \}, \quad (2.13)$$

on which the period map will give a \mathbb{C} -valued function. When X is a Sakai surface, Q^\perp is isomorphic to the root lattice of an affine root system and the values of the period map on a basis of simple roots are called the root variables. These appear as n -dependent parameters in discrete Painlevé equations [39].

Begin by taking a rational 2-form ω on X such that $-\text{div } \omega = D$. Let the support of D be $D_{\text{red}} = \bigcup_i D_i$, so ω defines a holomorphic symplectic form on $X - D_{\text{red}}$. This gives the mapping

$$\begin{aligned} \hat{\chi} : H_2(X - D_{\text{red}}; \mathbb{Z}) &\rightarrow \mathbb{C}, \\ \Gamma &\mapsto \int_\Gamma \omega. \end{aligned} \quad (2.14)$$

In order to use this to define a period map on Q^\perp , we begin with the long exact sequence of relative singular homology for the pair $(X, X - D_{\text{red}})$, which includes the following:

$$H_3(X; \mathbb{Z}) \rightarrow H_3(X, X - D_{\text{red}}; \mathbb{Z}) \xrightarrow{\partial^*} H_2(X - D_{\text{red}}; \mathbb{Z}) \xrightarrow{i^*} H_2(X; \mathbb{Z}) \xrightarrow{j^*} H_2(X, X - D_{\text{red}}; \mathbb{Z}).$$

Here i^* comes from the natural injection of cycles on $X - D_{\text{red}}$ into X , while j^* is induced by the quotient of cycles on X by cycles on $X - D_{\text{red}}$. We manipulate this exact sequence using the following facts:

- Lefschetz duality for the pair $(X, X - D_{\text{red}})$ gives

$$H_k(X, X - D_{\text{red}}; \mathbb{Z}) \cong H^{4-k}(D_{\text{red}}; \mathbb{Z}). \quad (2.15)$$

- Poincaré duality for D_{red} gives

$$H_1(D_{\text{red}}; \mathbb{Z}) \cong H^1(D_{\text{red}}; \mathbb{Z}). \quad (2.16)$$

- Under Poincaré duality for X , we have

$$H_3(X; \mathbb{Z}) \cong H^1(X; \mathbb{Z}) = 0, \quad (2.17)$$

because $b_1 = \text{rank } H^1(X; \mathbb{Z}) = \sum_{p+q=1} h^{p,q} = h^{1,0} + h^{0,1} = 2h^{0,1} = 0$, since X being rational implies that $h^{0,1} = 0$.

- Lefschetz duality as above gives $H_2(X, X - D_{\text{red}}; \mathbb{Z}) \cong H^2(D_{\text{red}}; \mathbb{Z}) \cong Q$, so under the Poincaré duality $H_2(X; \mathbb{Z}) \cong H^2(X; \mathbb{Z})$, in the long exact sequence above we have

$$\ker j^* \cong Q^\perp, \quad (2.18)$$

which is why the orthogonal complement of Q is significant in this construction.

Using these facts, from the sequence above we obtain

$$0 \rightarrow H_1(D_{\text{red}}; \mathbb{Z}) \rightarrow H_2(X - D_{\text{red}}; \mathbb{Z}) \rightarrow Q^\perp \rightarrow 0, \quad (2.19)$$

which through the mapping $\hat{\chi}$ on $H_2(X - D_{\text{red}}; \mathbb{Z})$ gives the following.

Definition 2.21 (period map [23, 39]). For X with effective anticanonical divisor D and choice of a rational two-form ω on X with $-\text{div } \omega = D$, the above construction gives

$$\chi : Q^\perp \rightarrow \mathbb{C} \quad \text{mod } \hat{\chi}(H_1(D_{\text{red}}; \mathbb{Z})), \quad (2.20)$$

which we call the period map of X . One can define χ as a \mathbb{C} -valued function by making a normalisation of the value of $\hat{\chi}$ on $H_1(D_{\text{red}}; \mathbb{Z})$ if this is nontrivial.

2.4.1. Computation of the period map. The computation of the period map proceeds along the same lines as in [23, Chapter I, Section 5] and [39, Lemma 21], which we recall here for completeness. The first step in the procedure is guaranteed to be possible for generalised Halphen surfaces, but for surfaces associated with non-integrable mappings does not come automatically either because the existence of an effective anticanonical divisor is not guaranteed or, even if this does exist, it is not obvious how to describe Q^\perp as a root lattice; we will address this later in this Section.

For an element $\alpha \in Q^\perp$ which can be expressed as the difference of the classes of two exceptional curves of the first kind,

$$\alpha = [C^1] - [C^0], \quad (2.21)$$

the computation of $\chi(\alpha)$ is done as follows:

- Find the unique component D_k of the anticanonical divisor which intersects both C^1 and C^0 , with multiplicity one, i.e. D_k such that

$$C^1 \cdot D_k = C^0 \cdot D_k = 1, \quad \text{and} \quad C^1 \cdot D_j = C^0 \cdot D_j = 0 \quad \text{for } j \neq k. \quad (2.22)$$

This unique component is guaranteed to exist by Riemann-Roch, Serre duality and the genus formula, as explained in [23, Chapter I, Section 0].

- Denote the points of intersection of these curves with D_k as $D_k \cap C^1$ and $D_k \cap C^0$. The value of $\chi(\alpha)$ is then computed using the residue formula (see [23, Chapter I, Section 5] for details) as

$$\chi(\alpha) = 2\pi i \int_{D_k \cap C^0}^{D_k \cap C^1} \text{Res}_{D_k} \omega. \quad (2.23)$$

It is important to note that the integral (2.23) will be computed in coordinates and therefore detects parameters in the mapping, which is fundamental to understanding the mechanism behind the full deautonomisation method.

2.4.2. Dynamics of the values of the period map. The crucial property of the period map for our purposes is how it interacts with the linear dynamics on $\text{Pic}(\mathcal{X})$. Take some linearly independent subset $\{\alpha_i\} \subset Q^\perp \subset \text{Pic}(\mathcal{X})$ whose span is closed under Φ , and let $M_{i,j}$ be the matrix representation of Φ with respect to this, i.e.

$$\Phi(\alpha_i) = \sum_j M_{i,j} \alpha_j. \quad (2.24)$$

For each n , take ω_n to be a rational 2-form on X_n chosen such that $\tilde{\varphi}_n^*(\omega_{n+1}) = \omega_n$. Then if we denote by χ_n and χ_{n+1} the period maps defined by ω_n and ω_{n+1} on X_n and X_{n+1} respectively, and the values of the period map of X_n by $a_i(n) = \chi_n(\iota_n^{-1}(\alpha_i))$, then we have

$$a_i(n+1) = \sum_j M_{i,j} a_j(n), \quad (2.25)$$

since $\chi_{n+1}(\iota_{n+1}^{-1}(\alpha_i)) = \chi_n(\tilde{\varphi}_n^* \circ \iota_{n+1}^{-1}(\alpha_i)) = \chi_n(\iota_n^{-1} \circ \Phi(\alpha_i)) = \sum_j M_{i,j} \chi_n(\iota_n^{-1}(\alpha_j))$. The second equality comes from the fact that $\tilde{\varphi}_n^*(\omega_{n+1}) = \omega_n$, and that $\tilde{\varphi}_n$ is biholomorphic and acts as a change of variables for the integrals.

The fact that the representation in (2.25) is given by exactly the same matrix as in (2.24) provides the sought after bridge between parameter evolution and dynamics on $\text{Pic}(\mathcal{X})$.

2.5. The question of existence of an effective anticanonical divisor. For Sakai surfaces associated with discrete Painlevé equations the existence of an effective anticanonical divisor is built into the definition. However, to carry out the period map construction in the non-integrable case, we must deal with the question of the existence of an effective anticanonical divisor on the surface X_n from a space of initial conditions for a mapping of type (c) in Proposition 2.14.

We will need the following fact relating to a space of initial conditions for a non-integrable mapping, the proof of which is found in [24].

Lemma 2.22 ([24, Lemma 4.15]). *For a mapping of type (c) in Proposition 2.14, denote by $v \in \text{Pic}_{\mathbb{R}}(\mathcal{X})$ the dominant eigenvector corresponding to the eigenvalue $\lambda > 1$ of Φ . Then the intersection product is negative definite on $v^{\perp} \cap \text{Pic}(\mathcal{X})$.*

Using techniques similar to those in [9] and [24], we have the following,

Theorem 2.23. *Let $\varphi_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a non-autonomous, non-integrable mapping which has a space of initial conditions, i.e. a mapping of type (c) in Proposition 2.14. Suppose there exists a sequence of rational 2-forms ω_n on \mathbb{P}^2 and constants $c_n \in \mathbb{C}^*$ such that*

$$\varphi_n^* \omega_{n+1} = c_n \omega_n. \quad (2.26)$$

Denoting the lift of ω_n to X_n by $\tilde{\omega}_n = (\rho_n \circ \sigma_n^{-1})^* \omega = (\sigma_n)_* \circ (\rho_n)^* \omega$, with its divisor having irreducible decomposition

$$\text{div}(\tilde{\omega}_n) = \sum_i m_i^{(n)} D_i^{(n)}, \quad (2.27)$$

then if under the identification of all $\text{Pic}(X_n)$ into the Picard lattice $\text{Pic}(\mathcal{X})$ we have, for all n ,

$$\iota_n^{-1}(D_i^{(n)}) = \iota_{n+1}^{-1}(D_i^{(n+1)}), \quad (2.28)$$

there exists a minimal space of initial conditions formed by surfaces which have effective anticanonical divisors.

Remark 2.24. In particular, if an autonomous mapping $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is such that $\psi^* \omega = c \omega$ for some rational two-form ω on \mathbb{P}^2 and some $c \in \mathbb{C}^*$, then any deautonomisation of it satisfies the conditions in Theorem 2.23.

Proof of Theorem 2.23. Let the space of initial conditions for the mapping φ_n be as in Definition 2.5. Let the decomposition of $\text{div}(\tilde{\omega}_n)$ on X_n into effective and anti-effective parts be

$$\text{div}(\tilde{\omega}_n) = D_n^+ - D_n^-, \quad (2.29)$$

where D^+ and D^- are effective divisors. Note that $-\text{div}(\tilde{\omega}_n)$ being effective is equivalent to $D_n^+ = 0$.

The idea now is to show that if $-\text{div}(\tilde{\omega}_n)$ is not effective then the space of initial conditions is not minimal. Indeed suppose $D_n^+ \neq 0$ and let its decomposition into irreducible components be

$$D_n^+ = \sum_i m_i C_i^{(n)}, \quad (2.30)$$

where $m_i > 0$ and $C_i^{(n)}$ are irreducible curves on X_n . Denote the classes of these by $[C_i^{(n)}] \in \text{Pic}(X_n)$. Then according to our assumptions the expressions for $[C_i^{(n)}]$ in terms of the geometric basis for $\text{Pic}(X_n)$ corresponding to π_n do not depend on n and we have the following well-defined elements of $\text{Pic}(\mathcal{X})$:

$$\mathcal{D}_i^+ = \iota_n^{-1}([C_i^{(n)}]), \quad \mathcal{D}^+ = \sum_i m_i \iota_n^{-1}([D]_i^+). \quad (2.31)$$

We will show that there exists some \mathcal{D}_i^+ that corresponds to the class of an exceptional curve of the first kind and which is periodic under Φ , and that its orbit gives a collection of classes of

exceptional curves of the first kind which are mutually disjoint, so by Lemma 2.9 the space of initial conditions is not minimal.

The fact that φ_n^* preserves the two-form ω_n in the sense of (2.26) implies that $\tilde{\varphi}_n^* \tilde{\omega}_{n+1} = c_n \tilde{\omega}_n$, and $\tilde{\varphi}_n^*(\operatorname{div}(\tilde{\omega}_{n+1})) = \operatorname{div}(\tilde{\omega}_n)$. Further, since $\tilde{\varphi}_n$ is an isomorphism, for any irreducible curve C on X_n we have

$$\operatorname{ord}(C, \operatorname{div}(\tilde{\omega}_n)) = \operatorname{ord}(\tilde{\varphi}_n(C), \operatorname{div}(\tilde{\omega}_{n+1})), \quad (2.32)$$

which allows us to deduce that Φ must permute the \mathcal{D}_i^+ in a way that preserves their multiplicity in the canonical class. In particular there exists a permutation σ such that $\Phi(\mathcal{D}_i^+) = \mathcal{D}_{\sigma(i)}^+$, so all \mathcal{D}_i^+ are in the part of $\operatorname{Pic}(\mathcal{X})$ periodic under Φ . Denoting by ℓ the order of the permutation σ , we have

$$\mathcal{D}_i^+ \cdot v = \Phi^\ell(\mathcal{D}_i^+) \cdot \Phi^\ell v = \lambda^\ell (\mathcal{D}_i^+ \cdot v), \quad (2.33)$$

and since $\lambda > 1$ we deduce that $\mathcal{D}_i^+ \in v^\perp \cap \operatorname{Pic}(\mathcal{X})$ for every i , so the intersection form is negative definite on the sublattice $\operatorname{span}_{\mathbb{Z}}\{\mathcal{D}_i^+\} \subset \operatorname{Pic}(\mathcal{X})$. This allows us to deduce that at least one of the \mathcal{D}_i^+ must correspond to an exceptional curve of the first kind, as follows. Taking the intersection of \mathcal{D}^+ with the canonical class in $\operatorname{Pic}(\mathcal{X})$ we have

$$\mathcal{D}^+ \cdot \mathcal{K}_{\mathcal{X}} = \mathcal{D}^+ \cdot \mathcal{D}^+ - \mathcal{D}^+ \cdot \mathcal{D}^- \leq \mathcal{D}^+ \cdot \mathcal{D}^+ < 0,$$

where the last inequality comes from the intersection form being negative definite on the span of the \mathcal{D}_i^+ . Since \mathcal{D}^+ corresponds to an effective class we then have that some \mathcal{D}_i^+ satisfies $\mathcal{D}_i^+ \cdot \mathcal{K}_{\mathcal{X}} < 0$. Since $\mathcal{D}_i^+ \in v^\perp \cap \operatorname{Pic}(\mathcal{X})$ we also deduce $\mathcal{D}_i^+ \cdot \mathcal{D}_i^+ < 0$. The genus formula then allows us to deduce that \mathcal{D}_i^+ must correspond to an exceptional curve of the first kind.

It remains to be shown that the elements of $\operatorname{Pic}(\mathcal{X})$ in the orbit of this \mathcal{D}_i^+ under Φ are mutually orthogonal. Letting $k \in \mathbb{Z}$ be such that $\Phi^k \mathcal{D}_i^+ = \mathcal{D}_j^+ \neq \mathcal{D}_i^+$, we again use the negative definiteness of the intersection form on the span of the \mathcal{D}_i^+ to show that

$$0 > \left(\mathcal{D}_i^+ + \Phi^k \mathcal{D}_i^+\right)^2 = -2 + 2\mathcal{D}_i^+ \cdot \Phi^k \mathcal{D}_i^+ \implies 1 > \mathcal{D}_i^+ \cdot \Phi^k \mathcal{D}_i^+ \geq 0, \quad (2.34)$$

so $\mathcal{D}_i^+ \cdot \Phi^k \mathcal{D}_i^+ = 0$ and the orbit of \mathcal{D}_i^+ gives a collection of mutually disjoint exceptional curves of the first kind which are permuted by the mapping.

Finally we note that after a minimisation, performed by blowing down these curves, we still have a space of initial conditions for the same φ_n in the sense of Definition 2.5, and we can apply the above procedure as many times as is necessary to obtain a minimal space of initial conditions formed of anticanonical surfaces. Note that the procedure is guaranteed to terminate since a space of initial conditions for a non-integrable mapping must have $\operatorname{rank} \operatorname{Pic}(\mathcal{X}) > 10$ according to Proposition 2.15. □

2.6. Bases for the orthogonal sublattice. The next problem which must be addressed is ensuring that on surfaces associated with non-integrable mappings, one can find appropriate elements of Q^\perp on which to compute the period map. For the remainder of this Section we assume all surfaces have effective anticanonical divisors.

Definition 2.25 (root basis of Q^\perp). Suppose a \mathbb{Z} -basis $\{\alpha_i\}$ for Q^\perp is such that each α_i can be expressed as the difference of two classes of exceptional curves of the first kind $\alpha_i = [C_i^1] - [C_i^0]$ and the numbers $c_{i,j} = 2 \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$ form the entries of a generalised Cartan matrix as defined in [21]. Then we call this a root basis for Q^\perp . Following [39], we will refer to the values of the period map on a root basis as root variables.

For each type of generalised Halphen surface in his classification, Sakai [39] found a root basis with a corresponding generalised Cartan matrix that is of affine type. In [41], Takenawa constructed the space of initial conditions for the Hietarinta-Viallet mapping [20], which has

an effective anticanonical divisor. Takenawa was also able to find a root basis in this case, corresponding to the generalised Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \quad (2.35)$$

which is of hyperbolic type $H_{71}^{(3)}$ [44]. Takenawa also constructed an example with root basis corresponding to the matrix

$$\begin{pmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix}, \quad (2.36)$$

which is of neither finite, affine, nor hyperbolic type.

For non-integrable mappings it is not immediately clear that we can always find a root basis, but for the purpose of full-deautonomisation it will be sufficient to show that one can always construct a linearly independent subset of Q^\perp that is closed under Φ and whose span includes the dominant eigenvector. To this end we define the following.

Definition 2.26 (sufficient subset of Q^\perp). For a space of initial conditions constructed from a mapping with unbounded degree growth, suppose a linearly independent subset $\{\beta_j\}$ of Q^\perp is such that the following two conditions hold:

- Each β_j can be expressed as the difference of two classes of exceptional curves of the first kind C_j^1 and C_j^0 , i.e. $\iota_n(\beta_j) = [C_j^1] - [C_j^0]$,
- Φ preserves $\text{span}_{\mathbb{Z}}\{\beta_j\}$.

If additionally the following requirement is met, which depends on whether the mapping is of type (b) or (c) of Proposition 2.14, then we call $\{\beta_j\}$ a sufficient subset:

- type (c) : The dominant eigenvector $v \in \text{Pic}_{\mathbb{R}}(\mathcal{X})$ of Φ is contained in $\text{span}_{\mathbb{R}}\{\beta_j\}$,
- type (b) : Some element in $\text{Pic}_{\mathbb{R}}(\mathcal{X})$ which is a generalised eigenvector for the eigenvalue 1 of Φ , but not an eigenvector, is contained in $\text{span}_{\mathbb{R}}\{\beta_j\}$.

