

On non negative multivariate additive processes in reliability

Sophie Mercier



Summary

- 1 Introduction
- 2 Non negative multivariate Lévy processes
- 3 Multivariate additive processes
- 4 Multivariate additive processes as hazard rate increment processes
- 5 Elements of conclusion and perspectives

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- 1 **Introduction**
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Introduction

- Classical model for cumulative deterioration: *univariate Lévy process*, such as gamma process¹, inverse gaussian process² or Wiener process^{3,4}
- Nowadays, several deterioration indicators are often monitored at the same time → need for *multivariate deterioration models*⁵
- Common model for multivariate cumulative deterioration: *multivariate Lévy process*
- Multivariate Lévy process suffers from several drawbacks → need for other model
- Suggestion: consider *multivariate additive process*

¹J.M. Van Noortwijk. “A Survey of the Application of Gamma Processes in Maintenance”. In: *Reliability Engineering & System Safety* 94.1 (2009), pp. 2–21.

²Xiao Wang and Dihua Xu. “An Inverse Gaussian Process Model for Degradation Data”. In: *Technometrics* 52.2 (2010), pp. 188–197.

³Zhengxin Zhang et al. “Degradation data analysis and remaining useful life estimation: A review on Wiener-process-based methods”. In: *European Journal of Operational Research* 271.3 (2018), pp. 775–796.

⁴W. Kahle, S. Mercier, and C. Paroissin. *Degradation Processes in Reliability*. Hoboken, NJ, USA: John Wiley & Sons, Inc., 2016.

⁵Xiaolin Wang et al. “Modeling multivariate degradation processes with time-variant covariates and imperfect maintenance effects”. In: *Applied Stochastic Models in Business and Industry* 37.3 (2021), pp. 592–611.

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Definition

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A non negative multivariate process $\mathbf{X} = (\mathbf{X}(t))_{t \geq 0}$ is said to be a *Lévy process* if:

- $\mathbf{X}(0) = 0$ a.s.,
- \mathbf{X} has independent and **homogeneous** increments,
- \mathbf{X} is stochastically continuous,
- \mathbf{X} is right-continuous with left-side limits.

The process \mathbf{X} is componentwise *non decreasing*.

J. Bertoin. *Lévy processes*. Vol. 121. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1996, pp. x+265.

R. Cont and P. Tankov. *Financial modelling with jump processes*. Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. xvi+535.

A.E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.

K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Vol. 68. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1999, pp. xii+486.

Jump structure (1/2)

- Let $\mathbf{X} = (X_1, \dots, X_d) = (X_1(t), \dots, X_d(t))_{t \geq 0}$ be a *non negative Lévy process* with bounded variations and no drift,
- \mathbf{X} is a *pure jump* process and for each $t > 0$, the distribution of the random vector $\mathbf{X}(t)$ is *infinitely divisible*.
- Let $\rho_t(d\mathbf{x})$ be the *Lévy measure* of $\mathbf{X}(t)$. Then $\rho_t(d\mathbf{x}) = t \rho(d\mathbf{x})$ with $\rho(d\mathbf{x}) = \rho_1(d\mathbf{x})$.
- Both the jump structure and distribution of the *process* \mathbf{X} are entirely characterized by the *Lévy measure* $\rho(d\mathbf{x})$.
- Technical assumption:

$$\int_{\mathbb{R}_+^d} \min(\|\mathbf{x}\|_\infty, 1) \rho(d\mathbf{x}) < +\infty$$

Jump structure (2/2)

- Let

$$\begin{aligned} J((0, t] \times B) &= \# \{s > 0 : (s, \Delta \mathbf{X}(s)) \in (0, t] \times B\} \\ &= \sum_{s>0} \delta_{(s, \Delta \mathbf{X}(s))} ((0, t] \times B) \end{aligned}$$

for all $t > 0$ and $B \in \mathcal{B}(\mathbb{R}_d^+ \setminus \{\mathbf{0}\})$, where $\Delta \mathbf{X}(s) := \mathbf{X}(s) - \mathbf{X}(s^-)$.

- Then $J(ds, d\mathbf{x})$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+^d$ with intensity measure $ds \rho(d\mathbf{x})$.
- $J((0, t] \times B)$ is Poisson distributed with $\mathbb{E}[J((0, t] \times B)] = t \rho(B)$.
- Also

$$\mathbf{X}(t) = \int_{(0, t]} \mathbf{x} J(ds, d\mathbf{x}) = \sum_{s \leq t} \mathbf{x} \delta_{(s, \Delta \mathbf{X}(s))}(d\mathbf{x})$$

→ series representation useful for simulation purpose
(inverse Lévy, Bondesson, thinning, rejection, ...) ^{10,11}.

¹⁰J. Rosiński. "Series Representations of Lévy Processes from the Perspective of Point Processes". In: *Lévy processes*. Springer, 2001, pp. 401–415.

¹¹L. Bondesson. "On simulation from infinitely divisible distributions". In: *Adv. in Appl. Probab.* 14.4 (1982), pp. 855–869.

