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Information, Insider Trading, Executive Reload Stock Options, Incentives, and Regulation

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Abstract

We introduce a theoretical model of executives with insider information (insider-executives) granted incentivizing executive stock options (ESO). We show that while insider-executives optimize their wealth, using their insider information nullifies ESO incentives, misaligning their and shareholders' interests. We offer realigning methods: granting insider-executives reload stock options (RSO) and imposing blackout trading periods (blackouts). Effective blackouts keep insider-executives incentivized without being overly restrictive, i.e., without reducing their welfare below that of outsiders. We introduce RSO pricing for insider-executives and offer policy implications: reestablishing the currently out-of-favor RSO, and allowing firms, not regulators, to set blackout periods on securities they issue.

JEL Codes: G11, G13, C02, C61

Keywords: Executive Stock Options, Insider Information, Constrained Portfolio Optimization, Non-Hedgeable, Non-Transferable, Reload, Enlarged Filtration

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1 Introduction

Following Pikovsky and Karatzas (1996), we introduce a theoretical model in which executives, privy to insider information that improves the predictability of future returns, optimize their personal wealth portfolios while being incentivized by non-transferable non-hedgeable American executive stock options—executive stock options (ESO). Future events that insiders have information on could be earnings announcements, mergers, acquisitions, capital restructuring, and research, and development progress and outcomes.

We demonstrate that, while trading their portfolios, insider-executives' use of their insider information, even if noisy, on stock return shocks occurring at a known future time generates infinite derived utility. For an example where utility is finite with other insider information, see Grorud and Pontier (1998, 1999). These utility gains nullify ESO incentives, consequently nullifying the alignment of these executives' and stockholders' interests.¹

We show that blackout trading periods (blackouts)² imposed on executives reduce their insider information benefits and may restore the ESO incentivizing mechanism if set effectively. We identify effective blackouts by defining blackout's lower and upper duration bounds. Lower bounds define the shortest blackout duration. If lower bounds are violated, insider-executives have an information advantage that would concern regulators, and the advantage can also make the ESO incentives negligible. Upper bounds define the longest blackout duration. If upper bounds are violated, then the instant exercise of all ESO once vested can happen, which eliminates the incentives that align these executives' and shareholders' interests. What's more, it may result in executives' derived utility under insider information to be less than outsiders' derived utility with the same initial total wealth but no blackouts. We call blackouts within the boundaries “effective”.

We note that idiosyncratic insider information requires blackouts that apply to a firm's stock

only, whereas insider information with systematic components might require blackouts that apply to all stocks.

However, the imposition of a blackout trading period, by itself, is not a satisfactory solution for several reasons. First, effective blackouts might not always exist. It may be possible for the upper bound to occur later than the lower bound, causing the effective blackout set to be a null one. Second, effective blackout duration depends on executives' specific attributes: wealth level and wealth composition, as well as the type and precision of the insider information. The type of insider information could be the terminal value of the risk source (equivalently the increments from the current value, which is the focus of this study), the upper bound at a fixed time, the local time at a fixed time, the last zero before a fixed time, a first hitting time, etc., or a combination of these. See Mansuy and Yor (2006, p. 34). Third, even for a particular executive, the required duration to make blackouts effective changes dynamically.³ Fourth, job termination dates have an overriding impact on effective blackout duration. Finally, blackout effectiveness depends on ESO allocations. Stronger incentivizing, say by allocating additional ESO, might render the blackout trading period excessively long, inducing the exercise of all incentivizing ESO, consequently causing executives to lose their sensitivity to the incentivizing effects. We call this the *ESO tolerance effect*, which we discuss in Section 7.

Due to all these effects, imposing blackouts cannot, by itself, resolve the incentivizing failure of ESO. We identify a mechanism that restores executives' incentives regardless of whether they have insider information: a combination of granting non-transferable non-hedgeable executive reload stock options (RSO) with "infinite reloads" and imposing blackouts. RSO are ESO that, upon exercise (reload, in this case), are paid for using the underlying stock (rather than cash) and converted to new at-the-money ESO. RSO with infinite reloads allow an unlimited number of reloads. We show that the optimal RSO exercise is when the underlying stock price hits a new high since the previous

reload. This result holds when option holders have insider information at any informational quality level and when RSO are non-transferable and non-hedgeable.

We introduce analytical (subjective and objective) pricing of non-transferable non-hedgeable ESO and RSO with infinite reloads for insider-executives. Subjective prices reflect the ESO or RSO values to the insider-executives, and objective prices reflect the ESO or RSO costs to firms. Please see Colwell, Feldman, and Hu (2015, p. 165). To accomplish the pricing, we merge two analytical approaches. The first approach is the constrained portfolio optimization techniques of Cvitanić and Karatzas (1992) and Karatzas and Kou (1996), which Colwell, Feldman and Hu (2015) develop to price non-transferable non-hedgeable American ESO. The second approach is the enlarged filtration technique. One of the earliest publications in this area is from Jeulin and Yor (1985), which includes papers by Chaleyet-Maurel and Jeulin, Jacod, Jeulin and Yor, and others. See also Al Hussaini and Elliott (1987). Some of the earliest papers on insider trading in finance are Kyle (1985) and Back (1992). However, the first papers to rigorously use the enlargement of filtrations approach include Pikovsky and Karatzas (1996), Elliott et al. (1997), Amendinger et al. (1998), and Baudoin (2003). Where relevant, we adopt the notation of Pikovsky and Karatzas (1996). Using this approach, we introduce a method of enlarging filtration to model the noisy insider information. This method does not seem to appear in the literature, although numerous models of insider information exist. Literature that studies “weak information,” in which the insider information is about the distribution of asset returns rather than their actual value, includes Baudoin and Nguyen-Ngoc (2004). Elliott and Jeanblanc (1998), and Grorud (2000), study asymmetric information in models with jumps. Hillairet (2005) and Hillairet and Jiao (2011) study insider information applied to credit derivatives. Imkeller (2002) uses progressive enlargement of filtrations, in which the additional information involves knowledge about a stopping time not accessible to uninformed traders. Biagini and Øksendal (2005) study insider trading with a more general formulation of enlarged filtration. Leon et al. (2003) use a

Malliavin calculus approach to maximize the expected utility of an insider with logarithmic utility. Amendinger et al. (2003) evaluate the monetary benefit of insider knowledge rather than calculating the expected utility gain due to insider trading. See also the book by Aksamit and Jeanblanc (2017).

Our results shed new light on the value and role of RSO. Our ability to analytically price non-transferable non-hedgeable RSO with infinite reloads for insider-executives, demonstrates that the value of an RSO with infinite reload is bounded above by the value of one stock. This finding should end calling RSO “money pumps” and change the Financial Accounting Standards Board’s attitude toward RSO. The Board responded to the RSO past pricing difficulty by requiring the granting firms to account for RSO as a separate award [See FAS123(R) paragraphs 24 to 26 and Saly et al. 1999]. Our results might deem this requirement unnecessary.

We run Monte Carlo simulations of six scenarios: executives with good/bad news insider information and outsider executives, each repeated under high/low-volatility regimes. Our main findings include these: (i) Subjective prices perceived by insider-executives, taking ESO’s non-transferability and non-hedgeability into account, are usually higher than firms’ granting costs, [note that with no insider information, subjective prices are typically lower than firms’ granting costs (objective prices). See, for example, Colwell, Feldman, and Hu (2015)], but when executives have insider information, ESO incentives could become weaker. Therefore, the overall granting efficiency⁴ of ESO to insider-executives, and more so in low-volatility regimes, could be low; (ii) Under high-volatility regimes, bad (good) news information typically strengthens (weakens) ESO granting efficiency. The simulation sensitivity analysis is consistent with our theoretical results.

Our results provide theoretical foundations for the following empirical research progression. Roulstone (2003) finds that insider trading laws increase executive compensation and share-based incentives. Denis and Xu (2013) show that those results are robust to alternative definitions of insider trading restrictions and enforcement and to panel regressions with country-fixed effects. Henderson

(2010) further studies the same relationship but focuses on Rule 10b5-1 and isolates the potential profits from portfolio optimization and informed trading. The evidence suggests that executives whose trading freedom increased using Rule 10b5-1 trading plans experienced reductions in other forms of pay to offset the potential gains from trading. Carpenter and Remmers (2001) and Aboody *et al.* (2008) examine whether insiders use private information to time the exercise of their ESO. Fu and Ligon (2010) investigate whether insider information motivates executives' early exercise upon vesting. Bettis *et al.* (2005) consider executives' insider role, calibrate the Carpenter (1998) utility-based model to get ESO values and incentives, and document the impact on insiders' exercise behavior. Brooks *et al.* (2010) find that the best-informed executives are inclined to exercise early. The operating performance of firms following exercises motivated by private information is significantly worse than that of firms in which the exercises were not motivated by private information.

Our findings have several policy implications. First, our ability to price executive RSO for insider-executives and to demonstrate RSO's essential role in executives' incentivizing suggest that it might have been premature for RSO to fall out of favor. Perceiving RSO as a money pump seems to have been a misunderstanding, and FABS's concerns regarding RSO pricing difficulty are now mitigated. Second, we demonstrate that careful use of a blackout is essential for keeping insider-executives' incentives on the one hand and fairness (no excessive restrictions) on the other. Third, we demonstrate how the combined use of RSO and blackouts is essential to keeping executives incentivized, thus, to effective corporate governance. Fourth, because of the conditional nature of effective blackouts and firms' superior personal information about their executives relative to the Securities Exchange Commission (SEC), firms should determine the effective blackouts for their executives. The SEC should regulate the blackout trading prohibitions. Indeed, it is the SEC's regulatory capacity to restrict executives from non-premediated trading in their firm's stocks if insider information is (only) idiosyncratic. If executives' information is systematic, the SEC may

extend the restrictions to trading retirement funds and other wealth components.