For non-autonomous mappings of type (b) of Proposition 2.14, it was shown in [24] that they correspond to Sakai surfaces, for which we can always find a sufficient subset in Q^\perp . However, in order to show that we can always find a sufficient subset for mappings of type (c) in Proposition 2.14 we will require the following Lemma consisting of several results from [26].

Lemma 2.27 ([26]). Denote the periodic part of $\text{Pic}(\mathcal{X})$ under Φ by

$$P_\Phi = \{\mathcal{F} \in \text{Pic}(\mathcal{X}) \mid \exists m \in \mathbb{Z} \setminus \{0\} \text{ such that } \Phi^m(\mathcal{F}) = \mathcal{F}\}, \quad (2.37)$$

and let $\bar{\mu}_\Phi(t) \in \mathbb{Q}[t]$ denote the minimal polynomial of Φ as a linear transformation of $\text{Pic}(\mathcal{X})/P_\Phi$. If there exists $F \in \text{Pic}(\mathcal{X})/P_\Phi$, $F \neq 0$, such that a polynomial $\psi(t) \in \mathbb{Q}[t]$ satisfies

$$\psi[\Phi]F = 0 \text{ mod } P_\Phi, \quad (2.38)$$

then $\psi(t)$ shares a factor with $\bar{\mu}_\Phi(t)$. For mappings of type (c) in Proposition 2.14, $\bar{\mu}_\Phi(t)$ is precisely the minimal polynomial of λ , the dominant eigenvalue of Φ , over \mathbb{Q} . We denote this minimal polynomial as $\mu_\lambda(t)$.

We will also require the following result, which follows from the classification of the anticanonical divisors for spaces of initial conditions for non-integrable mappings (Theorem A.17) that is given in Appendix A.

Proposition 2.28. Consider a mapping φ_n which is non-integrable, i.e. of type (c) in Proposition 2.14, with a minimal space of initial conditions provided by surfaces X_n which are anticanonical, i.e. X_n has an effective anticanonical divisor $D = \sum_j m_j D_j$. Then, there exists j such that

- $m_j = 1$,
- X_n has an exceptional curve of the first kind C such that $C \cdot D_i = \delta_{i,j}$.

We are now ready to prove the following.

Proposition 2.29. *For a minimal space of initial conditions for a non-integrable mapping, i.e. of type (c) in Proposition 2.14, one can always find a sufficient subset of Q^\perp .*

Proof. Let the surfaces forming the space of initial conditions be X_n , with $D = \sum_j m_j D_j$ as the effective anticanonical divisor of X_n . Then let $[C] \in \text{Pic}(X_n)$ be the class of the exceptional curve C of the first kind as in Proposition 2.28, which intersects exactly one component D_j of the anticanonical divisor $D = \sum_j m_j D_j$. The element of $\mathcal{D}_k = \iota_n^{-1}[D_k] \in \text{Pic}(\mathcal{X})$ corresponding to this class must lie in P_Φ , so let $\ell \in \mathbb{Z}_{\geq 0}$ be the smallest nonnegative integer such that $\Phi^\ell(\mathcal{D}_k) = \mathcal{D}_k$. Furthermore, let $\mathcal{C} = \iota_n^{-1}[C]$ be the element of $\text{Pic}(\mathcal{X})$ corresponding to the exceptional curve of the first kind C , which cannot lie in P_Φ because of the minimality of the space of initial conditions. Then we can construct a sufficient subset as follows. First let

$$\beta_0 = (\Phi^\ell - 1)\mathcal{C}, \quad (2.39)$$

which by construction corresponds to the difference of two classes of exceptional curves of the first kind. Then let $\beta_1 = \Phi\beta_0$, $\beta_2 = \Phi\beta_1$ and so on until the process terminates and some β_{h+1} is a linear combination of β_0, \dots, β_h , say $\Phi\beta_h + \sum_{i=0}^h a_i \beta_i = 0$. Then $\{\beta_0, \dots, \beta_h\}$ by construction satisfies the first two conditions in Definition 2.26, so it remains only to show that $v \in \text{span}_{\mathbb{R}}\{\beta_0, \dots, \beta_h\}$. From the construction of this subset we have that

$$\psi[\Phi](\Phi^\ell - 1)\mathcal{C} = 0, \quad (2.40)$$

where $\psi(t) = (t^{h+1} + \sum_{i=0}^h a_i t^i) \in \mathbb{Q}[t]$. Since we are in the third case of Proposition 2.14, Lemma 2.27 implies that ψ has $\mu_\lambda(t)$, the minimal polynomial over \mathbb{Q} of the dominant eigenvalue of Φ , as a factor. Therefore the characteristic polynomial of the restriction of Φ to $\text{span}_{\mathbb{R}}\{\beta_0, \dots, \beta_h\}$ has $\mu_\lambda(t)$ as a factor, so the dominant eigenvector v is contained in $\text{span}_{\mathbb{R}}\{\beta_0, \dots, \beta_h\}$ as required. \square

Definition 2.30 (sufficient deautonomisation). Consider a deautonomisation φ_n of an autonomous mapping ψ in the sense of Definition 2.18, with space of initial conditions given by surfaces X_n . For a sufficient subset $\{\beta_1, \dots, \beta_r\}$ of Q^\perp , consider the \mathbb{Z} -linear map

$$\begin{aligned} \chi : \text{span}_{\mathbb{Z}}\{\beta_1, \dots, \beta_r\} &\rightarrow (\mathbb{C} \bmod \hat{\chi}(H_1(D_{\text{red}}; \mathbb{Z})))^r, \\ \beta &\mapsto \chi_{X_n}(\iota_n(\beta)), \end{aligned} \quad (2.41)$$

where χ_{X_n} is the period map of X_n . If χ is injective, then we call the deautonomisation sufficient. This means that the linear system for the evolution with n , of the values of χ on the sufficient subset, will have the same characteristic polynomial as that of Φ restricted to $\text{span}_{\mathbb{Z}}\{\beta_1, \dots, \beta_r\}$.

In the case of type (c) mappings this characteristic polynomial will have $(t - \lambda)$ as a factor (where λ is the dominant eigenvalue of Φ), whereas for type (b) mappings it has $(t - 1)^2$ as a factor.

The construction of a sufficient deautonomisation depends significantly on the form in coordinates of the autonomous mapping. For mappings that take the form of mappings belonging to classes I-VI in the QRT-type classification, we give procedures for constructing sufficient deautonomisations in Appendix B. If a deautonomisation is sufficient in the above sense, this means that the evolution of values of the period map on the sufficient subset is guaranteed to give a characteristic polynomial with the dynamical degree of the mapping as a root, and we have the following main result of the paper.

Theorem 2.31. *If an autonomous mapping has a space of initial conditions and unbounded degree growth, and preserves a rational two-form, it has a minimal space of initial conditions formed of anticanonical surfaces. Then the period map parametrisation of the coefficients in a sufficient deautonomisation gives a linear system of difference equations for their evolution, the characteristic polynomial of which has the dynamical degree of the mapping as a root.*

3. A FAMILY OF MAPPINGS OF QRT CLASS III FORM

The first example we will illustrate in detail deals with a family of mappings that are of the form of mappings in QRT class III, i.e.

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = f(x_n), \quad (3.1)$$

where f is rational. From this point on we will make use of notation such as

$$x_{n-1} = \underline{x}, \quad x_n = x, \quad x_{n+1} = \bar{x}, \quad (3.2)$$

to denote up- and down-shifts in n of both variables and parameters.

We consider the family of equations

$$(\bar{x} + x)(x + \underline{x}) = \frac{\prod_{i=0}^m (x^2 - a_i^2)}{\prod_{j=1}^m (x^2 - b_j^2)}, \quad (3.3)$$

where $a_i \in \mathbb{C} \setminus \{0\}$, $i = 0, \dots, m$ and $b_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \dots, m$, for $m \in \mathbb{Z}_{\geq 1}$. Singularities of the mapping correspond to zeroes of the numerator or denominator of the rational function on the right-hand side of (3.3), and calculating along the same lines as in the examples in Section 1 we find that all of these singularities are confined and we have the singularity patterns

$$\{\pm a_i, \mp a_i\}, \quad \text{and} \quad \{\pm b_j, \infty, \mp b_j\}. \quad (3.4)$$

3.1. Full deautonomisation by singularity confinement. In this case it turns out that the following deautonomisation suffices to detect the dynamical degree of the mapping

$$(\bar{x} + x)(x + \underline{x}) = \frac{\prod_{i=0}^m (x - a_i)(x - d_i)}{\prod_{j=1}^m (x - b_j)(x - c_j)}. \quad (3.5)$$

Here a_i, b_j, c_j, d_i are now n -dependent and will be required to evolve such that the structure of confined singularities persists. We shall now derive these evolutions explicitly.

First consider the singularities that correspond to zeroes of the numerator of the right-hand side of (3.5). In order to have the same kind of confinement behaviour of these singularities as the patterns $\{\pm a_i, \mp a_i\}$ in the autonomous case, we require singularities which appear when some iterate x takes a value for which the numerator of the right-hand side vanishes, to lead to a next iterate \bar{x} that is a zero of the numerator of the up-shifted version of the equation. Considering first the singularity $x = a_k$, for some $k \in \{0, \dots, m\}$ with $\underline{x} = u$ free, we introduce a small parameter ε and compute

$$\underline{x} = u, \quad x = a_k + \varepsilon, \quad \bar{x} = -a_k + \mathcal{O}(\varepsilon), \quad (3.6)$$

and similarly for the singularity $x = d_k$,

$$\underline{x} = u, \quad x = d_k + \varepsilon, \quad \bar{x} = -d_k + \mathcal{O}(\varepsilon). \quad (3.7)$$

For these singularities to be confined in the same way after deautonomisation we impose that $x = a_k$ leads to $\bar{x} = \bar{d}_k$, and $x = d_k$ leads to $\bar{x} = \bar{a}_k$, which requires the confinement conditions

$$\bar{d}_k = -a_k, \quad \bar{a}_k = -d_k, \quad (3.8)$$

for which it can be verified by direct calculation that we indeed have the confined singularity patterns $\{a_k, \bar{d}_k\}$ and $\{d_k, \bar{a}_k\}$. This is tantamount to requiring that the lines blown down by the mapping $(\underline{x}, x) \mapsto (x, \bar{x})$ are sent to indeterminacies of the next iteration $(x, \bar{x}) \mapsto (\bar{x}, \bar{\bar{x}})$.

For the singularities $x = b_k$ and $x = c_k$ we have

$$\begin{aligned} \underline{x} &= u, \\ x &= b_k + \varepsilon, \\ \bar{x} &= \frac{\prod_i (b_k - a_i)(b_k - d_i)}{(b_k + u) \prod_{j \neq k} (b_k - b_j) \prod_j (b_k - c_j)} \varepsilon^{-1} + \mathcal{O}(1), \\ \bar{\bar{x}} &= -b_k + \sum_{j=1}^m (\bar{b}_j + \bar{c}_j) + \sum_{i=0}^m (\bar{a}_i + \bar{d}_i) + \mathcal{O}(\varepsilon), \end{aligned} \quad (3.9)$$

and a similar evolution for $x = c_k + \varepsilon$ obtained from (3.9) by interchanging the roles of b_k and c_k . We require that these singularities are confined in the same way as in the autonomous case, namely through \bar{x} taking a value which is a root of the denominator of the right-hand side in the twice up-shifted version of (3.5), corresponding to an indeterminacy (∞, \bar{b}_j) or (∞, \bar{c}_j) of the mapping $(\bar{x}, \bar{x}) \mapsto (\bar{x}, \bar{\bar{x}})$. Which of these indeterminacies the singularities $x = b_k$, $x = c_k$ are confined through can be chosen, without loss of generality, to correspond to the patterns $\{b_k, \infty, \bar{c}_k\}$ and $\{c_k, \infty, \bar{b}_k\}$ via the conditions

$$\begin{aligned}\bar{c}_k &= -b_k + \sum_{j=1}^m (\bar{b}_j + \bar{c}_j) + \sum_{i=0}^m (\bar{a}_i + \bar{d}_i), \\ \bar{b}_k &= -c_k + \sum_{j=1}^m (\bar{b}_j + \bar{c}_j) + \sum_{i=0}^m (\bar{a}_i + \bar{d}_i).\end{aligned}\tag{3.10}$$

For the purpose of full deautonomisation in this case it is sufficient (as explained in Appendix B) to take the parameters in the numerator of the right-hand side of (3.5) to be constant, after which the conditions (3.8) require

$$a_i = -d_i \in \mathbb{C} \setminus \{0\}, \text{ for all } n.\tag{3.11}$$

Then the remaining confinement conditions (3.10) give a linear system for the evolution of $\mathbf{b} = (b_1, \dots, b_m)^T$, $\mathbf{c} = (c_1, \dots, c_m)^T$:

$$\begin{pmatrix} \bar{\mathbf{b}} \\ \bar{\mathbf{c}} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m & \mathbf{1}_m & 0 & -\mathbf{I}_m \\ \mathbf{1}_m & \mathbf{1}_m & -\mathbf{I}_m & 0 \\ \mathbf{I}_m & 0 & 0 & 0 \\ 0 & \mathbf{I}_m & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix},\tag{3.12}$$

where \mathbf{I}_m is the $m \times m$ identity matrix, and $\mathbf{1}_m$ is the $m \times m$ matrix with all entries being equal to one. The matrix on the right-hand side is similar under column permutations to

$$M = \begin{pmatrix} \mathbf{1}_{2m} & -\mathbf{I}_{2m} \\ \mathbf{I}_{2m} & 0 \end{pmatrix},\tag{3.13}$$

so the characteristic polynomial of the linear system (3.12) coincides, up to cyclotomic factors, with that of M , which can be computed as a block determinant to be

$$\det(M - t\mathbf{I}_{4m}) = (t^2 + 1)^{2m-1} (t^2 - 2mt + 1).\tag{3.14}$$

Therefore we see that the characteristic polynomial has as a root

$$\lambda = m + \sqrt{m^2 - 1},\tag{3.15}$$

which by the results of Section 2 we can conclude to be the dynamical degree of the original equation (3.3), as we will go on to demonstrate in detail.

3.2. Space of initial conditions. We now illustrate how the mechanism by which the dynamical degrees of the family of mappings (3.3) were obtained above, fits into the framework of Section 2, beginning with the construction of the space of initial conditions for the autonomous mapping. We take the mapping initially on $\mathbb{P}^1 \times \mathbb{P}^1$ (see Remark 2.11) and perform a sequence of blowups to regularise it as an automorphism. We do this in the usual way by introducing $X = 1/x$, $Y = 1/y$ so $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by the four affine charts (x, y) , (X, y) , (x, Y) and (X, Y) and the equation (3.3) defines via $y = x$ the mapping

$$\begin{aligned}\varphi : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= -x + \frac{\prod_{i=0}^m (x^2 - a_i^2)}{(x+y) \prod_{j=1}^m (x^2 - b_j^2)}, \quad \bar{y} = x.\end{aligned}\tag{3.16}$$

In order to regularise this as an automorphism we require blowups of points given in coordinates by

$$\mathbf{a}_i^\pm : (x, y) = (\pm a_i, \mp a_i), \quad \mathbf{b}_i^\pm : (x, y) = (\infty, \pm b_i), \quad \tilde{\mathbf{b}}_i^\pm : (x, y) = (\pm b_i, \infty), \quad (3.17)$$

after which we have the rational surface X as shown in Figure 4.

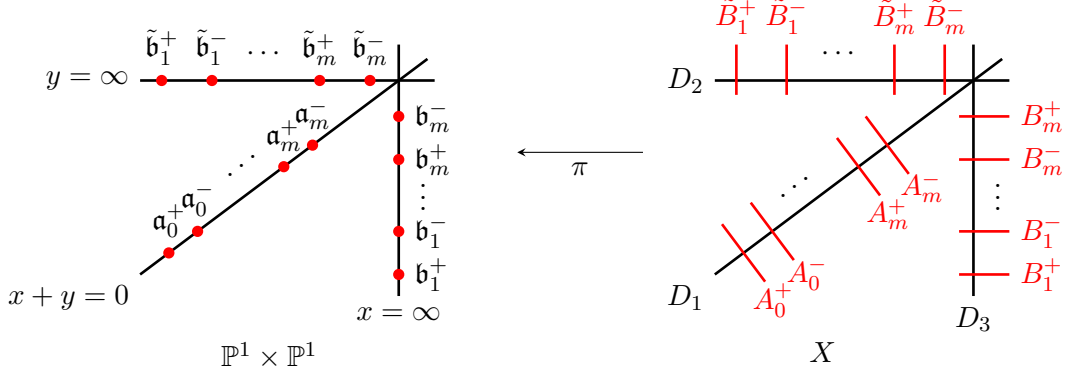


FIGURE 4. Space of initial conditions for Class III example (autonomous).

Denoting the composition of the blow-ups by $\pi : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and the exceptional divisors by $A_i^\pm = \pi^{-1}(\mathbf{a}_i^\pm)$, $B_j^\pm = \pi^{-1}(\mathbf{b}_j^\pm)$, $\tilde{B}_j^\pm = \pi^{-1}(\tilde{\mathbf{b}}_j^\pm)$, we have

$$\text{Pic}(X) = \mathbb{Z}\mathcal{H}_x + \mathbb{Z}\mathcal{H}_y + \sum_{i=0}^m (\mathbb{Z}\mathcal{A}_i^+ + \mathbb{Z}\mathcal{A}_i^-) + \sum_{j=1}^m (\mathbb{Z}\mathcal{B}_j^+ + \mathbb{Z}\mathcal{B}_j^- + \mathbb{Z}\tilde{\mathcal{B}}_j^+ + \mathbb{Z}\tilde{\mathcal{B}}_j^-), \quad (3.18)$$

where $\mathcal{H}_x = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1) \times 1)$ and $\mathcal{H}_y = \pi^*(1 \times \mathcal{O}_{\mathbb{P}^1}(1))$ correspond to classes of lines of constant x and y respectively, and we have used calligraphic script to indicate classes of exceptional divisors. Note that, as guaranteed by Theorem 2.23, the surface X has effective anticanonical divisor given by $D = D_1 + D_2 + D_3$, where D_1, D_2, D_3 are the proper transforms of the curves given by $x + y = 0$, $y = \infty$, and $x = \infty$ respectively. This is the pole divisor of the two-form on X given in the original coordinates by $\frac{dx \wedge dy}{x+y}$, which is preserved by the mapping.