Laplace transform and moments

- Laplace transform of $\mathbf{X}(t)$:

$$\mathcal{L}_{\mathbf{X}(t)}(\mathbf{y}) = \mathbb{E} \left(e^{-\langle \mathbf{y}, \mathbf{X}(t) \rangle} \right) = \exp \left[-t \int_{\mathbb{R}_+^d} \left(1 - e^{-\langle \mathbf{y}, \mathbf{x} \rangle} \right) \rho(d\mathbf{x}) \right] \text{ for all } \mathbf{y} \in \mathbb{R}_+^d.$$

- Let $\rho_i(dx_i)$ and $\rho_{i,j}(dx_i, dx_j)$ be the Lévy measures of the marginal Lévy processes $(X_i(t))_{t \geq 0}$ and $(X_i(t), X_j(t))_{t \geq 0}$, respectively.
- Then:

$$\mathbb{E}(X_i(t)) = t \int_{\mathbb{R}_+} x_i \rho_i(dx_i) = t \mathbb{E}(X_i(1)),$$

$$\text{var}(X_i(t)) = t \int_{\mathbb{R}_+} x_i^2 \rho_i(dx_i) = t \text{var}(X_i(1)),$$

$$\text{cov}(X_i(t), X_j(t)) = t \int_{\mathbb{R}_+} x_i x_j \rho_{i,j}(dx_i, dx_j) = t \text{cov}(X_i(1), X_j(1))$$

for all $t > 0$ and all $i \neq j$.

- All these moments are *linear functions* in t .
- Ratio $\mathbb{E}(X_i(t)) / \text{var}(X_i(t))$ and $\rho_{X_i(t), X_j(t)}$ are *independent* on time.

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 - Examples
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Definition

Definition

A multivariate process $\mathbf{X} = (\mathbf{X}(t))_{t \geq 0}$ is said to be an *additive process* if:

- $\mathbf{X}(0) = 0$ a.s.,
- \mathbf{X} has independent increments,
- \mathbf{X} is stochastically continuous,
- \mathbf{X} is right-continuous with left-side limits.

Remark

Whenever the increments are homogeneous, an additive process is a Lévy process.

K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Vol. 68. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1999, pp. xii+486.

E. Çinlar. *Probability and stochastics*. Vol. 261. Graduate Texts in Mathematics. Springer, New York, 2011.

Jump structure (1/2)

- Let $\mathbf{X} = (X_1, \dots, X_d) = (X_1(t), \dots, X_d(t))_{t \geq 0}$ be a *non negative multivariate additive process* with bounded variations and no drift.
- \mathbf{X} is componentwise *non decreasing*.
- \mathbf{X} is a *pure jump* process and for each $t > 0$, the distribution of the random vector $\mathbf{X}(t)$ is *infinitely divisible*.
- Let $\rho_t(d\mathbf{x})$ be its *Lévy measure*, with

$$\int_{\mathbb{R}_+^d} \min(\|\mathbf{x}\|_\infty, 1) \rho_t(d\mathbf{x}) < +\infty.$$

- The distribution of the *random vector* $\mathbf{X}(t)$ is characterized by its *Lévy measure* $\rho_t(d\mathbf{x})$.

Jump structure (2/2)

- Let

$$J((0, t] \times B) = \# \{s > 0 : (s, \Delta \mathbf{X}(s)) \in (0, t] \times B\}$$

for all $t > 0$ and $B \in \mathcal{B}(\mathbb{R}_+^d \setminus \{\mathbf{0}\})$.

- Then J is a Poisson random measure with intensity measure $\tilde{\rho}$ defined by

$$\tilde{\rho}((0, t] \times B) = \rho_t(B) \text{ for all } B \in \mathcal{B}(\mathbb{R}_+^d \setminus \{\mathbf{0}\}) \text{ and all } t > 0.$$

- $J((0, t] \times B)$ is Poisson distributed with $\mathbb{E}[J((0, t] \times B)] = \tilde{\rho}((0, t] \times B)$.
- Also

$$\mathbf{X}(t) = \int_{(0, t]} \mathbf{x} J(ds, d\mathbf{x}) = \sum_{s \leq t} \mathbf{x} \delta_{(s, \Delta \mathbf{X}(s))}(d\mathbf{x})$$

→ series representation, useful for simulation purpose.

- Both the jump structure and the distribution of the whole process \mathbf{X} are entirely characterized by the *intensity measure* $\tilde{\rho}$ ¹⁴.

Remark

In the specific case of a Lévy process: $\rho_t = t \rho$ for all $t > 0$ and $\tilde{\rho}(ds, d\mathbf{x}) = ds \rho(d\mathbf{x})$, where ρ is the Lévy measure of the process \mathbf{X} .

¹⁴K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Vol. 68. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1999, pp. xii+486.

