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Delphine Boucher, Jacques-Arthur Weil. Linear Differential Equations, Differential Galois Groups, First Integrals of Differential Systems. École thématique. Journées nationales de calcul formel, Luminy, France. 2007. hal-04117227

HAL Id: hal-04117227

<https://hal.science/hal-04117227>

Submitted on 5 Jun 2023

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Linear Differential Equations, Differential Galois Groups, First Integrals of Differential Systems

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Journées Nationales de Calcul Formel,
CIRM, Janvier 2007

Contents

I	Galois Theory of Linear Differential Equations	1
1	Differential Galois Theory	1
1.1	Differential Equations and Differential Systems	1
1.1.1	Differential Operators	1
1.1.2	Equivalent differential systems and differential mod- ules	1
1.1.3	First integrals of linear differential systems	3
1.2	The Differential Galois Group	3
1.3	Some Essential Properties	7
2	Second Order Differential Equations	11
2.1	Subgroups of $SL(2, C)$	11

2.1.1	Case I: Reducible Case	11
2.1.2	Case II: Imprimitive Case	12
2.1.3	Case III: Primitive case	13
2.2	The Kovacic algorithm	15
2.2.1	Symmetric powers	15
2.2.2	Algebraic Solutions of the Riccati Equation	16
3	Local and Global Differential Galois Theory	18
3.1	Local Solutions	18
3.1.1	Power Series Solutions	18
3.1.2	Exponents and Quasi-Series	19
3.1.3	Generalised Exponents	20
3.1.4	The Formal Local Galois group	21
3.2	Local and Global Algorithms	23
3.2.1	Rational Solutions	23
3.2.2	Radical and Global Solutions	23
3.3	Reducibility and Factorization	24
3.4	The Art of Computing Galois Groups	25
II	Integrability of Hamiltonian Differential Systems	28
4	Basic Facts on First Integrals of (Hamiltonian) Differential Systems	29
4.1	Variational equations and First Integrals	29
4.2	Hamiltonian Systems and Complete Integrability	30
4.3	Ziglin, Morales-Ramis	33
5	A criterion of non complete integrability of Hamiltonian systems	34
5.1	Examples	34
5.2	A non-integrability criterion	36
6	An example: the Friedman-Robertson-Walker model	38
6.1	The approach via constructive Morales-Ramis criteria	39
6.2	Higher variational approaches	40
7	Some properties to factorize symplectic linear differential systems	41
7.1	Factorization in practice	41
7.2	Properties of the eigenring	41
7.3	Properties of the exponents at the singularities	44
7.4	Application to the factorization of linear differential systems	45

Introduction

Linear differential equations are ubiquitous in mathematics and their applications. The Galois theory of linear differential equations aims at describing properties of solutions of linear differential equations by studying the algebraic relations between solutions (and their derivatives). This algebraic object also has some analytic interpretations; in a sense, one may think of the differential Galois group as measuring "what algebra sees of the dynamics" (Malgrange). Thanks to its algebraic nature, the differential Galois group is an object many of whose properties can be established/detected by symbolic computation.

Among the many applications of the algebraic theory of linear differential equations, we will highlight two families.

A first one stems from the fact that solutions of linear differential operators (" D -finite functions") have good "closure" properties: their sums, products, derivatives, etc are also D -finite. An extensive illustration of this aspect will be found in the lectures of J.-M. Maillard where applications to statistical mechanics are described.

A second family of applications is the study of non-linear differential systems. We are given a (non-linear) differential system and we assume that we know a solution curve; linearizing along this solution curve yields a first order linear differential system, the "variational system". Several properties of the original system, in particular existence of first integrals in some "good" class of functions, induce similar properties on the variational system; the algebraic theory of linear differential equations then yields methods and algorithms to detect such properties

This talk is split into two lectures. In the first part, we will focus on differential Galois groups and describe the basic foundations of the algorithms; this is adapted from notes for lectures by the second author in a CIMPA school in 2001. The second part will be devoted to detection of first integrals of non-linear differential systems by the variational approach; we will particularly emphasize on hamiltonian systems.

These notes are introductory material. Although proofs will be supplied, I'll refer to the original papers for technical aspects or up-to-date algorithms not covered here.

Part I

Galois Theory of Linear Differential Equations

1 Differential Galois Theory

Differential Galois theory is, as in the classical case, a tool to study the algebraic relations among solutions of linear differential equations; as we will see in the next lecture, it can also be used to study dynamical properties on solutions of some non-linear systems. A general reference for differential Galois theory is the fundamental book [vdPS03].

1.1 Differential Equations and Differential Systems

We consider a differential field k with a derivation that we note $'$ or ∂ if the context is not clear. For the rest of these notes, we assume that the field of constants C of k is algebraically closed (i.e any polynomial over C has all its roots in C) and of characteristic zero. Typical examples are \mathbb{Q} or \mathbb{C} .

1.1.1 Differential Operators

Let k denote a field (here $k = C(x)$) equipped with a derivation ∂ (naturally $\partial = \frac{d}{dx}$). One says that k is a differential field. We define the ring of differential operators as the set $k[\partial]$ with the (non-commutative) multiplication defined by $\partial \cdot a = a \cdot \partial + \partial(a)$ for $a \in k$. This multiplication corresponds to the composition of differential operators (compute the derivative of ay for a in k and an unknown function y).

Let L be a linear differential operator : $L = \partial^n + \dots + a_1 \partial + a_0, a_i \in k$.

K is a Picard-Vessiot extension associated to the homogeneous linear differential equation $L(y) = 0$ if K is generated over k by y_1, \dots, y_n and their derivatives where y_1, \dots, y_n are linearly independent solutions of $L(y) = 0$ and if the constant fields of k and K are equal.

1.1.2 Equivalent differential systems and differential modules

A linear differential equation is an equation of the form

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = 0$$

with the a_i being functions in k (not constants). Solving this equation is equivalent to solving the companion system

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

Conversely, suppose that (y_1, y_2, \dots, y_n) satisfy a homogeneous first order linear differential system $Y' = AY$ of size n . To find an equation associated with $Y' = AY$, we would like to find a system in companion form whose solutions are *equivalent* (or "isomorphic") to the ones of the original system (precisely: obtained with a change of variables $Y = PZ$ with P an invertible matrix with coefficients in k). This is done by the following cyclic vector process (see e.g [Bar93, vdPS03] for references and other methods). Consider $\Lambda \in k^n$ and let $z_1 = \Lambda Y = \lambda_1 y_1 + \cdots + \lambda_n y_n$. We compute $z_2 = z_1', \dots, z_{n+1} = z_1^{(n)}$ by using the relation $Y' = AY$. We obtain $n + 1$ linear expressions in the n variables y_i and so they are linearly dependant: this provides a linear differential equation $\mathcal{L}(z_1) = 0$ for z_1 . Letting $Z = (z_1, \dots, z_n)^t$, we now have a relation $Z = PY$ and $Z' = BY$; If the matrix P is invertible, Λ is called a *cyclic vector* for the system and $Z' = (BP^{-1})Z$ is a companion system, equivalent to the first one. It can be shown that the cyclic vectors form a Zariski open set, and almost all choices of Λ will fit [vdPS03].

Thus, in the following, everything that is stated for first order systems is valid for n -th order equations and vice-versa.

A useful tool to describe an equivalence class of differential systems (intrinsically) is the notion of differential module:

Definition 1.1. A differential module over (k, ∂) is finite-dimensional k -vector space $M \simeq k^n$ with an operator D satisfying a Leibniz rule:

$$\forall f \in k, v \in M : \quad D(f.v) = \partial(f).v + f.D(v)$$

Differential modules are sometimes also called "modules with a connection" in the litterature. To a differential system $Y' = AY$, one may canonically associate a differential module with a basis $e = (e_1, \dots, e_n)$ and the operator D acting by $D(e_i) = \sum_{j=1}^n -a_{j,i} e_j$. This way, a solution $Y = y_1 e_1 + \dots + y_n e_n$ of $Y' = AY$ is characterized by the relation $D(Y) = 0$

(check this) so that we may write $D = ' - A$.

It can be checked that two differential systems are equivalent if and only if they are associated to the same differential module (in fact, the change of variable that sends a system to the other can be viewed as a change of basis in the differential module).

The cyclic vector method above amounts to finding an element v such that $v, D(v), \dots, D^{(n-1)}(v)$ are a basis of the differential module M (hence the term “cyclic”).

1.1.3 First integrals of linear differential systems

Let U denote a fundamental matrix of solutions of $Y' = AY$ (i.e U is invertible and its columns are solutions of $Y' = AY$). A *linear first integral* of the system is a linear function \mathbb{R} such that $\mathbb{R}(Y)$ is a constant whenever Y satisfies $Y' = AY$. Let Y_i denote the columns of U and R_i denote the rows of U^{-1} . As $U^{-1}U = 1$, we see that $R_i \cdot Y_j = \delta_{i,j}$ (where $\delta_{i,j}$ is the kronecker symbol, $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ when $i \neq j$). It follows that the rows R_i define linear first integrals of the system. We see that the columns of ${}^tU^{-1}$ are coefficients of linear first integrals; they are also solutions of the *adjoint system* $Z' = -{}^tAZ$. We see that solutions of the adjoint (or *dual*) system yield first integrals of the system.

Similarly, one can show [Wei95, MR01] that polynomial first integrals will be homogeneous, and obtained from rational solutions of *symmetric powers* of the dual system. This simple fact is useful in the Morales-Ramis theory of non-integrability of Hamiltonian systems.

1.2 The Differential Galois Group

We will proceed as in classical Galois theory: first, we construct a field generated by all the solutions (and their derivatives)

Definition 1.2. A differential field extension $K \supset k$ is said to be a Picard-Vessiot extension of k (for $L(y) = 0$) if

1. $K = k(y_1, y_1', \dots, y_i^{(j)}, \dots, y_n^{(n-1)})$, where the y_i are a basis of solutions of $L(y) = 0$ (i.e K is the differential field generated¹ by the solutions of L).
2. K and k have the same field of constants.

¹Note that as $L(y_i) = 0$, we have $y_i^{(n)}$ and the higher derivatives in K , which really makes it a differential field

As the constant field of k is algebraically closed of characteristic 0, one can show that Picard-Vessiot extensions exist and are unique up to differential isomorphism ([vdPS03], the original proof goes back to Kolchin in 1948 [Kol99]). In the sequel, the term “solution” will always denote a solution in the Picard-Vessiot extension K .

Example 1.3. Let $k = \overline{\mathbb{C}}(x)$ and consider the equation $L(y) = y' - \frac{1}{3x}y$. We know that the solution is $x^{\frac{1}{3}}$ but let's construct the Picard-Vessiot extension (hence the solution) like we did in the preliminary lecture on Galois groups. We consider the ring $k[y]$ with the derivation $D = \frac{d}{dx} + \frac{1}{3x} \frac{d}{dy}$, where y is an indeterminate. In this construction, y satisfies $D(y) - \frac{1}{3x}y = 0$. However, it is easily checked that $D(\frac{y^3}{x}) = 0$ so this ring contains a new constant. Take the ideal $I = (y^3 - x)$; it is prime (in fact, it is maximal) and stable under the derivation. We now let $K = k[y]/I$. This is now a differential field and it now has no new constant (the “new constant” is included in the relation defined by I). This is now a Picard-Vessiot extension.

Note the similarity with the second construction of the splitting field in the preliminary Galois lecture.

In fact, the existence of Picard-Vessiot extension can be achieved through this construction (see [vdPS03, Mag94]). Consider the ring $R := k[X_{1,1}, \dots, X_{n,n}, W]$ where the $X_{i,j}$ are indeterminates and

$$W \cdot \det \begin{pmatrix} X_{1,1} & \dots & X_{1,n} \\ \vdots & & \vdots \\ X_{n,1} & \dots & X_{n,n} \end{pmatrix} = 1.$$

Extend the derivation on k to a derivation D on R by letting $D(X_{i,j}) = X_{i,j+1}$ for $1 \leq j < n$ and $D(X_{i,n}) = -\sum_{l=0}^{n-1} a_l X_{i,l+1}$. This way, we have formally realized that $L(X_{i,1}) = 0$. Let J denote a differential ideal in R (i.e $D(J) \subseteq J$) which is maximal among differential ideals. It can be shown that then J is a prime ideal. So R/J has no zero-divisor and we may let $K := \text{Frac}(R/J)$. Now K has no new constant: if $\frac{P}{Q}$ was a new constant, then $(P - Q)$ would be a differential ideal in R/J , contradicting the maximality of J . It follows that this construction yields a Picard-Vessiot extension K . The ideal J is called the *ideal of relations* among solutions.