Section 2 reviews legal essentials. Section 3 models insiders' constrained portfolio optimization. Section 4 discusses the insider-executives' optimal exercise policy. Section 5 makes policy recommendations on the regulation of blackout trading periods. Section 6 makes policy recommendations on designing efficient incentives for firms. Section 7 discusses simulation results. Section 8 concludes.

2 Legal essentials

Executives may use insider information in two trading styles: arbitrage, with which they profit using long and short positions to approach zero net investment and zero market exposure, and portfolio optimization, using insider information to improve portfolio processes.

Although legal obligations are not conditional on trading styles, insider arbitrage was effectively made illegal by anti-fraud decrees of the Securities Exchange Commission [see Section 10(b) and Rule 10b5 of the Securities Exchange Act of 1934, Trading Sanctions Act of 1984 (ITSA) because this style of trading is relatively easy to detect due to a large trading volume (see also Bainbridge 2013).

In the case of portfolio optimization, according to Rule 10b5-1 of the Act, insiders may use premeditated portfolio-optimization trading plans to avoid accusation. Unlike arbitrage, distinguishing between insiders' and outsiders' portfolio processes is practically impossible until optimal trading for insiders becomes extremely large near announcement time (see Pikovsky and Karatzas 1996, Eq. (2.3) and (2.10) on p. 1098-1099). Moreover, it is generally impossible to identify insider information arrival times and quality. Grorud and Pontier (1999) identified insider trading by constructing a statistical test to compare insiders' and outsiders' trading strategies. However, we note that distinguishing between trades motivated by insider information and those motivated by extreme beliefs is difficult under heterogeneous beliefs. Hence, being uninformed and being informed with very noisy

information are not technically distinguishable. However, we show in Proposition 6.1 that attaining infinite utility with noisy information is possible as long as it is not pure noise. Therefore, insiders can take advantage of Rule 10b5-1, which allows trading plans initiated before insider information arrivals because the rule is difficult to enforce.

Also, although insider trading facilitates rapid price discovery and enhances market informational efficiency, the price change caused by insider arbitrage is a one-off instant occurrence per information shock, and insiders are the only beneficiaries. In contrast, insider portfolio optimization is a sustained information release process, and there is time for profit sharing among insiders and outsiders. We can see this situation as a rationale for Rule 10b5, which prohibits insider arbitrage, and Rule 10b5-1, which allows insiders to execute premeditated trading plans.⁵

We focus in this study on the trading style of insider portfolio optimization. Specifically, we adopt the approach of Pikovsky and Karatzas (1996), in which insiders know the stock spot price that will prevail at a future time T^* . When they use this information, their optimal portfolio process yields higher returns. As the holding period approaches T^* , derived utility gains due to insider information, sufficiently increase due to insider information, to overwhelm the utility derived from their ESO, rendering the ESO and its incentivizing irrelevant.

Thus, an adequate blackout trading period is required. Section 306(a) and Regulation Blackout Trading Restriction (BTR) under the Sarbanes-Oxley Act of 2002 prohibit corporate executives and rank-and-file employees from engaging in transactions during a blackout in order to invalidate the necessary condition for insiders to achieve infinite derived utility, where the condition is the continuous trading in the neighborhood of T^* . For example, a blackout might be enacted to begin two weeks prior to the end of a fiscal quarter and end upon the completion of one full trading day after the public announcement of earnings for that quarter.

Regarding the insider trading restriction on exercising options, exercising through an “intra-

company” approach, i.e., executives providing value to the company in the form of cash or shares in exchange for more shares, is not a violation of Rule 10b5. Any other approach involving contemporaneous sales into the market is prohibited, e.g., the “broker-assisted cashless” exercise whereby, at the time of exercise, some or all the exercised shares are sold into the market, the requisite amount of the sale proceeds are used to pay the company for the exercise, and the holder keeps the net proceeds and any unsold shares. Even when taking the “intra-company” approach, the executives cannot sell the resulting shares during the blackout period (See Nathan and Hoffman 2013).

Furthermore, Statement of Financial Accounting Standards No.123 (revised 2004) [FAS123(R)], paragraph B69 to B72 (B80 to B82) precludes executives from hedging or transferring nonvested share-based compensations, including both nonvested share options and nonvested shares, to third parties; that is, their options are non-transferable non-hedgeable. Also, a short-equivalent position is forbidden, i.e., adopting trading strategies with a net short replicating position on the firm stock is not allowed.

Therefore, the insider trading liability during a blackout trading period, the intra-company exercise and the non-transferable non-hedgeable rules together form insider-executives’ portfolio constraints.

3 The model

Our model is designed to provide a legal basis for promulgating a blackout trading period. We show that if a blackout is not applied, insider portfolio optimization invalidates executives’ incentives. We focus on an executive who optimally trades until time $T \in [0, T^*)$ to maximize the expected utility generated from the terminal total wealth. We assume that the executive has (noisy) insider information about the firm’s stock value $S_1(T^*)$. The total wealth includes n non-transferable non-hedgeable ESO, and other personal wealth (defined as *outside wealth*) that can be freely traded in a

market with one risk-free asset and d primary assets, as long as not in a short selling position on firm stock against the non-hedgeable constraints with which executives must comply. As ESO are non-transfereable and non-hedgeable, they are “fixed” in the portfolio; hence, executives’ optimization is achieved by optimally trading on the outside wealth.

3.1 Noisy information

We assume executives observe a mixture of true information and noise over time rather than accurately knowing in advance the terminal value of a risk source. We model noise as an additional risk source associated with an imaginary primary asset that is non-tradable. In particular, we describe the original market, in which there is a traded bond whose price evolves according to the differential equation

$$(1) \quad dS_0(t) = S_0(t)r(t)dt, S_0(0) = 1,$$

where $r(t)$ is a scalar interest rate at time t . The uncertainty is driven by a $d + 1$ -dimensional standard Brownian motion, $W = (W_1, \dots, W_{d+1})$, in \mathbb{R}^{d+1} , defined on a complete probability space on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} := (\mathcal{F}_t, t \geq 0)$ is the completed right continuous natural filtration defined for any t by $\mathcal{F}_t = \sigma(W(s); 0 \leq s \leq t)$, with fixed time span $[0, T^*]$ for some finite $T^* > 0$. The primary asset prices $S_i, i = 1, \dots, d$ follow the dynamics

$$(2) \quad dS_i(t) = S_i(t)[b_i(t)dt + \sum_{j=1}^i \sigma_{i,j}(t)dW_j(t)], S_i(0) = s_i, i = 1, \dots, d.$$

Without loss of generality, we assume that S_1 is the firm stock price. The price of an imaginary asset S_{d+1} that serves as noise follows the dynamics

$$(3) \quad dS_{d+1}(t) = S_{d+1}(t)[dW_{d+1}(t)], S_{d+1}(0) = s_{d+1};$$

that is, without loss of generality, we can set for any t $b_{d+1}(t) = 0$ and $\sigma_{d+1,d+1}(t) = 1$. To explain why the setting is without loss of generality, we use the constrained portfolio optimization technique. We discuss the details in Section 4.3. Here $\sigma(t) = (\sigma_{i,j}(t), 1 \leq j \leq i \leq d+1)$ is a volatility matrix. The sub-matrix for $i, j \leq d$ is the lower unit triangular of the Cholesky decomposition of the positive definite variance-covariance matrix of the primary assets' return vector $dS_i(t)/S_i(t), i \leq d$. As a consequence S_1 is a function of W_1 only. We also assume that $\sigma_{i,d+1} = 0$ and $b(t) := (b_1(t), \dots, b_{d+1}(t))^T$ is a drift rate vector. We assume that r, σ and b are \mathbb{F} progressively measurable processes. The market price of risk is a process θ defined as

$$(4) \quad \theta(t) := \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}],$$

where $\mathbf{1} = (1, \dots, 1)^T$, and we assume that $E[\int_0^{T^*} \|\theta(t)\|^2 dt] < \infty$. At time t , we assume that the executives can observe process G , the mixture of true information $W_1(T^*) - W_1(t)$ and noise $W_{d+1}(T^*) - W_{d+1}(t)$ on top of the realized value of $W_1(t)$; meaning

$$(5) \quad G(t) := W_1(t) + \lambda[W_1(T^*) - W_1(t)] + \sqrt{1 - \lambda^2}[W_{d+1}(T^*) - W_{d+1}(t)], \quad t \in [0, T^*]$$

with constant information quality coefficient $\lambda \in [0, 1]$. The initial information is $G(0) = \lambda W_1(T^*) + \sqrt{1 - \lambda^2} W_{d+1}(T^*)$. A greater λ indicates a higher precision of insider information. Then, to insider-executives, the complete probability space where Brownian motion W is defined is $(\Omega, \mathcal{G}, \mathbb{P})$; the probability measure \mathbb{P} is unchanged; and the filtration is enlarged from \mathbb{F} to $\mathbb{G} = (\mathcal{G}_t, 0 \leq t \leq T^*)$ with $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(G(0))$. Note that $G(t)$ is both \mathcal{F}_{T^*} and \mathcal{G}_t -measurable.

So, according to Jacod (1985) and thanks to Ghorud and Pontier (1998) Proposition 3.5 we have that $\tilde{W}(t) = W(t) - \int_0^t a(s)ds$ is a (\mathbb{P}, \mathbb{G}) -martingale, where $a(s)$ is a vector and $\langle q^y, W_i \rangle_t = \int_0^t q_s^y a_i(s)ds$. Substituting $y = \lambda W_1(T^*) + \sqrt{1 - \lambda^2} W_{d+1}(T^*)$ will give the process a as we see in Proposition 3.1 below.