Laplace transform and moments

- Laplace transform of $\mathbf{X}(t)$:

$$\mathcal{L}_{\mathbf{X}(t)}(\mathbf{y}) = \mathbb{E} \left(e^{-\langle \mathbf{y}, \mathbf{X}(t) \rangle} \right) = \exp \left[- \int_{\mathbb{R}_+^d} \left(1 - e^{-\langle \mathbf{y}, \mathbf{x} \rangle} \right) \rho_t(d\mathbf{x}) \right]$$

for all $\mathbf{y} \in \mathbb{R}_+^d$.

- Let $\rho_{t,i}(dx_i)$ and $\rho_{t,i,j}(dx_i, dx_j)$ be the Lévy measures of the marginal r.v. $X_i(t)$ and $(X_i(t), X_j(t))$, respectively.
- Then:

$$\mathbb{E}(X_i(t)) = \int_{\mathbb{R}_+} x_i \rho_{t,i}(dx_i),$$

$$\text{var}(X_i(t)) = \int_{\mathbb{R}_+} x_i^2 \rho_{t,i}(dx_i),$$

$$\text{cov}(X_i(t), X_j(t)) = \int_{\mathbb{R}_+} x_i x_j \rho_{t,i,j}(dx_i, dx_j) \text{ for all } t > 0 \text{ and all } i \neq j.$$

- Ratio $\mathbb{E}(X_i(t)) / \text{var}(X_i(t))$ or $\rho_{X_i(t), X_j(t)}$ are *dependent* on time, except if $\rho_t(d\mathbf{x}) = \Lambda(t) \rho_1(d\mathbf{x})$ that is if $\tilde{\rho}(dt, d\mathbf{x}) = \lambda(t) dt \rho(d\mathbf{x})$.

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Time-scaled Lévy processes

- Let $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a *time-scaling function* such that:
 - λ is continuous,
 - $\lambda(t) > 0$ for all $t > 0$,
 - $\Lambda(t) = \int_0^t \lambda(s) ds < +\infty$ for all $t > 0$,
 - $\lim_{t \rightarrow \infty} \Lambda(t) = \infty$.
- Let $\mathbf{Y} = (\mathbf{Y}(t))_{t \geq 0}$ be a multivariate non negative Lévy process with Lévy measure ρ .
- We set $\mathbf{X}(t) = \mathbf{Y}(\Lambda(t))$ for all $t \geq 0$.
- Then \mathbf{X} is a *time-scaled Lévy process* and it is a specific *additive process* with intensity measure $\tilde{\rho}(ds, d\mathbf{x}) = \lambda(s) ds \rho(d\mathbf{x})$.
- Hence, the ratio $\mathbb{E}(X_i(t)) / \text{var}(X_i(t))$ and $\rho_{X_i(t), X_j(t)}$ are *independent* on time

-> *Time-scaling does not enlarge that much the modeling ability of a Lévy process.*

Extended Lévy process

- Let $\mathbf{Y} = (\mathbf{Y}(t))_{t \geq 0}$ be a *Lévy process* with Lévy measure ρ .
- We set

$$\mathbf{X}(t) = \int_{(0,t]} \mathbf{b}(u) d\mathbf{Y}(u) = \left(\int_{(0,t]} b_1(u) dY_1(u), \dots, \int_{(0,t]} b_d(u) dY_d(u) \right)$$

for all $t \geq 0$, where $\mathbf{b}(u) = (b_1(u), \dots, b_d(u))$ for all $u \geq 0$ is a componentwise positive and continuous function on $(\mathbb{R}_+^*)^d$.

- $\mathbf{X} = (\mathbf{X}(t))_{t \geq 0}$ is said to be an *extended Lévy process*.
- \mathbf{X} is an additive process and if $\rho(d\mathbf{x}) = \rho(\mathbf{x}) d\mathbf{x}$ (to simplify), its *intensity measure* is

$$\tilde{\rho}(ds, d\mathbf{u}) = ds \rho \left(\frac{u_1}{b_1(s)}, \dots, \frac{u_d}{b_d(s)} \right) \frac{1}{b_1(s) \times \dots \times b_d(s)} d\mathbf{u}.$$

Remark

Can be extended to time-scaled Lévy process.

Extended Lévy process, references

- Univariate extended gamma process^{15,16,17,18}
- Extended multivariate Lévy process¹⁹
- Simulation based on series expansion²⁰, where they consider $\mathbf{X}(t) = \int_{(0,t]} \mathbf{b}(t, u) d\mathbf{Y}(u)$ (shot noise process but not an additive process).

¹⁵E. Çinlar. “On a generalization of gamma processes”. In: *J. Appl. Probab.* 17 (1980), pp. 467–480.

¹⁶RL. Dykstra and P. Laud. “A Bayesian nonparametric approach to reliability”. In: *The Annals of Statistics* 9.2 (1981), pp. 356–367.

¹⁷M. Guida, F. Postiglione, and G. Pulcini. “A time-discrete extended gamma process for time-dependent degradation phenomena”. In: *Reliability Engineering & System Safety* 105.0 (2012), pp. 73–79.

¹⁸Z. Al Masry, S. Mercier, and G. Verdier. “Approximate simulation techniques and distribution of an extended Gamma process”. In: *Methodology and Computing in Applied Probability* 19 (2017), pp. 213–235.