Definition 1.4. We call a differential k -automorphism of K an automorphism g of K which leaves k fixed and which commutes with the derivation, i.e:

1. $\forall y \in K, g(y)' = g(y')$

$$2. \forall y \in k, g(y) = y$$

The differential Galois group $G = \text{Gal}_\partial(K/k)$ of a differential extension $K \supset k$ is the group of differential k -automorphisms of K .

The differential Galois group G of $L(y) = 0$ is defined as the differential Galois group of K/k , where K is a Picard-Vessiot extension of k for L .

Consider the n -dimensional C -vector space V of solutions of $L(y) = 0$ in K , generated by the y_i over C . Let $g \in G$ and let y denote a solution of $L(y) = 0$. Then, $g(y)$ is also a solution. Indeed, as $a_i \in k$, we have $g(a_i) = a_i$ and :

$$\begin{aligned} L(g(y)) &= g(y)^{(n)} + a_{n-1}g(y)^{(n-1)} + \dots + a_0g(y) \\ &= g(y^{(n)}) + g(a_{n-1})g(y^{(n-1)}) + g(a_0)g(y) \\ &= g(L(y)) = g(0) = 0. \end{aligned}$$

As any solution is a linear combination of the y_i , we deduce that there exists constants $c_{i,j}$ such that $g(y_j) = \sum_i c_{i,j}y_i$. As an automorphism is fully determined by its action on the generators y_i of K , this gives us a faithful representation of G as a subgroup of the group $GL(n, C)$ of invertible $n \times n$ matrices. In fact, one can show that G is a *linear algebraic group*, i.e the entries $c_{i,j}$ of the matrices are defined as solutions of a set of algebraic equations; the reason for this is the fact (intuitively clear, but still deep) that G can also be viewed as the set of such automorphisms that preserve all the relations among the solutions.

Recall the construction of the Picard-Vessiot extension; G can be viewed as the set of matrices with a right action on the $X_{i,j}$ that preserve the ideal of relations J , and this is what makes it a linear algebraic group. In what follows, we will identify G to this representation as a group of matrices acting on solutions.

A little bit on linear algebraic groups Before proceeding with Galois theory, a quick incursion into linear algebraic groups.

Recall that an affine algebraic variety V over C is defined as the set of solution of some polynomial equations (e.g straight lines, a circle, conics, etc..). The ideal $I(V)$ associated to the variety is the set of polynomials that vanish at every point of the variety. Conversely, to any ideal we associate the variety of points that annul all polynomials of the ideal. The variety is called irreducible if $I(V)$ is prime. As every ideal is the intersection of a finite

number of prime ideals, we see that any affine variety is the union of a finite number of irreducible varieties.

A linear algebraic group G is a group of $n \times n$ matrices whose entries form an algebraic variety of C^{n^2+1} . The reason for adding one dimension is to include the condition that the determinant is not zero: if we add one variable, the condition $\det(g) \neq 0$ is rephrased as $u \cdot \det(g) - 1 = 0$ which is now a polynomial equation. This way, we see that $GL(n, C)$ is a linear algebraic group.

If G is a linear algebraic group and G is irreducible as a variety, we will say that it is *connected* (for the Zariski topology). If G is not connected, then G is the finite union of irreducible ("connected") varieties. Among those, the one that contains the identity element is called the *connected component of the identity* in G , and denoted by G° . One can show that G° is a normal subgroup of finite index in G .

A linear algebraic group is said to *virtually* have a property if G° has that property; for example, G is called *virtually abelian* if G° is abelian (as a group).

The *dimension* of G is defined as the transcendence degree of G° over C .

Example 1.5. Here are classical examples of linear algebraic groups.

1. $GL(n, C)$ and $SL(n, C)$ (defined by $\det(g) = 1$).
2. The group of upper triangular matrices T (defined by $T_{i,j} = 0$ for $j < i$).
3. Let I_n denote the identity matrix of size n and the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

The set of matrices M that satisfy ${}^t M \cdot J \cdot M = J$ (this relation induces a finite set of polynomial relations on the entries of M) is called the Symplectic group $Sp(2n, C)$ and will be central in the applications to symplectic mechanics.

4. Any finite group of matrices (check this!)

To measure properties of the connected component of the identity G° , one uses its Lie algebra. The Lie algebra of a linear algebraic group is the tangent space at the identity (this makes sense: the group is an algebraic variety, so there is a natural tangent space). This is simply computed with the epsilon-trick (see [Put99a, vdPS03]) as follows. Let ϵ denote an object

satisfying $\epsilon^2 = 0$ (think of ϵ as being ideally small). A matrix M is in the tangent space at the identity if and only if $Id + \epsilon M$ satisfies the equations of the group (or “is a $C[\epsilon]$ -point of the variety”). This defines the Lie algebra $Lie(G)$.

The dimension of G as a variety equals the dimension of $Lie(G)$ as a vector space. If you compute the Lie algebra for the groups in the above examples (easy computation), you will find that $sl(n, C) := Lie(SL(n, C))$ is the set of matrices with zero trace, that the Lie algebra of T is T itself, that $sp(2n, C) := Lie(Sp(2n, C))$ is the set of matrices M satisfying ${}^tMJ + JM = 0$ (or $M = JS$, with S any symmetrical matrix), and that the Lie algebra of a finite group is equal to 0.

1.3 Some Essential Properties

As in classical Galois theory, there is a Galois correspondence between algebraic subgroups of G and differential subfields of K . We will admit here a weak version of this correspondence which will be enough for our purposes (see e.g [vdPS03] for a full proof).

Theorem 1.6 (Galois normality). *Let K denote a Picard-Vessiot extension of k , let G be its differential Galois group, and let $z \in K$. Then:*

$$z \in k \iff \forall g \in G, g(z) = z.$$

A first application of this concerns unimodularity of the Galois group: recall that a matrix is called unimodular if its determinant is equal to 1, and that the group of unimodular matrices is denoted by $SL(n, C)$.

Define the *Wronskian* matrix $W = (y_i^{(j-1)})_{i,j=1..n}$ and the Wronskian determinant $w = \det(W)$.

Exercise 2.

1. Show that y_1, \dots, y_n are linearly independent over C if and only if the Wronskian determinant w is not equal to zero.
2. Show that $w' = a_1 w$
3. Show that, $\forall g \in G, g(w) = w \cdot \det(g)$ (Hint: show that g acts on W by multiplication on the right).

◇

unimodular

Lemma 1.7. *There exists $f \in k$ such that $a_{n-1} = \frac{f'}{f}$ if and only if $G \subset SL(n, C)$.*

Proof. Assume that $G \subset SL(n, C)$. By the above exercise, we have $g(w) = w$ for all g in G hence the normality theorem shows that $w \in k$ and it is thus a solution in k of $w' = a_{n-1}w$.

Conversely, if there exists f in k such that $a_{n-1} = \frac{f'}{f}$, then we must have $w = c.f$ with $c \in C$ (because K contains no new constants) so $w \in k$ and thus the relation $g(w) = w \cdot \det(g)$ implies that $\det(g) = 1$. \square

Note that we can always arrange that the Galois group be unimodular by letting $y = z \cdot e^{\int \frac{a_{n-1}}{n}}$, we see that z satisfies a linear differential over k where we have no term in $z^{(n-1)}$ any more, hence the wronskian is a constant (necessarily in k) and the Galois group is unimodular.

If $z \in K$, the orbit of z under G , noted $\text{Orb}_G(z)$ is the set of elements of the form $g(z)$ for some g in G .

algebrique

Proposition 1.8. *Let $z \in K$. Then z is algebraic of degree m over k if and only if $\text{Orb}_G(z)$ has exactly m elements.*

All solutions of $L(y) = 0$ are algebraic if and only if G is a finite group.

Proof. Assume that z is algebraic; let Q be its minimum polynomial. Let

$$P = \prod_{g \in G} (Y - g(z)) \in K[Y].$$

Of course, $P(z) = 0$ and we now show that the coefficients are in k . Let $g_0 \in G$. As left multiplication by g_0 is a bijection of G , we have

$$\prod_{g \in G} (Y - g_0 \cdot g(z)) = \prod_{\tilde{g} \in G} (Y - \tilde{g}(z)).$$

So the coefficients of P are fixed by the group and hence, by Galois normality, they are all in k . The roots of P are exactly $\text{Orb}_G(z)$ by construction. As Q is the minimum polynomial of z , Q is a divisor of P so the roots of Q are in $\text{Orb}_G(z)$. Now, the image of a root of Q (here: z) by any element of G is again a root of Q . We conclude that there are as many roots of Q as elements in $\text{Orb}_G(z)$, hence $\deg(Q) = \text{card}(\text{Orb}_G(z))$.

Conversely, if $\text{Orb}_G(z)$ has exactly m elements, call g_1, \dots, g_m elements of G such that $\text{Orb}_G(z) = \{g_1(z), \dots, g_m(z)\}$. As above, we form $P = \prod_{i=1..m} (Y - g_i(z))$. Let $g \in G$; as it is an automorphism, $g(g_i(z)) \neq g(g_j(z))$ when $i \neq j$ so g acts as a transitive permutation on the $g_i(z)$. It follows that P has coefficients in k . Now, if it was reducible, z would be a zero of a factor and its orbit would hence have less than m element. We conclude that z is algebraic of degree m .

Now, if G is a finite group, the above shows that any element in the Picard-Vessiot extension is algebraic; conversely, if all solutions are algebraic, then the y_i have only a finite number of possible images under G so, as an automorphism of K is defined by its action on the generators of K , the group G must be finite. \square

Exercise 3. Let $z \in K$ be algebraic of degree m . Let $\text{Stab}_G(z) := \{g \in G, g(z) = z\}$ denote the stabilizer of z in G . Show that m is the index in G of $\text{Stab}_G(z)$ (i.e the cardinal of the quotient). \diamond

exponentiel

Proposition 1.9. *An non-zero element z of K is exponential over k if and only if, for all $g \in G$, there exists a constant $c_g \in C$ such that $g(y) = c_g \cdot y$.*

Proof. Let $g \in G$.

$$\left(\frac{g(z)}{z}\right)' = \frac{g(z')}{z} - \frac{z'}{z^2}g(z) = \frac{g(z)}{z} \left(g\left(\frac{z'}{z}\right) - \frac{z'}{z}\right).$$

So, $g(z)/z$ is a constant for all $g \in G$ if and only if z'/z is left fixed by all $g \in G$. By Galois normality, this is indeed equivalent with the fact that $z'/z \in k$. \square

Note that in fact, this means that an element z of K is exponential if and only if the straight-line $z.k$ is globally invariant under G .

Exercise 4.[Put99c]

We consider the equation $y' = ay$ with $a \in \overline{C}(x)$.

1. Show that any proper algebraic subgroup of \overline{C}^* is finite and cyclic.
2. By considering the possible Galois groups, show that an algebraic solution $y' = ay$ must satisfy $y^m = f$ where $f \in \overline{C}(x)$ (i.e y is *radical* over $\overline{C}(x)$). Show that the equation $y' = ay$ has an algebraic solution if and only if there exists a positive integer m such that the equation $f' = maf$ has a solution $f \in \overline{C}(x)$.
3. For $m \in \mathbb{N}$, show that the equation $f' = maf$ has a rational solution if and only if $a = \sum_i \frac{n_i}{m(x-x_i)}$ with $n_i \in \mathbb{Z}$ having their gcd prime to m . What is the Galois group in this case.
4. Deduce from this a method which decides if the equation $y' = ay$ has an algebraic solution, computes it, and gives the differential Galois group.