Through calculations in charts, we find that the mapping acts by pullback on $\text{Pic}(X)$ as follows.

$$\varphi^* : \begin{cases} \mathcal{H}_x \mapsto (2m+1)\mathcal{H}_x + \mathcal{H}_y - \sum_{i=0}^m \mathcal{A}_i^+ - \sum_{i=0}^m \mathcal{A}_i^- - \sum_{j=1}^m \tilde{\mathcal{B}}_j^+ - \sum_{j=1}^m \tilde{\mathcal{B}}_j^-, \\ \mathcal{H}_y \mapsto \mathcal{H}_x, \quad \mathcal{A}_i^\pm \mapsto \mathcal{H}_x - \mathcal{A}_i^\mp, \quad \mathcal{B}_j^\pm \mapsto \mathcal{H}_x - \tilde{\mathcal{B}}_j^\pm, \quad \tilde{\mathcal{B}}_j^\pm \mapsto \mathcal{B}_j^\mp. \end{cases} \quad (3.19)$$

While it is possible to compute the characteristic polynomial of the matrix of φ^* with respect to this basis for $\text{Pic}(X)$, it will be more convenient to do this in terms of the root basis we will introduce after deautonomisation.

3.3. Space of initial conditions for deautonomised version. In what follows we will use a straightforward adaptation of the formulation of the space of initial conditions for a non-autonomous mapping of Section 2 to account for families of surfaces obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$, rather than \mathbb{P}^2 (see Remark 2.11). In particular the notions of geometric bases for $\text{Pic}(X_n)$ and identification of Picard groups into a single \mathbb{Z} -module via $\iota_n : \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(X_n)$ are defined when X_n is obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ in the natural way.

Consider the deautonomised equation (3.5) defining the mapping

$$\begin{aligned} \varphi_n : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= -x + \frac{\prod_{i=0}^m (x - a_i)(x - d_i)}{(x + y) \prod_{j=1}^m (x - b_j)(x - c_j)}, \quad \bar{y} = x, \end{aligned} \quad (3.20)$$

subject to the confinement conditions derived above, namely

$$\begin{aligned} \bar{a}_i &= -d_i, & \bar{d}_i &= -a_i, \\ \bar{b}_k &= -c_k + \sum_{j=1}^m (b_j + c_j) + \sum_{i=0}^m (a_i + d_i), \\ \bar{c}_k &= -b_k + \sum_{j=1}^m (b_j + c_j) + \sum_{i=0}^m (a_i + d_i). \end{aligned} \quad (3.21)$$

To construct the space of initial conditions we introduce the surface X_n obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the following points, as shown in Figure 5:

$$\begin{aligned} \mathbf{a}_i &: (x, y) = (a_i, -a_i), & \mathbf{e}_i &: (x, y) = (d_i, -d_i), & i &= 0, \dots, m, \\ \mathbf{b}_j &: (x, y) = (\infty, b_j), & \mathbf{c}_j &: (x, y) = (\infty, c_j), \\ \tilde{\mathbf{b}}_j &: (x, y) = (b_j, \infty), & \tilde{\mathbf{c}}_j &: (x, y) = (c_j, \infty), & j &= 1, \dots, m. \end{aligned} \quad (3.22)$$

Denoting the composition of the blow-ups by $\pi_n : X_n \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, and the exceptional divisors by $A_i = \pi_n^{-1}(\mathbf{a}_i)$, $E_i = \pi_n^{-1}(\mathbf{e}_i)$, $B_j = \pi_n^{-1}(\mathbf{b}_j)$, $C_j = \pi_n^{-1}(\mathbf{c}_j)$, $\tilde{B}_j = \pi_n^{-1}(\tilde{\mathbf{b}}_j)$, $\tilde{C}_j = \pi_n^{-1}(\tilde{\mathbf{c}}_j)$, we identify all $\text{Pic}(X_n)$ into a single \mathbb{Z} -module

$$\text{Pic}(\mathcal{X}) = \mathbb{Z}\mathcal{H}_x + \mathbb{Z}\mathcal{H}_y + \sum_{i=0}^m (\mathbb{Z}A_i + \mathbb{Z}E_i) + \sum_{j=1}^m (\mathbb{Z}B_j + \mathbb{Z}C_j + \mathbb{Z}\tilde{B}_j + \mathbb{Z}\tilde{C}_j), \quad (3.23)$$

where $\iota_n(\mathcal{H}_x) = \pi_n^*(\mathcal{O}_{\mathbb{P}^1}(1) \times 1)$, $\iota_n(\mathcal{A}_i) = [A_i]$ and so on. The intersection product gives the symmetric bilinear form on $\text{Pic}(\mathcal{X})$ defined by $\mathcal{H}_x \cdot \mathcal{H}_y = 1$, $\mathcal{H}_x \cdot \mathcal{H}_x = \mathcal{H}_y \cdot \mathcal{H}_y = 0$, with all generators that correspond to exceptional divisors of blow-ups being of self-intersection -1 and orthogonal to all other generators. Each surface X_n has an effective anticanonical divisor given by $D = D_1 + D_2 + D_3 \in |-\mathcal{K}_{X_n}|$, where D_1, D_2, D_3 are the proper transforms of the curves given by $x + y = 0$, $y = \infty$, and $x = \infty$ respectively.

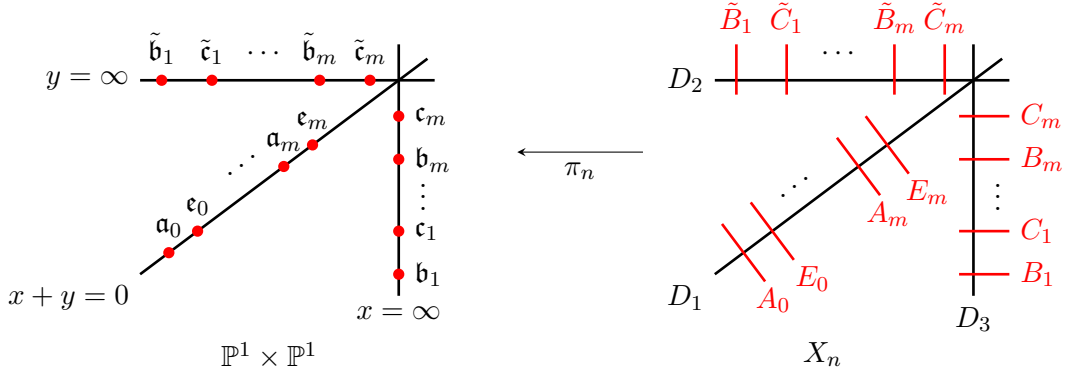


FIGURE 5. Space of initial conditions for Class III example (deautonomised).

Proposition 3.1. *With the confinement conditions (3.21), the mapping φ_n becomes an isomorphism $\tilde{\varphi}_n = \pi_{n+1}^{-1} \circ \varphi_n \circ \pi_n : X_n \rightarrow X_{n+1}$, and its pullback induces the following lattice automorphism $\Phi = \iota_n^{-1} \circ \varphi_n^* \circ \iota_{n+1}$ of $\text{Pic}(\mathcal{X})$:*

$$\Phi : \begin{cases} \mathcal{H}_x \mapsto (2m+1)\mathcal{H}_x + \mathcal{H}_y - \sum_{i=0}^m A_i - \sum_{i=0}^m E_i - \sum_{j=1}^m \tilde{B}_j - \sum_{j=1}^m \tilde{C}_j, & \mathcal{H}_y \mapsto \mathcal{H}_x, \\ A_i \mapsto \mathcal{H}_x - E_i, & E_i \mapsto \mathcal{H}_x - A_i, & B_j \mapsto \mathcal{H}_x - \tilde{B}_j, & C_j \mapsto \mathcal{H}_x - \tilde{C}_j, \\ \tilde{B}_j \mapsto C_j, & \tilde{C}_j \mapsto B_j. \end{cases} \quad (3.24)$$

Remark 3.2. This means that the mapping (3.5) with the confinement conditions (3.21) is a deautonomisation of (3.3) in the sense of Definition 2.18, via

$$\begin{aligned} \kappa : \text{Pic}(\mathcal{X}) &\rightarrow \text{Pic}(X), \\ \mathcal{A}_i &\mapsto \mathcal{A}_i^+, \quad \mathcal{E}_i \mapsto \mathcal{A}_i^-, \quad \mathcal{B}_j \mapsto \mathcal{B}_j^+, \quad \mathcal{C}_j \mapsto \mathcal{B}_j^-, \quad \tilde{\mathcal{B}}_j \mapsto \tilde{\mathcal{B}}_j^+, \quad \tilde{\mathcal{C}}_j \mapsto \tilde{\mathcal{B}}_j^-. \end{aligned} \quad (3.25)$$

3.4. Root basis and period map. With the space of initial conditions and linear transformation Φ in hand, we now calculate the period map of X_n and confirm the mechanism by which the dynamical degree appeared in the confinement conditions. We take the rational two-form to be that given in the initial affine charts for $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$\omega = k \frac{dx \wedge dy}{x+y} = -k \frac{dX \wedge dy}{X(1+Xy)} = -k \frac{dx \wedge dY}{Y(xY+1)} = k \frac{dX \wedge dY}{XY(X+Y)}, \quad (3.26)$$

with $k \in \mathbb{C}^*$ being arbitrary at this stage, and denote its lift to X_n under π_n by

$$\tilde{\omega}_n = \pi_n^* \omega. \quad (3.27)$$

We have $-\text{div}(\tilde{\omega}_n) = D_1 + D_2 + D_3$, and the elements of $\text{Pic}(\mathcal{X})$ corresponding to the irreducible components are

$$\mathcal{D}_1 = \mathcal{H}_x + \mathcal{H}_y - \sum_{i=0}^m \mathcal{A}_i - \sum_{i=0}^m \mathcal{E}_i, \quad \mathcal{D}_2 = \mathcal{H}_y - \sum_{j=1}^m \tilde{\mathcal{B}}_j - \sum_{j=1}^m \tilde{\mathcal{C}}_j, \quad \mathcal{D}_3 = \mathcal{H}_x - \sum_{j=1}^m \mathcal{B}_j - \sum_{j=1}^m \mathcal{C}_j, \quad (3.28)$$

so we denote their span and its orthogonal complement respectively by

$$Q = \text{span}_{\mathbb{Z}} \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}, \quad Q^\perp = \{\mathcal{F} \in \text{Pic}(\mathcal{X}) \mid \mathcal{F} \cdot \mathcal{D}_i = 0\}. \quad (3.29)$$

To find a root basis for Q^\perp in the sense of Definition 2.25, we note that when $m = 1$, X_n becomes a Sakai surface of (additive) type $A_2^{(1)*}$ in Sakai's classification scheme so we can use the root basis there, which is formed of simple roots for an affine root system of type $E_6^{(1)}$ (see [39, Appendix A] or [22, 8.2.16]). It turns out that this choice can be generalised to the $m > 1$ case as follows.

Proposition 3.3. *We have a root basis for $Q^\perp \subset \text{Pic}(\mathcal{X})$ given by*

$$Q^\perp = \sum_{i=0}^{2m} \mathbb{Z}\alpha_i + \sum_{j=1}^{2m-1} \mathbb{Z}\beta_j + \sum_{j=1}^{2m-1} \mathbb{Z}\gamma_j + \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2, \quad (3.30)$$

where

$$\begin{aligned} \varepsilon_1 &= \mathcal{H}_y - \mathcal{A}_0 - \mathcal{B}_1, & \varepsilon_2 &= \mathcal{H}_x - \mathcal{A}_0 - \tilde{\mathcal{B}}_1, \\ \alpha_0 &= \mathcal{A}_0 - \mathcal{E}_0, & \beta_1 &= \mathcal{B}_1 - \mathcal{C}_1, & \gamma_1 &= \tilde{\mathcal{B}}_1 - \tilde{\mathcal{C}}_1, \\ \alpha_1 &= \mathcal{E}_0 - \mathcal{A}_1, & \beta_2 &= \mathcal{C}_1 - \mathcal{B}_2, & \gamma_2 &= \tilde{\mathcal{C}}_1 - \tilde{\mathcal{B}}_2, \\ \alpha_2 &= \mathcal{A}_1 - \mathcal{E}_1, & \beta_3 &= \mathcal{B}_2 - \mathcal{C}_2, & \gamma_3 &= \tilde{\mathcal{B}}_2 - \tilde{\mathcal{C}}_2, \\ & \vdots & & \vdots & & \vdots \\ \alpha_{2m-1} &= \mathcal{E}_{m-1} - \mathcal{A}_m, & \beta_{2m-2} &= \mathcal{C}_{m-1} - \mathcal{B}_m, & \gamma_{2m-2} &= \tilde{\mathcal{C}}_{m-1} - \tilde{\mathcal{B}}_m, \\ \alpha_{2m} &= \mathcal{A}_m - \mathcal{E}_m, & \beta_{2m-1} &= \mathcal{B}_m - \mathcal{C}_m, & \gamma_{2m-1} &= \tilde{\mathcal{B}}_m - \tilde{\mathcal{C}}_m. \end{aligned} \quad (3.31)$$

Proof. In this case $\mathcal{K}_{\mathcal{X}} \notin Q^\perp$ and we have no overlap between Q and Q^\perp , so since $\text{rank Pic}(\mathcal{X}) = 6m + 4$ we need to find $6m + 1$ elements to construct a root basis. We begin by taking the obvious choices of differences of pairs of classes of exceptional divisors intersecting the components D_1, D_2, D_3 , which provides $6m - 1$ elements and leads to $\alpha_0, \dots, \alpha_{2m}, \beta_1, \dots, \beta_{2m-1}$ and $\gamma_1, \dots, \gamma_{2m-1}$ as above. For the additional two elements required to form a root basis, we take cues from the known choice in the $m = 1$ case and set ε_1 to be the difference of the class $\mathcal{H}_y - \mathcal{A}_0$ of the proper transform of the line $y = -a_0$ and the class \mathcal{B}_1 of B_1 , which both intersect D_3 . Similarly we choose ε_2 to be the difference of the class $\mathcal{H}_x - \mathcal{A}_0$ of the proper transform

of the line $x = a_0$ and the class $\tilde{\mathcal{B}}_1$ of \tilde{B}_1 , which both intersect D_2 . The fact that these choices lead to a root basis in the sense of Definition 2.25 is immediate, and in Figure 6 we present the graph corresponding to the generalised Cartan matrix given by the intersection numbers of the elements, i.e. nodes corresponding to elements, with edges joining them if the pair of elements has intersection 1. We remark that this graph already appeared in the work of Looijenga [23] as the intersection configuration of a root basis for a surface with an anticanonical cycle of length three - this root basis will appear again in a way more directly related to Looijenga's work in Section 4. \square

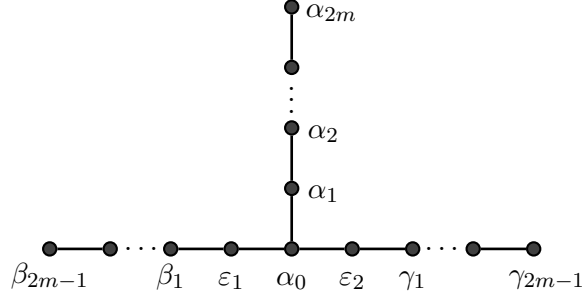


FIGURE 6. Intersection diagram of root basis for Q^\perp in Class III example

We now compute the period mapping on this basis via the procedure outlined in Section 2. Note that in this case D_{red} is the union of three copies of \mathbb{P}^1 meeting at a single point so $H_1(D_{\text{red}}; \mathbb{Z}) = 0$, and the only normalisation we need in order to define χ as a \mathbb{C} -valued function is a choice of the constant k appearing in ω as in (3.26). A convenient choice leads to the following, in which the period map is that on X_n and we slightly abuse notation by writing α_i etc. when more precisely we mean their corresponding elements $\iota_n(\alpha_i) \in \text{Pic}(X_n)$.

Lemma 3.4. *The root variables for the root basis in Proposition 3.3 are given by*

$$\begin{aligned}
\chi(\alpha_0) &= a_0 - d_0, & \chi(\epsilon_1) &= -a_0 - \underline{b}_1, & \chi(\epsilon_2) &= -a_0 + b_1, \\
\chi(\alpha_1) &= d_0 - a_1, & \chi(\beta_1) &= \underline{b}_1 - \underline{c}_1, & \chi(\gamma_1) &= -b_1 + c_1, \\
\chi(\alpha_2) &= a_1 - d_1, & \chi(\beta_2) &= \underline{c}_1 - \underline{b}_2, & \chi(\gamma_2) &= -c_1 + b_2, \\
\chi(\alpha_3) &= d_1 - a_2, & \chi(\beta_3) &= \underline{b}_2 - \underline{b}_2, & \chi(\gamma_3) &= -b_2 + c_2, \\
&\vdots & & \vdots & & \vdots \\
\chi(\alpha_{2m-1}) &= d_{m-1} - a_m, & \chi(\beta_{2m-2}) &= \underline{c}_{m-1} - \underline{b}_m, & \chi(\gamma_{2m-2}) &= -c_{m-1} + b_m, \\
\chi(\alpha_{2m}) &= a_m - d_m, & \chi(\beta_{2m-1}) &= \underline{b}_m - \underline{c}_m, & \chi(\gamma_{2m-1}) &= -b_m + c_m.
\end{aligned} \tag{3.32}$$

Proof. In order to compute the period mapping using the procedure outlined in Section 2, we first note that the residues of $\tilde{\omega}_n$ along the components of the anticanonical divisor D are given in coordinates by

$$\begin{aligned}
\text{Res}_{D_1} \tilde{\omega}_n &= \text{Res}_{s=0} k \frac{dx \wedge ds}{s} = -kdx, \\
\text{Res}_{D_2} \tilde{\omega}_n &= \text{Res}_{Y=0} -k \frac{dx \wedge dY}{Y(xY + 1)} = kdx, \\
\text{Res}_{D_3} \tilde{\omega}_n &= \text{Res}_{X=0} -k \frac{dX \wedge dy}{X(1 + Xy)} = -kdy,
\end{aligned} \tag{3.33}$$

where $s = x + y$. To calculate $\chi(\alpha_i)$ for $i = 0, \dots, 2m$ we note that we can express $\mathcal{A}_j - \mathcal{E}_k = [A_j] - [E_k]$ as the difference of two exceptional curves of the first kind which intersect D_1 , which is the proper transform of the line $x + y = 0$. Therefore using the residue computed above we

have

$$\chi(\mathcal{A}_j - \mathcal{E}_k) = 2\pi i \int_{E_k \cap D_1}^{A_j \cap D_1} \text{Res}_{D_1} \tilde{\omega}_n = 2\pi i \int_{x=d_k}^{x=a_j} -k dx = -2\pi i k (a_j - d_k). \quad (3.34)$$

Similarly we find

$$\begin{aligned} \chi(\tilde{\mathcal{B}}_j - \tilde{\mathcal{C}}_k) &= 2\pi i \int_{\tilde{\mathcal{C}}_k \cap D_2}^{\tilde{\mathcal{B}}_j \cap D_2} \text{Res}_{D_2} \tilde{\omega}_n = 2\pi i \int_{x=c_k}^{x=b_j} k dx = 2\pi i k (b_j - c_k), \\ \chi(\mathcal{B}_j - \mathcal{C}_k) &= 2\pi i \int_{C_k \cap D_3}^{B_j \cap D_3} \text{Res}_{D_3} \tilde{\omega}_n = 2\pi i \int_{y=c_k}^{y=b_j} -k dy = -2\pi i k (b_j - c_k), \end{aligned} \quad (3.35)$$

which allows us to deduce the values of χ on all elements of the root basis except for ε_1 and ε_2 , but these are computed similarly. For example for ε_1 , we use its expression as the difference of classes $\mathcal{H}_y - \mathcal{A}_0$ and \mathcal{B}_1 of exceptional curves intersecting D_3 , and note that $\mathcal{H}_y - \mathcal{A}_0$ corresponds to the proper transform of the line $y = -a_0$ so we compute

$$\chi(\varepsilon_1) = 2\pi i \int_{y=b_1}^{y=-a_0} -k dy = 2\pi i k (a_0 + b_1), \quad (3.36)$$

and ε_2 is dealt with similarly. Finally we choose k such that $2\pi i k = -1$, and we have the result. \square

With this lemma in hand, we can compute how the values of the period map on the root basis evolve with n as dictated by the confinement conditions (3.21). Denoting vectors of elements of the root basis by $\alpha_e = (\alpha_0, \alpha_2, \dots, \alpha_{2m})^T$, $\alpha_o = (\alpha_1, \alpha_3, \dots, \alpha_{2m-1})^T$, $\beta_o = (\beta_1, \beta_3, \dots, \beta_{2m-1})^T$, $\beta_e = (\beta_2, \beta_4, \dots, \beta_{2m})^T$, $\gamma_o = (\gamma_1, \gamma_3, \dots, \gamma_{2m-1})^T$, and $\gamma_e = (\gamma_2, \gamma_4, \dots, \gamma_{2m})^T$, the following is obtained by direct calculation.