Proposition 3.1 *Since hypothesis H' is satisfied (the law of the initial information $G(0)$ is absolutely continuous), there exists a compensating process $\{a(t)\}$ such that*

$$\tilde{W}(t) := W_t - \int_0^t a(s)ds$$

is a Brownian motion on $(\Omega, \mathcal{G}, \mathbb{P}, \mathbb{G})$. Let q_t^y the \mathcal{F}_t conditional density of $G(0)$, namely

$$q_t^y = \frac{1}{\sqrt{2\pi(T^* - t)}} \exp\left(-\frac{[y - (\lambda W_1(t) + \sqrt{1 - \lambda^2} W_{d+1}(t))]^2}{2(T^* - t)}\right),$$

which satisfies $dq_t^y = q_t^y \alpha(t, y) dW(t)$. Then $a(t) = \alpha(t, G(0))$.

Using Grorud and Pontier (1998), Proposition 4.1, on $\mathcal{G}_t, \forall t \in [0, T^*)$, the change of probability measure

$$\left. \frac{dQ^{\mathbb{G}}}{d\mathbb{P}} \right|_{\mathcal{G}_t} = e^{-\int_0^t \theta_a(s) dW(s) - \frac{1}{2} \int_0^t \|\theta_a(s)\|^2 ds}$$

with

$$(6) \quad \theta_a(t) = \theta(t) + a(t)$$

is the Radon-Nikodym derivative changing the probability measure from \mathbb{P} to $Q^{\mathbb{G}}$ under which the discounted stock price $e^{-\int_0^t r(s)ds} S(t)$ is a martingale and under the probability measure $Q^{\mathbb{G}}$ the σ -algebras \mathcal{F}_t and $\sigma(G(0))$ are independent.

Our model changes the nature of insider information; it is a modification of Equation (3.1) in Pikovsky and Karatzas (1996, p. 1103) and has three intuitively appealing properties. First, insiders continuously observe a stochastic noisy process of the risk source rather than one T^* -noisy signal. Observing the process G continuously or observing the initial $G(0)$ give the same enlarged filtration. Second, the noisy insider information under this setting is unbiased, i.e., for all $t < T^*$,

$E^{\mathbb{P}}[G(t)|\mathcal{F}_t^W] = W_1(t)$. Third, the quality and precision of insider information increases proportionally to the reciprocal of the remaining time, up to the date at which the information is disclosed, i.e., $Var^{\mathbb{P}}[G(t)|\mathcal{F}_t^W] = T^* - t$.

An alternative formulation of insider information is one in which insiders know an interval to which a future date's stock price belongs. In this case, insiders enjoy only finite derived utility. See, e.g., Hillairet (2005), Hillairet and Jiao (2011), or D'Auria and Salmeron (2020). Our model shows the possibility that insiders with noisy information can still achieve infinite derived utility, which indicates that even if executives' insider information is noisy, the firm still needs to pursue an effective incentivizing mechanism.

3.2 Insiders' price of risk

Pikovsky and Karatzas (1996) studied several models of insider information, including where insiders observe future returns, future prices, or noisy future returns. For these models, they showed that the price of risk for insiders is the process $\theta_a(t) = \theta(t) + a(t)$, where $a(t)$ corresponds to the particular type of information, and $\theta(t)$ is the price of risk to outsiders. For the information $G(t)$ set given by Equation (5), we have the corollary below.

Corollary 3.2 *The compensating process a , given insider information G in Eq. (5), is with $t < T^*$*

$$\begin{aligned}
 a_1(t) &= \frac{\lambda[\lambda(W_1(T^*) - W_1(t)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(t))]}{T^* - t}, \\
 (7) \quad a_i(t) &= 0, \quad i = 2, 3, \dots, d, \\
 a_{d+1}(t) &= \frac{\sqrt{1 - \lambda^2}[\lambda(W_1(T^*) - W_1(t)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(t))]}{T^* - t}.
 \end{aligned}$$

Proof: Given insider information $G = G(0)$, the \mathcal{F}_t -conditional density of G is that of $\lambda W_1(T^*) + \sqrt{1 - \lambda^2} W_{d+1}(T^*)$ given \mathcal{F}_t^W :

$$q_t^y = \frac{1}{\sqrt{2\pi} (T^* - t)} \exp\left(-\frac{[y - (\lambda W_1(t) + \sqrt{1 - \lambda^2} W_{d+1}(t))]^2}{2(T^* - t)}\right).$$

Define the function $f^y : (t, x) \rightarrow \frac{1}{\sqrt{2\pi} (T^* - t)} \exp\left(-\frac{[y - (\lambda x_1 + \sqrt{1 - \lambda^2} x_{d+1})]^2}{2(T^* - t)}\right)$, for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^{d+1}$.

Using Itô's formula, we have

$$dq_t^y = \frac{\partial f^y}{\partial t} dt + \sum_{i=1}^{d+1} \frac{\partial f^y}{\partial x_i} dW_i(t) + \frac{1}{2} \sum_{i=1}^{d+1} \frac{\partial^2 f^y}{\partial (x_i)^2} dt.$$

The derivatives are given by

$$\begin{aligned} \frac{\partial f^y}{\partial x_1} &= f^y \frac{\lambda[y - (\lambda x_1 + \sqrt{1 - \lambda^2} x_{d+1})]}{T^* - t}, \\ \frac{\partial f^y}{\partial x_i} &= 0, i = 2, 3, \dots, d, \\ \frac{\partial f^y}{\partial x_{d+1}} &= f^y \frac{\sqrt{1 - \lambda^2}[y - (\lambda x_1 + \sqrt{1 - \lambda^2} x_{d+1})]}{T^* - t}, \quad \partial_t f + \Delta f = 0, \end{aligned}$$

and $a(t) = \alpha(t, G(0))$ according to Proposition 3.1 which yields the result. •

3.3 Portfolio process

Assume insider-executives cannot affect market prices but can dynamically choose a \mathbb{R}^{d+1} valued \mathbb{G} -progressively measurable portfolio-proportion process $\pi = (\pi_1, \dots, \pi_{d+1})^\top$, with $\int_0^T \|\pi(t)\|^2 dt < \infty$, almost surely. [See, e.g., Remark 3.6.10, Karatzas and Shreve (1998).] Namely, they decide at any time $t \in [0, T^*)$ the proportion $\pi_i(t, \cdot)$ of their wealth $X(t)$ to invest in the i th primary asset, $1 \leq i \leq d$, based on their enlarged information \mathcal{G}_t . Let us ignore the executive stock options for the

moment. Then, the wealth process X corresponding to the portfolio process π follows

$$(8) \quad dX(t) = X(t)[r(t)dt + \pi^\top(t)\sigma(t)(\theta_a(t)dt + d\tilde{W}(t))], \quad X(0) = x_0 > 0,$$

where \tilde{W} is the Brownian motion under insider-executives' risk-neutral probability measure Q^G , defined in Proposition 3.1:

$$(9) \quad \tilde{W}(t) := W(t) + \int_0^t (\theta(s) - \theta_a(s))ds,$$

where θ_a is defined in equation (6), with a defined by Corollary 3.2.

Now suppose the executive is granted $N(0)$ ESO at date 0, each having payoff $B(t)$ if exercised at date t . Then, the executive's wealth process becomes

$$(10) \quad dX^{x,n}(t) = X^{x,n}(t)[rdt + \pi^\top(t)\sigma(t)(\theta_a(t)dt + d\tilde{W}(t))] + \dot{n}(t)B(t)dt, \quad X^{x_0,n_0}(0) \equiv x_0 \in \mathbb{R}^+, \\ dN(t) = -\dot{n}(t)dt, \quad N(0) \equiv n_0 \in \mathbb{N}.$$

Here, \dot{n} is the rate at which the options are exercised (which will be chosen to optimize the executive's expected utility from terminal wealth). So, $\int_0^T \dot{n}_s ds = n_0$, and $\int_t^T \dot{n}_s ds = n$, meaning that there are n options remaining at date t . In the following section, we use equation (10) to derive the Hamilton-Jacobi-Bellman (HJB) for the optimisation problem.

3.4 The first order conditions of the Hamilton-Jacobi-Bellman Equation

We show how the First Order Conditions (FOCs) from the Hamilton-Jacobi-Bellman (HJB) equation determine whether to exercise options (early) if continuous partial exercise is allowed. We do not directly consider ESO non-transferability and non-hedgeability at this stage, although those features are assumed. In Section 4, we explicitly formulate those features as portfolio constraints, which

will not change the theoretical results in this section, just altering some parameters of the wealth dynamics.

We take three steps. In Step 1, we derive the FOCs for optimal controls, i.e., the portfolio process π and the exercise rate \dot{n} . In Step 2, we explicitly derive the equations for $\partial_x \mathbb{J}$ and $\partial_n \mathbb{J}$, which are the key derivatives that determine the early exercise policy. In Step 3, we use the explicit FOCs to show how the early exercise decision depends on a *subjective option price*, which is related to the *marginal indifference prices* discussed by Davis (1994), Davis (1997), and Karatzas and Kou (1996).