5. Application: compute the Galois group for $y' = y$, for $y' = \frac{1}{4x}y$, and for $y' = \frac{\alpha}{x}y$.

◇

Exercise 5.[Logarithms]

1. Consider the field $K = \overline{\mathbb{C}}(x, \log(x))$. Show that K is a Picard-Vessiot extension of $\overline{\mathbb{C}}(x)$ corresponding to a homogeneous linear differential equation of order 2 and that its Galois group is conjugate to $G_a = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, c \in \overline{\mathbb{C}} \right\}$.
2. Show that G_a is abelian, isomorphic to the additive group $(\overline{\mathbb{C}}, +)$, and that its only algebraic subgroups are itself and $\{Id\}$.
3. Show that the reasoning of (i) applies for any element f such that $f' = a \in \overline{\mathbb{C}}(x)$. Deduce from this that either f is transcendental or f is in $\overline{\mathbb{C}}(x)$ (theorem of Liouville).

◇

These two exercises show that when one adjoins to k an exponential or an integral, the Galois group of the extension is abelian. Recall that the Liouvillian functions are the elements of fields obtained by adjoining successively exponentials, integrals, or algebraic elements to $\overline{\mathbb{C}}(x)$.

Exercise 6. We say that a field K is a purely Liouvillian extension of $\overline{\mathbb{C}}(x)$ if it is constructed via a tower of fields $\overline{\mathbb{C}}(x) = K_0 \subset K_1 \subset \dots \subset K_N = K$ where K_{i+1} is obtained from K_i by adjoining either an exponential or an integral. Show that the differential Galois group of a purely Liouvillian extension K is solvable.

◇

If we allow arbitrary algebraic extensions, their Galois group G is finite and needs not be abelian any more (in general, it is not!). However, G° is then reduced to the identity. This gives us the first step to the following theorem of Kolchin (which we will admit)

Theorem 1.10. *A linear differential has a basis of liouvillian solutions if and only if its differential Galois group G is virtually solvable.*

Kolchin actually proved the Lie-Kolchin theorem: G° is solvable if and only if its matrices can be put simultaneously in triangular form. The above theorem then follows without too many difficulties. To get convinced, you may study an exemple of this situation where a linear differential equation

has a solvable differential Galois group.

exp-red

Exercise 7. Consider the differential equation

$$L(y) = y'' + (-g(x) - 2)y' + (g(x) + 1)y = 0$$

with g a non-zero rational function in $\overline{\mathbb{C}}(x)$.

1. Show that it admits $y_1 = e^x$ as a solution.
2. Using variation of constants, compute a second solution from y_1 and integrals.
3. Show that the differential Galois group of L is solvable.
4. Show that the differential Galois group of L is either

$$\left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, c \in \overline{\mathbb{C}}^* \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}, c \in \overline{\mathbb{C}}^*, d \in \overline{\mathbb{C}} \right\}$$

and give a criterion to decide between either case.

◇

2 Second Order Differential Equations

We will now show how to use the differential Galois group for solving linear differential equations. For simplicity, we focus on second order equations. The idea is to first classify the possible Galois groups and then, on the basis of this classification, to study the corresponding properties of the solutions. We consider the differential equation $L(y) = y'' - a_1 y' - a_0 y = 0$.

2.1 Subgroups of $SL(2, \mathbb{C})$

Recall from lemma ^{unimodular} 1.7 that we can assume that the differential Galois group is unimodular, i.e a subgroup of $SL(2, \overline{\mathbb{C}})$. The subgroups of $SL(2, \overline{\mathbb{C}})$ are classified and we now go through this construction.

2.1.1 Case I: Reducible Case

Definition 2.1. Let G be a linear group acting on a vector space. We say that (the action of) G is reducible if there exists a non-trivial subspace $W \subset V$ such that $G(W) \subset W$.

In our case, $\dim(V) = 2$ so W has to be of dimension 1 and the matrices in the Galois group are triangular (or diagonal). By proposition 1.9 (and the remark right after it), this is equivalent with the fact that the differential equation has an exponential solution. We also see that the group is diagonal if there are (at least) two exponential solutions (this case is handled further in exercise 9). In this case, W admits a complement subspace that is also stable under G and the group is then said to be *completely reducible*.

To the equation $L(y) = 0$, we associate the Riccati equation, i.e the equation satisfied by $u = \frac{y'}{y}$. Simple computation shows that $u' = a_0 + a_1 u - u^2$. We see that the equation is reducible if and only if the Riccati equation has a rational solution.

If G is reducible, then L itself is reducible, in the sense that it can be written as a composition of two operators (check this in exercise 7); you may check that L has a right factor $\partial - u$ if and only if u is a solution of the Riccati equation.

If G does not act reducibly, it is called *irreducible*.

2.1.2 Case II: Imprimitive Case

Definition 2.2. Let G be an irreducible group acting on a vector space V . We say that G is *imprimitive* if there exist subspaces V_i such that $V = V_1 \oplus \dots \oplus V_r$ and G permutes transitively the V_i :

$$\forall i = 1, \dots, r \quad \forall g \in G, \quad \exists j \in \{1, \dots, r\} : g(V_i) = V_j.$$

We then say that V_1, \dots, V_r form a *system of imprimitivity* for G .

In our case, we must have $r = 2$ and $\dim(V_1) = \dim(V_2) = 1$. The matrices have the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \quad \text{with } a, b \in \mathbb{C}^*.$$

Lemma 2.3. Assume that G is irreducible. Then the Riccati equation has an algebraic solution of degree 2 if and only if G is imprimitive.

Proof. Let P denote the minimum polynomial of an algebraic Riccati solution of degree 2. Let u_1, u_2 be the roots of P . As u_1, u_2 satisfy the Riccati equation, there exists solutions y_i of $L(y) = 0$ such that $y'_i/y_i = u_i$. As G

permutes the u_i , it permutes the lines $V_i = C. \langle y_i \rangle$ and the V_i form a system of imprimitivity.

Reciprocally, let $V_i = C. \langle y_i \rangle$ be a system of imprimitivity. If we let $u_i = y'_i/y_i$, then G permutes the u_i and proposition 1.8 shows that they are algebraic of degree 2 and conjugate (you may also check that the symmetric functions in the u_i are left fixed by G). \square

2.1.3 Case III: Primitive case

Definition 2.4. *If G is irreducible and not imprimitive, we say that it is primitive.*

One can show ([SU93a]) that an equation whose Galois group is an infinite primitive subgroup of $SL(n, C)$ does not have Liouvillian solutions; And if the group is finite, all solutions are algebraic (proposition 1.8) and hence Liouvillian.

In the case of $SL(2, C)$, there are three primitive groups (e.g [Kov86, SU93b, SU93a]) The tetrahedral group ($A_4^{SL_2}$) of order 24, the octahedral group ($S_4^{SL_2}$) of order 48, and the icosahedral group ($A_5^{SL_2}$) of order 120. We will now review them

The tetrahedral group $A_4^{SL_2}$ of order 24 is generated by matrices

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{3}(2\xi - 1) \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

where ξ denotes a primitive sixth root of unity, i.e $\xi^2 - \xi + 1 = 0$. The subgroup $\langle M_1 \rangle$ generated by M_1 has order 6. Assume the second order operator L has $A_4^{SL_2}$ as its Galois group. Let y_1, y_2 denote a basis of the solution space on which these matrices act and have this form. We see that the line generated by y_1 is left fixed by $\langle M_1 \rangle$; this imposes that $u_1 := \frac{y'_1}{y_1}$ is left fixed by $\langle M_1 \rangle$ (recall proposition 1.9) so its orbit under G has length at most 4 (in fact, direct computation shows that it has length exactly 4). By the Galois correspondenc (proposition 1.8), it follows that u_1 is algebraic of degree 4. We conclude that when the Galois group is $A_4^{SL_2}$, the Riccati equation has a solution which is algebraic of degree 4.

Another way to see this result is the following. Computing the conjugates of y_1 under G shows that the polynomial $Y_1 (2 Y_2 + Y_1) (4 Y_2^2 - 2 Y_1 Y_2 + Y_1^2)$ is a semi-invariant of the Galois group (i.e it is sent to a multiple of itself by

the group acting by linear substitution, right action). It can also be checked that its cube is an invariant. We will use this below to compute the minimum polynomial of u_1 .

The octaedral group $S_4^{SL_2}$ of order 48, is generated by matrices

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad 1/2 \xi (\xi^2 + 1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

where ξ denotes a primitive eighth root of unity, i.e $\xi^4 + 1 = 0$. The subgroup $\langle M_1 \rangle$ generated by M_1 has order 8 and reasoning as above shows that the riccati solution u_1 is algebraic of degree $6 = \frac{48}{8}$.

The group admits the semi-invariant $Y_1^5 Y_2 - Y_1 Y_2^5$, whose square is an invariant.

The icosaedral group $A_5^{SL_2}$ of order 120. is generated by matrices

$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}$$

where ξ denotes a primitive tenth root of unity, i.e $\xi^4 - \xi^3 + \xi^2 - \xi + 1 = 0$, and $\phi = \frac{1}{5}(\xi^3 - \xi^2 + 4\xi - 2)$ and $\psi = \frac{1}{5}(\xi^3 + 3\xi^2 - 2\xi + 1)$. The subgroup $\langle M_1 \rangle$ generated by M_1 has order 10 and reasoning as above shows that the riccati solution u_1 is algebraic of degree $12 = \frac{120}{10}$.

The group admits the invariant $Y_1^{11} Y_2 - 11 Y_1^6 Y_2^6 - Y_1 Y_2^{11}$.

In the cases of these three groups, all solutions are algebraic and hence all solutions of the Riccati equation are of course also algebraic. By developing on propositions 1.8 et 1.9, the study of those groups (see [?, ?] or push further the above calculations) shows that :

- For $A_4^{SL_2}$: The Riccati has algebraic solutions of degrees 4,6 ou 12.
- For $S_4^{SL_2}$: The Riccati has algebraic solutions of degrees 6,8,12, ou 24.
- For $A_5^{SL_2}$: The Riccati has algebraic solutions of degrees 12,20,30, ou 60.

Lastly, if $G = SL(2, C)$, the differential equation does NOT have liouvilian solutions. In view of the applications to hamiltonian mechanics ([?]),

we see that the only subgroup(s) of $SL(2, C)$ that will yield obstructions to integrability are ... only $SL(2, C)$ itself.

This classification work is summarized in Kovacic's theorem

kovacic

Theorem 2.5 ([Kov86]). *Let $L(y) = 0$ be a linear differential equation with coefficients in k . It has Liouvillian solutions if and only if it has solutions of the form $y = e^{\int u}$ where u is an algebraic solution of degree 1 (reducible case I), degree 2 (Imprimitive case II), or degree in $\{4, 6, 12\}$ (Primitive case III) of the associated Riccati equation.*

2.2 The Kovacic algorithm

We now give a (simplified) version of the algorithm of Kovacic for solving second order linear differential equations.

2.2.1 Symmetric powers

Let $P = U^m + b_{m-1}U^{m-1} + \dots + b_0$ be the minimum polynomial of an algebraic solution of the Riccati equation. Let u_1, \dots, u_m be the roots of P , and y_i the solutions of $L(y) = 0$ of which they are logarithmic derivatives. We then have

$$b_{m-1} = -(u_1 + \dots + u_m) = -\left(\frac{y'_1}{y_1} + \dots + \frac{y'_m}{y_m}\right) = -\frac{(\prod y_i)'}{\prod y_i}.$$

The coefficient $-b_{m-1}$ is thus the logarithmic derivative of a product of m solutions of $L(y) = 0$.

sympow

Lemma 2.6. *Let y_1, y_2 be a basis of solutions of $L(y) = 0$. There exists a linear differential equation $L^{\otimes m}$ whose solution space is the set of homogeneous polynomials of degree m in y_1, y_2 with coefficients in C .*

Proof. Let y be a generic solution of $L(y) = 0$. Let $z = y^m$. Compute z', z'', \dots, z^{m+1} by always replacing y'' by its expression given by $L(y) = 0$. The $z^{(i)}$ are linear combinations of monomials of degree m in y, y' . These monomials form a k -vector space of dimension $m+1$; if we have $m+2$ elements of such a space, they are linearly dependent, so $z, z', z'', \dots, z^{m+1}$ satisfy a linear dependence relation over k that we note $L^{\otimes m}(z) = 0$: what we know is that it has order at most $m+1$.