Proposition 3.5. *The evolution with n of the root variables for the surface X_n is given by*

$$\bar{\chi} = M_\Phi \cdot \chi, \quad (3.37)$$

$$\begin{aligned} \text{where } \chi &= (\chi(\varepsilon_1), \chi(\varepsilon_2), \chi(\alpha_o), \chi(\alpha_e), \chi(\beta_e), \chi(\beta_o), \chi(\gamma_e), \chi(\gamma_o))^T, \\ \bar{\chi} &= (\bar{\chi}(\bar{\varepsilon}_1), \bar{\chi}(\bar{\varepsilon}_2), \bar{\chi}(\bar{\alpha}_o), \bar{\chi}(\bar{\alpha}_e), \bar{\chi}(\bar{\beta}_e), \bar{\chi}(\bar{\beta}_o), \bar{\chi}(\bar{\gamma}_e), \bar{\chi}(\bar{\gamma}_o))^T, \end{aligned}$$

$$\text{and } M_\Phi = \begin{pmatrix} 0 & -1 & * & * & * & * & * & * \\ 1 & 2m & * & * & * & * & * & * \\ 0 & 0 & -I_{m+1} & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_m \\ 0 & 0 & 0 & 0 & I_m & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m & 0 & 0 \end{pmatrix} \quad (3.38)$$

and where $\chi, \bar{\chi}$ are the period maps of the surfaces X_n, X_{n+1} respectively, and bars on root basis elements indicate their counterparts in $\text{Pic}(X_{n+1})$ under the identification ι_{n+1} .

As expected, the evolution of the root variables is given by exactly the same matrix as that of Φ on Q^\perp with respect to the root basis. Indeed, by direct computation using the expressions in Proposition 3.3 for the root basis in terms of generators of $\text{Pic}(\mathcal{X})$ and Φ as it appears in Proposition 3.1, we see that the evolution of the root basis for Q^\perp under Φ takes the form

$$\Phi : \delta \mapsto M_\Phi \delta, \quad (3.39)$$

where $\delta = (\varepsilon_1, \varepsilon_2, \alpha_o, \alpha_e, \beta_e, \beta_o, \gamma_e, \gamma_o)^T$. Further, it can be verified by direct calculation that all elements of the root basis aside from ε_1 and ε_2 are in the periodic part P_Φ of $\text{Pic}(\mathcal{X})$ under Φ , so the minimal polynomial with the dynamical degree of the mapping as a root can be obtained as follows.

Proposition 3.6. *The transformation of Q^\perp/P_Φ , induced by Φ is given with respect to the basis $\{\varepsilon_1 + P_\Phi, \varepsilon_2 + P_\Phi\}$ by the matrix*

$$\begin{pmatrix} 0 & -1 \\ 1 & 2m \end{pmatrix}, \quad (3.40)$$

whose characteristic polynomial is $(t^2 - 2mt + 1)$. In particular the dynamical degree of the mapping is equal to the largest root $\lambda = m + \sqrt{m^2 - 1}$ of this polynomial.

Remark 3.7. The fact that, in order to read off the dynamical degree of the mapping from the confinement conditions (3.21), it was sufficient to keep a_i and d_i constant as in (3.11) is precisely because the root variables that are fixed because of this choice, correspond to elements of $Q^\perp \cap P_\Phi$. Keeping these parameters fixed still gives a sufficient deautonomisation in the sense of Definition 2.30.

4. A FAMILY OF MAPPINGS OF QRT CLASS IV FORM

We next consider the family of mappings of the form

$$(\bar{x}x - 1)(xx - 1) = \frac{\prod_{i=0}^m (x - a_i)(x - a_i^{-1})}{\prod_{j=1}^m (x - b_j)(x - b_j^{-1})}, \quad (4.1)$$

where $a_i \in \mathbb{C} \setminus \{0\}$, $i = 0, \dots, m$ and $b_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \dots, m$, for $m \in \mathbb{Z}_{\geq 1}$. Similarly to the example in Section 3, we have singularities corresponding to zeroes of the numerator and denominator of the rational function on the right-hand side of (4.1), and these are confined with singularity patterns $\{a_i^{\pm 1}, a_i^{\mp 1}\}$, and $\{b_j^{\pm 1}, \infty, b_j^{\mp 1}\}$.

4.1. Full deautonomisation by singularity confinement. To determine the dynamical degree of the mapping (4.1), we take only a sufficient deautonomisation (see Appendix B) given by

$$(\bar{x}x - z)(xx - z) = \frac{\prod_{i=0}^m (x - a_i)(x - d_i)}{\prod_{j=1}^m (x - b_j)(x - c_j)}, \quad (4.2)$$

where the n -dependence of z , a_i , d_i is to be determined, and the rest of the parameters on the right-hand side are taken to be constant:

$$\bar{b}_j = b_j, \quad \bar{c}_j = c_j. \quad (4.3)$$

For the singularities $x = b_j$, and $x = c_j$ to be confined in the same way as in the autonomous mapping (4.1) we can choose to require patterns $\{b_j, \infty, c_j\}$ and $\{c_j, \infty, b_j\}$, which for constant b_j and c_j leads to

$$c_j = b_j^{-1}, \quad \text{for } j = 1, \dots, m. \quad (4.4)$$

Hence these patterns are exactly the same as in the autonomous case, i.e. $\{b_j^{\pm 1}, \infty, b_j^{\mp 1}\}$. We enforce confinement of the singularities $x = a_i$ and $x = d_i$ respectively through the patterns $\{a_i, \bar{d}_i\}$ and $\{d_i, \bar{a}_i\}$, by imposing the conditions

$$\bar{a}_i = \frac{z}{d_i}, \quad \bar{d}_i = \frac{z}{a_i}. \quad (4.5)$$

Finally we note that to properly deautonomise equation (4.1), we must ensure $x = 0$ is not a singularity. To see that this is the case before deautonomisation, note that at $x = 0$ the right-hand side of (4.1) is equal to 1, so when recast as a mapping (4.10) a cancellation occurs so that there is no factor of x in the denominator of the rational function giving \bar{x} . To preserve this property under deautonomisation we require

$$z\bar{z} = \prod_{i=0}^m a_i d_i, \quad (4.6)$$

so in particular from this condition and its up-shifted version we have

$$\bar{z}z = \prod_{i=0}^m \bar{a}_i \bar{d}_i = \prod_{i=0}^m (zd_i^{-1})(za_i^{-1}) = \frac{z^{2m+1}}{z}, \quad (4.7)$$

from which we deduce the equation governing the n -dependence of z to be

$$\bar{z}z = z^{2m}. \quad (4.8)$$

Note that this constraint is multiplicative, as opposed to the additive constraint found for the mappings in Section 3. However, upon linearisation it leads to the same characteristic polynomial as in Section 3:

$$t^2 - 2mt + 1, \quad (4.9)$$

the largest root of which is again $\lambda = m + \sqrt{m^2 - 1}$, which we will confirm to be the dynamical degree of the mapping (4.1).

4.2. Space of initial conditions. We now construct the space of initial conditions for the autonomous equation (4.1), recast as the mapping

$$\begin{aligned} \varphi : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= \frac{1}{x} + \frac{\prod_{i=0}^m (x - a_i)(x - a_i^{-1})}{x(xy - 1) \prod_{j=1}^m (x - b_j)(x - b_j^{-1})}, \quad \bar{y} = x, \end{aligned} \quad (4.10)$$

and we will recycle notation for blow-up points, exceptional divisors, etc. from Section 3. For this mapping we blow up the points

$$\mathfrak{a}_i^\pm : (x, y) = (a_i^{\pm 1}, a_i^{\mp 1}), \quad \mathfrak{b}_i^\pm : (x, y) = (\infty, b_i^{\pm 1}), \quad \tilde{\mathfrak{b}}_i^\pm : (x, y) = (b_i^{\pm 1}, \infty), \quad (4.11)$$

after which we have the rational surface X as shown in Figure 7. This surface has effective anticanonical divisor $D = D_1 + D_2 + D_3 \in |-\mathcal{K}_X|$, where D_1, D_2, D_3 are the proper transforms of the curves given by $xy = 1$, $y = \infty$, and $x = \infty$ respectively. This is the pole divisor of the two-form on X given in the original coordinates by $\frac{dx \wedge dy}{xy-1}$, which is preserved by the mapping.

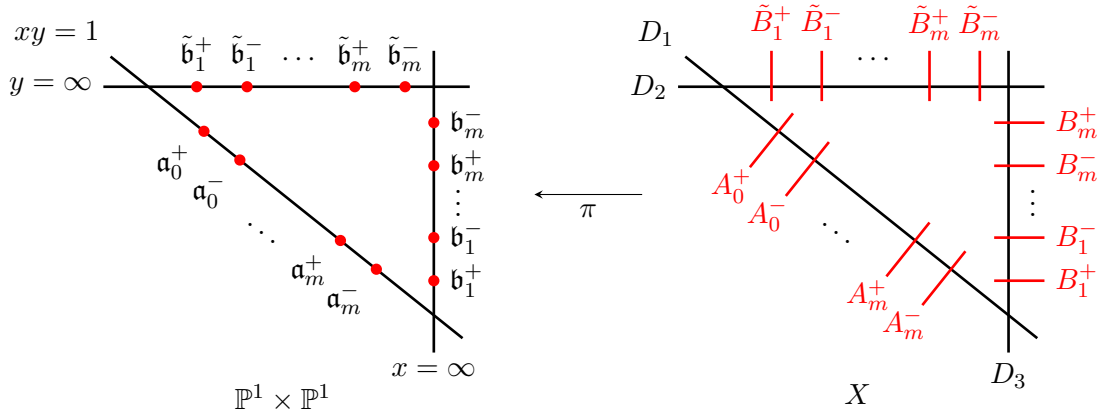


FIGURE 7. Space of initial conditions for Class IV example (autonomous).

Through calculations in charts, we find that the mapping acts by pullback on $\text{Pic}(X)$ in exactly the same way as the example from Class III studied in Section 3, as given by (3.19). While this immediately implies that the dynamical degree must also be the same, the way in which the confinement conditions must be linearised before the characteristic polynomial for Φ can be read off in this case is different. This is related to the geometry of the anticanonical divisor D as we will show below.

4.3. Space of initial conditions for deautonomised version. Consider the deautonomised equation (4.2), with confinement conditions as derived above, defining the mapping

$$\begin{aligned} \varphi_n : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= \frac{z}{x} + \frac{\prod_{i=0}^m (x - a_i)(x - d_i)}{x(xy - z) \prod_{j=1}^m (x - b_j)(x - b_j^{-1})}, \quad \bar{y} = x, \end{aligned} \quad (4.12)$$

with parameters satisfying

$$\bar{b}_i = b_i, \quad \bar{a}_i = \frac{z}{d_i}, \quad \bar{d}_i = \frac{z}{a_i}, \quad zz = \prod_{i=0}^m a_i d_i. \quad (4.13)$$

We construct the surface X_n by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the following points, as shown in Figure 8:

$$\begin{aligned} \mathbf{a}_i : (x, y) &= (a_i, z a_i^{-1}), & \mathbf{e}_i : (x, y) &= (d_i, z d_i^{-1}), & i &= 0, \dots, m \\ \mathbf{b}_j^\pm : (x, y) &= (\infty, b_j^{\pm 1}), & \tilde{\mathbf{b}}_j^\pm : (x, y) &= (b_j^{\pm 1}, \infty), & j &= 1, \dots, m. \end{aligned} \quad (4.14)$$

We identify all Picard groups of the surfaces X_n into the single lattice $\text{Pic}(\mathcal{X})$ as before and obtain the following.

Proposition 4.1. *With the confinement conditions (4.13), the mapping φ_n in (4.12) becomes an isomorphism $\tilde{\varphi}_n = \pi_{n+1}^{-1} \circ \varphi_n \circ \pi_n : X_n \rightarrow X_{n+1}$, and its pullback induces the following lattice automorphism $\Phi = \iota_n^{-1} \circ \varphi_n^* \circ \iota_{n+1}$ of $\text{Pic}(\mathcal{X})$:*

$$\Phi : \begin{cases} \mathcal{H}_x \mapsto (2m+1)\mathcal{H}_x + \mathcal{H}_y - \sum_{i=0}^m \mathcal{A}_i - \sum_{i=0}^m \mathcal{E}_i - \sum_{j=1}^m \tilde{\mathcal{B}}_j^+ - \sum_{j=1}^m \tilde{\mathcal{B}}_j^-, & \mathcal{H}_y \mapsto \mathcal{H}_x, \\ \mathcal{A}_i \mapsto \mathcal{H}_x - \mathcal{E}_i, & \mathcal{E}_i \mapsto \mathcal{H}_x - \mathcal{A}_i, & \mathcal{B}_j^\pm \mapsto \mathcal{H}_x - \tilde{\mathcal{B}}_j^\pm, & \tilde{\mathcal{B}}_j^\pm \mapsto \mathcal{B}_j^\mp. \end{cases} \quad (4.15)$$

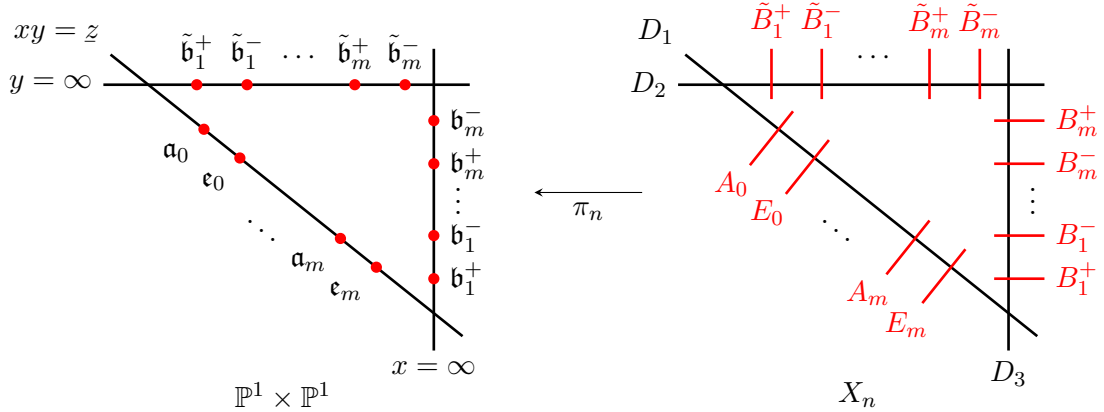


FIGURE 8. Space of initial conditions for Class IV example (deautonomised).