¹⁹S. Mercier and C. Sangüesa. “A general multivariate lifetime model with a multivariate additive process as conditional hazard rate increment process”. In: *Metrika* 86.1 (2023), pp. 91–129.

²⁰S. Yuan and R. Kawai. “Numerical aspects of shot noise representation of infinitely divisible laws and related processes”. In: *Probability Surveys* 18 (2021), pp. 201–271, **70 p.**

Extended Lévy process, example (1/2)

- Let \mathbf{Y} be a bivariate Lévy process with Lévy measure ρ .
- We have

$$\begin{aligned}\text{var}(X_1(t)) &= \int_{\mathbb{R}_+} x_1^2 \rho_{t,1}(dx_1) = \int_{\mathbb{R}_+} x_1^2 \int_0^t \tilde{\rho}(ds, dx_1) \\ &= \int_{\mathbb{R}_+} \int_0^t x_1^2 \rho\left(\frac{x_1}{b_1(s)}\right) \frac{1}{b_1(s)} ds dx_1 \\ &= \int_0^t b_1^2(s) ds \int_{\mathbb{R}_+} u_1^2 \rho(u_1) du_1 \\ &= \text{var}(Y_1(1)) \int_0^t b_1^2(s) ds\end{aligned}$$

- In the same way

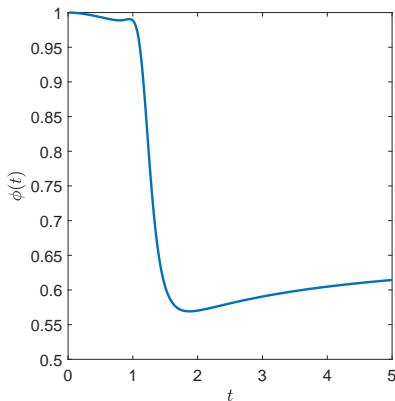
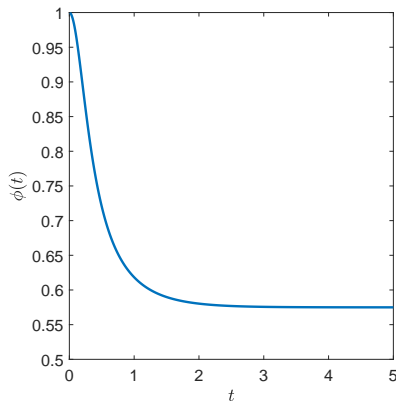
$$\text{cov}(X_1(t), X_2(t)) = \text{cov}(Y_1(1), Y_2(1)) \int_0^t b_1(s) b_2(s) ds$$

- Hence $\rho_{(X_1(t), X_2(t))} = \rho_{(Y_1(1), Y_2(1))} \phi(t)$ with

$$\phi(t) = \frac{\int_0^t b_1(s) b_2(s) ds}{\left(\int_0^t b_1^2(s) ds\right)^{\frac{1}{2}} \left(\int_0^t b_2^2(s) ds\right)^{\frac{1}{2}}}.$$

Extended Lévy process, example (2/2)

We take $b_1(s) = e^s$, $b_2(s) = e^{10s}$ (left) and $b_1(s) = 1 + s$, $b_2(s) = 1 + 10s$ (right).



→ $\rho_{X_1(t), X_2(t)}$ is *dependent* on time and can have any behaviour.

The same for $\mathbb{E}(X_1(t)) / \text{var}(X_1(t))$ or $\mathbb{E}(X_1(t)) / \mathbb{E}(X_2(t)) \dots$

A shock model (1/2)

- Shocks arrive according to a *non homogeneous Poisson process* $(N_t)_{t \geq 0}$ with intensity function $\lambda(\cdot)$ and points $\{T_n, n \in \mathbb{N}^*\}$
- Let $Q(t, d\mathbf{u})$ be a *transition probability kernel* from $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ to $(\mathbb{R}_+^k, \mathcal{B}(\mathbb{R}_+^k))$
- Let $(\mathbf{U}_n)_{n \in \mathbb{N}^*}$ be a sequence of non negative k -dimensional random vectors such that given the T_n 's, $n = 1, 2, \dots$, the \mathbf{U}_n 's, $n = 1, 2, \dots$ are conditionally independent with conditional distribution $Q(T_n, d\mathbf{u})$
- Let $\mathbf{h} : (\mathbb{R}_+ \times \mathbb{R}_+^k, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+^k)) \rightarrow (\mathbb{R}_+^d, \mathcal{B}(\mathbb{R}_+^d))$ be a measurable function with

$$\mathbf{h}(t, \mathbf{u}) = (h_1(t, \mathbf{u}), \dots, h_d(t, \mathbf{u})).$$

We set

$$\mathbf{X}(t) = \sum_{n=1}^{N_t} \mathbf{h}(T_n, \mathbf{U}_n) \text{ for all } t > 0$$

(shot noise process).

Remark

Specific case: non-homogeneous compound Poisson process.