Let A be the differential ring $K[X_1, X_2]$ where the derivation is given by $X'_1 = X'_2 = 0$. By construction, we have $L^{\otimes m}((X_1 y_1 + X_2 y_2)^m) = 0$. We easily infer that any monomial of degree m in y_1, y_2 is a solution of

$L^{\otimes m}(z) = 0$. But, if these monomials were linearly dependent, then y_1, y_2 would be linearly dependent (any homogeneous polynomial in two variables factors as a product of linear factors over \mathbb{C}): So they form a vector space of dimension $m + 1$, then $L^{\otimes m}(z)$ is order $m + 1$, and its solution space is precisely that vector space. \square

We can calculate the linear dependence between z and z' using standard linear algebra, but a faster method is the following:

Exercise 8. [[BMW97]] Let $L(y) = y'' + ay' + by = 0$. we define recursively a sequence of operators L_i by :

$$\begin{cases} L_0(y) &= y, \\ L_1(y) &= y', \\ L_{i+1}(y) &= L_i(y)' + iaL_i(y) + i(m-i+1)bL_{i-1}(y). \end{cases}$$

1. Let y be a solution of $L(y) = 0$. Show by induction that

$$L_i(y^m) = m(m-1) \cdots (m-i+1)y^{m-i}(y')^i.$$

2. Deduce that $L_{m+1} = L^{\otimes m}$.

\diamond

2.2.2 Algebraic Solutions of the Riccati Equation

Theorem 2.7. *The Riccati equation has a solution algebraic of degree at most m if and only if the symmetric power $L^{\otimes m}(z) = 0$ has an exponential solution.*

Proof. If the Riccati equation has an algebraic solution of degree m , we have seen that the coefficient b_{m-1} of its minimum polynomial is the logarithmic derivative of an exponential solution of $L^{\otimes m}(z) = 0$.

Conversely, let z be an exponential solutions of $L^{\otimes m}(z) = 0$. Lemma 2.6^{is sym} shows that there exists a polynomial $Q(y_1, y_2)$ homogeneous of degree m such that $z = Q(y_1, y_2)$. Let v be the logarithmic derivative of z . As $Q(y_1, y_2)$ factors as a product of linear factors over \mathbb{C} , let u_1, \dots, u_m be the logarithmic derivatives of these factors. A linear combination of solutions is a solution so the u_i are Riccati solutions. For any $g \in G$, as $g(v) = v$, $g(u_i)$ must be one of the u_j : by proposition 1.8, it follows that the u_i are algebraic of degree at most m . \square

If the u_i do not have degree m , you may check that the product P of their minimum polynomials will be of degree m and its coefficient b_{m-1} will be given by $b_{m-1} = -v'/v$.

In [UW96], it is shown how, in many cases, one can also ask for rational solutions of $L^{\odot m}(y)$, which simplifies the algorithm.

Recall that the Riccati equation associated with $y'' + a_1y' + a_0y$ is $u' = -a_0 - a_1u - u^2$. Differentiating the identity $P(u) = 0$ and replacing u' by its expression, we obtain a polynomial relation of degree $m + 1$ for u . The remainder of the Euclidean division of this polynomial of degree $m + 1$ by P must be zero which gives us the following recursion to obtain all coefficients b_i once b_{m-1} is known.

$$(\sharp)_m : \quad \begin{cases} b_m = 1 \\ b_{i-1} = \frac{-b'_i + b_{m-1}b_i + a_1(i-m)b_i + a_0(i+1)b_{i+1}}{m-i+1}, & m-1 \geq i \geq 0 \\ b_{-1} = 0 \end{cases}$$

Finally, we obtain the following algorithm to calculate algebraic solutions of Riccati equations:

For $m \in \{1, 2, 4, 6, 12\}$:

- Compute $L^{\odot m}(y)$
- Seek exponential solutions f
- If there are some: let $b_{m-1} = -\frac{f'}{f}$, compute the other coefficients b_i of P by using $(\sharp)_m$, and return P
else, proceed with next m .

If no solution is found this way, there are no Liouvillian solutions.

compred

Exercise 9. Consider the differential equation $L(y) = y'' - ry = 0$.

1. Write the Riccati equation $R(u) = 0$ satisfied by $u = y'/y$.
2. Show that $L^{\odot 2}(y) = y''' - 4ry' - 2r'y''$.
3. We assume that R has an algebraic solution of degree 2.

(a) Show that its minimum polynomial has the form

$$P = u^2 - \frac{f'}{f}u + \frac{f''}{2f} - r$$

where $L^{\odot 2}(f) = 0$.

- (b) Show that $f \operatorname{disc}(P) = c$ where c is a constant.
- (c) We assume that $L^{\otimes 2}(f) = 0$ has a solution $f \in \overline{\mathbb{C}}(x)$ and that $\operatorname{disc}(P) \neq 0$. Show that the Riccati equation has one or two rational solutions; Compute them, and deduce that L admits the liouvillian solutions $y = \sqrt{f} e^{\pm \int \frac{\sqrt{c}}{2f}}$.
- (d) Conversely, show that if $L(y) = 0$ has two linearly independent exponential solutions, then the matrices of the Galois group are diagonal matrices and $L^{\otimes 2}(y) = 0$ has a rational solution (recall that the Galois group is unimodular)
- (e) Application : solve $y'' - \frac{c}{16x^2}y = 0$ où $c \in \overline{\mathbb{C}}$.

◇

3 Local and Global Differential Galois Theory

A powerful method to obtain information on differential Galois groups is to first compute some local information at the singularities, and then try to patch (or glue) together these local informations. This is how one will compute rational solutions, exponential solutions, factorizations, etc. We will now sketch some (easiest) algebraic aspects of this local approach, and how to realize actual computations with it.

3.1 Local Solutions

3.1.1 Power Series Solutions

series

We make the coefficients of our differential equation polynomials: $L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$ where the a_i are now polynomials (in fact, analytic would be enough). If $a_n(x_0) \neq 0$, then Cauchy's theorem shows that the equation has a basis of analytic solutions around zero.

Computing these power series is achieved the following way. Let $T = x - x_0$ if $x_0 \in \overline{\mathbb{C}}$ or $T = \frac{1}{x}$ if $x_0 = \infty$. Perform the change of variables $x \mapsto T$ in the equation and plug $\sum_i c_i T^i$ into the equation: you will obtain a recurrence relation for the c_i and they will be uniquely determined by their first n terms (exercise: prove this cleanly).

To fix notations, let's make this recursion explicit. The operator L can be viewed as an endomorphism of the infinite dimensional vector space $\overline{\mathbb{C}}[[x]]$. Assume that $a_i = \sum_{j=1}^{m_i} a_{i,j} x^j$. We write the action of L on a basis of $\overline{\mathbb{C}}[[x]]$:

we get $L(x^N) = \sum_{i=0}^n \sum_{j=0}^{m_i} N(N-1) \dots (N-i+1) a_{i,j} x^{N+j-i}$. Now,

$$L\left(\sum_{N=0}^{\infty} c_N x^N\right) = \sum_{N=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^{m_i} N(N-1) \dots (N-i+1) c_N a_{i,j} x^{N+j-i}. \quad (3.1) \quad \boxed{\text{pre-rec}}$$

So, grouping powers of x , we obtain a recurrence relation

$$(\mathcal{R}_N) : \sum_{l=0}^m E_l(N) c_{N-l} = 0 \quad (3.2) \quad \boxed{\text{recurrence}}$$

with E_l being a polynomial. In particular, if v is the *valuation* of the power series (the smallest integer such that $c_v \neq 0$), we must have $E_0(v) = 0$ (all the c_{v-l} are zero).

If $a_n(x_0) \neq 0$, then direct computation shows that $E_0(N) = N(N-1) \dots (N-(n-1))$.

Exercise 10. Use this to show the formal part of Cauchy's theorem i.e the existence of a basis of solutions in power series. \diamond

exponents

3.1.2 Exponents and Quasi-Series

We now turn to the singular case. For notational convenience, we assume that the considered singular point is zero (i.e $a_n(0) = 0$).

We say that a solution y is a *quasi-series* if it is of the form $y = x^\alpha \phi$ with ϕ analytic and $\alpha \in \overline{\mathbb{C}}$. If $\alpha \in \mathbb{Z}$, this is a *Laurent* series; if $\alpha \in \mathbb{Q}$, this is a *Puiseux* series. If further ϕ has valuation zero (i.e its constant term is not zero), then α is well defined and we call it the *exponent* of the quasi-series y .

Example 3.1. *The Euler homogeneous equation ([Inc44]).*

Consider the equation $L(y) = x^2 y'' + c_1 x y' + c_2 y = 0$ with c_1, c_2 constants (this equation is often called, as many other equations, an *Euler equation*). Computation shows that $L(x^\alpha) = E_0(\alpha) x^\alpha$ with $E_0(\alpha) = \alpha(\alpha-1) + c_1 \alpha + c_2$. We see that there is a solution of the form x^α if and only if $E_0(\alpha) = 0$.

If E_0 has two distinct roots α_1, α_2 , then we have distinct solutions x_1^α and x_2^α (check that they are linearly independent over \mathbb{C}).

If E_0 has a double root α_1 , we have $E_0(\alpha_1) = E'_0(\alpha_1) = 0$. We differentiate the relation $L(x^\alpha) = E_0(\alpha) x^\alpha$ with respect to α (note that $\frac{\partial}{\partial \alpha}(x^\alpha) = x^\alpha \log(x)$): we obtain $L(x^\alpha \log(x)) = (E'_0(\alpha) + \log(x) E_0(\alpha)) x^\alpha$, from which it follows that $L(x_1^\alpha \log(x)) = 0$ so a second solution is $x_1^\alpha \log(x)$.

Exercise 11. Show that the equation $L(y) := 2xy'' + 3y' + 2y = 0$ has a basis of quasi-series solutions at $x = 0$. Hint: Compute $L(x^{\alpha+N})$ for an

arbitrary integer N and show that $L(\sum_{N=0}^{\infty} c_N x^{\alpha+N}) = 0$ if and only if we both have that $E_0(\alpha) = 0$ for some polynomial E_0 and the c_N are solutions of a recursion relation (compute it). \diamond

To make this general, look at equation ^{pre-rec}3.1 and replace N by $\alpha + N$ in there. Relation ^{recurrence}3.2 then becomes

$$(\mathcal{R}_N) : \sum_{l=0}^m E_l(\alpha + N) c_{N-l} = 0 \quad (3.3) \quad \text{recurrence2}$$

A necessary condition for the existence of a quasi-series solution with exponent α is that $E_0(\alpha) = 0$ (from the case $N = 0$ in recursion ^{recurrence2}3.3).

Definition 3.2.

The polynomial E_0 is called the indicial polynomial of L at zero.

The roots of E_0 are called the exponents of L at zero.

If E_0 has degree exactly n , then zero is called a regular singularity; otherwise, E_0 has degree strictly less than n and zero is an irregular singularity.

Exercise 12. Let α be an exponent at zero such that, for all $i \in \mathbb{N}$, $\alpha + i$ is Not an exponent. Prove that L admits a quasi-series solution of exponent α . \diamond

More generally, if the singularity is regular singular, then (see [?] or [Inc44]) either there is a basis of quasi-series solutions, or there is a basis formed of quasi-series and of solutions of the form $x^{\alpha_1} \phi_1, x^{\alpha_1 - n_1} (\phi_2 + x^{n_1} \phi_1 \log(x)), \dots$ (this may happen in the case when two exponents differ by an integer). Moreover, the power series ϕ_i are analytic. So in this case, the formal theory and the analytical theory coincide.

3.1.3 Generalised Exponents

local

If the singularity zero is not regular, then quasi-series and logarithms are clearly not enough to define solutions. For example, consider the equation $x^2 y' + y = 0$: the solution $e^{\frac{1}{x}}$ can not be written as a quasi-series at zero, so we need to add exponentials to our formal local objects.

Definition 3.3. An element $e_i \in \overline{\mathbb{C}}[x^{-\frac{1}{r}}]$ is called a generalized exponent if there is a formal solution of the form $e^{\int \frac{e_i}{x}} \phi_i$ where $\phi_i \in \overline{\mathbb{C}}[[x]][e_i, \log(x)]$ and the valuation (with respect to x , not counting the log) of ϕ_i is equal to zero.

If $r > 1$, then r is called the ramification index of the generalized exponent.

Note that exponents themselves are generalized exponents: indeed $x^\alpha = e^{\int \frac{\alpha}{x}}$.