Let us begin to derive the FOCs for optimal controls. Consider an insider-executive's portfolio that includes ESO with a continuous partial exercise feature. We define an exercise rate process \dot{n} as the number of ESO exercised per unit time elapsed, therefore we have

$$(11) \quad dN(t) = -\dot{n}(t)dt, \quad N(0) \in \mathbb{N},$$

where $N(0)$ is the number of ESO initially granted to executives. From now on, working conditionally to \mathcal{G}_t , we denote $X(t) = x, N(t) = n$. The optimal portfolio process $\pi^{\mathbb{G}^*}$ and exercise process $(\dot{n})^*$ jointly solve

$$(12) \quad \mathbb{J}^{\mathbb{G}} := \mathbb{J}^{\mathbb{G}}(t, X_*^{x,n}(t), N(t), T) := \operatorname{ess\,sup}_{(\pi^{\mathbb{G}}, \dot{n}) \in \mathcal{A}^{\mathbb{G}}(t, x, K, T) \times \mathcal{N}(t)} E^{\mathbb{P}}[U(X_{\pi^{\mathbb{G}}, \dot{n}}^{x,n}(T)) | \mathcal{G}_t],$$

where $X_*^{x,n}$ denotes the wealth with cash endowment x , n units of ESO granted and not exercised at time t ; X_* is generated by the optimal portfolio process $\pi^{\mathbb{G}^*}$ and exercise process $(\dot{n})^*$; and $\mathcal{A}^{\mathbb{G}}(t, x, K, T)$ is the class of \mathbb{R}^{d+1} valued \mathcal{G} -progressively measurable portfolio processes, $\pi^{\mathbb{G}}(t, \omega)$, satisfying the conditions

- (i) $\pi^{\mathbb{G}}(t, \omega) \in (-\infty, \infty)$ for $dt \otimes d\mathbb{P}$ a.e.,
- (ii) $E^{\mathbb{P}}[\max(-U(X_{\pi^{\mathbb{G}}, \dot{n}}^{x,n}(T)), 0)] < \infty$,

- (iii) for initial capital $x \in (0, \infty)$, $X_{\pi^c, \dot{n}}^{x,n}(t) \geq 0$, for $t \in [0, T]$ dt almost surely.

$\mathcal{N}(t)$ is the class of \mathbb{R}_0^+ valued \mathcal{G} -progressively measurable exercise processes, $\dot{n}(t, \omega)$, satisfying the conditions

- (i) $\dot{n}(t, \omega) \in (0, \infty)$ for $dt \otimes dP$ a.e.,
- (ii) $\int_0^T \dot{n}(t) dt = n_0$.

Our goal here is to prove the following theorem where the set of controls \dot{n} is constrained. We prove below that this constrained set is not empty.

Definition 3.3 *Let \mathcal{N} represent the set of controls \dot{n} satisfying*

- $\int_t^T \dot{n}(s, n, \omega) ds = n$,
- $\lim_{\varepsilon \rightarrow 0} \int_t^T \frac{\dot{n}^*(s, n+\varepsilon) - \dot{n}^*(s, n)}{\varepsilon} B(s) \xi^{1, \pi^*}(s, T)$ exists,
- the support of $\dot{n}(s, n, \omega)$ is $\{(s, \omega) : B(s, \omega) > \kappa(s, x, n, T)\}$, where the ratio κ is defined to be equal to the following stochastic process, depending on initial values (x, n) :

$$(13) \quad \kappa(t, x, n, T) := \frac{\partial_n \mathbb{J}(t, x, n, T)}{\partial_x \mathbb{J}(t, x, n, T)}.$$

Theorem 3.4 *We assume that $\mathbb{J}^{\mathbb{G}}$ is C^2 and we restrain the set of controls \dot{n} as \mathcal{N} defined in Definition 3.3. Then, early exercise is optimal on the event*

$$\{(t, \omega) | \{B(t) > \hat{C}(t, T)\}\}$$

where

$$(14) \quad \hat{C}(t, T) = \int_t^T \frac{E^{\mathbb{P}} [U'(X_*^{x,n}(t, T)) \varpi(s) B(s) \xi^{1, \pi^*}(s, T) | \mathcal{G}_t]}{E^{\mathbb{P}} [U'(X_*^{x,n}(t, T)) \xi^{1, \pi^*}(t, T) | \mathcal{G}_t]} ds.$$

We give an explicit expression for ϖ , given certain assumptions, in Proposition 3.9. We begin the proof of this result with the following lemma.

Lemma 3.5 *The Hamilton-Jacobi-Bellman equation for an insider's optimization problem is given by*

$$H(t, x, n, b, p, M) = \max_{\pi, \dot{n}} \left[(-x(r_t + \pi^\top(t)\sigma(t)(\theta_a(t)) + \dot{n}(t)B(t)) p_1 - \dot{n}(t)p_2 - \frac{1}{2}M_{11}x^2\|\pi^\top(t)\sigma(t)\|^2) \right],$$

where $p_1 = \partial_x \mathbb{J}^G$, $p_2 = \partial_n \mathbb{J}^G$, $M_{11} = \partial_{xx}^2 \mathbb{J}^G$.

Proof: Recall that using Girsanov Theorem with $\mathbb{P}^a = Z_a \mathbb{P}$, the new Brownian motion (cf. Proposition 3.1) is

$$(15) \quad d\tilde{W}(s) = dW(s) - a(s)ds.$$

If the firm gives the executive N_0 ESO, then the ‘outside’ wealth process not including non-exercised ESO is given by (10)

$$(16) \quad \begin{aligned} dX^{x,n}(t) &= X^{x,n}(t)[r(t)dt + \pi^\top(t)\sigma(t)(\theta_a(t)dt + d\tilde{W}(t))] + \dot{n}(t)B(t)dt, \quad X^{x_0, n_0}(0) \equiv x_0 \in \mathbb{R}^+, \\ dN(t) &= -\dot{n}(t)dt, \quad N(0) \equiv n_0 \in \mathbb{N} \end{aligned}$$

under the constraint $\int_t^T \dot{n}(s)ds = n$.

When the used strategy is (π, \dot{n}) , the unique explicit solution is

$$(17) \quad \begin{aligned} X_{\pi, \dot{n}}^{x,n}(t, T) &= \exp \left(\int_t^T (r(s) + \pi^\top(s)\sigma(s)\theta_a(s) - \frac{1}{2}\|\pi^\top(s)\sigma(s)\|^2)ds + \pi^\top(s)\sigma(s)d\tilde{W}(s) \right) \times \\ &\left[x + \int_t^T \dot{n}(s)B(s) \exp \left\{ - \left(\int_t^s (r(u) + \pi^\top(u)\sigma(u)\theta_a(u) - \frac{1}{2}\|\pi^\top(u)\sigma(u)\|^2) du + \pi^\top(u)\sigma(u)d\tilde{W}(u) \right) \right\} ds \right], \end{aligned}$$

where for fixed t we write $N(t) = n$ and $X^{x,n}(t) = x$ since we need the n in the superscript so that we can calculate $\partial_n \mathbb{J}^G$.

Next, let us define the following process:

$$(18) \quad \xi^{x,\pi}(t, u) := x \exp \left(\int_t^u (r(s) + \pi^\top(s)\sigma(s)\theta_a(s) - \frac{1}{2}\|\pi^\top(s)\sigma(s)\|^2)ds + \pi^\top(s)\sigma(s)d\tilde{W}(s) \right)$$

so $\xi^{x,\pi}(t, t) = x$.

When the optimal control (π^*, \dot{n}^*) is used, we denote at time u

$$(19) \quad X_*^{x,n}(t, u) := \xi^{1,\pi^*}(t, u) \left[x + \int_t^u \dot{n}^*(s) \frac{B(s)}{\xi^{1,\pi}(t, s)} ds \right] = \xi^{x,\pi^*}(t, u) + \int_t^u \dot{n}^*(s) B(s) \xi^{1,\pi^*}(s, u) ds.$$

With a log utility function the aim is to solve the following optimization problem:

$$\begin{aligned} \mathbb{J}^G(t, x, n, T) = \max_{\pi, \dot{n}} \left\{ E^{\mathbb{P}} \left[\int_t^T (r_s + \pi^\top(s)\sigma(s)\theta_a(s) - \frac{1}{2}\|\pi^\top(s)\sigma(s)\|^2)ds + \pi^\top(s)\sigma(s)d\tilde{W}(s) \right. \right. \\ \left. \left. + \log \left[x + \int_t^T \dot{n}^*(s) \frac{B(s)}{\xi^{1,\pi}(t, s)} ds \right] \middle| \mathcal{G}_t \right] \right\}. \end{aligned}$$

Note that $x \mapsto \mathbb{J}^G(t, x, n, T)$ is non-decreasing, so $\partial_x \mathbb{J}^G \geq 0$.

We can now present the Hamilton-Jacobi-Bellman equation (HJB) by applying Pham (2009), Section 3.4 (with no consumption):

$$(20) \quad \begin{aligned} H(t, x, n, b, p, M) \\ = \max_{\pi, \dot{n}} \left[(-x(r_t + \pi^\top(t)\sigma(t)\theta_a(t)) + \dot{n}^*(s)B(t)) p_1 - \dot{n}(t)p_2 - \frac{1}{2}M_{11}x^2\|\pi^\top(t)\sigma(t)\|^2 \right] \end{aligned}$$

where $p_1 = \partial_x \mathbb{J}^G$, $p_2 = \partial_n \mathbb{J}^G$, $M_{11} = \partial_{xx}^2 \mathbb{J}^G$. •

From the above lemma, we can now provide the First Order Conditions for this problem.