A shock model (2/2)

Proposition

Under technical assumptions, $\mathbf{X} = (\mathbf{X}_t)_{t \geq 0}$ is an additive process with intensity measure

$$\tilde{\rho}(ds, d\mathbf{x}) = \lambda(s) ds Q_{\mathbf{h}(s, \cdot)}(s, d\mathbf{x}),$$

where

$$Q_{\mathbf{h}(s, \cdot)}(s, B) = \int \mathbf{1}_B(\mathbf{h}(s, \mathbf{u})) Q(s, d\mathbf{u})$$

for all $B \in \mathcal{B}(\mathbb{R}_+^k)$, that is $Q_{\mathbf{h}(s, \cdot)}(s, d\mathbf{x})$ stands for the *push-forward measure* of $Q(s, d\mathbf{u})$ by the function $\mathbf{h}(s, \cdot)$.

Remark

It is possible to include possibly fatal shocks, with time-dependence and dependence between components for the probability of the fatality of a shock.

S. Mercier and C. Sangüesa. "A general multivariate lifetime model with a multivariate additive process as conditional hazard rate increment process". In: *Metrika* 86.1 (2023), pp. 91–129.

A bivariate shock model, example (1/2)

- We take:

- $\mathbf{h}(T_n, \mathbf{U}_n) = \left((1 + T_n)^{\alpha_1} U_n^{(1)}, (1 + T_n)^{\alpha_2} U_n^{(2)} \right),$
- $U_n^{(1)} = V_n^{(1)} + V_n^{(3)}$ and $U_n^{(2)} = V_n^{(2)} + V_n^{(3)},$
- $\left[V_n^{(1)} | T_n \right] \stackrel{\mathcal{D}}{=} \left[V_n^{(2)} | T_n \right] \sim \Gamma(\beta T_n, 1)$ and $\left[V_n^{(3)} | T_n \right] \sim \Gamma((1 - \beta) T_n, 1),$
- $V_n^{(1)}, V_n^{(2)}, V_n^{(3)}$ are conditionally independent given $T_n.$

- Hence $\left[U_n^{(1)} | T_n \right] \stackrel{\mathcal{D}}{=} \left[U_n^{(2)} | T_n \right] \sim \Gamma(T_n, 1).$

- We get

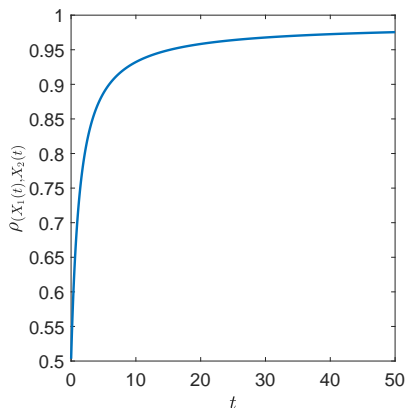
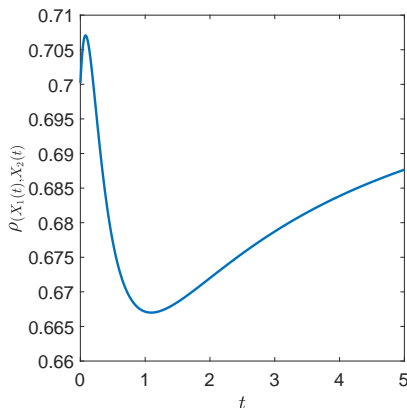
$$\begin{aligned} \text{Var}(X_i(t)) &= g(3 + 2\alpha_i, t) - g(2 + 2\alpha_i, t), \\ \text{cov}(X_1(t), X_2(t)) &= g(3 + \alpha_1 + \alpha_2, t) - (1 + \beta) g(2 + \alpha_1 + \alpha_2, t) \\ &\quad + \beta g(1 + \alpha_1 + \alpha_2, t) \end{aligned}$$

with

$$g(\alpha) = \frac{(1+t)^\alpha - 1}{\alpha}.$$

A shock model, example (2/2)

We take $\alpha_1 = 10, \alpha_2 = 0.5, \beta = 0.3$ (left) and $\alpha_1 = 1.25, \alpha_2 = 0.5, \beta = 0.5$ (right).



→ $\rho_{X_1(t), X_2(t)}$ is *dependent* on time and can have any behaviour.

The same for $\mathbb{E}(X_1(t)) / \text{var}(X_1(t))$ or $\mathbb{E}(X_1(t)) / \mathbb{E}(X_2(t)) \dots$

First elements of conclusion about additive processes

- Additive processes are constructed from Poisson random measures in a *very similar way* as Lévy processes
- Unlike (time-scaled) Lévy processes, they allow *time-dependent ratios* $\mathbb{E}(X_i(t)) / \text{var}(X_i(t))$, $\mathbb{E}(X_i(t)) / \mathbb{E}(X_j(t))$, ... and *time-dependent correlation coefficients* $\rho_{X_i(t), X_j(t)}$
→ *increased modeling ability* of additive processes with respect to Lévy processes.
- They can be used for modeling:
 - *multivariate cumulative deterioration*,
 - *hazard rate process*, in a Bayesian fashion.