To compute generalized exponents, one looks for formal Puiseux series solutions for the Riccati equation associated with L (i.e solutions in $C((x^{\frac{1}{r}}))$ for some $r \in \mathbb{N}$) and keeps only the parts of such solutions whose valuation is less or equal to -1 ; the degrees of the generalized exponents can be measured from the *Newton polygon* of L at zero.

One can show ([vdPS03, ?]) that one can compute a basis of formal solutions of the form $e^{\int \frac{e_i}{x}} \phi_i$ where $\phi_i \in \overline{C}[[x]][e_i, \log(x)]$.

3.1.4 The Formal Local Galois group

formal-group

We still assume that we work at zero (otherwise, take a local parameter $t = x - x_0$ at $x_0 \in \overline{C}$ or $t = \frac{1}{x}$ at infinity and work with t). We consider the field $\overline{C}((x))$ as our base field. The *formal local Galois group* \hat{G}_0 at zero is defined as the differential Galois group of a Picard-Vessiot extension of $\overline{C}((x))$ for L .

Because we know the structure of the formal solutions, we may describe the structure of the formal local Galois group: for each i , we may write $e^{\int \frac{e_i}{x}} = x^{\alpha_i} e^{P_i}$ (with P_i of negative degree in x).

The *formal local monodromy* is defined as the Galois group over $\overline{C}((x))$ of

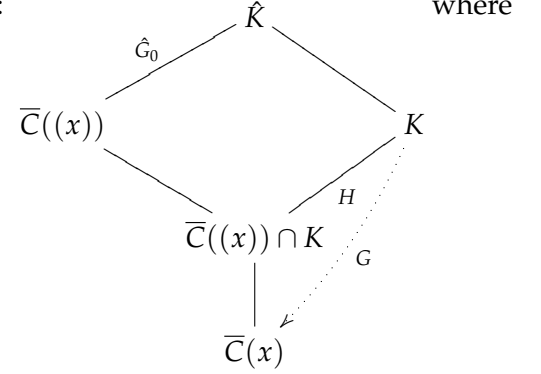
1. either $\overline{C}((x))(x^{\alpha_1}, \dots, x^{\alpha_n})$ if there are no logarithms in the solutions (in which case it is a torus)
2. or $\overline{C}((x))(x^{\alpha_1}, \dots, x^{\alpha_n}, \log(x))$ if there are logarithms in the solutions (in this case, it contains a unipotent element)

The *exponential Torus* is defined as the Galois group of $\overline{C}((x))(e^{P_1}, \dots, e^{P_n})$. One readily sees that these two groups generate the formal local Galois group; moreover, they can be easily computed from the given of local solutions.

Lemma 3.4. *The formal local Galois group can be embedded into a subgroup of the differential Galois group of L over $\overline{C}(x)$.*

Proof. We use the fact that $\overline{C}(x)$ can be embedded in $\overline{C}((x))$ so we view it as

a subfield. Consider the following Kaplansky diagram:



H is the differential Galois group of K . Galois theory shows that H is a subgroup of K . Then, the above diagram shows ([?]) that the Galois groups H and \hat{G}_0 are isomorphic. Thus, \hat{G}_0 can be viewed as a subgroup of G . \square

If we now take for our base field the field $\overline{\mathbb{C}}(\{x\})$ of convergent power series, we define the *local Galois group* G_0 as the differential Galois group of a Picard-Vessiot extension of $\overline{\mathbb{C}}(\{x\})$. A Kaplansky diagram again shows that $\hat{G}_0 \subset G_0 \subset G$.

At a regular singularities, we have $\hat{G}_0 = G_0$ and the Schlesinger density theorem ([?]) shows that the global Galois group is generated by its local Galois groups.

At irregular singularities, though, then new phenomena may occur (Stokes phenomenon) and in this case $\hat{G}_0 \subsetneq G_0$.

A very simple illustration of a link between local and global information is given in the following

finite

Proposition 3.5. *Assume that the Global galois group is finite. Then the exponents at all singularities are rational.*

Proof. The local Galois group is embedded in the global Galois group and hence finite. The exponential torus is infinite so there cannot be irregular points and all points must be regular. Now, if there are logarithms, the monodromy contains an additive subgroup and is infinite (alternatively: a logarithm is transcendental hence not algebraic, contradicting the fact that the group is finite). So the monodromy must be diagonal. But, because it is finite, it is cyclic and hence the exponents must be rational. \square

We note that this result is proved more naturally using Puiseux expansions of algebraic functions, but this proof gives light on the power of the Galois theoretic tools.

3.2 Local and Global Algorithms

3.2.1 Rational Solutions

ratsols

Let \mathcal{S} denote the set of singular points (i.e the zeroes of a_n and possibly infinity). We search for a method to check if our differential equation has a rational solution. Let y be a rational function. Then y can be written as $y = \prod_{x_i \in \mathcal{S}} (x - x_i)^{\alpha_i} \cdot (p_m x^m + p_{m-1} x^{m-1} + \dots + p_0)$. So to compute y , we need to find the α_i , the degree m , and the coefficients p_i . Expansion in Laurent series (or partial fraction decomposition) shows that the α_i must be exponents of L at x_i . Now, expansion at infinity (expand in powers of $\frac{1}{x}$) shows that there must exist an integer exponent α_∞ at infinity such that $m = -\alpha_\infty - \sum_{x_i \in \mathcal{S}} \alpha_i$. We thus obtain the following algorithm, whose solutions are a basis of rational solutions (if any) of $L(y) = 0$:

1. Select the minimal integer exponents α_i at all singularities, including ∞ . If one singularity does not have integer exponents, then STOP.
2. Let $m := -\alpha_\infty - \sum_{x_i \in \mathcal{S}} \alpha_i$. if m is not positive, then STOP.
Plug $y = \prod_{x_i \in \mathcal{S}} (x - x_i)^{\alpha_i} \cdot (p_m x^m + p_{m-1} x^{m-1} + \dots + p_0)$ into the equation
3. solve the resulting linear system in the p_i .

3.2.2 Radical and Global Solutions

The same reasoning applies to radical solutions, i.e the exponents may be used also to compute solutions having some power which is rational: one can similarly prove (see e.g [?, ?]) that there is a radical solution only if there are rational exponents e_i at all singular points $x_i \in \mathcal{S}$ such that $m := -e_\infty - \sum_{x_i \in \mathcal{S}} e_i$ is a positive integer. The solution would again be $y = P \cdot \prod_{x_i \in \mathcal{S}} (x - x_i)^{e_i}$ with P of degree m . Plugging this expression $L(y) = 0$ with indeterminate coefficients of P gives a linear system for the coefficients of P , any (non-zero) solution of this system leading to a solution of L ; Note that unlike the case of rational solutions, there may be different combinations of the e_i to be tested. Also, note that if some factors of a_n are irreducible polynomials, then we may have to compute with a splitting field of those to check for combinations, and this can make the algorithm more costly.

For the more general case of exponential solutions, the process is similar, though a little bit more technical, see [?] or [vdPS03].

Now we would like to see how to use these tools in the Kovacic algorithm. We introduce the following useful trick from [?]:

Note that if we have two formal solutions $e^{\int \frac{e_i}{x}} \phi_i$, then their product is $e^{\int \frac{e_1+e_2}{x}} \phi_1 \phi_2$, hence $e_1 + e_2$ is a generalized exponent for the symmetric square of L . In general, it is easy to verify that the expressions $ie_1 + (m-i)e_2$ form the generalized exponents of $L^{\otimes m}$. So we can check necessary conditions on rational (or radical) solutions of $L^{\otimes m}$ without having to compute this equation.

3.3 Reducibility and Factorization

- Let $G \subset GL(n, C)$ be a linear algebraic group. The representation in $GL(n, C)$ of the group G is said to be *reducible* if there exists a non-trivial subspace W which is left globally invariant by G ; it is called *completely reducible* if any subspace W of C^n which is left globally invariant by G admits a complement V in C^n which is invariant under G . In our case (characteristic zero), this is equivalent with being reductive (i.e all linear representations are completely reducible).
- A differential operator is called *reducible* if it can be written as a product $L = L_1 L_2$ with $L_i \in k[\partial]$ but not in k . Right-hand factors correspond to subspaces of the solution space that are stable under the differential Galois group. The ring of differential operators is seen being left and right euclidean ring, one can define (effectively) notions of greatest common right divisors (gcrd) and least common left multiples (lclm). A differential operator is called *completely reducible* if it is the least common (left) multiple of irreducible operators (equivalently: if its Galois group is reductive).

In terms of linear differential system, here is another possible formulation.

- A system $[A] Y' = AY$ with $A \in \mathcal{M}_n(k)$ is said to be reducible if there exists a matrix $P \in GL_n(k)$ such that the change of variable $Y = PZ$ transforms the system $[A]$ into a system of the form

$$Z' = \begin{pmatrix} \times & 0 \\ \times & F \end{pmatrix} Z$$

where F is a square matrix of dimension k (it corresponds to a factor). Otherwise the system is said to be irreducible.

The system is said to be completely reducible if it can be transformed into a block diagonal system

$$Z' = \begin{pmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_s \end{pmatrix} Z$$

where each system $v' = F_i v$ is irreducible.

3.4 The Art of Computing Galois Groups

We now have tools to smoothly use and apply the Kovacic algorithm (and generalisations like [SU97, vHRUW99]). We will show on examples how to combine all these tools together to obtain differential Galois groups.

Example 3.6. Consider the Airy equation $L(y) = y'' - xy = 0$. The only singularity is infinity. The local (generalised) exponents are $-\frac{1}{4} \pm 2\sqrt{x}^3$ at infinity. Because of the ramification at infinity, we see that the equation cannot have exponential solutions. Now we look at the second symmetric power.

Looking at sums of the exponents, we see that the only rational exponent of the second symmetric power $L^{\odot 2}$ will be $(\frac{1}{2})$ at infinity. This cannot be the degree of a polynomial. so there cannot exists a radical solution the the second symmetric power.

Now, as the equation is irregular at infinity, the group cannot be finite by proposition 3.5: This excludes case 3 of the Kovacic algorithm. Finally, the only possibility is that the Galois group is $SL(2, \mathbb{C})$.

Example 3.7. Consider the equation

$$L(y) = y'' + \left(\frac{1}{4} (x-1)^{-1} + \frac{5}{4} (x-1)^{-2} + \frac{3}{16} x^{-2} \right) y = 0$$

The local (generalised) exponents are $(\frac{1}{4}, \frac{3}{4})$ at zero, the roots of $X^2 - 2X + 5$ at 1 and $-\frac{1}{4} \pm 2\sqrt{-x}$ at infinity. Because of the ramification at infinity, we see that the equation cannot have exponential solutions. Now we look at the second symmetric power.

Looking at sums of the exponents, we see that the only rational exponents of the second symmetric power $L^{\odot 2}$ will be $(\frac{1}{2}, 1, \frac{3}{2})$ at zero, (1) at 1 and $(-\frac{1}{2})$ at infinity. Taking the lowest e_0, e_1, e_∞ possible, we have $-e_\infty - e_1 - e_0 = -1 < 0$

and the latter cannot be the degree of a polynomial so there cannot exist a radical solution the the second symmetric power.

Now, as the equation is irregular at infinity (and does not have rational exponents at 1), the group cannot be finite by proposition 3.5: ^{finite} this excludes case 3 of the Kovacic algorithm. Finally, the only possibility is that the Galois group is $SL(2, \mathbb{C})$.

Exercise 13. Consider the equation $L(y) = x^3 y'' + (x^2 + x)y' - y = 0$

1. Show that the exponents at ∞ are $(0, 0)$ and that the generalized exponents at 0 are $(1, -\frac{1}{x})$.
2. Show that $e^{\frac{1}{x}}$ is an exponential solution, the only one (up to scalar multiplication).
Show that there is a unique power series solution at zero, and that it is divergent (and Gevrey). Compute the formal local Galois group at 0.
3. Show that the formal solutions at infinity are of the form $y_{1,\infty}$ and $y_{1,\infty} \log(x) + y_{2,\infty}$ where the $y_{i,\infty}$ are power series in $\frac{1}{x}$. Compute the formal local Galois group at infinity.
4. Compute the global Galois group, and compare it with the formal local Galois groups.

◇

Exercise 14. The Whittaker equation $L(y) = y'' - (\frac{1}{4} + \frac{12}{\lambda})y = 0$. The exponents at zero are the roots of $\lambda X^2 - \lambda X - 12$ and at infinity $\pm \frac{x}{2}$.