4.4. Root basis and period map. Again we calculate the period map of X_n and confirm the mechanism by which the dynamical degree appeared in the confinement conditions, including the linearisation that was required to obtain a difference equation from which the minimal polynomial for the dynamical degree could be read off. The rational two-form preserved by the mapping in this case is given in charts for $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$\omega_n = k \frac{dx \wedge dy}{xy - z} = -k \frac{dX \wedge dY}{X(Y - Xz)} = -k \frac{dx \wedge dY}{Y(x - Yz)} = k \frac{dX \wedge dY}{XY(1 - XYz)}, \quad (4.16)$$

with $k \in \mathbb{C}^*$, and we again denote its lift to X_n under π_n by

$$\tilde{\omega}_n = \pi_n^* \omega_n. \quad (4.17)$$

Direct calculation shows that the two-form is preserved by the mapping, i.e.,

$$\frac{d\bar{x} \wedge d\bar{y}}{\bar{x}\bar{y} - z} = \frac{dx \wedge dy}{xy - z}, \quad (4.18)$$

and even though ω_n on $\mathbb{P}^1 \times \mathbb{P}^1$ depends on n , we are still within the remit of Theorem 2.23. Indeed X_n has effective anticanonical divisor $-\operatorname{div}(\tilde{\omega}_n) = D_1 + D_2 + D_3$, and the elements of $\operatorname{Pic}(\mathcal{X})$ corresponding to the irreducible components are given by

$$\mathcal{D}_1 = \mathcal{H}_x + \mathcal{H}_y - \sum_{i=0}^m \mathcal{A}_i - \sum_{i=0}^m \mathcal{E}_i, \quad \mathcal{D}_2 = \mathcal{H}_y - \sum_{j=1}^m \tilde{\mathcal{B}}_j^+ - \sum_{j=1}^m \tilde{\mathcal{B}}_j^-, \quad \mathcal{D}_3 = \mathcal{H}_x - \sum_{j=1}^m \mathcal{B}_j^+ - \sum_{j=1}^m \mathcal{B}_j^-. \quad (4.19)$$

Defining the subsets $Q, Q^\perp \subset \operatorname{Pic}(\mathcal{X})$ as above, we find the same root basis as in Section 3, but changes in notation mean that \mathcal{B}_j and \mathcal{C}_j are replaced with \mathcal{B}_j^+ and \mathcal{B}_j^- respectively, and their tilded versions similarly, i.e.

$$\begin{aligned} \varepsilon_1 &= \mathcal{H}_y - \mathcal{A}_0 - \mathcal{B}_1^+, & \varepsilon_2 &= \mathcal{H}_x - \mathcal{A}_0 - \tilde{\mathcal{B}}_1^+, \\ \alpha_0 &= \mathcal{A}_0 - \mathcal{E}_0, & \beta_1 &= \mathcal{B}_1^+ - \mathcal{B}_1^-, & \gamma_1 &= \tilde{\mathcal{B}}_1^+ - \tilde{\mathcal{B}}_1^-, \\ \alpha_1 &= \mathcal{E}_0 - \mathcal{A}_1, & \beta_2 &= \mathcal{B}_1^- - \mathcal{B}_2^+, & \gamma_2 &= \tilde{\mathcal{B}}_1^- - \tilde{\mathcal{B}}_2^+, \\ \alpha_2 &= \mathcal{A}_1 - \mathcal{E}_1, & \beta_3 &= \mathcal{B}_2^+ - \mathcal{B}_2^-, & \gamma_3 &= \tilde{\mathcal{B}}_2^+ - \tilde{\mathcal{B}}_2^-, \\ & \vdots & & \vdots & & \vdots \\ \alpha_{2m-1} &= \mathcal{E}_{m-1} - \mathcal{A}_m, & \beta_{2m-2} &= \mathcal{B}_{m-1}^- - \mathcal{B}_m^+, & \gamma_{2m-2} &= \tilde{\mathcal{B}}_{m-1}^- - \tilde{\mathcal{B}}_m^+, \\ \alpha_{2m} &= \mathcal{A}_m - \mathcal{E}_m, & \beta_{2m-1} &= \mathcal{B}_m^+ - \mathcal{B}_m^-, & \gamma_{2m-1} &= \tilde{\mathcal{B}}_m^+ - \tilde{\mathcal{B}}_m^-. \end{aligned} \quad (4.20)$$

In particular the generalised Cartan matrix for this basis is the same as in Section 3 encoded by the graph in Figure 6. The key difference between this example and that of the previous section is the parametrisation by the period map of the coefficients in the mapping and the centres of blow-ups, because of the fact that the anticanonical divisor is a cycle and $H_1(D_{\text{red}}; \mathbb{Z}) \cong \mathbb{Z}$. In particular the period map is defined by the surface only modulo $\hat{\chi}(H_1(D_{\text{red}}; \mathbb{Z})) \cong \mathbb{Z}$, which can be seen in terms of the residues of $\tilde{\omega}_n$ along the irreducible components,

$$\begin{aligned} \operatorname{Res}_{D_1} \tilde{\omega}_n &= \operatorname{Res}_{s=0} k \frac{dx \wedge ds}{xs} = -k \frac{dx}{x}, \\ \operatorname{Res}_{D_2} \tilde{\omega}_n &= \operatorname{Res}_{Y=0} -k \frac{dx \wedge dY}{Y(x - Yz)} = k \frac{dx}{x}, \\ \operatorname{Res}_{D_3} \tilde{\omega}_n &= \operatorname{Res}_{X=0} -k \frac{dX \wedge dy}{X(y - Xz)} = -k \frac{dy}{y}, \end{aligned} \quad (4.21)$$

where $s = xy - z$. Note that evaluating integrals of these residues will lead to logarithmic functions of parameters in the mapping. Computing along the same lines as in the proof of Lemma 3.4 and imposing appropriate normalisations of both k and the value $\hat{\chi}(\ell)$ for a generator ℓ of $H_1(D_{\text{red}}; \mathbb{Z})$ as in [39], we find the following.

Lemma 4.2. *The root variables of X_n for the root basis above are related to the parameters in the mapping (4.12) according to*

$$\begin{aligned}
e^{\chi(\alpha_0)} &= a_0 d_0^{-1}, & e^{\chi(\varepsilon_1)} &= z a_0^{-1} b_1^{-1}, & e^{\chi(\varepsilon_2)} &= a_0^{-1} b_1, \\
e^{\chi(\alpha_1)} &= d_0 a_1^{-1}, & e^{\chi(\beta_1)} &= b_1^2, & e^{\chi(\gamma_1)} &= b_1^{-2}, \\
e^{\chi(\alpha_2)} &= a_1 d_1^{-1}, & e^{\chi(\beta_2)} &= b_1^{-1} b_2^{-1}, & e^{\chi(\gamma_2)} &= b_1 b_2, \\
e^{\chi(\alpha_3)} &= d_1 a_2^{-1}, & e^{\chi(\beta_3)} &= b_2^2, & e^{\chi(\gamma_3)} &= b_2^{-2}, \\
&\vdots & & \vdots & & \vdots \\
e^{\chi(\alpha_{2m-1})} &= d_{m-1} a_m^{-1}, & e^{\chi(\beta_{2m-2})} &= b_{m-1}^{-1} b_m^{-1}, & e^{\chi(\gamma_{2m-2})} &= b_{m-1} b_m, \\
e^{\chi(\alpha_{2m})} &= a_m d_m^{-1}, & e^{\chi(\beta_{2m-1})} &= b_m^2, & e^{\chi(\gamma_{2m-1})} &= b_m^{-2}.
\end{aligned} \tag{4.22}$$

The evolution of the root variables is again given by the matrix of Φ with respect to the root basis, and we can also explain how the evolution of the parameter z gave the scalar difference equation (4.8) from which the dynamical degree could be read off after linearisation. Expressing \underline{z} in terms of the root variables of X_n we have

$$\underline{z} = e^{\chi(\varepsilon_1) - \chi(\varepsilon_2) + \chi(\beta_1)}, \tag{4.23}$$

and in light of this we introduce

$$\begin{aligned}
v_1 &= \varepsilon_1 - \varepsilon_2 + \beta_1, \\
v_2 &= \Phi(v_1),
\end{aligned} \tag{4.24}$$

and see that

$$\Phi(v_2) = -v_1 + 2mv_2 + G, \tag{4.25}$$

where $G = 2\beta_1 - \sum_{j=2}^{2m-1} (2m-j)\beta_j + 2\gamma_1 - \sum_{j=2}^{2m-1} (2m-j)\gamma_j$. Noting that all β_j, γ_j are in the intersection of Q^\perp with the periodic part P_Φ of $\text{Pic}(\mathcal{X})$ under Φ , we can verify the matrix of the induced transformation of Q^\perp/P_Φ with respect to the basis $\{v_1 + P_\Phi, v_2 + P_\Phi\}$ to be

$$\begin{pmatrix} 0 & 1 \\ -1 & 2m \end{pmatrix}, \tag{4.26}$$

so again we can read off the dynamical degree from its characteristic polynomial. The fact that this characteristic polynomial appeared so neatly from the evolution of z is because in our chosen deautonomisation the parameters b_j are kept constant, which leads to

$$e^{\chi(G)} = 1, \tag{4.27}$$

and therefore, through the period map, equation (4.25) becomes

$$\bar{z} = e^{\bar{\chi}(\bar{v}_2)} = e^{-\chi(v_1) + 2m\chi(v_2) + \chi(G)} = z^{2m} \underline{z}^{-1}, \tag{4.28}$$

which is precisely the confinement condition from which we read off the dynamical degree.

5. A FAMILY OF MAPPINGS OF QRT CLASS I FORM

The next example we will consider is the family of equations (1.12) introduced in the Introduction, namely

$$\bar{x} + x = \frac{1}{x^{2m}}, \tag{5.1}$$

where $m \in \mathbb{Z}_{\geq 1}$, which are of the form of Class I QRT-type mappings. This mapping admits the confined singularity pattern $\{0, \infty^{2m}, 0\}$, and if we take the deautonomisation

$$\bar{x} + x = \frac{1}{x^{2m}} - \frac{b}{x}, \tag{5.2}$$

the n -dependence of b required for the singularity confinement to persist is $\bar{b} - 2mb + \underline{b} = 0$, from which we can read off minimal polynomial $t^2 - 2mt + 1$ for the dynamical degree of the mapping (5.1), namely

$$\lambda = m + \sqrt{m^2 - 1}. \quad (5.3)$$

5.1. Space of initial conditions. The space of initial conditions for equation (5.1) was constructed in [26], which we recall here with some added details which will be necessary for deautonomisation and the computation of the period map. Consider the mapping

$$\begin{aligned} \varphi : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= \frac{1 - yx^{2m}}{x^{2m}}, \quad \bar{y} = x, \end{aligned} \quad (5.4)$$

defined by equation (5.1). In contrast to the examples studied above, here, to construct the space of initial conditions we will have to blow up infinitely near points, i.e. those lying on the exceptional divisors of previous blow-ups. For this reason we will need to introduce notation for charts covering exceptional divisors according to the following convention: after blowing up a point \mathfrak{p}_i given in some affine chart (x, y) by

$$\mathfrak{p}_i : (x, y) = (x_*, y_*), \quad (5.5)$$

the exceptional divisor E_i of the blowup of \mathfrak{p}_i is covered by two affine coordinate charts (u_i, v_i) and (U_i, V_i) given by

$$\begin{aligned} x = u_i v_i + x_*, \quad y = v_i + y_*, & \quad x = V_i + x_*, \quad y = U_i V_i + y_*, \\ u_i = \frac{x - x_*}{y - y_*}, \quad v_i = y - y_*, & \quad \text{and} \quad U_i = \frac{y - y_*}{x - x_*}, \quad V_i = x - x_*. \end{aligned} \quad (5.6)$$

In particular the exceptional divisor E_i has in these charts the local equations $v_i = 0$ and $V_i = 0$.

For the case at hand we require $8m$ blow-ups in total, $4m$ each over the points $(x, y) = (0, \infty)$ and $(x, y) = (\infty, 0)$ which we outline now. Firstly letting $\mathfrak{p}_1 : (x, y) = (0, \infty)$ and introducing coordinates for the exceptional divisor as above we require first $2m$ blowups of points $\mathfrak{p}_{k+1} : (U_k, V_k) = (0, 0)$ for $k = 1, \dots, 2m - 1$. Note that each of these points \mathfrak{p}_{k+1} lies on the proper transform of the line $y = \infty$ under the blowups of $\mathfrak{p}_1, \dots, \mathfrak{p}_k$. Then we find the point $\mathfrak{p}_{2m+1} : (U_{2m}, V_{2m}) = (1, 0)$ away from the proper transform of $y = \infty$, and after blowing this up we require another sequence of blow-ups of points $\mathfrak{p}_{2m+2}, \dots, \mathfrak{p}_{4m}$ given in coordinates by $\mathfrak{p}_{2m+k+1} : (u_k, v_k) = (0, 0)$, $k = 1, \dots, 2m - 1$, though these points are away from the proper transforms of $y = \infty$ or any of the previous exceptional divisors. Similarly for $\tilde{\mathfrak{p}}_1 : (x, y) = (\infty, 0)$ we initially blow up $2m$ points on the proper transform of the line $x = \infty$ given by $\tilde{\mathfrak{p}}_{k+1} : (\tilde{u}_k, \tilde{v}_k) = (0, 0)$, where we have used tildes to denote the coordinates introduced in blowing up points $\tilde{\mathfrak{p}}_i$ to distinguish them from those coming from blow-ups of \mathfrak{p}_i . Then similarly we find $\tilde{\mathfrak{p}}_{2m+1} : (\tilde{u}_{2m}, \tilde{v}_{2m}) = (1, 0)$ followed by a sequence $\tilde{\mathfrak{p}}_{2m+2}, \dots, \tilde{\mathfrak{p}}_{4m}$ given by $\tilde{\mathfrak{p}}_{2m+k+1} : (\tilde{u}_k, \tilde{v}_k) = (0, 0)$, $k = 1, \dots, 2m - 1$. The resulting surface is shown in Figure 9, where E_i and \tilde{E}_i are exceptional divisors from the blow-ups of \mathfrak{p}_i and $\tilde{\mathfrak{p}}_i$ respectively, and we have introduced notation D_i, \tilde{D}_i for the curves which will be shown to be the components of an anticanonical divisor of X whose classes in $\text{Pic}(X)$ are

$$\begin{aligned} \mathcal{D}_0 &= \mathcal{H}_y - \sum_{i=1}^{2m} \mathcal{E}_i, \quad \mathcal{D}_i = \mathcal{E}_i - \mathcal{E}_{i+1}, \quad i = 1, \dots, 4m - 1, \\ \tilde{\mathcal{D}}_0 &= \mathcal{H}_x - \sum_{i=1}^{2m} \tilde{\mathcal{E}}_i, \quad \tilde{\mathcal{D}}_i = \tilde{\mathcal{E}}_i - \tilde{\mathcal{E}}_{i+1}, \quad i = 1, \dots, 4m - 1. \end{aligned} \quad (5.7)$$

The fact that the mapping becomes an automorphism of X is verified by calculation in charts.

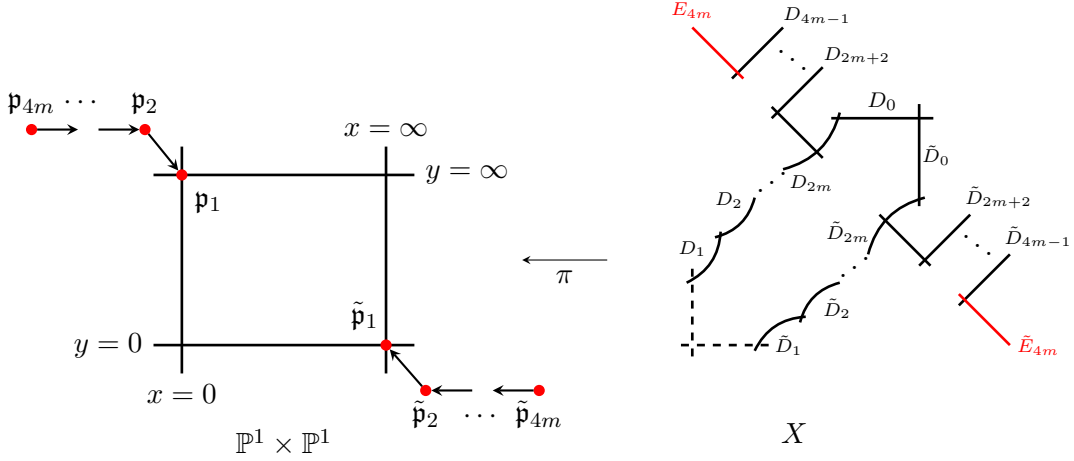


FIGURE 9. Space of initial conditions for Class I example

5.2. **Space of initial conditions for deautonomised version.** We take the deautonomised mapping (5.2) as

$$\begin{aligned} \varphi_n : \mathbb{P}^1 \times \mathbb{P}^1 &\rightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= \frac{1 - bx^{2m-1} - yx^{2m}}{x^{2m}}, \quad \bar{y} = x, \end{aligned} \quad (5.8)$$

with the confinement condition

$$\bar{b} - 2mb + b = 0. \quad (5.9)$$

The points to be blown up to construct the surface X_n that provides a space of initial conditions for mapping (5.8), are given by the same expressions in coordinates as for the autonomous case with the exception of the final points \mathbf{p}_{4m} and $\tilde{\mathbf{p}}_{4m}$, which for the non-autonomous mapping are

$$\mathbf{p}_{4m} : (u_{4m-1}, v_{4m-1}) = (b, 0), \quad \tilde{\mathbf{p}}_{4m} : (\tilde{u}_{4m-1}, \tilde{v}_{4m-1}) = (\bar{b}, 0). \quad (5.10)$$

Denoting again the exceptional divisors arising from the blow-ups by $E_i = \pi_n^{-1}(\mathbf{p}_i)$ and $\tilde{E}_i = \pi_n^{-1}(\tilde{\mathbf{p}}_i)$, we identify all $\text{Pic}(X_n)$ into the single \mathbb{Z} -module

$$\text{Pic}(\mathcal{X}) = \mathbb{Z}\mathcal{H}_x + \mathbb{Z}\mathcal{H}_y + \sum_{i=1}^{4m} \mathbb{Z}\mathcal{E}_i + \sum_{i=1}^{4m} \mathbb{Z}\tilde{\mathcal{E}}_i, \quad (5.11)$$

where $[E_i] = \iota_n(\mathcal{E}_i)$ and $[\tilde{E}_i] = \iota_n(\tilde{\mathcal{E}}_i)$.

Proposition 5.1. *With the confinement condition (5.9), the mapping φ_n in (5.8) becomes an isomorphism $\tilde{\varphi}_n = \pi_{n+1}^{-1} \circ \varphi_n \circ \pi_n : X_n \rightarrow X_{n+1}$, and its pullback induces the following lattice automorphism $\Phi = \iota_n^{-1} \circ \varphi_n^* \circ \iota_{n+1}$ of $\text{Pic}(\mathcal{X})$:*

$$\Phi : \left\{ \begin{aligned} \mathcal{H}_x &\mapsto 2m\mathcal{H}_x + \mathcal{H}_y - \sum_{i=1}^{4m} \mathcal{E}_i, & \mathcal{H}_y &\mapsto \mathcal{H}_x, & \mathcal{E}_i &\mapsto \tilde{\mathcal{E}}_i, & \tilde{\mathcal{E}}_i &\mapsto \mathcal{H}_x - \mathcal{E}_{4m+1-i}. \end{aligned} \right. \quad (5.12)$$

5.3. **Root basis and period map.** The rational two-form ω on $\mathbb{P}^1 \times \mathbb{P}^1$ preserved by φ_n is given by

$$\omega = kdx \wedge dy = -k \frac{dx \wedge dY}{Y^2} = -k \frac{dX \wedge dy}{X^2} = k \frac{dX \wedge dY}{X^2 Y^2}, \quad (5.13)$$

where $k \in \mathbb{C}^*$, and we denote its lift to X_n by $\tilde{\omega}_n = \pi_n^* \omega$. The pole divisor of $\tilde{\omega}_n$ is effective and is given in terms of the curves D_j, \tilde{D}_j by

$$-\text{div } \tilde{\omega}_n = 2D_0 + \sum_{j=1}^{2m} jD_j + \sum_{j=1}^{2m-1} (2m-j)D_{2m+j} + 2\tilde{D}_0 + \sum_{j=1}^{2m} j\tilde{D}_j + \sum_{j=1}^{2m-1} (2m-j)\tilde{D}_{2m+j}. \quad (5.14)$$

To construct a root basis in this case we perform a similar procedure to that outlined in the proof of Proposition 2.29. The singularity pattern $\{0, \infty^{2m}, 0\}$ corresponds to the movement of the following exceptional curves of the first kind under the mapping according to

$$[x = 0] \longrightarrow \tilde{E}_{4m} \longrightarrow E_{4m} \longrightarrow [y = 0], \quad (5.15)$$

where $[x = 0]$ and $[y = 0]$ indicate proper transforms under the blowups. Passing to $\text{Pic}(\mathcal{X})$ we have

$$\mathcal{H}_x - \mathcal{E}_1 \xleftarrow{\Phi} \tilde{\mathcal{E}}_{4m} \xleftarrow{\Phi} \mathcal{E}_{4m} \xleftarrow{\Phi} \mathcal{H}_y - \tilde{\mathcal{E}}_1, \quad (5.16)$$

so we take $\mathcal{H}_y - \tilde{\mathcal{E}}_1$ and see that the component of D it intersects corresponds to $\tilde{\mathcal{D}}_1 = \tilde{\mathcal{E}}_1 - \tilde{\mathcal{E}}_2$, which has period 4 under Φ , so we compute

$$\Phi^4(\mathcal{H}_y - \tilde{\mathcal{E}}_1) = 2m\mathcal{H}_x + \mathcal{H}_y - \sum_{i=1}^{4m} \mathcal{E}_i - \tilde{\mathcal{E}}_1. \quad (5.17)$$

and obtain an element of Q^\perp given by

$$\alpha_1 = (\Phi^4 - 1)(\mathcal{H}_y - \tilde{\mathcal{E}}_1) = 2m\mathcal{H}_x - \sum_{i=1}^{4m} \mathcal{E}_i. \quad (5.18)$$

Applying Φ to α_1 we see that a root basis can be found as follows.