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The point

- d identical components subject to a *common stressful external environment*,
- lifetime of each component: characterized by its *hazard (failure) rate*,
- impact of the environment: *increases* the d components hazard rates,
- the d hazard rate increments are *dependent* and *increase* over time,
- they are jointly modelled by a *d -dimensional non negative additive process*.

S. Mercier and C. Sangüesa. "A general multivariate lifetime model with a multivariate additive process as conditional hazard rate increment process". In: *Metrika* 86.1 (2023), pp. 91–129.

Notations and assumptions

- $h_i(t)$: *intrinsic hazard rate* function of the i -th component, $i = 1, \dots, d$ (out of the stressing environment),
- $H_i(t)$: *cumulative intrinsic hazard rate* function of the i -th component, $i = 1, \dots, d$,
- $\mathbf{X} = (X_1, \dots, X_d) = (X_1(t), \dots, X_d(t))_{t \geq 0}$: non negative multivariate additive process with bounded variations and no drift,
- *conditional hazard rate* of the i -th component given \mathbf{X} :

$$h_{X_i}(t) = h_i(t) + X_i(t)$$

for all $t > 0$, all $i = 1, \dots, d$,

- given \mathbf{X} , the components are independent.

The process \mathbf{X} is called *Conditional Hazard Rate Increment (CHRI) process*.

Remark

All the dependence between components is due to the dependence between the X_i 's, $i = 1, \dots, d$.

The multivariate lifetime

- τ_i : lifetime of the i -th component, $i = 1, \dots, d$,
- $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$: the multivariate lifetime,
- $\bar{F}_{\boldsymbol{\tau}}(\mathbf{s})$: the multivariate survival function with

$$\bar{F}_{\boldsymbol{\tau}}(\mathbf{s}) = \mathbb{P} \left[\bigcap_{i=1}^d \{\tau_i > \mathbf{s}_i\} \right]$$

for all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}_+^d$.

The point: computation of $\bar{F}_{\boldsymbol{\tau}}(\mathbf{s})$.

Preliminary analysis

We have:

$$\begin{aligned}\bar{F}_\tau(\mathbf{s}) &= \mathbb{P}\left(\bigcap_{i=1}^d \{\tau_i > \mathbf{s}_i\}\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(\bigcap_{i=1}^d \{\tau_i > \mathbf{s}_i\} \mid \mathbf{X}\right)\right] \\ &= \mathbb{E}\left[\prod_{i=1}^d \underbrace{\mathbb{P}(\tau_i > \mathbf{s}_i \mid \mathbf{X})}_{e^{-H_{X_i}(\mathbf{s}_i)}}\right] \quad (\text{conditional independence assumption})\end{aligned}$$

with

$$H_{X_i}(\mathbf{s}_i) = \int_0^{\mathbf{s}_i} h_{X_i}(t) dt = \int_0^{\mathbf{s}_i} (h_i(t) + X_i(t)) dt = H_i(\mathbf{s}_i) + \int_0^{\mathbf{s}_i} X_i(t) dt$$

and

$$\bar{F}_\tau(\mathbf{s}) = \mathbb{E}\left[\prod_{i=1}^d e^{-H_i(\mathbf{s}_i)} e^{-\int_0^{\mathbf{s}_i} X_i(t) dt}\right] = e^{-\sum_{i=1}^d H_i(\mathbf{s}_i)} \times \mathbb{E}\left[e^{-\sum_{i=1}^d \int_0^{\mathbf{s}_i} X_i(t) dt}\right].$$

Main result

Theorem

Let $\mathbf{X} = (\mathbf{X}(t))_{t \geq 0}$ be a multivariate non negative additive process with bounded variations, no drift and intensity measure $\tilde{\rho}(dt, d\mathbf{x})$.

Then:

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s}) - K(\mathbf{s})}$$

for all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}_+^d$ where $H(\mathbf{s}) = \sum_{i=1}^d H_i(s_i)$ and

$$K(\mathbf{s}) = \int_{(0, \infty) \times (\mathbb{R}_+^d \setminus \{\mathbf{0}\})} \left(1 - e^{-\sum_{i=1}^d (s_i - t)^+ x_i}\right) \tilde{\rho}(dt, d\mathbf{x}).$$

Extends 23, 24, 25, 26

²³Y. Kebir. "On hazard rate processes". In: *Naval Research Logistics* 38.6 (1991), pp. 865–876. ISSN: 0894-069X; 1520-6750/e.

²⁴J.H. Cha and J. Mi. "On a stochastic survival model for a system under randomly variable environment". In: *Methodology and Computing in Applied Probability* 13.3 (2011), pp. 549–561.

²⁵S. Mercier and H.H. Pham. "A bivariate failure time model with random shocks and mixed effects". In: *Journal of Multivariate Analysis* 153 (2017), pp. 33–51.

²⁶M.L. Wenocur. "A reliability model based on the gamma process and its analytic theory". English. In: *Advances in Applied Probability* 21.4 (1989), pp. 899–918. ISSN: 0001-8678.