Corollary 3.6 *The two First Order Conditions are as follows. For the portfolio process, we have*

$$(21) \quad \pi_v^{\mathbb{G}^*}(t) = -\frac{\partial_x \mathbb{J}^{\mathbb{G}}}{x \partial_{xx}^2 \mathbb{J}^{\mathbb{G}}} (\sigma_t^\top)^{-1} \theta_a(t)$$

and for the early exercise decision

$$(22) \quad \dot{n}(t, n) = 0 \text{ on the set } \{B(t) \partial_x \mathbb{J}^{\mathbb{G}}(t, x, n, T) \leq \partial_n \mathbb{J}^{\mathbb{G}}(t, x, n, T)\}.$$

Alternatively, it is optimal to exercise on the set

$$(23) \quad B(t) > \frac{\partial_n \mathbb{J}^{\mathbb{G}}(t, x, n, T)}{\partial_x \mathbb{J}^{\mathbb{G}}(t, x, n, T)} =: \kappa(t, x, n, T)$$

with the constraint for all (t, n)

$$(24) \quad \int_t^T \dot{n}(s, n) \mathbf{1}_{\{B(s) > \frac{\partial_n \mathbb{J}^{\mathbb{G}}(s, x, n, T)}{\partial_x \mathbb{J}^{\mathbb{G}}(s, x, n, T)}\}} ds = n.$$

Remark 3.7 *Here it could be noticed that there exists a process a_n such that this optimal control can be expressed as $\dot{n}^*(s, n) = a_n(s) \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}}$, even if we are unable to provide an explicit expression for a_n here.*

Proof: Since there is no interaction between the two parameters π and \dot{n} , we can separately maximize (under the specified constraints)

$$(25) \quad \pi \rightarrow -x \pi^\top(t) \sigma(t) \theta_a(t) p_1 - \frac{1}{2} M_{11} x^2 \|\pi^\top(t) \sigma(t)\|^2 = -x \pi^\top(t) \sigma(t) \theta_a(t) \partial_x \mathbb{J}^{\mathbb{G}} - \frac{1}{2} \partial_{xx}^2 \mathbb{J}^{\mathbb{G}} x^2 \|\pi^\top(t) \sigma(t)\|^2$$

$$(26) \quad \dot{n} \rightarrow \dot{n}(t) B(t) \partial_x \mathbb{J}^{\mathbb{G}} - \dot{n}(t) \partial_n \mathbb{J}^{\mathbb{G}}.$$

The first maximization follows easily from differentiating the HJB equation (it is a concave function of π) with respect to the vector π .

Concerning optimal \dot{n} in HJB, we have to optimize a linear application, under the constraints on \dot{n} , namely, $\dot{n} \geq 0$ and $\int_t^T \dot{n}(s)ds = n$. The result is

$$(27) \quad \dot{n}(t) = 0 \text{ on the set } \{B(t)\partial_x \mathbb{J}^{\mathbb{G}}(t, x, n) \leq \partial_n \mathbb{J}^{\mathbb{G}}(t, x, n)\}$$

and on the complement set $\{B(t)\partial_x \mathbb{J}^{\mathbb{G}}(t, x, n) > \partial_n \mathbb{J}^{\mathbb{G}}(t, x, n)\}$ \dot{n}^* has to satisfy the constraint (24).

•

To complete the proof of Theorem 3.4, it remains to show that

$$(28) \quad \partial_x \mathbb{J}^{\mathbb{G}}(t, x, n, T) = E^{\mathbb{P}} [U' (X_*^{x,n}(t, T)\xi^{1,\pi^*}(t, T)) | \mathcal{G}_t]$$

and under the additional assumptions provided in Theorem 3.4, there exists ϖ such that

$$(29) \quad \partial_n \mathbb{J}^{\mathbb{G}}(t, x, n, T) = E^{\mathbb{P}} \left[U' (X_*^{x,n}(t, T)) \int_t^T \varpi(s)B(s)\xi^{1,\pi^*}(s, T)ds \middle| \mathcal{G}_t \right].$$

In the following lemma, we prove equation (28); then in Proposition 3.9, we produce a preliminary result regarding $\partial_n \mathbb{J}^{\mathbb{G}}$ under additional conditions that equation (29) holds. Under these additional conditions, we can also write an explicit expression for ϖ .

Lemma 3.8 *First, we have,*

$$\partial_x \mathbb{J}^{\mathbb{G}}(t, x, n, T) = E^{\mathbb{P}} [U' (X_*^{x,n}(t, T)\xi^{1,\pi^*}(t, T)) | \mathcal{G}_t].$$

Next, assuming $\dot{n} \in \mathcal{N}$, there exists ϖ such that

$$(30) \quad \partial_n X = \int_t^T \varpi(s)B(s)\xi^{1,\pi^*}(s, T)ds.$$

Proof: First, recall equation (19):

$$(31) \quad X_*^{x,n}(t, u) = \xi^{1,\pi^*}(t, u) \left[x + \int_t^u \dot{n}^*(s) \frac{B(s)}{\xi^{1,\pi}(t, s)} ds \right] = \xi^{x,\pi^*}(t, u) + \int_t^u \dot{n}^*(s) B(s) \xi^{1,\pi^*}(s, u) ds.$$

Intuitively, the option payoff $B(s)$, paid at time s is optimally invested in the portfolio $\xi^{1,\pi^*}(s, u)$ until date u . So, in simple terms, multiplying by ξ gives the future value. We now calculate $\partial_x \mathbb{J}^G$ and $\partial_n \mathbb{J}^G$.

For $\partial_x \mathbb{J}^G$, note that from equation (31), $\partial_x X_*^{x,n}(t, u) = \xi^{1,\pi^*}(t, u)$, so

$$(32) \quad \partial_x \mathbb{J}^G = E^{\mathbb{P}} [U'(X_*^{x,n}(t, T)) \xi^{1,\pi^*}(t, T) | \mathcal{G}_t].$$

This proves equation (28), the denominator in equation (14).

Let us briefly remark that $U' > 0$ for any differentiable utility function, and ξ is an exponential process so $\xi > 0$, which implies that $\partial_x \mathbb{J}^G > 0$. In case of log utility, we get

$$(33) \quad U'(X_*^{x,n}(t, T)) \xi^{1,\pi^*}(t, T) = \frac{\xi^{1,\pi^*}(t, T)}{X_*^{x,n}(t, T)} > 0.$$

In order to complete the proof of Theorem 3.4, we need to derive an expression for $\partial_n \mathbb{J}^G$, which appears in (27), so we have to calculate $\partial_n X$. Recall equation (19) at time T :

$$X_*^{x,n}(t, T) = \xi^{x,\pi^*}(t, T) + \int_t^T \dot{n}^*(s) B(s) \xi^{1,\pi^*}(s, T) ds.$$

For $\varepsilon > 0$, we would like to consider $\frac{1}{\varepsilon} (X_*^{x,n+\varepsilon}(t, T) - X_*^{x,n}(t, T))$; however, the optimal exercise rate may be a function of n , and so we write, for example, $\dot{n}^*(s, n + \varepsilon)$. Note that $\partial_n X$ is the limit of

$$\frac{1}{\varepsilon} (X_*^{x,n+\varepsilon}(t, T) - X_*^{x,n}(t, T)) = \frac{1}{\varepsilon} \int_t^T (\dot{n}^*(s, n + \varepsilon) - \dot{n}^*(s, n)) B(s) \xi^{1,\pi^*}(s, T) ds,$$

as ε approaches zero. The limit exists in the case of $\dot{n} \in \mathcal{N}$ (defined in Definition 3.3). Using the

facts that $\int_t^T \dot{n}^*(s, n) ds = n$ and $\int_t^T \dot{n}^*(s, n + \varepsilon) ds = n + \varepsilon$ we find that $\forall \varepsilon > 0$

$$\frac{1}{\varepsilon} \int_t^T (\dot{n}^*(s, n + \varepsilon) - \dot{n}^*(s, n)) ds = 1.$$

So, $\frac{1}{\varepsilon} \int_t^T (\dot{n}^*(s, n + \varepsilon) - \dot{n}^*(s, n)) B(s) \xi^{1, \pi^*}(s, T) ds$ is some sort of *weighted average* of the values $B(s) \xi^{1, \pi^*}(s, T)$ for $s \in [t, T]$. Thus, we write according to Definition 3.3

$$(34) \quad \partial_n X = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^T (\dot{n}^*(s, n + \varepsilon) - \dot{n}^*(s, n)) B(s) \xi^{1, \pi^*}(s, T) ds = \int_t^T \varpi(s) B(s) \xi^{1, \pi^*}(s, T) ds,$$

which ends the proof. •

The following actually proves that $\dot{n}(s, n, \omega) = a_n(s, \omega) \mathbf{1}_{\{B(s, \omega) > \kappa(s, \omega, x, n, T)\}}$ satisfies Definition 3.3, recalling a_n introduced in Remark 3.7.

Proposition 3.9 *Under the assumptions that $\mathbb{J}^{\mathbb{G}} \in C^2$ and*

$$(35) \quad \dot{n}(s, n, \omega) = a_n(s, \omega) \mathbf{1}_{\{B(s, \omega) > \kappa(s, \omega, x, n, T)\}} \text{ satisfies Definition 3.3 with } n \mapsto a_n \text{ of class } C^1,$$

the requested $\partial_n(X_*^{x, n})(t, T)$ is given by

$$(36) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^T (\dot{n}^*(s, n + \varepsilon) - \dot{n}^*(s, n)) B(s) \xi^{1, \pi^*}(s, T) ds \\ &= \int_t^T \left[\frac{a_n(s, \omega) \partial_n \kappa(s, x, n, T)}{B(s)} + \partial_n a_n(s, \omega) \right] B(s) \xi^{1, \pi^*}(s, T) ds \end{aligned}$$

meaning that in this case $\varpi(s, x, n) = \frac{a_n(s, \omega) \partial_n \kappa(s, x, n, T)}{B(s)} + \partial_n a_n(s, \omega)$.