Proposition 5.2. *We have a root basis for Q^\perp given by*

$$\alpha_1 = 2m\mathcal{H}_x - \sum_{i=1}^{4m} \mathcal{E}_i, \quad \alpha_2 = 2m\mathcal{H}_y - \sum_{i=1}^{4m} \tilde{\mathcal{E}}_i. \quad (5.19)$$

In Figure 10 we give the intersection diagram of the irreducible components of the anticanonical divisor as well as the diagram corresponding to the generalised Cartan matrix formed by the numbers $c_{i,j} = 2 \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$, which here is

$$\begin{pmatrix} 2 & -2m \\ -2m & 2 \end{pmatrix}. \quad (5.20)$$

In particular in the $m = 1$ case X_n is a Sakai surface of type $D_7^{(1)}$, and the the root basis in Proposition 5.2 is a basis of simple roots for an affine root system of type $A_1^{(1)}$, where the $|\alpha|^2=4$ non-standard length of roots does not affect the fact that the matrix with entries $c_{i,j} = 2 \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$ becomes the usual generalised Cartan matrix of type $A_1^{(1)}$.

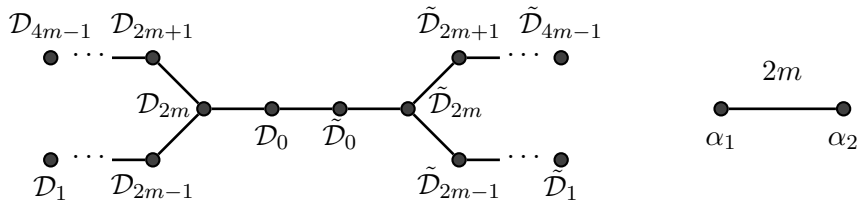


FIGURE 10. Intersection diagrams of anticanonical divisor components and root basis for Class I example

The evolution of the root basis under Φ is given by

$$\Phi : \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 2m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad (5.21)$$

and we see the factor $(t^2 - 2mt + 1)$ from the characteristic polynomial of Φ on $\text{Pic}(\mathcal{X})$ as the characteristic polynomial of its restriction to Q^\perp , and we can deduce that the dynamical degree of the mapping is $\lambda = m + \sqrt{m^2 - 1}$. To see this on the level of the confinement condition on

b we use the expressions for the root basis as differences of classes of exceptional curves of the first kind and find the root variables for the root basis in Proposition 5.2 to be

$$\chi(\alpha_1) = b, \quad \chi(\alpha_2) = -b. \quad (5.22)$$

Therefore the confinement condition $\bar{b} = 2mb - b$ corresponds to evolution of the root variables according to the same matrix as Φ on the root basis as in (5.21), i.e.

$$\bar{\chi}(\bar{\alpha}_1) = b = -\chi(\alpha_2), \quad \bar{\chi}(\bar{\alpha}_2) = -\bar{b} = b - 2mb = \chi(\alpha_1) + 2m\chi(\alpha_2), \quad (5.23)$$

and we see how the parameter evolution reflects the dynamical degree.

6. A FAMILY OF CLASS V-TYPE MAPPINGS WITH SHORT PATTERNS

The last example we will study in detail is one to which other integrability tests based on singularity analysis cannot be applied directly. Consider the autonomous equation

$$\left(\frac{\bar{x} + x - 2z}{\bar{x} + x} \right) \left(\frac{x + \bar{x} - 2z}{x + \bar{x}} \right) = \frac{((x-z)^2 - a_1^2)((x-z)^2 - a_2^2)}{(x^2 - b_1^2)(x^2 - b_2^2)}, \quad (6.1)$$

where the parameters z, a_i, b_j are constants. This mapping has the singularity confinement property, with singularity patterns

$$\{z \pm a_i, z \mp a_i\}, \quad \{\pm b_j, \mp b_j\}, \quad (6.2)$$

for $i, j = 1, 2$. Because of the absence of singularity patterns of length ≥ 3 , neither Halburd's method [17] nor its 'express' version [37] can be used to compute the dynamical degree of this mapping - see [37, Section 5] for an explanation of why this is the case. However full deautonomisation is still effective, as we will demonstrate.

6.1. Full deautonomisation by singularity confinement. To illustrate full deautonomisation here we will consider the family of non-autonomous mappings

$$\left(\frac{\bar{x} + x - 2z}{\bar{x} + x} \right) \left(\frac{x + \bar{x} - 2z}{x + \bar{x}} \right) = \frac{\prod_{i=1}^m (x - z - a_i)(x - z - d_i)}{\prod_{j=1}^m (x^2 - b_j^2)}, \quad (6.3)$$

for some integer $m \geq 1$, where z, a_i and d_i are allowed to evolve with n but b_j are kept constant:

$$\bar{b}_j = b_j. \quad (6.4)$$

The family of mappings (6.3) contains a sufficient deautonomisation of the autonomous equation (6.1) when $m = 2$. We will show that full deautonomisation can be used to determine the dynamical degrees of the whole family of non-autonomous mappings (6.3), which in particular recovers that of the autonomous case (6.1).

With the parameters b_j constant we automatically have the singularity patterns $\{\pm b_j, \mp b_j\}$, but in order for the remaining singularities to be confined similarly to the autonomous case we must require the confinement conditions

$$\bar{z} + \bar{a}_i = z - d_i, \quad \bar{z} + \bar{d}_i = z - a_i, \quad (6.5)$$

under which the mapping admits the singularity patterns $\{z + a_i, \bar{z} + \bar{d}_i\}$ and $\{z + d_i, \bar{z} + \bar{a}_i\}$. To completely preserve the singularity structure and obtain a deautonomisation of the mapping in the sense of Definition 2.18 we must also ensure that $x = \infty$ is not a singularity, just as in the case of the autonomous mapping (6.1). This requires, at each n the constraint

$$2z + 2\bar{z} - \sum_{i=1}^m (z + a_i) - \sum_{i=1}^m (z + d_i) = 0. \quad (6.6)$$

Similarly to in Section 4 we combine this condition and its up-shift to find

$$\bar{z} - 2(m-1)z + z = 0, \quad (6.7)$$

from which we read off the characteristic polynomial $t^2 - 2(m-1)t + 1$ with largest root

$$\lambda = m - 1 + \sqrt{m(m-2)}, \quad (6.8)$$

which is equal to the dynamical degree of the mapping when $m \geq 2$. When $m = 1$ the mapping is periodic, with period 4, and therefore has bounded degree growth.

6.2. Space of initial conditions for non-autonomous mapping. The non-autonomous equation (6.3) defines the mapping

$$\begin{aligned} \varphi_n : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \\ (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= -x - \frac{2z(x+y-2z)g(x)}{(x+y)(f(x)-g(x))+2zg(x)}, \quad \bar{y} = x, \end{aligned} \quad (6.9)$$

where $f(x) = \prod_{i=1}^m (x-z-a_i)(x-z-d_i)$, $g(x) = \prod_{j=1}^m (x^2-b_j^2)$,

subject to the confinement conditions derived previously:

$$\bar{z} + \bar{a}_i = z - d_i, \quad \bar{z} + \bar{d}_i = z - a_i, \quad 2z + 2\bar{z} - \sum_{i=1}^m (z+a_i) - \sum_{i=1}^m (z+d_i) = 0. \quad (6.10)$$

The points to be blown up to construct the space of initial conditions are given by

$$\mathbf{a}_i : (x, y) = (z+a_i, z-a_i), \quad \mathbf{e}_i : (x, y) = (z+d_i, z-d_i) \quad \mathbf{b}_j^\pm : (x, y) = (\pm b_j, \mp b_j), \quad (6.11)$$

and we denote the exceptional divisors by $A_i = \pi^{-1}(\mathbf{a}_i)$, $E_i = \pi^{-1}(\mathbf{e}_i)$, and $B_i^\pm = \pi^{-1}(\mathbf{b}_i^\pm)$. The surface X_n has an effective anticanonical divisor $D = D_1 + D_2$, where D_1 and D_2 are the two rational curves given in coordinates by

$$D_1 : x + y = 0, \quad D_2 : x + y - z = 0, \quad (6.12)$$

which intersect at a single point $(x, y) = (\infty, \infty)$ with multiplicity two. These correspond to the elements of $\text{Pic}(\mathcal{X})$

$$D_1 = \mathcal{H}_x + \mathcal{H}_y - \sum_{i=1}^m \mathcal{B}_i^+ - \sum_{i=1}^m \mathcal{B}_i^-, \quad D_2 = \mathcal{H}_x + \mathcal{H}_y - \sum_{i=1}^m \mathcal{A}_i - \sum_{i=1}^m \mathcal{E}_i, \quad (6.13)$$

where $\mathcal{A}_i = \iota_n([A_i])$, $\mathcal{E}_i = \iota_n([E_i])$, and $\mathcal{B}_i^\pm = \iota_n([B_i^\pm])$; we give a schematic representation of X_n in Figure 11.

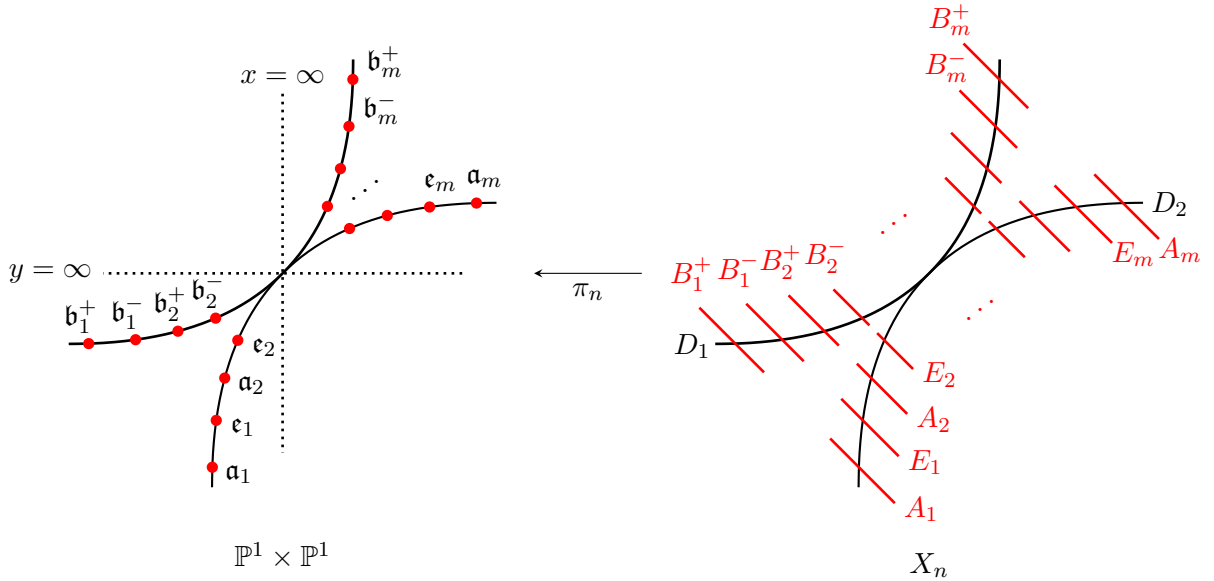


FIGURE 11. Space of initial conditions for Class V example (deautonomised).

Proposition 6.1. *With the confinement conditions (6.10), the mapping φ_n in (6.9) becomes an isomorphism $\tilde{\varphi}_n = \pi_{n+1}^{-1} \circ \varphi_n \circ \pi_n : X_n \rightarrow X_{n+1}$, and its pullback induces the following lattice automorphism $\Phi = \iota_n^{-1} \circ \varphi_n^* \circ \iota_{n+1}$ of $\text{Pic}(\mathcal{X})$:*

$$\Phi : \begin{cases} \mathcal{H}_x \mapsto 2m\mathcal{H}_x + \mathcal{H}_y - \sum_{i=1}^m (\mathcal{A}_i + \mathcal{E}_i) - \sum_{j=1}^m (\mathcal{B}_j^+ + \mathcal{B}_j^-), \\ \mathcal{H}_y \mapsto \mathcal{H}_x, \quad \mathcal{A}_i \mapsto \mathcal{H}_x - \mathcal{E}_i, \quad \mathcal{E}_i \mapsto \mathcal{H}_x - \mathcal{A}_i, \quad \mathcal{B}_i^\pm \mapsto \mathcal{H}_x - \mathcal{B}_i^\mp. \end{cases} \quad (6.14)$$

The characteristic polynomial of Φ as written in Proposition 6.1 can be verified to have a factor of $t^2 - 2(m-1)t + 1$, meaning that the dynamical degree of the mapping is equal to 1 if $m = 1$ or $m = 2$, and otherwise is equal to the largest root $\lambda > 1$ of this polynomial as in (6.8). We remark this family of mappings contains all three types from the classification in Proposition 2.14. When $m = 2$ the autonomous mapping preserves an elliptic fibration, with its deautonomisation being a discrete Painlevé equation of (additive) surface type $A_1^{(1)*}$ [39], and when $m = 1$ the degree growth of the equation is bounded.

6.3. Root basis and period map. The non-autonomous mapping preserves the n -dependent two-form ω_n on $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\omega_n = kz \frac{dx \wedge dy}{(x+y)(x+y-z)}, \quad (6.15)$$

where $k \in \mathbb{C}^*$, since under the mapping φ_n given by (6.9) we have

$$kz \frac{d\bar{x} \wedge d\bar{y}}{(\bar{x} + \bar{y})(\bar{x} + \bar{y} - z)} = kz \frac{dx \wedge dy}{(x+y)(x+y-z)}. \quad (6.16)$$

The anticanonical divisor in this case is then

$$-\text{div } \tilde{\omega}_n = D = D_1 + D_2, \quad (6.17)$$

and the fact that D_1 and D_2 intersect at a single point with multiplicity two means that $H_1(D_{\text{red}}; \mathbb{Z}) = 0$, which is reflected in the parametrisation of the coefficients by root variables being additive.

Proposition 6.2. *We have a root basis for Q^\perp given by*

$$Q^\perp = \sum_{i=0}^{4m-1} \mathbb{Z}\alpha_i, \quad (6.18)$$

where

$$\begin{aligned} \alpha_0 &= \mathcal{H}_x - \mathcal{H}_y, \\ \alpha_1 &= \mathcal{E}_1 - \mathcal{E}_2, \quad \alpha_2 = \mathcal{E}_2 - \mathcal{E}_3, \quad \dots \quad \alpha_{m-1} = \mathcal{E}_{m-1} - \mathcal{E}_m, \\ \alpha_m &= \mathcal{E}_m - \mathcal{A}_1, \\ \alpha_{m+1} &= \mathcal{A}_1 - \mathcal{A}_2, \quad \alpha_{m+2} = \mathcal{A}_2 - \mathcal{A}_3, \quad \dots \quad \alpha_{2m-1} = \mathcal{A}_{m-1} - \mathcal{A}_m, \\ \alpha_{2m} &= \mathcal{H}_y - \mathcal{A}_m - \mathcal{B}_1^+, \\ \alpha_{2m+1} &= \mathcal{B}_1^+ - \mathcal{B}_2^+, \quad \alpha_{2m+2} = \mathcal{B}_2^+ - \mathcal{B}_3^+, \quad \dots \quad \alpha_{3m-1} = \mathcal{B}_{m-1}^+ - \mathcal{B}_m^+, \\ \alpha_{3m} &= \mathcal{B}_m^+ - \mathcal{B}_1^-, \\ \alpha_{3m+1} &= \mathcal{B}_1^- - \mathcal{B}_2^-, \quad \alpha_{3m+2} = \mathcal{B}_2^- - \mathcal{B}_3^-, \quad \dots \quad \alpha_{4m-1} = \mathcal{B}_{m-1}^- - \mathcal{B}_m^-. \end{aligned} \quad (6.19)$$

We give the intersection diagram of this root basis in Figure 12, which when $m = 2$ reduces to the Dynkin diagram associated with the symmetry type $E_7^{(1)}$ of the corresponding discrete Painlevé equation.

Remark 6.3. The diagram in Figure 12 also appeared in the work of Looijenga [23] through the root basis for a rational surface with an anticanonical cycle of length two, i.e. two rational curves intersecting at two distinct points each with multiplicity one. In a way similar to how the

examples in Sections 3 and 4 share the same intersection diagram of their root bases but have additive and multiplicative natures of their root variables due to differing ranks of $H_1(D_{\text{red}}; \mathbb{Z})$, the example in the present section has a multiplicative counterpart. Explicitly the family of non-autonomous mappings is given by

$$\left(\frac{\bar{x}x - z^2}{\bar{x}x - 1} \right) \left(\frac{xx - z^2}{xx - 1} \right) = \frac{\prod_{i=1}^m (xz^{-1} - a_i)(xz^{-1} - a_i^{-1})}{\prod_{j=1}^m (x - b_j)(x - b_j^{-1})}, \quad (6.20)$$

and it has the form of Class VI mappings in the QRT-type classification. The confinement conditions, space of initial conditions and period map for this multiplicative counterpart of the present example can be worked out along the same lines as in the other sections.