Time-scaled Lévy process as CHRI process (1/2)

Corollary

Let \mathbf{X} be a time-scaled Lévy process with $\mathbf{X}(t) = \mathbf{Y}(\Lambda(t))$, and let ρ be the Lévy measure of \mathbf{Y} .

Then

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s}) - K(\mathbf{s})}$$

with

$$\begin{aligned} K(\mathbf{s}) &= \int_F \left(1 - e^{-\sum_{i=1}^d (s_i - t)^+ x_i}\right) \lambda(t) dt \rho(d\mathbf{x}) \\ &= - \int_0^\infty \ln \left(\mathcal{L}_{\mathbf{Y}(1)} \left((s_1 - t)^+, \dots, (s_d - t)^+ \right) \right) \lambda(t) dt. \end{aligned}$$

for all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}_+^d$.

Time-scaled Lévy process as CHRI process (2/2)

Corollary

Let \mathbf{X} be a time-scaled Lévy process with $\mathbf{X}(t) = \mathbf{Y}(\Lambda(t))$, and let ρ be the Lévy measure of \mathbf{Y} .

Assume the dependence between the marginal processes of \mathbf{Y} to be modeled by a Lévy copula \mathbf{C} .

Then

$$\bar{F}_{\tau}(\mathbf{s}) = \exp \left[\int_{\mathbb{R}_+^2} \mathbf{C}(U_1(s_1), U_2(s_2)) \gamma_{\mathbf{v}, \lambda}(\mathbf{s}) ds_1 ds_2 \right] \bar{F}_{\tau_1}(v_1) \bar{F}_{\tau_2}(v_2)$$

with

$$U_i(s_i) = \int_{s_i}^{\infty} \rho_i(s_i) ds_i \text{ for } i = 1, 2,$$

$$\gamma_{\mathbf{v}, \lambda}(\mathbf{s}) = \int_0^{\min(v_1, v_2)} (v_1 - t)(v_2 - t) e^{-(v_1 - t)s_1 - (v_2 - t)s_2} \lambda(t) dt$$

for all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}_+^d$.

Univariate CHRI Lévy process, example (1/2)

Let $h(v) = 0$ for all $v \geq 0$.

- *Homogeneous univariate gamma process*: $X(t) \sim \Gamma(at, b)$ with

$$f_{\Gamma(at,b)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx} \mathbf{1}_{\mathbb{R}_+}(x)$$

Then:

$$\bar{F}_\tau(v) = e^{at} \left(\frac{b}{b+v} \right)^{a(b+v)},$$

$$h_\tau(v) = a \ln \left(\frac{b+v}{b} \right).$$

- *Homogeneous inverse Gaussian process*: $X(t) \sim IG(at, b)$ with

$$f_{IG(at,b)}(x) = \frac{at}{(2\pi x^3)^{\frac{1}{2}}} \exp \left(-\frac{b^2}{2} x + abt - \frac{(at)^2}{2} \frac{1}{x} \right) \mathbf{1}_{\mathbb{R}_+^*}(x)$$

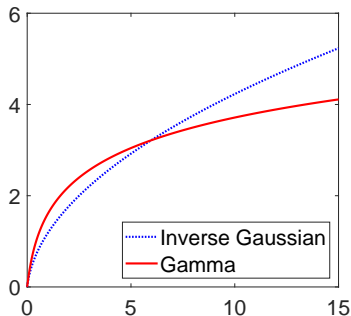
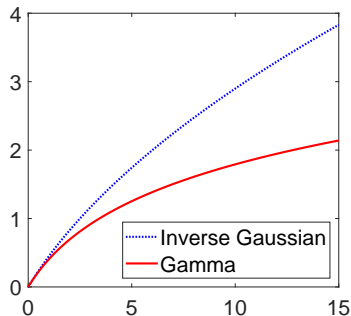
Then:

$$\bar{F}_\tau(v) = \exp \left(-a \left(\frac{1}{3} \left((b^2 + 2v)^{\frac{3}{2}} - b^3 \right) - bv \right) \right)$$

$$h_\tau(v) = a \left(\sqrt{b^2 + 2v} - b \right).$$

Univariate CHRI Lévy process, example (2/2)

- Both gamma and inverse Gaussian processes share the same mean $\mathbb{E}(X(t)) = at/b$.
- Plots of $h_\tau(v)$ for $a = 1$ and $b = 2$ (left), $b = 0.25$ (right):



Gamma: $h_\tau(v) \underset{v \rightarrow \infty}{\sim} a \ln(v)$,

Inverse Gaussian: $h_\tau(v) \underset{v \rightarrow \infty}{\sim} a\sqrt{2v}$.