Proof: Below, we omit the dependence on $\omega \in \Omega$. Given our assumption (35) about \dot{n}^* ,

$$\int_t^T \dot{n}^*(s, n + \varepsilon) B(s) \xi^{1, \pi^*}(s, T) ds = \int_t^T a_{n+\varepsilon}(s) \mathbf{1}_{(\kappa(s, x, n+\varepsilon, T), \infty)}(B(s)) \times B(s) \xi^{1, \pi^*}(s, T) ds.$$

So,

$$\begin{aligned} (37) \quad & \frac{1}{\varepsilon} \int_t^T (\dot{n}^*(s, n + \varepsilon) - \dot{n}^*(s, n)) B(s) \xi^{1, \pi^*}(s, T) ds \\ &= \frac{1}{\varepsilon} \int_t^T (a_{n+\varepsilon}(s) \mathbf{1}_{\{B(s) > \kappa(s, x, n+\varepsilon, T)\}} - a_n(s) \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}}) \times B(s) \xi^{1, \pi^*}(s, T) ds \\ &= \frac{1}{\varepsilon} \int_t^T a_{n+\varepsilon}(s) (\mathbf{1}_{\{B(s) > \kappa(s, x, n+\varepsilon, T)\}} - \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}}) \times B(s) \xi^{1, \pi^*}(s, T) ds \\ &\quad + \frac{1}{\varepsilon} \int_t^T (a_{n+\varepsilon}(s) - a_n(s)) \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}} \times B(s) \xi^{1, \pi^*}(s, T) ds. \end{aligned}$$

The second term satisfies

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_t^T (a_{n+\varepsilon}(s) - a_n(s)) \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}} \times B(s) \xi^{1, \pi^*}(s, T) ds \\ &= \int_t^T \partial_n a_n(s) \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}} \times B(s) \xi^{1, \pi^*}(s, T) ds. \end{aligned}$$

The first part,

$$(38) \quad \frac{1}{\varepsilon} \int_t^T a_{n+\varepsilon}(s) (\mathbf{1}_{\{B(s) > \kappa(s, x, n+\varepsilon, T)\}} - \mathbf{1}_{\{B(s) > \kappa(s, x, n, T)\}}) \times B(s) \xi^{1, \pi^*}(s, T) ds,$$

we denote as $I_\varepsilon^1 + I_\varepsilon^2$, where I_ε^1 is integrated over the set $\{\kappa(s, x, n + \varepsilon, T) < \kappa(s, x, n, T)\}$ and I_ε^2 is integrated over the set $\{\kappa(s, x, n + \varepsilon, T) > \kappa(s, x, n, T)\}$. We now look for the limit of (38) when

$\varepsilon \rightarrow 0$ on the event $\{\kappa(s, x, n + \varepsilon, T) < \kappa(s, x, n, T)\}$; that is, we look for the limit of

$$I_\varepsilon^1 := \frac{1}{\varepsilon} \int_t^T a_{n+\varepsilon}(s) \mathbf{1}_{[\kappa(s, x, n+\varepsilon, T), \kappa(s, x, n, T)]}(B(s)) \times B(s) \xi^{1, \pi^*}(s, T) ds.$$

The integrand belongs to the interval $\frac{a_{n+\varepsilon}(s)}{\varepsilon} [\kappa(s, x, n+\varepsilon, T), \kappa(s, x, n, T)] \xi^{1, \pi^*}(s, T)$ and so the integral belongs to the interval

$$I_\varepsilon^1 \in \left[\int_t^T a_{n+\varepsilon}(s) \frac{\kappa(s, x, n + \varepsilon, T)}{\varepsilon} \xi^{1, \pi^*}(s, T) ds, \int_t^T a_{n+\varepsilon}(s) \frac{\kappa(s, x, n, T)}{\varepsilon} \xi^{1, \pi^*}(s, T) ds \right].$$

The limit of the integrand when $\varepsilon \rightarrow 0$ is a Dirac measure since the interval goes to the point (using the assumption that $n \mapsto a_n$ is continuous) $\int_t^T a_n(s) \partial_n \kappa(s, x, n, T) \xi^{1, \pi^*}(s, T) ds$. Similarly, on the event $\{\kappa(s, x, n + \varepsilon, T) > \kappa(s, x, n, T)\}$, we get $\lim_{\varepsilon \rightarrow 0} I_\varepsilon^2 = - \int_t^T a_n(s) \partial_n \kappa(s, x, n, T) \xi^{1, \pi^*}(s, T) ds$, meaning that if $\mathbb{J} \in C^2$, the limit of this second part exists, equal to

$$\int_t^T a_n(s) \partial_n \kappa(s, x, n, T) \xi^{1, \pi^*}(s, T) ds = \int_t^T a_n(s) \frac{\partial_{nn}^2 \mathbb{J}^{\mathbb{G}} \partial_x \mathbb{J}^{\mathbb{G}} - \partial_{xn}^2 \mathbb{J}^{\mathbb{G}} \partial_n \mathbb{J}^{\mathbb{G}}}{(\partial_x \mathbb{J}^{\mathbb{G}})^2}(s, x, n, T) \xi^{1, \pi^*}(s, T) ds.$$

This completes the proof of the proposition using (30) in Lemma 3.8. •

We are now able to prove Theorem 3.4.

Proof: Given inequality (23) and equation (32), it remains to find the numerator of the right-hand side of inequality (23). Under the assumptions of Proposition 3.9,

$$\begin{aligned} \partial_n \mathbb{J}^{\mathbb{G}} &= \partial_n E^{\mathbb{P}} [U(X_*^{x, n}(t, T)) | \mathcal{G}_t] = E^{\mathbb{P}} [U'(X_*^{x, n}(t, T)) \partial_n X_*^{x, n}(t, T) | \mathcal{G}_t] \\ &= E^{\mathbb{P}} \left[U'(X_*^{x, n}(t, T)) \int_t^T \varpi(s) B(s) \xi^{1, \pi^*}(s, T) ds | \mathcal{G}_t \right]. \end{aligned}$$

Thus, as desired,

$$\begin{aligned}
\frac{\partial_n \mathbb{J}(t, x, n, T)}{\partial_x \mathbb{J}(t, x, n, T)} &= \frac{E^{\mathbb{P}} \left[U'(X_*^{x,n}(t, T)) \int_t^T \varpi(s) B(s) \xi^{1, \pi^*}(s, T) ds \mid \mathcal{G}_t \right]}{E^{\mathbb{P}} \left[U'(X_*^{x,n}(t, T)) \xi^{1, \pi}(t, T) \mid \mathcal{G}_t \right]} \\
&= \int_t^T \frac{E^{\mathbb{P}} \left[U'(X_*^{x,n}(t, T)) \varpi(s) B(s) \xi^{1, \pi^*}(s, T) \mid \mathcal{G}_t \right]}{E^{\mathbb{P}} \left[U'(X_*^{x,n}(t, T)) \xi^{1, \pi}(t, T) \mid \mathcal{G}_t \right]} ds \\
&=: \hat{C}(t, T).
\end{aligned}$$

•

4 Option Replication and Portfolio Constraints

Apart from equation (19) a second way to understand the optimal wealth process is to use an approach due to Ingersoll (2006). (See also Colwell, Feldman, and Hu, 2015.) Instead of granting the options outright, we imagine that the firm gives the executive enough money to replicate the options and then requires the executive to perform the replication. The executive also has short-sale constraints, so if we write Δ_R for the hedge ratio needed to replicate the ESO, then the executive must always hold at least $n\Delta_R$ shares of their firm's stock until exercise. This introduces a *constraint* on their portfolio. One key fact is that both approaches—formulating the outside wealth not including ESO, and formulating the wealth including initial cash endowment replicating ESO—produce the same terminal wealth, $X(T, T)$.

This approach helps us with two problems. First, recall that Theorem 3.4 gives us an expression for the condition under which the insider-exercises options. When we consider portfolio constraints, we can re-write this condition in terms of *subjective* or *marginal indifference prices* for the ESO (see Section 4.1), which provides us with a more intuitive understanding of the early exercise decision. Second, this approach to the optimal wealth process also helps us see how the insider's optimal

expected utility differs from an otherwise identical outsider (see below Section 4.4).

4.1 Insider portfolio constraints

We now present the problem of finding an optimal portfolio when the portfolio is subject to constraints. For the general case, as in Cvitanić and Karatzas (1992), for $(t, \omega) \in [0, T] \times \Omega$ we let $K(t, \omega)$ be a closed, convex, nonempty subset of \mathbb{R}^{d+1} . Here K represents the constraint on an executive's portfolio; that is, their portfolio must satisfy $\pi(t, \omega) \in K(t, \omega)$ for $(t, \omega) \in [0, T] \times \Omega$. Let \mathcal{H} denote the Hilbert space of \mathbb{G} -progressively measurable processes v with values in \mathbb{R}^{d+1} and with the inner product $\langle v_1, v_2 \rangle := E \int_0^T (v_1(t))^\top v_2(t) dt$. Let δ be the support function of the convex set $-K(t, \omega)$, defined on \mathcal{H} :

$$(39) \quad \delta(v(t, \omega)) = \delta(v(t, \omega) | K(t, \omega)) := \sup_{\rho \in K} (-\rho^\top v(t, \omega)) : \mathbb{R}^{d+1} \rightarrow \mathbb{R} \cup \{+\infty\}; (t, \omega) \in [0, T] \times \Omega,$$

where $v(t, \omega) := (v_1(t, \omega), \dots, v_{d+1}(t, \omega))^\top$. Let the convex cone

$$(40) \quad \tilde{K}(t, \omega) := \{v(t, \omega) \in \mathbb{R}^{d+1}; \delta(v(t, \omega) | K(t, \omega)) < \infty\}$$

denote the effective domain of the support function. Here, K represents the portfolio constraint faced by an executive whose firm's stock is S_1 , while $\delta(v(t))$ will be used to solve the constrained portfolio problem.

The constraints placed on an insider-executive's portfolio include holding non-transferable non-hedgeable ESO; a trivial constraint (i.e., no constraint) on other primary assets; and non-tradability on an imaginary primary asset for noise.

Colwell, Feldman, and Hu (2015) used a replication argument to translate portfolios with non-transferable non-hedgeable derivatives into portfolios of primary assets (only) with stochastic portfolio constraints. Here we assume that ESO are European. (In Section 5, we consider American ESO).