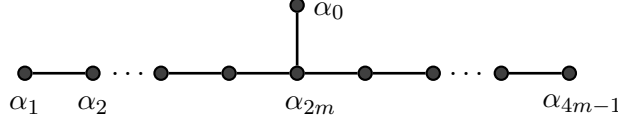


FIGURE 12. Intersection diagram of root basis for Q^\perp in Class V example

The evolution of the root basis under Φ as in Proposition 6.1 can be computed to be

$$\Phi : \begin{pmatrix} \alpha_0 \\ \alpha_{2m} \\ \boldsymbol{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} 2m-1 & 2m & * \\ -1 & -1 & * \\ 0 & 0 & U \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_{2m} \\ \boldsymbol{\alpha} \end{pmatrix} \quad (6.21)$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{2m-1}, \alpha_{2m+1}, \dots, \alpha_{4m-1})^T$ and U is a known square matrix of size $4m-2$ satisfying $U^2 = I_{4m-2}$. We therefore see the factor $t^2 - 2(m-1)t + 1$ in the characteristic polynomial of Φ as expected and confirm the dynamical degree of the original mapping and its deautonomised version as read off from the confinement conditions.

To compute the period map we note that the residues of $\tilde{\omega}_n$ along the components of the anticanonical divisor are given in coordinates by

$$\begin{aligned} \text{Res}_{D_1} \tilde{\omega}_n &= \text{Res}_{s_1=0} k\tilde{z} \frac{dx \wedge ds_1}{s_1(s_1 - \tilde{z})} = kdx, \\ \text{Res}_{D_2} \tilde{\omega}_n &= \text{Res}_{s_2=0} k\tilde{z} \frac{dx \wedge ds_2}{(s_2 + \tilde{z})s_2} = -kdx, \end{aligned} \quad (6.22)$$

where $s_1 = x + y$ and $s_2 = x + y - \tilde{z}$. Therefore, with k chosen for convenience to be 1, the root variables of X_n for the root basis in Proposition 6.2 are related to the parameters in the mapping (6.9) as follows, and the evolution of these root variables with the confinement conditions can be checked directly to match with the action of Φ on the root basis.

$$\begin{aligned} \chi(\alpha_0) &= 2\tilde{z}, \\ \chi(\alpha_1) &= -d_1 + d_2, \quad \chi(\alpha_2) = -d_2 + d_3, \quad \dots \quad \chi(\alpha_{m-1}) = -d_{m-1} + d_m, \\ \chi(\alpha_m) &= -d_m + a_1, \\ \chi(\alpha_{m+1}) &= -a_1 + a_2, \quad \chi(\alpha_{m+2}) = -a_2 + a_3, \quad \dots \quad \chi(\alpha_{2m-1}) = -a_{m-1} + a_m, \\ \chi(\alpha_{2m}) &= -\tilde{z} + a_m - b_1, \\ \chi(\alpha_{2m+1}) &= b_1 - b_2, \quad \chi(\alpha_{2m+2}) = b_2 - b_3, \quad \dots \quad \chi(\alpha_{3m-1}) = b_{m-1} - b_m, \\ \chi(\alpha_{3m}) &= b_m + b_1, \\ \chi(\alpha_{3m+1}) &= -b_1 + b_2, \quad \chi(\alpha_{3m+2}) = -b_2 + b_3, \quad \dots \quad \chi(\alpha_{4m-1}) = -b_{m-1} + b_m. \end{aligned} \quad (6.23)$$

The parameter z that satisfies the recurrence whose characteristic polynomial encodes the dynamical degree corresponds to the root variable $\chi(\alpha_0)$. Computing the action of Φ on α_0 we find

$$(\Phi^2 - 2(m-1)\Phi + 1)\alpha_0 = 0, \quad (6.24)$$

which through the period mapping gives the confinement condition on z , and we see again how the dynamical degree appears in the confinement conditions.

7. CONCLUSION

In summary, in this paper we have provided an explanation (as summarised in Theorem 1.1) of the correspondence between confinement conditions for a vast class of birational mappings of the plane and their dynamical degree. As we explained, this correspondence underlies the efficacy of the method of full deautonomisation by singularity confinement, as an integrability test for such mappings. While the problem of effectiveness of the anticanonical divisor class means that the method is not guaranteed to work for birational mappings in general, examples with a space of initial conditions but no effective anticanonical divisor are rare, and Theorem 2.23 can be regarded as an affirmative answer to a non-autonomous counterpart to the conjecture initially posed by Gizatullin, in the case of mappings preserving a rational two-form.

Another interpretation of the results in this paper is as a generalisation of Takenawa's attempt [42] towards a version of the Sakai theory of discrete Painlevé equations for non-integrable mappings, involving rational surfaces associated via Q^\perp to root systems of indefinite type. In particular non-integrable mappings from the class to which Theorem 2.23 applies provide a suite of new examples with indefinite type root bases, extending some which appeared in Looijenga's study of rational surfaces with anticanonical cycle [23]. While we have not confirmed this, it is natural to expect that the symmetry groups of the families of surfaces constructed in this paper can be described in terms of the Weyl group associated to Q^\perp . Further, while a classification of the kinds of effective anticanonical divisors of surfaces forming the space of initial conditions for non-integrable mappings cannot be given by a finite list as in the discrete Painlevé case, we have given a local classification of the intersection configuration of irreducible components in Appendix A.

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Data availability statement. There is no associated data.

APPENDIX A. CLASSIFICATION OF ANTICANONICAL DIVISORS FOR CONFINING NON-INTEGRABLE MAPPINGS

In this Appendix we provide a classification of anticanonical divisors (Theorem A.17) for spaces of initial conditions that correspond to non-integrable mappings, i.e. mappings with dynamical degree greater than 1. As mentioned in [24], the theory of spaces of initial conditions for mappings of the plane allows for two description that are, however, essentially the same. While the description introduced in the main part of this paper is more intuitive, the other one, by a rational surface and a Cremona isometry, is more compatible with the general theory needed in such a classification. Therefore, in this appendix, we shall use the description by Cremona isometries. The following are our assumptions and notations that we shall use in this appendix:

- X is a rational surface.
- σ is a Cremona isometry. That is, σ is a \mathbb{Z} -linear transformation of $\text{Pic}(X)$ that satisfies
 - $\sigma\mathcal{K}_X = \mathcal{K}_X$,
 - $(\sigma F_1) \cdot (\sigma F_2) = F_1 \cdot F_2$ for any $F_1, F_2 \in \text{Pic}(X)$,
 - if $F \in \text{Pic}(X)$ is effective, then so is σF .
- $\lambda > 1$: the maximum eigenvalue of σ , which is irrational.
- $v \in \text{Pic}_{\mathbb{R}}(X)$: the dominant eigenvector of σ , which we may assume is nef.

- X is minimal as a space of initial conditions.
- The anticanonical class $-\mathcal{K}_X$ is effective. That is, we have

$$D = \sum_{j=1}^N m_j D_j \in |-\mathcal{K}_X|,$$

where $D_j \subset X$ is irreducible, $m_j \in \mathbb{Z}_{>0}$ and $D_i \neq D_j$ ($i \neq j$).

- $D_{\text{red}} = D_1 \cup \cdots \cup D_N$.

Lemma A.1. D is connected.

Proof. The proof of [39, Proposition 6] is still valid in our case. \square

Lemma A.2. $g_a(D_j) \leq 1$ for any j . Moreover, if D has a component D_j such that $g_a(D_j) \geq 1$, then $D = D_j$. In particular, if D is not irreducible, then each component of D is a smooth rational curve.

Proof. Take a geometric basis $\mathcal{H}, \mathcal{E}_1, \dots, \mathcal{E}_K \in \text{Pic}(X)$ and let $\pi: X \rightarrow \mathbb{P}^2$ be the corresponding blow-down. Using this basis, each $[D_j]$ is expressed as

$$[D_j] = c_{j0}\mathcal{H} + c_{j1}\mathcal{E}_1 + \cdots + c_{jK}\mathcal{E}_K.$$

Here, $c_{j0} \geq 0$ since D_j is effective. Moreover, since $-\mathcal{K}_X = 3\mathcal{H} - \mathcal{E}_1 - \cdots - \mathcal{E}_K$, we have

$$3 = \sum_j m_j c_{j0},$$

which implies $c_{j0} \leq 3$.

From here on, we fix j . If $c_{j0} = 0$, then D_j is the strict transform of some exceptional curve of π . Therefore, D_j is rational and $g_a(D_j) = 0$. Let us consider the remaining case, i.e. $c_{j0} = 1, 2, 3$. In this case, $C := \pi(D_j) \subset \mathbb{P}^2$ is a curve in \mathbb{P}^2 and its class is $[C] = \mathcal{O}_{\mathbb{P}^2}(c_{j0})$. Therefore, by the genus formula on \mathbb{P}^2 , we have $g_a(C) = \frac{1}{2}(c_{j0} - 2)(c_{j0} - 1) \leq 1$. Since taking the strict transform of a curve with respect to a birational morphism does not increase its arithmetic genus, we conclude that $g_a(D_j) \leq g_a(C) \leq 1$.

If $g_a(D_j) = 1$, then $c_{j0} = 3$ and $m_j = 1$. In this case, by the genus formula, we have

$$D_j \cdot \sum_{i \neq j} m_i D_i = D_j \cdot (-\mathcal{K}_X - [D_j]) = 0,$$

which implies $D = D_j$ since D is connected. \square

Lemma A.3. If D is irreducible, then $g_a(D) = 1$.

Proof. Since $-\mathcal{K}_X - [D] = 0$ in this case, we have $g_a(D) = 1$ by the genus formula. \square

Lemma A.4. If D is not irreducible and $m_j = 1$, then $D_j \cdot \sum_{i \neq j} m_i D_i = 2$.

Proof. In this case, $g_a(D_j) = 0$ since D is not irreducible. By the genus formula, we have

$$2 = D_j \cdot (-\mathcal{K}_X - [D_j]) = D_j \cdot \sum_{i \neq j} m_i D_i.$$

\square

Lemma A.5. D_j is orthogonal to v for all j . Moreover, there exists $\ell \in \mathbb{Z}_{>0}$ such that $\sigma^\ell[D_j] = [D_j]$ for all j .

Proof. Since v is nef and D_j is effective, we have

$$v \cdot [D_j] \geq 0.$$

Multiplying both sides by m_j and taking the sum for j , we have

$$v \cdot (-\mathcal{K}_X) \geq 0.$$

However, Since $v \cdot (-\mathcal{K}_X) = 0$, $v \cdot D_j$ must be 0 for all j . Therefore, $D_j \in v^\perp \cap \text{Pic}(X)$. Since σ preserves the intersection form on the lattice $v^\perp \cap \text{Pic}(X)$, which is negative definite, there exists $\ell > 0$ such that $\left(\sigma|_{v^\perp \cap \text{Pic}(X)}\right)^\ell = \text{id}_{v^\perp \cap \text{Pic}(X)}$. \square

Lemma A.6. $D_j^2 < 0$.

Proof. Clear from $D_j \in v^\perp \cap \text{Pic}(X)$. \square

Lemma A.7. D_j is not an exceptional curve of the first kind. In particular, $D_j^2 \leq -2$.

Proof. Since X is minimal, there does not exist an exceptional curve of the first kind C such that $C \cdot v = 0$ [24, Proposition 4.17]. Since D_j is orthogonal to v , D_j is not an exceptional curve of the first kind. \square

Lemma A.8. For any nontrivial linear combination $F = \sum_j c_j [D_j]$ with $c_j \in \mathbb{R}$, we have $F^2 < 0$.

Proof. The intersection form is negative definite on the lattice $v^\perp \cap \text{Pic}(X)$. Therefore, the form is negative definite on the real vector space $V := (v^\perp \cap \text{Pic}(X)) \otimes \mathbb{R}$. Since $F \in V \setminus \{0\}$, we have $F^2 < 0$. \square

Lemma A.9. For any $a_j \in \mathbb{Z}_{\geq 0}$, we have

$$h^0\left(\sum_j a_j D_j\right) = 1,$$

which implies that D is the only effective divisor that belongs to the class $-\mathcal{K}_X$. In particular, σ acts on the set $\{D_1, \dots, D_N\}$ as a permutation.

Proof. By Lemma A.8, the matrix $(D_i \cdot D_j)_{i,j}$ is negative definite. Hence, we have $h^0\left(\sum_j a_j D_j\right) = 1$ [24, Proposition A.9]. \square

Lemma A.10. Let $E \subset X$ be an exceptional curve of the first kind. Then, there exists i such that $m_j D_j \cdot E = \delta_{i,j}$. In particular, D has a component of multiplicity 1.

Proof. By the genus formula, we have $E \cdot (-\mathcal{K}_X) = 1$. Since

$$1 = \sum_i m_i D_i \cdot E$$

and $m_i D_i \cdot E \geq 0$, there exists i such that $m_j D_j \cdot E = \delta_{i,j}$. \square

Lemma A.11. Let $m_j = 1$ and $D_j \cdot D_k > 0$. Then, only one of the following holds:

- (1) $D_j \cdot D_k = 1$, $m_k = 1$.
- (2) $D_j \cdot D_k = 1$, $m_k = 2$.
- (3) $D_j \cdot D_k = 2$, $m_k = 1$.

Proof. Clear from

$$0 < m_k D_j \cdot D_k \leq D_j \cdot \sum_{i \neq j} m_i D_i = 2.$$

\square

Lemma A.12. If $D_1 \cdot D_2 = 2$ and $m_1 = m_2 = 1$, then $D = D_1 + D_2$. In this case, there are only two possible configurations of $D = D_1 + D_2$:

- (1) D_1 and D_2 intersect at a point with multiplicity 2.
- (2) D_1 and D_2 intersect transversely at two points.

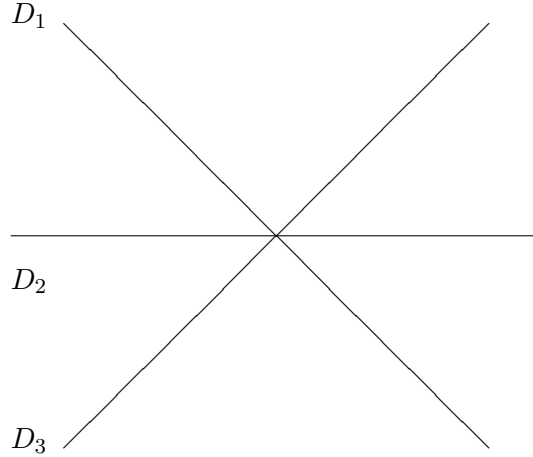
Proof. By Lemma A.11, we have

$$2 = D_1 \cdot D_2 \leq \sum_{i \neq 1} m_i D_1 \cdot D_i = 2,$$

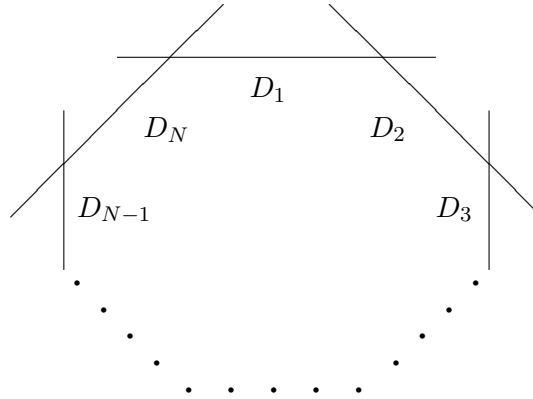
which implies that $D_2 \cdot D_i = 0$ for all $i \neq 1, 2$. Replacing D_2 with D_1 , we have $D_1 \cdot D_i = 0$ for all $i \neq 1, 2$. Therefore, $D_1 + D_2$ is a connected component of D . Since D is connected, D must coincide with $D_1 + D_2$. \square

Lemma A.13. *If $D_1 \cdot D_2 = 1$ and $m_1 = m_2 = 1$, then only the following types of configurations are possible:*

- (1) $D = D_1 + D_2 + D_3$ and the configuration has only one intersection:



- (2) $N \geq 3$ and $D = D_1 + \dots + D_N$ forms a cycle of \mathbb{P}^1 (after reordering the indices):



Proof. By Lemma A.11, we have

$$2 = D_2 \cdot \sum_{i \neq 2} m_i D_i = D_2 \cdot \left(D_1 + \sum_{i \neq 1, 2} m_i D_i \right) = 1 + D_2 \cdot \sum_{i \neq 1, 2} m_i D_i,$$

which implies that D has only one component D_ℓ other than D_1, D_2 such that $D_2 \cdot m_\ell D_\ell = 1$. In particular, $m_\ell = 1$ and $D_2 \cdot D_\ell = 1$. We may assume that $\ell = 3$.

If D_1, D_2, D_3 intersect at one point, then none of D_1, D_2, D_3 intersects $D - D_1 - D_2 - D_3$. Since D is connected, D must be $D_1 + D_2 + D_3$ in this case.

Let us consider the other possibilities, i.e. when $D_1 \cap D_2 \cap D_3 = \emptyset$. In this case, D_2 does not intersect any D_j other than D_1, D_3 . Using Lemma A.11 for $j = 3$, there exists a component D_ℓ other than D_2, D_3 such that $D_3 \cdot D_\ell = 1$ and $m_\ell = 1$. If $\ell = 1$, then $D = D_1 + D_2 + D_3$. If not, we may assume that $\ell = 4$.

Using Lemma A.11 for $j = 4$, there exists a component D_ℓ other than D_3, D_4 such that $D_4 \cdot D_\ell = 1$ and $m_\ell = 1$. Repeating this procedure, one can find a sequence D_1, D_2, \dots, D_N that forms a cycle. \square

Lemma A.14. *Suppose that D has a cycle $D_{j_1}, \dots, D_{j_\ell}$ for $\ell \geq 3$, i.e.*

$$D_{j_i} \cdot D_{j_k} \begin{cases} > 0 & (i = k \pm 1 \pmod{\ell}) \\ < 0 & (i = j) \\ = 0 & (\text{otherwise}) \end{cases}$$

for $i, k = 1, \dots, \ell$. Then, $D = D_{j_1} + \dots + D_{j_\ell}$ and the configuration of D must be one of case (2) in Lemma A.13. Moreover, if $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) \geq 1$ and D is not irreducible, then the configuration of D must be either (2) of Lemma A.12 or one of (2) in Lemma A.13.

Proof. If $m_{j_i} = 1$ for some i , then, by Lemma A.13, the configuration of D must be one of (2) in Lemma A.13. Let us assume that $m_{j_1}, \dots, m_{j_\ell} \geq 2$ and deduce a contradiction.