Extended Lévy process as CHRI process

Corollary

Let $\mathbf{X} = (\mathbf{X}(t))_{t \geq 0}$ be an extended Lévy process with

$$\mathbf{X}(t) = \int_{(0,t]} \mathbf{b}(u) d\mathbf{Y}(u)$$

and let ρ be the Lévy measure of \mathbf{Y} . Then

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s}) - K(\mathbf{s})}$$

with

$$\begin{aligned} K(\mathbf{s}) &= \int_F \left(1 - e^{-\sum_{i=1}^d (s_i - t)^+ b_i(t) x_i} \right) dt \rho(d\mathbf{x}) \\ &= - \int_0^\infty \ln \left(\mathcal{L}_{\mathbf{Y}(1)} \left((s_1 - t)^+ b_1(t), \dots, (s_d - t)^+ b_d(t) \right) \right) dt \end{aligned}$$

for all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}_+^d$.

Extended Lévy process as CHRI process, example

- Let $(Z_i(t))_{t \geq 0}$, $i = 1, 2, 3$ be three independent univariate time-scaled gamma processes with $Z_i(t) \sim \Gamma(a_i \wedge(t), 1)$.
- We set

$$\begin{cases} Y_1(t) = Z_1(t) + Z_3(t) \\ Y_2(t) = Z_2(t) + Z_3(t) \end{cases}$$

- Then $\mathbf{Y} = (Y_1, Y_2)$ is a bivariate time-scaled Lévy process.
- With $\mathbf{X}(t) = \int_{(0,t]} \mathbf{b}(u) d\mathbf{Y}(u)$, we get

$$\bar{F}_\tau(\mathbf{s}) = e^{-H(\mathbf{s}) - K(\mathbf{s})}$$

with

$$\begin{aligned} K(\mathbf{s}) = & a_1 \int_0^{s_1} \lambda(t) \ln(1 + (s_1 - t) b_1(t)) dt + a_2 \int_0^{s_2} \lambda(t) \ln(1 + (s_2 - t) b_2(t)) dt \\ & + a_3 \int_0^{\max(s_1, s_2)} \lambda(t) \ln(1 + (s_1 - t)^+ b_1(t) + (s_2 - t)^+ b_2(t)) dt \end{aligned}$$

Shock models

Proposition

For the shock model (under additional technical assumptions), we have

$$\bar{F}_\tau(\mathbf{v}) = e^{-H(\mathbf{v})} \exp \left[- \int_F \left(1 - e^{-\sum_{i=1}^d (v_i - s)^+ h_i(t, \mathbf{x})} \right) \lambda(s) ds Q(s, d\mathbf{x}) \right]$$

for all $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}_+^d$.

Extends 27, 28

²⁷ S. Mercier and H.H. Pham. "A bivariate failure time model with random shocks and mixed effects". In: *Journal of Multivariate Analysis* 153 (2017), pp. 33–51.

²⁸ J.H. Cha and J. Mi. "On a stochastic survival model for a system under randomly variable environment". In: *Methodology and Computing in Applied Probability* 13.3 (2011), pp. 549–561.

Summary

- 1 Introduction
- 2 Non negative multivariate Lévy processes
- 3 Multivariate additive processes
- 4 Multivariate additive processes as hazard rate increment processes
- 5 Elements of conclusion and perspectives**

Elements of conclusion

- Additive processes allow an *increased modeling ability* with respect to (time-scaled) Lévy processes.
- They can be used for modeling for modeling *hazard rate process* in a Bayesian fashion → *full form* expressions for the distribution of the multivariate lifetime.
- They can also be used for modeling *cumulative deterioration* but in that case, technical tools are required for:
 - the computation of the joint distribution of $\mathbf{X}(t)$ (not available in full form, as for Lévy processes)
 - statistical inference of \mathbf{X} .

Perspectives

- For the computation of the joint distribution of $\mathbf{X}(t)$:
 - *Simulation procedures* based on series expansions could be explored^{29,30}.
- For the statistical *parametric* inference of \mathbf{X} based on *low frequency* observation:
 - *Generalized Method of Moments (GMM)* based on the Laplace transform could be explored³¹.
- In the context of a *high frequency* observation (mostly continuous monitoring):
 - One could think about extending some results for non parametric estimation of the Lévy measure^{32,33} of a Lévy process in the additive process context (to be looked at!).

²⁹S. Yuan and R. Kawai. “Numerical aspects of shot noise representation of infinitely divisible laws and related processes”. In: *Probability Surveys* 18 (2021), pp. 201–271.

³⁰Z. Al Masry, S. Mercier, and G. Verdier. “Approximate simulation techniques and distribution of an extended Gamma process”. In: *Methodology and Computing in Applied Probability* 19 (2017), pp. 213–235.










³¹Z. Al Masry, S. Mercier, and G. Verdier. “Generalized method of moments for an extended gamma process”. In: *Communications in Statistics-Theory and Methods* 47.15 (2018), pp. 3687–3714.

³²A. Bücher and M. Vetter. “Nonparametric inference on Lévy measures and copulas”. In: *Ann. Statist.* 41.3 (2013), pp. 1485–1515.

³³P. Tankov. “Lévy Copulas: Review of Recent Results”. In: *The Fascination of Probability, Statistics and their Applications: In Honour of Ole E. Barndorff-Nielsen*. Ed. by Mark Podolskij et al. Springer, Cham, 2016, pp. 127–151.

THANK YOU FOR YOUR ATTENTION!








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