Then the non-transferable non-hedgeable constraint on ESO can be represented as a constraint on the firm's stock as

$$(41) \quad [n\Delta_R(t)S_1(t)/X(t), \infty),$$

where n is the number of ESO granted; $\Delta_R(t)$ is the hedge ratio that allows the executive to replicate the ESO; $S_1(t)$ is the stock price underlying the ESO; and $X(t)$ is the insider-executive's total wealth in dollars. Hence, the full portfolio constraint is $K(t, \omega) = [n\Delta_R(t)\frac{S_1(t)}{X(t)}, \infty) \times (-\infty, \infty)^{d-1} \times [0, 0]$.

4.2 Insiders' constrained portfolio optimization

Postponing our discussion of American features of ESO until Section 5, we denote the utility function as U , the value of process X at time t as x , and denote the terminal time of portfolio optimization as $T \in [0, T^*)$.

Insiders' derived utility is

$$(42) \quad J^{\mathbb{G}}(t, x, n, T) := \operatorname{ess\,sup}_{\pi^{\mathbb{G}} \in \mathcal{A}^{\mathbb{G}}(t, x, K, T)} E^{\mathbb{P}}[U(X_{\pi^{\mathbb{G}}}^{x, n}(T)) | \mathcal{G}_t],$$

where $\mathcal{A}^{\mathbb{G}}(t, x, K, T)$ is the class of \mathbb{R}^{d+1} valued \mathcal{G} -progressively measurable portfolio processes, $\pi^{\mathbb{G}}(t, \omega)$, satisfying the conditions

- (i) $\pi^{\mathbb{G}}(t, \omega) \in K$ for $dt \otimes dP$ a.e.,
- (ii) $E^{\mathbb{P}}[\max(-U(X_{\pi^{\mathbb{G}}}^{x, n}(T)), 0)] < \infty$,
- (iii) for initial capital $x \in (0, \infty)$, $X_{\pi^{\mathbb{G}}}^{x, n}(t) \geq 0$, for $t \in [0, T]$ dt almost surely.

Similarly, an outsider's derived utility is

$$(43) \quad J^{\mathbb{F}}(x, t, T) := \operatorname{ess\,sup}_{\pi^{\mathbb{F}} \in \mathcal{A}^{\mathbb{F}}(t, x, K, T)} E^{\mathbb{P}}[U(X_{\pi^{\mathbb{F}}}^{x, n}(T)) | \mathcal{F}_t],$$

where $\mathcal{A}^{\mathbb{F}}(t, x, K, T)$ is the class of \mathbb{R}^{d+1} valued \mathbb{F} - progressively measurable portfolio processes, $\pi^{\mathbb{F}}(t, \omega)$, satisfying the same conditions (i)(ii)(iii) replacing \mathbb{G} by \mathbb{F} .

Note that $\mathcal{A}^{\mathbb{G}}(t, x, K, T)$ and $\mathcal{A}^{\mathbb{F}}(t, x, K, T)$ are defined as the sets of admissible portfolio processes from time t to T , to insiders and outsiders, respectively, with the same initial total wealth composition, the same market value x , and the same portfolio constraints $K(t, \omega)$. Here, $X^{\mathbb{G}} := X_{\pi^{\mathbb{G}}}^{x, n}$ is the wealth process given by Equation (8) corresponding to the portfolio process $\pi^{\mathbb{G}}$ and wealth x at time t ; $X^{\mathbb{F}} := X_{\pi^{\mathbb{F}}}^{x, n}$ is the solution of the same equation corresponding to the portfolio process $\pi^{\mathbb{F}}$ and the same initial wealth; $X^{\mathbb{G}}$ and $X^{\mathbb{F}}$ represent the insiders' and outsiders' total wealth dynamic, respectively.

4.3 Solution of insiders' constrained portfolio optimization

Cvitanić and Karatzas (1992) and Karatzas and Kou (1996) solved the constrained primary assets portfolio optimization problem. By adjusting the drift rates, they transformed the original market into an auxiliary one in which the portfolio constraints automatically hold. The problem becomes a classical unconstrained portfolio optimization, and they show the condition under which the unconstrained solution in the auxiliary market equals the constrained solution in the original market. Colwell, Feldman, and Hu (2015) included ESO in the portfolio by using a replication argument to translate ESO into primary assets and risk-free assets and by analytically pricing non-transferable non-hedgeable American ESO using the constrained portfolio optimization technique.

Pikovskiy and Karatzas (1996) proved that the insiders' price of risk, θ_a , is of the form

$$(44) \quad \theta_a(t) = \theta(t) + a(t),$$

where the outsiders' price of risk, θ , is an \mathbb{F} -adapted process and $a(u)$ is independent of \mathcal{F}_u under the probability measure $Q^{\mathbb{G}}$ (defined in Section 3.1). We further adjust the drift rate owing to the

portfolio constraint $K(t, \omega)$ specified in Section 4.1 to transform the insiders' constrained optimization in the original market into a classical unconstrained optimization in the auxiliary market.

From now on, we assume a log utility. Thanks to Eq. (8.5) and (8.6) in Cvitanić and Karatzas (1992, p. 777), an insider's optimal portfolio process is

$$(45) \quad \pi_v^{\mathbb{G}^*}(t) = [\sigma(t)\sigma^\top(t)]^{-1}[b_{a,v}^{\mathbb{G}}(t) - r_v^{\mathbb{G}}(t)\mathbf{1}],$$

where

$$(46) \quad b_{a,v}^{\mathbb{G}}(t) = b(t) + \sigma(t)a(t) + v^{\mathbb{G}}(t) + \delta(v^{\mathbb{G}}(t))\mathbf{1} \text{ and } r_v^{\mathbb{G}}(t) = r(t) + \delta(v^{\mathbb{G}}(t))$$

are the drift rate and the risk-free rate, respectively, in the auxiliary market; $v^{\mathbb{G}}(t)$ is the drift rate adjustment vector owing to the portfolio constraints to transform insiders' perceived market to an auxiliary one; and the scalar support function δ of the convex set $-K$ is defined in Section 4.1. Then, an insider's price of risk with portfolio constraints, (respectively, an outsider executives' price of risk under the same constraints but setting $a = 0$) are

$$(47) \quad \theta_{a,v}(t) := \sigma^{-1}(t)[b_{a,v}^{\mathbb{G}}(t) - r_v^{\mathbb{G}}(t)\mathbf{1}] ; \theta_{0,v}(t) = \sigma^{-1}(t)[b(t) + v^{\mathbb{F}}(t) - r(t)\mathbf{1}] = \theta(t) + \sigma^{-1}(t)v^{\mathbb{F}}(t),$$

instead of θ_a and θ_0 , which are the market prices of risk without portfolio constraints. Here, $v^{\mathbb{F}}(t)$ is the drift rate adjustment vector, owing to the same portfolio constraints, that transforms outsiders' perceived market to an auxiliary one, as discussed in Cvitanić and Karatzas (1992).

Because $a(t) = (a_1(t), 0, \dots, 0, a_{d+1}(t))^\top$, it follows that

$$(48) \quad \begin{aligned} \sigma(t)a(t) &= (\sigma_{11}a_1 + \sigma_{1,d+1}a_{d+1}, \sigma_{21}a_1 + \sigma_{2,d+1}a_{d+1}, \dots, \sigma_{d+1,1}a_1 + \sigma_{d+1,d+1}a_{d+1})^\top \\ &= (\sigma_{11}a_1, \sigma_{21}a_1, \dots, \sigma_{d1}a_1, a_{d+1})^\top, \end{aligned}$$

where we have used the fact that $\sigma_{i,d+1}(t) = \sigma_{d+1,i}(t) = 0$ for $i \neq d+1$, and $\sigma_{d+1,d+1} = 1$. This implies that insider knowledge about $W_1(T^*)$ can, in general, give executives knowledge about other

correlated stocks. It is possible to assume that the insider knowledge is idiosyncratic and, therefore, uncorrelated to other stocks, but we omit the details.

A similar calculation holds as we discuss in (51). Also,

$$\sigma(t)v^{\mathbb{G}}(t) = (\sigma_{11}v_1^{\mathbb{G}}, \sigma_{21}v_1^{\mathbb{G}}, \dots, \sigma_{d1}v_1^{\mathbb{G}}, v_{d+1}^{\mathbb{G}})^T, \text{ etc.}$$

If we set $a(t) = 0$, then the insider's constrained portfolio optimization becomes an outsider's one; if we further set $v^{\mathbb{G}}(t) = 0$, then it degenerates into a classical unconstrained portfolio optimization.

We next describe how $v^{\mathbb{G}}(t)$ and the functional form of δ are determined. Recalling equation (41) we have the constraint

$$(49) \quad K(t, \omega) = [n\Delta_R(t) \frac{S_1(t)}{X(t)}, \infty) \times (-\infty, \infty)^{d-1} \times [0, 0],$$

where Δ_R represents the hedge ratio that the executive would use to replicate the executive stock options. By Definition (39), the corresponding support function is

$$\delta(v(t)|K) := \sup_{\rho \in K} (-\rho^\top v) = -n\Delta_R(t) \frac{S_1(t)}{X(t)} \times v_1(t),$$

defined on the effective domain of δ , represented as $\tilde{K} = \{v(t) \in \mathbb{R}^{d+1}; \delta(v(t)|K) < \infty\} = [0, \infty) \times [0, 0]^{d-1} \times (-\infty, \infty)$. This implies that $v_2 = \dots = v_d = 0$ for our problem.