1. Show (using the above examples) that for generic values of λ the Galois group is $SL(2, \mathbb{C})$.
2. Show that the exponents at zero are rational if and only if $\lambda = \frac{12}{n(n-1)}$ with $n \in \mathbb{Q}$.
3. In this case, the equation is $y'' - \frac{1}{4} - \frac{n(n-1)}{x^2}$. Show that the exponents at zero are $(n, 1-n)$ and $\pm \frac{x}{2}$ at infinity. Prove that the Galois group is $SL(2, \mathbb{C})$ unless n is an integer.
4. Perform the change of variables $y(x) = e^{\frac{x}{2}} f(x)$. Search for f as a power series: its coefficients u_N satisfy the recursion $Nu_N - (n+N) * (n-1-N)u(N+1) = 0$ and $u(0) = 0$. Conclude that when n is an integer, f is a polynomial and hence L has one (in fact, two) exponential solutions.

◇

In this exercise, we see that these tools give strong necessary conditions. However, question (4) shows that when there are parameters and the necessary conditions are satisfied, then it is not that easy to decide if there actually exists a solution. In this case, it was feasible; in general, it is not (see [Bou99]) and even sometimes undecidable.

Still, in the applications to Hamiltonian mechanics, we encounter many systems where "mysteriously" reasonings like the above (and many other tricks) allow one to say a lot about non-integrability of entire families of equations. This will be the topic of the next lecture.

Part II

Integrability of Hamiltonian Differential Systems

The *Hamiltonian systems* are differential systems which describe the equations of motion of mechanical systems whose mechanical energy (the Hamiltonian) is conserved. One says that the mechanical energy is a *first integral* for the Hamiltonian system. The notion of *complete integrability* of a Hamiltonian system refers to the given of sufficiently many independent first integrals (see definitions below).

In recent decades, many significant improvements have been made in the research regarding complete (meromorphic) integrability of Hamiltonian systems after the pioneering papers of Ziglin ([Zig83a, Zig83b], [Zig83a, Zig83b], 1982) relating integrability to properties of the monodromy group of a *variational equation* (or *variational system*) along a known solution. Churchill, Rod and Singer ([CRS95, BCRS96], 1996) and Morales and Ramis ([MRR01a], 1998; [MRRS05], 2005) then recast and extended these ideas in terms of the differential Galois group of this *variational equation*.

To quickly summarize, this culminated in the following criterion of Morales and Ramis (from [MRR01a]) :

Let (S) be a Hamiltonian system, $x_0(t)$ be a particular solution of (S) and G be the differential Galois group of the normal variational equation of (S) computed along the solution $x_0(t)$. If the system (S) is completely integrable, then the connected component of the identity in the group G , denoted G^0 , is an abelian group.

This theorem was generalized to 'higher variational equations' in [MRRS05]. At first sight, it may not be obvious that the conclusion of the statement (i.e. G^0 abelian) is any easier to test than its premises. For second order equations, many methods are known after the work of Kovacic [Kov86], see also [UW96], [DLR92], ...). They enabled many results of non integrability, let us cite only some of them : [Tsy01b, Tsy01a, MSS01, Mac02, Aud03, MP03, MPW04, MP04, MP05, Aud01, Aud02, Aud03]

The goal of these notes is to provide tools that seem to be practically efficient in handling families of Hamiltonian systems. We would like to promote the following idea. The variational system has an infinitesimally symplectic matrix. This fact induces a number of structural properties on the system that can be used to simplify computations. So our approach is to

try to exploit and preserve as much as possible this symplectic structure. Another point is that integrability questions are asked on families of Hamiltonians depending on parameters. This induces families of variational equations which also depend on parameters and this is an obstruction to algorithms that test Morales-Ramis theorem and the criterium we deduced from it. However, we will show below that the symplectic structure allows one to confront a part of this obstruction. We will emphasize on systems of order greater than 2.

The topic of this lecture is at the crossing point of two theories for which we assume the reader has a working knowledge (or is ready to accept some statements and references of literature): Integrability of Hamiltonian systems (see [Aud01, Aud02] [MRR01a], [MR01, MRR01b], [Chu02], [MRRS05] ...) and Differential Galois theory (see the previous lecture, and the remarkable reference book [vdPS03], or also [Sin90b], [Mag94]...).

4 Basic Facts on First Integrals of (Hamiltonian) Differential Systems

section2

In this first section we recall basic facts on Hamiltonian systems, differential galois theory and the Ziglin and Morales-Ramis theorems ([MR01],[MRR01a]).

4.1 Variational equations and First Integrals

Definition 4.1. Let K be a field of characteristic zero. Let us consider $F = (f_1, \dots, f_n)$ where f_i is of class C^k on an open set U of K^n . A dynamical system is a system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1(t), \dots, x_n(t)), \quad i = 1, \dots, n$$

which can be written, in vectorial notation as

$$\frac{dx}{dt} = F(x)$$

We will focus on conserved quantities for such systems:

Definition 4.2. A function $G : U \rightarrow \mathbb{R}$ is a first integral of the dynamical system $\dot{x} = F(x)$ if, for all solution $x(t)$ of the system,

$$\frac{d}{dt} G(x(t)) = 0.$$

The first integrals give the geometry of the solution curve. Indeed the solutions of the system lie on the hypersurfaces $G = cte$.

To answer questions regarding the existence of analytic first integrals, Poincaré studied the solutions which are infinitesimally close to a given particular solution of the differential system. The behavior of these solutions is given by a homogeneous linear differential system, the *variational system*, which is obtained by linearization of the differential system along the particular solution.

Definition 4.3. *The variational system along a solution $x_0(t)$ of a differential system is the linear differential system:*

$$h'(t) = \text{Jac}(F)(x_0(t)) h(t)$$

where $\text{Jac}(F)$ denotes the Jacobian matrix of F at $x_0(t)$.

Let $G : U \rightarrow \mathbb{R}$ and let k in \mathbb{N} such that $(dG)_x = \dots = (d^{k-1}G)_x = 0$. The junior part G° of G is $d^k G$ (see [Aud01] for a nicer definition)

weak-ziglin

Lemma 4.4 (Ziglin, weak form). *If (S) admits m algebraically independent first integrals f_1, \dots, f_m , then it admits m algebraically independent first integrals g_1, \dots, g_m such that their junior parts $g_1^\circ, \dots, g_m^\circ$ are algebraically independent. Furthermore they are solutions to the variational system.*

4.2 Hamiltonian Systems and Complete Integrability

Definition 4.5. Let $n \in \mathbb{N}^*$, $x = (x_1, \dots, x_{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$.

A Hamiltonian system on a non empty domain U of \mathbb{R}^{2n} is a system of differential equations of the form:

$$\begin{cases} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}(q, p) \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}(q, p) \\ i = 1, \dots, n \end{cases}$$

where $H : U \rightarrow \mathbb{R}$ is the Hamiltonian function.

The variables p_i and q_i are conjugate variables. The positive integer n is called the number of degrees of freedom.

Remark 15 : A Hamiltonian system can also be written in the following form:

$$x'(t) = J \nabla H(x(t))$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I is the order n identity matrix and $\nabla H(x)$ is the gradient of H at x .

◇ The Hamiltonian H is conserved, it is a first integral for the hamiltonian system. Indeed, for each solution x of the system, $dH(x(t)) = 0$.

A first integral G of the Hamiltonian system is also characterized by the equality

$$\{G(x), H(x)\} = 0$$

(or $\{G(x), H(x)\} = \{H(x), G(x)\}$)

where

$$\{G_1, G_2\} = \sum_{i=1}^n \left(\frac{\partial G_1}{\partial p_i} \frac{\partial G_2}{\partial q_i} - \frac{\partial G_1}{\partial q_i} \frac{\partial G_2}{\partial p_i} \right) = \langle \nabla G_1(x), J \nabla G_2(x) \rangle$$

is the Poisson bracket of G_1 and G_2 .

Definition 4.6. A Hamiltonian system with number of degrees of freedom equal to n is called completely integrable if it has n first integrals G_1, \dots, G_n such that:

- G_1, \dots, G_n are functionally independant
($\nabla G_1, \dots, \nabla G_n$ are linearly independant)
- G_1, \dots, G_n are in involution: for all solution x of the Hamiltonian system,
 $\{G_i(x), G_j(x)\} = \{G_j(x), G_i(x)\}$
(G_1, \dots, G_n commute for the Poisson bracket).

It is necessary to give a more precise sense to this notion of complete integrability by asking which class of functions we want the first integrals to belong to (analytic, algebraic, meromorphic functions, ...).

To answer the question of the analytic complete integrability, Poincaré studied the solutions which are infinitesimally close to a particular solution of the Hamiltonian system. The behavior of these solutions is given by a homogeneous linear differential system, the *variational system*, which is obtained by linearization of the Hamiltonian system along the particular solution.

Definition 4.7. The variational system along a solution $x_0(t)$ of a Hamiltonian system is the linear differential system:

$$h'(t) = J \mathcal{H}(H, x_0(t)) h(t)$$

where $\mathcal{H}(H, x_0(t))$ is the hessian of H at $x_0(t)$.

Remark 16 : The matrix $A = J \mathcal{H}(H, x_0(t))$ belongs to the simple Lie algebra $sp(2n, \mathbb{C}) = \{M \in Gl(2n, \mathbb{C}) / {}^t M J + J M = 0\}$ which is the Lie Algebra of the symplectic group $Sp(2n, \mathbb{C}) = \{g \in Gl(2n, \mathbb{C}) / {}^t g J g = J\}$. One says that A is an *infinitesimally symplectic* matrix. \diamond

As $x_0(t)$ is a particular solution of the variational system along the solution $x_0(t)$, one can reduce it and get a *normal variational system* of order $2n - 2$. During the last twenty years many results on the *meromorphic* non integrability of the Hamiltonian systems were deduced from the study of this (normal) variational system.

Exercise 17. We consider here the example of the system of HÖnon-Heiles from [MRRS05].

The initial Hamiltonian is

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{3}x_1^3 + \frac{1}{2}x_1x_2^2$$

A particular solution of it is ([MRRS05]) :

$$x_0(t) = \left(\frac{1}{2} - \frac{3}{2} \tanh\left(\frac{t}{2}\right), 0, -\frac{3}{2} \tanh\left(\frac{t}{2}\right) \left(1 - \tanh^2\left(\frac{t}{2}\right)\right), 0 \right)$$

The variational system along this particular solution is :

$$h'(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 + 3 \tanh^2(\frac{t}{2}) & 0 & 0 & 0 \\ 0 & -\frac{3}{2} + \frac{3}{2} \tanh^2(\frac{t}{2}) & 0 & 0 \end{pmatrix} h(t)$$

If one wants rational coefficients one makes the change $x = \tanh(t/2)$ and one gets

$$h'(x) = \begin{pmatrix} 0 & 0 & -\frac{2}{x^2-1} & 0 \\ 0 & 0 & 0 & -\frac{2}{x^2-1} \\ -\frac{2(3x^2-2)}{x^2-1} & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{pmatrix} h(x)$$

To reduce it to a normal variational system one makes the symplectic change of variable (see [MRR01a] or annex of [Bou00b], [Bou00a, BW03]) $h = PY$ where the first column of the matrix P is $x'_0(t)$ and P satisfies ${}^t P J P - J = 0$. One gets an equivalent linear differential system from which one extracts the rows and columns 2 and 4. One gets the normal variational system

$$Y'(x) = \begin{pmatrix} 0 & 3 \\ \frac{2}{x^2-1} & 0 \end{pmatrix} Y(x)$$

◇

In 1982, Ziglin ([Zig83a, Zig83b], [Zig83a, Zig83b]) established necessary conditions of meromorphic complete integrability on the *monodromy group* of the variational system. In 1995, Baider, R.C. Churchill, Rod and M. Singer ([CRS95, BCRS96]) and in 1998, J.J. Morales and J.P. Ramis ([MRR01a]) established necessary conditions of meromorphic complete integrability on the *differential Galois group* of the variational system.

4.3 Ziglin, Morales-Ramis

We may now restate lemma ^{weak-ziglin}4.4 in differential Galois terms (this is not Ziglin's original formulation, but rather the one from [CRS95, BCRS96] or [Aud01]).