Take a geometric basis $\mathcal{H}, \mathcal{E}_1, \dots, \mathcal{E}_K \in \text{Pic}(X)$ and the corresponding blow-down $\pi: X \rightarrow \mathbb{P}^2$. Using this basis, each $[D_{j_i}]$ is expressed as

$$[D_{j_i}] = c_{i0}\mathcal{H} + c_{i1} + \mathcal{E}_1 + \dots + c_{iK}\mathcal{E}_K.$$

Since D_{j_i} is effective, c_{i0} must be nonnegative. Since

$$[m_1 D_{j_1} + \dots + m_\ell D_{j_\ell}] \leq [D] = -\mathcal{K}_X = 3\mathcal{H} - \mathcal{E}_1 - \dots - \mathcal{E}_N,$$

$m_1, \dots, m_\ell \geq 2$ and

$$[m_1 D_{j_1} + \dots + m_\ell D_{j_\ell}] = (m_1 c_{10} + \dots + m_\ell c_{\ell 0})\mathcal{H} + (\mathcal{E}\text{-part}),$$

at most one of $c_{10}, \dots, c_{\ell 0}$ is 1 and the others are 0.

Let us consider the case where $c_{10} = 1, c_{20} = \dots = c_{\ell 0} = 0$. Let $H = \pi(D_{j_1})$, which is a line in \mathbb{P}^2 . In this case, $\pi(D_{j_2}), \dots, \pi(D_{j_\ell})$ are points on H and any intersection among $D_{j_1}, \dots, D_{j_\ell}$ is transversal. Therefore, when considered as a graph, $D_{j_1}, \dots, D_{j_\ell} \subset X$ form a tree with D_{j_1} at its root, which contradicts the choice of $D_{j_1}, \dots, D_{j_\ell}$ since if a tree graph consists of copies of \mathbb{P}^1 and every intersection has multiplicity 1, then it does not have a cycle.

If $c_{10} = \dots = c_{\ell 0} = 0$, then $\pi(D_{j_1}), \dots, \pi(D_{j_\ell})$ are the same point of \mathbb{P}^2 . Therefore, $D_{j_1}, \dots, D_{j_\ell}$ form a tree and any intersection has multiplicity 1, which leads to a contradiction in the same way as above. \square

Lemma A.15. *If $D_1 \cdot D_2 \geq 2$, then $D_1 \cdot D_2 = 2$ and $m_1 = m_2 = 1$. In particular, the configuration of D is one of those in Lemma A.12.*

Proof. Take a geometric basis $\mathcal{H}, \mathcal{E}_1, \dots, \mathcal{E}_K \in \text{Pic}(X)$ and the corresponding blow-down $\pi: X \rightarrow \mathbb{P}^2$. Using this basis, $[D_1]$ and $[D_2]$ are expressed as

$$[D_i] = c_{i0}\mathcal{H} + c_{i1} + \mathcal{E}_1 + \dots + c_{iK}\mathcal{E}_K$$

for $i = 1, 2$. We may assume that $c_{10} \geq c_{20}$. Since $c_{i0} \geq 0$ and $m_1 c_{10} + m_2 c_{20} \leq 3$, we have $(c_{10}, c_{20}) = (0, 0), (1, 0), (1, 1), (2, 1), (2, 0)$ or $(3, 0)$.

We show that only the cases $(c_{10}, c_{20}) = (3, 0)$ and $(2, 1)$ are possible. If $c_{10} = c_{20} = 0$, then D_1 and D_2 are both the strict transforms of some exceptional curves of π . Therefore, $D_1 \cdot D_2 = 0$ or 1, which is a contradiction. If $c_{10} = c_{20} = 1$, then $\pi(D_1)$ and $\pi(D_2)$ are both lines in \mathbb{P}^2 . Therefore, $\pi(D_1) \cdot \pi(D_2) = 1$, which contradicts the fact that $\pi(D_1) \cdot \pi(D_2) \geq D_1 \cdot D_2 = 2$. If $c_{10} = 1, 2$ and $c_{20} = 0$, then $\pi(D_1)$ is a smooth curve in \mathbb{P}^2 and D_2 is the strict transform of some exceptional curve of π . Therefore, $D_1 \cdot D_2 = 0$ or 1, which is a contradiction. Therefore, we have $(c_{10}, c_{20}) = (3, 0), (2, 1)$. In particular, m_1 must be 1.

Next, we show that $D_1 \cdot D_2 = 2$. If $(c_{10}, c_{20}) = (2, 1)$, then $D_1 \cdot D_2 \leq \pi(D_1) \cdot \pi(D_2) = 2$. Since $D_1 \cdot D_2 \geq 2$ by the assumption, we have $D_1 \cdot D_2 = 2$. Let us consider the case $(c_{10}, c_{20}) = (3, 0)$. Let $C = \pi(D_1) \subset \mathbb{P}^2$ and let $P = \pi(D_2) \in \mathbb{P}^2$. Since $c_{10} = 3$, the arithmetic genus of C is 1. Then, C is either a smooth elliptic curve, a rational curve with a node or a rational curve with a cusp. In particular, for any $Q \in C$, the multiplicity of C at Q is at most 2. On the other hand, the multiplicity of C at P is at least $D_1 \cdot D_2 \geq 2$. Therefore, C has multiplicity 2 at P and $D_1 \cdot D_2 = 2$.

Since $D_1 \cdot D_2 = 2$ and $m_1 = 1$, it follows from Lemma A.11 that $m_k = 1$. Hence, the configuration of D_1 and D_2 must be one of those in Lemma A.12. \square

Lemma A.16. *If D_1, D_2, D_3 intersect at one point, then $m_1 = m_2 = m_3 = 1$ and $D = D_1 + D_2 + D_3$. In particular, the configuration of D is (1) in Lemma A.13.*

Proof. By Lemma A.15, we have

$$D_1 \cdot D_2 = D_1 \cdot D_3 = D_2 \cdot D_3 = 1.$$

Take a geometric basis $\mathcal{H}, \mathcal{E}_1, \dots, \mathcal{E}_K \in \text{Pic}(X)$ and the corresponding blow-down $\pi: X \rightarrow \mathbb{P}^2$. Using this basis, $[D_1], [D_2], [D_3]$ are expressed as

$$[D_i] = c_{i0}\mathcal{H} + c_{i1} + \mathcal{E}_1 + \dots + c_{iK}\mathcal{E}_K$$

for $i = 1, 2, 3$. We may assume that $c_{10} \geq c_{20} \geq c_{30}$.

First, we show that m_1 or m_2 is 1. Since $c_{i0} \geq 0$ and $m_1c_{10} + m_2c_{20} + m_3c_{30} \leq 3$, we have

$$(c_{10}, c_{20}, c_{30}) = (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0), (2, 1, 0) \text{ or } (3, 0, 0).$$

If $c_{10} = 0$, then D_1, D_2, D_3 are all the strict transforms of some exceptional curves of π , which contradicts the assumption that D_1, D_2, D_3 intersect at a point. If $(c_{10}, c_{20}, c_{30}) = (1, 0, 0)$, then $H := \pi(D_1) \subset \mathbb{P}^2$ is a line and $P := \pi(D_2) = \pi(D_3) \in H \subset \mathbb{P}^2$ is a point. In this case, since H has multiplicity 1 at P , it is impossible for D_1, D_2, D_3 to intersect at one point, which is again a contradiction. Therefore, we have $c_1 + c_2 \geq 2$, which implies that either m_1 or m_2 is 1.

We may assume that $m_1 = 1$. Using Lemma A.4, we have

$$\begin{aligned} 2 &= \sum_{i \neq 1} m_i D_i \cdot D_1 \\ &= m_2 D_2 \cdot D_1 + m_3 D_3 \cdot D_1 + \sum_{i \neq 1, 2, 3} m_i D_i \cdot D_1 \\ &= m_2 + m_3 + \sum_{i \neq 1, 2, 3} m_i D_i \cdot D_1, \end{aligned}$$

which implies that $m_2 = m_3 = 1$. Therefore, by Lemma A.13, we have $D = D_1 + D_2 + D_3$. \square

Theorem A.17.

$$\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) \leq 2.$$

Moreover:

- (1) *If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 2$, then D is irreducible and is a smooth elliptic curve.*
- (2) *If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 1$ and D is irreducible, then D is a rational curve with one nodal singularity.*
- (3) *If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 1$ and D is not irreducible, then each component of D is a smooth rational curve. In this case, there are two possible configurations of D :*
 - (a) *$D = D_1 + D_2$ has two intersection points and their multiplicities are both 1.*
 - (b) *$D = D_1 + \dots + D_N$ forms a cycle as in (2) of Lemma A.13.*
- (4) *If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 0$ and D is irreducible, then D is a rational curve with one cusp.*
- (5) *If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 0$ and D is not irreducible, then each component of D is a smooth rational curve. In this case, there are three possible configurations of D :*
 - (a) *$D = D_1 + D_2$ has only one intersection point and its multiplicity is 2.*
 - (b) *$D = D_1 + D_2 + D_3$ has only one intersection point as in (1) of Lemma A.13.*

- (c) All the intersections among D_1, \dots, D_N are transversal and, if considered as a graph, D_1, \dots, D_N does not have a cycle.
 In this case, if $m_j = 1$, then there exists k such that $m_k = 2$ and $D_j \cdot D_i =$
- $$\begin{cases} 1 & (i = k) \\ 0 & (i \neq j, k). \end{cases}$$

Proof. First, we consider the case where D is irreducible. In this case, by Lemma A.3, the arithmetic genus of D is 1. Therefore, D is either a smooth elliptic curve, a rational curve with one nodal singularity, or a rational curve with one cusp. Which case D belongs to is determined by $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) \in \{0, 1, 2\}$.

From here on, we assume that D is not irreducible. In this case, by Lemma A.2, all the components are smooth rational curves. Therefore, by Lemma A.14, the rank of $H_1(D_{\text{red}}; \mathbb{Z})$ is 0 or 1.

Let us consider the case where there exist $k \neq j$ such that $D_j \cdot D_k \geq 2$. In this case, by Lemma A.15, the configuration of D is one in Lemma A.12. Which case D belongs to is determined by $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) \in \{0, 1\}$.

If there exist j, k, ℓ such that $D_j \cap D_k \cap D_\ell \neq \emptyset$, then, by Lemma A.16, the configuration of D is (1) of Lemma A.13.

Let us consider the remaining cases, i.e. all the intersections among D_1, \dots, D_N are transversal. If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 1$, then D has a cycle $D_{j_1}, \dots, D_{j_\ell}$ for $\ell \geq 3$. Therefore, by Lemma A.14, the configuration of D is one of (2) in Lemma A.13. If $\text{rank } H_1(D_{\text{red}}; \mathbb{Z}) = 0$, then D_1, \dots, D_N do not have a cycle as a graph. The last statement follows from Lemmas A.11 and A.13. \square

APPENDIX B. SUFFICIENT DEAUTONOMISATIONS FOR QRT-TYPE MAPPINGS

In what follows we will specify, for three-point mappings from each of the classes I-VI in the classification of QRT-type mappings as it appears in [16, Appendix], how to deautonomise a mapping with the singularity confinement property in a way that is sufficiently general for the dynamical degree to be detected by the confinement conditions. Note that the deautonomisations given here are general enough to be sufficient for all possible mappings from the relevant class with the singularity confinement property, but can be specialised further in special cases as we have seen in the examples above.

B.1. Class I. Consider the three-point equation

$$\bar{x} + x = f(x), \tag{B.1}$$

where f is rational. This falls into the remit of Theorem 2.23, since when recast as a mapping

$$\begin{aligned} (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= f(x) - y, \quad \bar{y} = x, \end{aligned} \tag{B.2}$$

it preserves the rational two-form

$$d\bar{x} \wedge d\bar{y} = dx \wedge dy. \tag{B.3}$$

We consider f to be of the form

$$f(x) = \frac{1}{x^k} \frac{P_M(x)}{Q_N(x)}, \tag{B.4}$$

where $k \in \mathbb{Z}_{\geq 0}$ and P_M and Q_N are polynomials of degree M and N respectively with no factors of x . A sufficient deautonomisation is given by taking

$$f(x) = \frac{a_0 + a_1x + \dots + a_Mx^M}{x^k(x - \beta_1)(x - \beta_2) \dots (x - \beta_N)}, \tag{B.5}$$

where all $a_0, \dots, a_M, \beta_1, \dots, \beta_N$ are allowed to evolve with n .

B.2. Class II. Consider the three-point equation

$$\bar{x}\underline{x} = f(x), \quad (\text{B.6})$$

where f is rational. This falls into the remit of Theorem 2.23, since when recast as a mapping

$$\begin{aligned} (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= \frac{f(x)}{y}, \quad \bar{y} = x, \end{aligned} \quad (\text{B.7})$$

it preserves the rational two-form

$$\frac{d\bar{x} \wedge d\bar{y}}{\bar{x}\bar{y}} = \frac{dx \wedge dy}{xy}. \quad (\text{B.8})$$

Let f be of the form

$$f(x) = x^k \frac{P_M(x)}{Q_N(x)}, \quad (\text{B.9})$$

where $k \in \mathbb{Z}$ and P_M and Q_N are polynomials of degree M and N respectively with no factors of x . A sufficient deautonomisation is given by taking

$$f(x) = Ax^k \frac{(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_M)}{(x - \beta_1)(x - \beta_2) \dots (x - \beta_N)}, \quad (\text{B.10})$$

where all $\alpha_1, \dots, \alpha_M, \beta_1, \dots, \beta_N$ and A are allowed to evolve with n .

B.3. Class III. Consider the three-point equation

$$(\bar{x} + x)(x + \underline{x}) = f(x), \quad (\text{B.11})$$

where f is rational. This falls into the remit of Theorem 2.23, since when recast as a mapping

$$\begin{aligned} (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= -x + \frac{f(x)}{x + y}, \quad \bar{y} = x, \end{aligned} \quad (\text{B.12})$$

it preserves the rational two-form

$$\frac{d\bar{x} \wedge d\bar{y}}{\bar{x} + \bar{y}} = \frac{dx \wedge dy}{x + y}. \quad (\text{B.13})$$

Let f be of the form

$$f(x) = \frac{P_M(x)}{Q_N(x)}, \quad (\text{B.14})$$

where P_M and Q_N are polynomials of degree M and N respectively. A sufficient deautonomisation is given by taking

$$f(x) = A \frac{(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_M)}{(x - \beta_1)(x - \beta_2) \dots (x - \beta_N)}, \quad (\text{B.15})$$

where β_1, \dots, β_N are allowed to evolve with n , but $\alpha_1, \dots, \alpha_M$ and A can be taken to be constant.

B.4. Class IV. Consider the three-point equation

$$(\bar{x}x - 1)(x\underline{x} - 1) = f(x), \quad (\text{B.16})$$

where f is rational. This falls into the remit of Theorem 2.23, since when recast as a mapping

$$\begin{aligned} (x, y) &\mapsto (\bar{x}, \bar{y}), \\ \bar{x} &= \frac{1}{x} + \frac{f(x)}{xy - 1}, \quad \bar{y} = x, \end{aligned} \quad (\text{B.17})$$

it preserves the rational two-form

$$\frac{d\bar{x} \wedge d\bar{y}}{\bar{x}\bar{y} - 1} = \frac{dx \wedge dy}{xy - 1}. \quad (\text{B.18})$$

Let f be of the form

$$f(x) = \frac{P_M(x)}{Q_N(x)}, \quad (\text{B.19})$$

where P_M and Q_N are polynomials of degree M and N respectively. A sufficient deautonomisation is given by

$$(\bar{x}x - z)(x\bar{x} - w) = A \frac{(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_M)}{(x - \beta_1)(x - \beta_2) \dots (x - \beta_N)}, \quad (\text{B.20})$$

where z, w, A and $\alpha_1, \dots, \alpha_M$ are allowed to evolve with n , but β_1, \dots, β_N can be taken to be constant.

B.5. Class V. Consider the three-point equation

$$\left(\frac{\bar{x} + x - 1}{\bar{x} + x} \right) \left(\frac{x + \bar{x} - 1}{x + \bar{x}} \right) = f(x), \quad (\text{B.21})$$

where f is rational. This falls into the remit of Theorem 2.23, since when recast as a mapping

$$(x, y) \mapsto (\bar{x}, \bar{y}), \quad (\text{B.22})$$

$$\frac{\bar{x} + x - 1}{\bar{x} + x} = \frac{x + y}{x + y - 1} f(x), \quad \bar{y} = x,$$

it preserves the rational two-form

$$\frac{d\bar{x} \wedge d\bar{y}}{(\bar{x} + \bar{y} - 1)(\bar{x} + \bar{y})} = \frac{dx \wedge dy}{(x + y - 1)(x + y)}. \quad (\text{B.23})$$

Let f be of the form

$$f(x) = \frac{P_M(x)}{Q_N(x)}, \quad (\text{B.24})$$

where P_M and Q_N are polynomials of degree M and N respectively. A sufficient deautonomisation is given by

$$\left(\frac{\bar{x} + x - z}{\bar{x} + x} \right) \left(\frac{x + \bar{x} - w}{x + \bar{x}} \right) = A \frac{(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_M)}{(x - \beta_1)(x - \beta_2) \dots (x - \beta_N)}, \quad (\text{B.25})$$

where z, w, A and $\alpha_1, \dots, \alpha_M$ are allowed to evolve with n , but β_1, \dots, β_N can be taken to be constant.

B.6. Class VI. Consider the three-point equation

$$\left(\frac{\bar{x}x - z^2}{\bar{x}x - 1} \right) \left(\frac{x\bar{x} - z^2}{x\bar{x} - 1} \right) = f(x), \quad (\text{B.26})$$

where f is rational and z is constant. This falls into the remit of Theorem 2.23, since when recast as a mapping

$$(x, y) \mapsto (\bar{x}, \bar{y}), \quad (\text{B.27})$$

$$\frac{\bar{x}x - z^2}{\bar{x}x - 1} = \frac{xy - 1}{xy - z^2} f(x), \quad \bar{y} = x,$$

it preserves the rational two-form

$$\frac{d\bar{x} \wedge d\bar{y}}{(\bar{x}\bar{y} - z^2)(\bar{x}\bar{y} - 1)} = \frac{dx \wedge dy}{(xy - z^2)(xy - 1)}. \quad (\text{B.28})$$

Let f be of the form

$$f(x) = \frac{P_M(x)}{Q_N(x)}, \quad (\text{B.29})$$

where P_M and Q_N are polynomials of degree M and N respectively. A sufficient deautonomisation is given by

$$\left(\frac{\bar{x}x - z}{\bar{x}x - 1} \right) \left(\frac{x\bar{x} - w}{x\bar{x} - 1} \right) = A \frac{(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_M)}{(x - \beta_1)(x - \beta_2) \dots (x - \beta_N)}, \quad (\text{B.30})$$

where z , w , A and $\alpha_1, \dots, \alpha_M$ are allowed to evolve with n , but β_1, \dots, β_N can be taken to be constant.

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