Theorem 4.8 (Ziglin, strong form). *Consider a differential system $(S) : x' = F(x)$ and the variational system $[A] : Y' = AY$ along a solution $x_0(t)$ of (S) . If (S) admits m algebraically independent meromorphic first integrals, then $\text{Gal}([A])$ admits m algebraically independent rational invariants.*

For completely integrable hamiltonian systems, the additional involution property translates into the celebrated result of Morales and Ramis:

theoRM

Theorem 4.9 (J.J. Morales, J.P. Ramis). *Consider a hamiltonian system (S) and the variational system $[A] : Y' = AY$ along a solution $x_0(t)$ of (S) . If the system (S) is completely integrable, then $\text{Gal}([A])$ is virtually abelian.*

In next section we present a criterion deduced from Morales-Ramis theorem to get sufficient conditions of non complete integrability of a Hamiltonian system (see [Bou00b, Bou00a, BW03] for a preliminary version of it).

5 A criterion of non complete integrability of Hamiltonian systems

int-criteria

The analysis of the local solutions around singularities can give information on the integrability of nonlinear systems of ordinary differential equations such as Hamiltonian systems (Kovalevskaya and Painlevé test). However, the criterion we deduce from Morales Ramis theorem requires not only a local study of the normal variational equation but also a global study of it. In next subsection we give introductory examples illustrating that the study of the local solutions for the variational system will not be sufficient in general to give an obstruction to the virtual abelianity of the corresponding Galois group. In the second subsection, we give a non integrability criterion based on local and global considerations.

5.1 Examples

Before stating the criterion, let us focus on two introducing examples.

- Let us consider the first order linear differential symplectic system

$$Y' = \begin{pmatrix} 0 & \frac{1}{x} \\ 0 & 0 \end{pmatrix} Y$$

which can be also written

$$xY'(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y(x)$$

This system has formal solutions at zero with logarithmic terms as the local monodromy is given by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (whose single eigenvalue is 0 and who is not diagonalizable). However its differential Galois group is additive, so abelian.

One can notice that this system is equivalent to the linear differential equation

$$xy''(x) + y'(x) = 0$$

(using the cyclic vector $(0, 1)$).

Its space of solutions is generated by 1 and $\ln(x)$. The presence of logarithmic terms in the formal solutions does not suffice to imply the non virtual abelianity of the group G . Furthermore, one notices here

that the equation was reducible without being completely reducible. Indeed,

$$x\partial^2 + \partial = (x\partial + 1)\partial \neq \partial(x\partial + 1).$$

This is the crucial point that is at the origin of the proposition ^{prop2}5.2 and theorem ^{new}5.3 (or corollary ^{new2}5.5).

- Let us consider the normal variational system of the example 1

$$Y'(x) = \begin{pmatrix} 0 & 3 \\ \frac{2}{x^2-1} & 0 \end{pmatrix} Y(x)$$

The system can be written

$$(x-1)Y'(x) = \begin{pmatrix} 0 & 3(x-1) \\ \frac{2}{x+1} & 0 \end{pmatrix} Y(x) = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -\frac{1}{2} & 0 \end{pmatrix} (x-1) + \dots \right] Y(x)$$

So the monodromy matrix at 1 is given by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and there are logs at 1.

As the exponents at 1 and -1 are both equal to 0, the exponential solutions of the system correspond to the polynomial solutions. The exponents at infinity are 2 and -3 so one searches a polynomial solution of degree 3. One finds (using package ISOLDE in Maple) $^t(z(z^2 - 1), z^2 - \frac{1}{3})$ so the system is reducible.

One can also see the normal variational system as an equivalent linear differential equation

$$y''(x) - \frac{6}{x^2-1} y(x) = 0$$

The exponents at the singularities are 0 and 1 at the points -1 and 1 ; -3 and 2 at the point infinity, so an exponential solution is of the form $p(x)$ where p is a polynomial of degree 3. One finds one single solution $z^3 - z$ so the system is reducible without being completely reducible.

Again, the differential Galois group of this normal variational system is abelian, so the presence of logarithmic terms does not suffice to conclude to the non virtual abeliannity and Morales-Ramis theorem

does not enable to conclude here. Furthermore, this special example was given a complete answer in [MRRS05] where the use of higher variational equations enabled to conclude to the non complete integrability of the system.

5.2 A non-integrability criterion

We propose a criterion where added assumptions about the factorization of the equation suffice to make the group non virtually abelian. We derive this criterion using both the presentation of linear differential equations and the presentation of linear differential systems. First we need a lemma (established independently in [Bou00b, ?]).

lem1 **Lemma 5.1** ([Bou00b, BW03, ?]). *Let $G \subset GL(n, \mathbb{C})$ be a completely reducible linear algebraic group acting on $V = \mathbb{C}^n$. The following assertions are equivalent:*

- (a) G° is solvable,
- (b) G° is diagonalizable,
- (c) G° is abelian.

Proof. The implications (b) \Rightarrow (c) and (c) \Rightarrow (a) are immediate. Now assume that (a) holds. First we assume that V is irreducible under G . Because G° is solvable, it is triangularizable. In particular, all its elements have a common eigenvector v_1 . Let $g \in G$ and $h \in G^\circ$. Because G° is normal in G , we have $h(g(v_1)) = g(\tilde{h}(v_1))$ with $\tilde{h} \in G^\circ$. But, as v_1 is a common eigenvector for G° , we have $\tilde{h}(v_1) = \chi_{\tilde{h}} \cdot v_1$ with $\chi_{\tilde{h}} \in \mathbb{C}$. But now $h(g(v_1)) = g(\chi_{\tilde{h}} \cdot v_1) = \chi_{\tilde{h}} \cdot g(v_1)$ so all $g(v_1)$ are eigenvectors of G° . The linear space spanned by the $g(v_1)$ (for $g \in G$) is a subspace of V invariant under G . By irreducibility, it is equal to V . Because it is generated by the $g(v_1)$, G° acts diagonally on it.

Now, if V is reducible, it is a direct sum of subspaces which are irreducible under G , and we apply inductively the above reasoning to these irreducible summands. \square

If G is the differential Galois group of a linear differential equation $L(y) = 0$, then it is easier to find sufficient conditions of non-diagonality of G° using the local information that we can read in the formal solutions

prop2 **Proposition 5.2.** *Let $L(y) = 0$ be a homogeneous linear differential equation with Galois group G . Assume that the equation $L(y) = 0$ is completely reducible.*

1. If it has formal solutions around some point which contain logarithmic terms, then the connected component of the identity in the group G is not an abelian group.
2. If it has a non-trivial Stokes multiplier around some point (an irregular singularity), then the connected component of the identity in the group G is not an abelian group.

Proof. Let L be a differential operator of degree n with coefficients in a field k with differential Galois group G .

Let us assume that L is irreducible. If the group G^0 is abelian, then according to Lemma 5.1, it is diagonalizable. However, if a logarithmic term appears locally, then the corresponding local group has a non trivial unipotent subgroup ([?]). So the group G^0 contains a non trivial unipotent subgroup, which contradicts the diagonality of G^0 .

If L is completely reducible one concludes in the same way as in the proof of Lemma 5.1.

Part (b) is proved similarly because the presence of a Stokes multiplier induces a unipotent element in the differential Galois group (see [?] or [?] for definition and properties of Stokes multipliers). \square

We extend now the proposition 5.2 to a more general one:

Theorem 5.3. *Let L be a linear differential operator such that $L = L_1 L_2 \dots L_s$. If there exists i in $\{1, \dots, s\}$ such that L_i is completely reducible and the linear differential equation $L_i(y) = 0$ has formal solutions with logarithmic terms, then the differential Galois group G of the equation $L(y) = 0$ is not virtually abelian.*

Proof. If $s = 1$ one concludes using proposition 5.2.

All we now need is to prove that if there exists i in $\{1 \dots s\}$ such that the differential Galois group of $L_i(y) = 0$ is not virtually abelian then the differential Galois group of $L(y) = 0$ is also not virtually abelian. It is enough to prove this in the case of two factors ($s = 2$), the result then being an easy induction. So we assume that $L = L_1.L_2$.

Lemma 5.4. *Let K denote a Picard-Vessiot of k associated to L . If F is an intermediate differential Picard-Vessiot extension ($k \subset F \subset K$), then virtual abelianity of $\text{Gal}_d(K/k)$ implies that of $\text{Gal}_d(F/k)$.*

Proof. From [?] (lemma 5.10 page 38), we know that there is a natural surjection from $\text{Gal}_d(K/k)$ to $\text{Gal}_d(F/k)$. In particular, the abelianity of $\text{Gal}_d(K/k)^\circ$ implies that of $\text{Gal}_d(F/k)^\circ$. \square

We now finish the proof of the theorem. As the solutions of L_2 are solutions of L , K contains a Picard-Vessiot extension for L_2 and the result follows for L_2 from the lemma.

Now, L_2 can be viewed as a morphism from $Sol(L)$ to $Sol(L_1)$. Hence, K contains a full set of solutions of L_1 and we may apply the lemma again to conclude for L_1 .

We now conclude applying proposition ^{prop2} 5.2 to the L_i . □

Using the point of view of the systems, we get :

new2

Corollary 5.5. *Let $Y' = AY$ be a linear differential system such that it is equivalent to a block-triangular system $Z' =$*

$$Z' = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ \times & \ddots & 0 & \\ \times & \times & B_{s-1} & 0 \\ \times & \times & \times & B_s \end{pmatrix} Z$$

If there exists i in $\{1, \dots, s\}$ such that the system $v'_i = B_i v_i$ is completely reducible and has formal solutions with logarithmic terms then the differential Galois group G of the system $Y' = AY$ is not virtually abelian.

Remark 18 : As we already had noted in Proposition ^{prop2} 5.2 the conclusion of the above theorem still holds if we replace the presence of logarithms by the presence of a non trivial Stokes multiplier (or any recognizable non-diagonalizable element) at an irregular singularity. The reason is that these Stokes matrices (see [?] for example) induce a unipotent element in the differential Galois group and hence yield an obstruction to diagonalizability. ◇

6 An example: the Friedman-Robertson-Walker model

Let us consider the Hamiltonian of the system FRW (Friedmann-Robertson-Walker) cosmological model

$$H = \frac{1}{2} (\dot{a}^2 + \dot{\phi}^2) + \frac{1}{2} a^2 + \frac{1}{2} \phi^2 - \frac{1}{2} m^2 a^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + \frac{1}{4} \Lambda a^4$$

where a is the scale factor of the universe; Λ is the cosmological constant; ϕ is the scalar field with self-coupling constant λ and with mass m (which we will assume to be non zero).

6.1 The approach via constructive Morales-Ramis criteria

- We first select a particular solution X_0 to the hamiltonian system associated to H :

$$X_0 = (0, \phi_0, 0, \dot{\phi}_0)$$

where

$$\phi_0^2 = -\frac{2}{\lambda} \wp(t) - \frac{2}{3\lambda}$$

and $\wp(t) = \wp(t, g_2, g_3)$ is the Weierstrass function with parameters

$$g_2 = \frac{4}{3} + 4h\lambda \text{ and } g_3 = \frac{8}{27} + \frac{4}{3}h\lambda$$

- The variational equation can then be seen as the direct sum of two Lamé equations (the tangential part and the normal part of the variational equation). After the change of variable $x = \wp(t)$, we get the following normal variational equation:

$$(E) \quad \frac{4}{27} (3x-2)(3x+1)^2 y''(x) + \frac{2}{3} (3x-1)(3x+1) y'(x) + \left(\frac{2m^2}{\lambda} x + 1 + \frac{2m^2}{3\lambda} \right) y(x) = 0$$

- There are logarithmic terms in some formal solution of (E) at the singular point $-1/3$. Namely, the formal solutions at that point are $s_1 = \sqrt{3x+1} (1 + \dots)$ and $-\frac{m^2}{2\lambda} s_1 \ln(3x+1) + s_2$ where s_2 is a formal series.
- The equation (E) is irreducible if

$$\lambda \neq -\frac{2m^2}{(n+2)(n+1)}$$

with $n \in \mathbb{N}$.

Indeed, the equation (E) is irreducible if, and only if, it possesses no exponential solution

$$y(x) = (3x+1)^{-1/2} (3x-2)^e p(x)$$

where $p(x)$ is a polynomial of degree d ; $e = 0$ or $\frac{1}{2}$ (exponent at $2/3$) and

$$d - \frac{1}{2} + e + e_\infty = 0$$

with e_∞ , exponent at infinity.

A necessary condition for the existence of such a solution is that the indicial equation at infinity has a root $-d + \frac{1}{2}$ or $-d$ with d natural integer. This implies

$$\lambda = -\frac{2m^2}{2d(2d-1)}$$

or

$$\lambda = -\frac{2m^2}{2d(2d+1)}$$

which is equivalent to

$$\lambda = -\frac{2m^2}{(n+2)(n+1)} \text{ with } n \in \mathbb{N}$$

- Using theorem ^{new}5.3 and Morales-Ramis theorem, we conclude that if

$$\lambda \neq -\frac{2m^2}{(n+2)(n+1)}$$

with $n \in \mathbb{N}$ then the FRW system is not completely integrable.

6.2 Higher variational approaches

If

$$\lambda = -\frac{2m^2}{(n+2)(n+1)}$$

with $n \in \mathbb{N}$ then the variational equation is a direct sum of two Lamé-Hermite type equations. One then needs to go to *higher variational equations*. The variational equation is an approximation at order 1 for the linearization. Approximation at order 2, 3, etc, yields successive linear differential systems (with growing size). Their Galois groups G_k form a tower of groups; the result of Morales-Ramis-Simo is that each of these G_k must be virtually abelian.

Now, as the variational equation is a direct sum of two Lamé-Hermite type equations, one may reduce the Morales-Ramis-Simo criterion to a local study of the differential at zero. This local study of the third variational equations enables to conclude that the system is not completely integrable for $n \in \{2, \dots, 10\}$ (an additional global study is not required in this special case).

There are two particular cases : $\Lambda = \lambda$ when $n = 0$ and $n = 1$. For both these special cases an additional polynomial first integral I_n is found:

- $I_0 = \phi \dot{a} - a \dot{\phi}$
- $I_1 = \dot{a} \dot{\phi} + a \phi - \frac{m^2}{3}(a \phi^3 + a^3 \phi)$

If $n \geq 11$, we conjecture that the FRW system is not completely integrable.

7 Some properties to factorize symplectic linear differential systems

fact

The theorem ^{new}5.3 can be applied very easily when the coefficients of the normal variational system (or equation) are in $\mathbb{Q}(x)$ as there exist efficient algorithms to compute formal solutions of linear differential equations (systems) and to factorize linear differential operators (systems). However the equations we deal with here are *parameterized* equations and the existing algorithms are not adapted to this situation (problems of undecidability). In the following we see how the symplectic structure of the normal variational system can help the factorization of the normal variational system and hence the application of the criterion. This symplectic structure can be read very simply on the matrix of the normal variational system this is why we choose to keep the system instead of transforming it to a linear differential equation (using a cyclic vector computation).

7.1 Factorization in practice

The factorization of linear differential systems was studied and implemented in [Gri90, Bro96b]. Another family of algorithms uses the notion of eigenring [Sin96], [BP98]

In this section we first give a property of the eigenring in the symplectic case, which enables a simpler computation of it. Then we see that despite the presence of parameters one can look easily for some elements of the eigenring.

Let k denote a differential field (e.g $k = \mathbb{C}(x)$) and let A be infinitesimally symplectic.

7.2 Properties of the eigenring

The differential Galois group G of the differential system $Y' = AY$ is a subgroup of the symplectic group ([Aud01] page 65). Let V denote the

solution space of $Y' = AY$. If U is a fundamental solution matrix, then $J.U$ is a fundamental solution matrix for the dual system $Y' = -{}^tAY$. Hence, J is the matrix of the *musical isomorphism* between V and its dual V^* , i.e these are isomorphic G -modules.

The *Eigenring* of A is the set $\mathcal{E}(A) := \{P \in GL(2n, k) | P' = AP - PA\}$. It describes ([Sin96]) the ring $End_G(V)$ of G -endomorphisms of V ; in particular, the presence of a non-trivial element in the Eigenring provides a factorization of the differential system.

Now, as $End(V) = V \otimes V^*$ and $V \simeq_G V^*$, we see that $End(V) \simeq_G V \otimes V = Sym^2(V) \oplus Ext^2(V)$. The image of the identity in this morphism is the symplectic form (hence an element of $Ext^2(V)$) that is preserved by the symplectic group. In fact, this isomorphism can be made even more explicit:

eigenring

Proposition 7.1. *Let A be an infinitesimally symplectic matrix (${}^tAJ + JA = 0$) and let $\mathcal{E}(A)$ be its eigenring. Then*

$$\mathcal{E}(A) = \mathcal{E}_+(A) \oplus \mathcal{E}_-(A)$$

where

$$\mathcal{E}_+(A) = \{T \in \mathcal{E}(A), {}^tTJ + JT = 0\}$$

$$\mathcal{E}_-(A) = \{T \in \mathcal{E}(A), {}^tTJ - JT = 0\}.$$

Proof. Let A be an infinitesimally symplectic matrix (${}^tAJ + JA = 0$) and let T be in $\mathcal{E}(A)$. We first need the following lemma:

Lemma 7.2. *Each matrix of M of $M_n(\mathbb{C})$ can be uniquely written as the sum of a matrix M_1 such that $M_1J - J{}^tM_1 = 0$ and a matrix M_2 such that $M_2J + J{}^tM_2 = 0$ (M_2 is an infinitesimally symplectic matrix).*

Proof. Let M be in $M_n(\mathbb{C})$ and let us consider the two matrices $M_1 = -\frac{1}{2}(MJ + J{}^tM)J$ and $M_2 = -\frac{1}{2}(MJ - J{}^tM)J$.

One checks:

$$M = M_1 + M_2,$$

$$M_1J - J{}^tM_1 = 0,$$

$$M_2J + J{}^tM_2 = 0 \text{ (} M_2 \text{ is an infinitesimally symplectic matrix).}$$

□

According to the lemma, the matrix T can be written as a unique sum of two matrices T_1 and T_2 such that

$$T_1J - J{}^tT_1 = T_2J + J{}^tT_2 = 0.$$

It suffices to prove now that both T_1 and T_2 are in $\mathcal{E}(A)$.

As T is in $\mathcal{E}(A)$, it satisfies

$$T' = AT - TA$$

so

$$T'_1 + T'_2 = (AT_1 - T_1A) + (AT_2 - T_2A). \quad (7.4) \quad \boxed{\text{eqT1T2}}$$

A quick computation enables to see

$$T'_1 J - J^t T'_1 = (AT_1 - T_1A)J - J^t(AT_1 - T_1A) = 0$$

and

$$T'_2 J + J^t T'_2 = (AT_2 - T_2A)J + J^t(AT_2 - T_2A) = 0.$$

According to the previous lemma, the equality $\stackrel{\text{eqT1T2}}{(7.4)}$ enables to conclude:

$$T'_1 = AT_1 - T_1A$$

and

$$T'_2 = AT_2 - T_2A$$

so T_1 and T_2 are in $\mathcal{E}(A)$ and one has written the matrix T as the direct sum of two elements of $\mathcal{E}_-(A)$ and $\mathcal{E}_+(A)$. □

Remark 19 : If we define the Eigenring not over k but over a Picard-Vessiot extension K then the elements of $\text{Sym}^2(V)$ correspond to $\mathcal{E}_+(A)$ and those of $\text{Ext}^2(A)$ correspond to $\mathcal{E}_-(A)$. We recover, in particular, the well-known fact that the Lie algebra of the symplectic group is isomorphic to the symmetric square of its faithful representation of order $2n$. ◇

The elements of the eigenring of A correspond to rational solutions of the tensor system

$$Z' = (A \otimes I - I \otimes^t A) Z$$

(see [?] or [Sin96]).

If $A = (a_{i,j})_{1 \leq i,j \leq n}$ (with $n = 2m$ even), then the matrix $A \otimes I - I \otimes^t A$ is the $n^2 \times n^2$ matrix $(a_{i,j}I) - \text{diag}({}^t A)$. At first sight, one needs to compute a rational vector solution with n^2 components. However, thanks to proposition eigenring 7.1, one can reduce the number of unknown components. Indeed elements of $\mathcal{E}_+(A)$ (resp. $\mathcal{E}_-(A)$) have $m^2 + m(m+1)$ (resp. $m^2 + m(m-1)$) unknowns instead of $4m^2$. So one can reduce the number of unknowns while searching rational solutions of the tensor system.

7.3 Properties of the exponents at the singularities

To compute rational solutions of the tensor system

$$Z' = (A \otimes I - I \otimes {}^t A) Z$$

one needs to compute the exponents at the singular points of this system. When the matrix A depends on parameters then these exponents may also depend on the parameters, which makes the search of rational solutions difficult (in particular there are problems of indecidability). However if the matrix A is infinitesimally symplectic then some of the exponents of the tensor system will not depend on the parameters. In this case the search of the rational solutions of the tensor system (i.e. the elements of the eigenring of A) can be effectively achieved.

Proposition 7.3. *Let us consider a symplectic linear differential system. Let us assume that it is regular singular at the point 0 and that it can be written:*

$$xY'(x) = A(x)Y(x)$$

where the matrix $A(x)$ is an infinitesimally symplectic matrix defined at the point 0.

Then the exponents at the point 0 can be gathered in the following way : $(e_1, -e_1, e_2, -e_2)$.

Proof. The exponents at 0 are the eigenvalues of the matrix $A(0)$. But this matrix is infinitesimally symplectic (${}^t A(0) J + J A(0) = 0$) so its eigenvalues can be gathered in the following way : $(e_1, -e_1, e_2, -e_2)$. □

So in this particular case the structure of the matrix of the system enables to conclude that two sums of two exponents are equal to 0 and so are independant of the parameters.

As the exponents at the singularities of the tensor system are sums of exponents of the initial system, some of the exponents for the tensor system are independant of the parameters (even if all the exponents of the initial system depend on the parameters).

This proposition can be also seen with the point of view of the linear differential equations.

exponents

Proposition 7.4. *Let $L(y) = 0$ be a linear differential homogeneous equation and let G be its differential Galois group. Assume that the group G is symplectic. Let p be a singular point, and let r denote the ramification index at this point (note: if the point is regular singular, then $r = 1$).*

The generalized exponents at p are pairwise gathered in the following way : $(e, m - e)$ where m is a rational in $\frac{1}{r}\mathbb{Z}$.

Proof. Let us assume that we work at the point 0 and let $y(x) = x^e \Sigma(x)$ be a formal solution at the point 0 with $\Sigma(x) \in \bar{\mathbb{Q}}[\ln(x)]((x^{\frac{1}{r}}))$.

Let g be in the formal local Galois group G_0 at the point 0; then there exists c such that $g(x^e) = cx^e$ and c is an eigenvalue for g . But the group G is symplectic so $\frac{1}{c}$ is also an eigenvalue and there exists \tilde{e} such that $g(x^{\tilde{e}}) = \frac{1}{c} x^{\tilde{e}}$.

Then $g(x^{e-\tilde{e}}) = x^{e-\tilde{e}}$ and $x^{e-\tilde{e}} \in \bar{\mathbb{Q}}((x^{\frac{1}{r}}))$. So the difference between the two exponents, $e - \tilde{e}$, belongs to \mathbb{Q} . Furthermore if there is no ramification ($r = 1$), $e - \tilde{e}$ is an integer. \square

7.4 Application to the factorization of linear differential systems

If constructions like exterior powers, exponents in local solutions are obtained from the original local solutions by sums (and adding integers).

For example, in the second exterior powers, the exponents will be (modulo an integer) sums pairwise of exponents of the original solutions; symplecticity guarantees that some of these sums will not depend on parameters, thus making computations easier. Similar observations hold for other constructions (on the dual, because symplecticity makes it isomorphic to the original system, on the Endomorphism ring of the differential module, etc).

This makes factorization of symplectic parametrized differential systems much easier because *some* factors will be constructively computable. This turns out to be powerful in applications - although the success is still not fully explained.

Remark 20 : If the original Hamiltonian system admits a first integral, independent from the Hamiltonian whose initial form is linear (resp. quadratic), then the differential Galois group admits an invariant of degree 1 (resp. 2). In both case, this introduces an invariant element $Sym^2(V)$ and, by the above result, in the eigenring. So in such a case, the system will be reducible.

Similar observations can be found in the book of M. Audin. \diamond

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