Let the variance-covariance matrix be denoted by $\sigma(t)\sigma^\top(t) = \Psi(t)$ and note that $\Psi^{-1}(t)$ is sometimes known as the precision matrix. The following notations are useful:

$$(50) \quad h(t) := \Psi^{-1}(t) ; g(t) := \sigma^{-1}(t)$$

with elements respectively $h_{i,j}(t)$ and $g_{i,j}(t)$.

Thanks to Equation (11.4) in Cvitanić and Karatzas (1992, p. 790), , given our value for $\delta(v(t))$

and replacing $\theta(t)$ with $\theta_a(t)$, under log-utility, we have

$$\begin{aligned}
(51) \quad v^{\mathbb{G}}(t) &= \underset{v=(v_1, v_2, \dots, v_{d+1}) \in \tilde{K}}{\operatorname{argmin}} [2\delta(v(t)|K) + \|\theta_a(t) + \sigma^{-1}(t)v\|^2] \\
&= \underset{v=(v_1, v_2, \dots, v_{d+1}) \in \tilde{K}}{\operatorname{argmin}} \left[-\frac{2n\Delta_R(t)S_1(t)v_1(t)}{X(t)} + \|\sigma^{-1}(t)[b(t) + \sigma(t)a(t) - r(t)\mathbf{1} + v]\|^2 \right].
\end{aligned}$$

Lemma 4.1 *The solution to Equation (51) is*

$$(52) \quad v_1^{\mathbb{G}}(t) = \frac{1}{h_{1,1}(t)} \max \left(\frac{n\Delta_R(t)S_1(t)}{X(t)} - g_{11}a_1(t) - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t), 0 \right),$$

$$(53) \quad v_{d+1}^{\mathbb{G}}(t) = r(t) - a_{d+1}(t),$$

while from Definition (40) of \tilde{K} , we get $v_i^{\mathbb{G}}(t) = 0$ for $i = 2, \dots, d$.

As a direct consequence, since $b_{d+1} = 0$ and $\sigma_{d+1, d+1} = 1$, then the $d + 1$ component of the optimal portfolio $\pi_v^{\mathbb{G}^*}$ is null.⁶

Proof: Recall that in the matrix $\sigma(t)$, $\sigma_{i, d+1}(t) = \sigma_{d+1, i}(t) = 0$ for $i \neq d + 1$. This result also holds for $\Psi(t) = \sigma(t)\sigma^\top(t)$, as well as for $\Psi^{-1}(t) = h(t)$; that is, $h_{i, d+1}(t) = h_{d+1, i}(t) = 0$, for $i \neq d + 1$.

Moreover, $h_{d+1, d+1}(t) = 1$. Next, note that

$$\begin{aligned}
& -\frac{2n\Delta_R(t)S_1(t)v_1}{X(t)} + \|\sigma^{-1}(t)[b(t) + \sigma(t)a(t) - r(t)\mathbf{1} + v]\|^2 \\
&= -\frac{2n\Delta_R(t)S_1(t)v_1}{X(t)} + (b(t) + \sigma(t)a(t) - r(t)\mathbf{1} + v)^\top h(t)(b(t) + \sigma(t)a(t) - r(t)\mathbf{1} + v).
\end{aligned}$$

The function to be minimized is convex and differentiable. So the solution is either a boundary condition or one for which the derivative equals zero; so, taking the derivative with respect to v_1

gives

$$\begin{aligned}
& -\frac{2n\Delta_R(t)S_1(t)}{X(t)} + 2(b(t) + \sigma(t)a(t) - r(t)\mathbf{1} + v^\top)h(t)e_1 \\
& = -2\frac{n\Delta_R(t)S_1(t)}{X(t)} + 2(b(t) - r(t)\mathbf{1})^\top h(t)e_1 + 2(\sigma(t)a(t))^\top h(t)e_1 + 2v^\top(t)h(t)e_1 \\
& = -2\frac{n\Delta_R(t)S_1(t)}{X(t)} + 2(b(t) - r(t)\mathbf{1})^\top h(t)e_1 + 2(e_1)^\top (\sigma^\top(t))^{-1}a(t) + 2v^\top(t)h(t)e_1 \\
& = -2\frac{n\Delta_R(t)S_1(t)}{X(t)} + 2\sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) + 2g_{11}(t)a_1(t) + 2v_1(t)h_{11}(t) = 0,
\end{aligned}$$

where $e_j = (0, \dots, 1, \dots, 0)^\top$ is a vector with 1 in the j^{th} row and zeroes elsewhere. Here we have used the fact that, for a general matrix M , we have, $(e_i)^\top M e_j = M_{ij}$. Solving for $v_1(t)$ gives Equation (52). Finally, because $\tilde{K} = [0, \infty) \times 0^{d-1} \times (-\infty, \infty)$, we see that $v_1(t)$ must remain non-negative.

Differentiating with respect to v_{d+1} and setting the derivative equal to zero, we find that

$$\begin{aligned}
& 2([b(t) + \sigma(t)a(t) - r(t)\mathbf{1}]^\top h(t)e_{d+1} + 2v(t)h(t)e_{d+1}) \\
& = 2(b_{d+1} + \sigma_{d+1,d+1}a_{d+1} - r)h_{d+1,d+1}(t) + 2v_{d+1}(t)h_{d+1,d+1}(t) = 2(a_{d+1}(t) - r(t)) + 2v_{d+1}(t) = 0,
\end{aligned}$$

which proves Equation (53) since $b_{d+1} = 0$.

Setting $\alpha = 0$ in Equations (52) and (53), we get $v^\mathbb{F}(t)$. •

4.4 Decomposition of utility increment owing to insider information

We discuss how informational advantage can improve insiders' derived utility by decomposing the increment of the derived utility, gained due to insider information, into two components: a substantial one and a perceived one.

The derived utility increment due to having insider information is

$$\Delta J(x, t, T) := J^{\mathbb{G}}(x, t, T) - J^{\mathbb{F}}(x, t, T).$$

Under log-utility, it can be decomposed into two components:

- (i) a *substantial increment*, $\Delta \mathcal{J}(x, t, T) := E^{\mathbb{P}}[\Delta J(x, t, T) | \mathcal{F}_t]$, owing to insiders' ability to improve the optimal portfolio process conditional on an enlarged information set,
- (ii) a *perceived component*, $\Delta J(x, t, T) - \Delta \mathcal{J}(x, t, T)$, caused by insiders' and outsiders' differing perceptions. If the American features of ESO are further considered (see Section 5), the perceived component affects insiders' decisions on choosing the optimal exercise time (or optimal exercise rate if partial exercise is allowed) of their ESO. Because ESO are non-transferable non-hedgeable, exercising ESO relaxes the portfolio constraints, which alters the insider's optimal constrained portfolio process and, accordingly, causes a substantial impact.

As an illustration, solving the stochastic differential Equation (8) for the insiders who hold non-transferable non-hedgeable European ESO, we get

$$\begin{aligned} X_{\pi_v^{\mathbb{G}}}^{x,n}(T) &= x \exp \left[\int_t^T \left(r(u) - \frac{1}{2} \|\sigma^\top(u) \pi_v^{\mathbb{G}}(u)\|^2 + (\pi_v^{\mathbb{G}}(u))^\top \sigma(u) \theta_a(u) \right) du + \int_t^T (\pi_v^{\mathbb{G}}(u))^\top \sigma(u) d\tilde{W}(u) \right], \\ (54) \quad &= x \exp \left[\int_t^T \left(r(u) + \frac{1}{2} \|\theta_a(u)\|^2 - \frac{1}{2} \|\theta_a(u) - \sigma^\top(u) \pi_v^{\mathbb{G}}(u)\|^2 \right) du + \int_t^T (\pi_v^{\mathbb{G}}(u))^\top \sigma(u) d\tilde{W}(u) \right]. \end{aligned}$$

Recalling equation (45), for log-utility, the insiders' optimal constrained portfolio process is given by

$$(55) \quad \pi_v^{\mathbb{G}}(u) = [\sigma^\top(u)]^{-1} \theta_{a,v}(u).$$

Recalling (47): $\theta_{a,v}(u) = \theta(u) + a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) = \theta_a(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u)$, it follows that $\theta_a(u) - \theta_{a,v}(u) = -\sigma^{-1}(u)v^{\mathbb{G}}(u)$. Substituting into Equation (54) and computing the optimal value $J^{\mathbb{G}}(t, x, n, T) =$

$E^{\mathbb{P}}[\log X_*^{x,n}(T)|\mathcal{G}_t]$ yields

$$(56) \quad J^{\mathbb{G}}(t, x, n, T) := \ln(x) + E^{\mathbb{P}}[\int_t^T (r(u) + \frac{1}{2}\|\theta_a(u)\|^2)du|\mathcal{G}_t] - E^{\mathbb{P}}[\int_t^T (\frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2)du|\mathcal{G}_t]$$

$$= \ln(x) + E^{\mathbb{P}}[\int_t^T (r(u) + \frac{1}{2}\|\theta(u)\|^2)du|\mathcal{F}_t] + \Delta J^{asy} + \Delta J^{sym} - E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2)du|\mathcal{G}_t],$$

where $\Delta J^{asy} := E^{\mathbb{P}}[\int_t^T \theta^\top(u)a(u)du|\mathcal{G}_t]$ and $\Delta J^{sym} := E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|a(u)\|^2du|\mathcal{G}_t]$ and using the fact that r and θ are \mathbb{F} adapted.

Remark 4.2 As \tilde{W} is a \mathcal{G}_t -Brownian motion, taking expectation of (54) conditional on \mathcal{G}_t , we get the expected instantaneous return of insider's optimal constrained portfolio, which we assume is higher than the riskfree rate, i.e. $E^{\mathbb{P}}[(r(u) + \frac{1}{2}\|\theta_a(u)\|^2)|\mathcal{G}_t] - \frac{1}{2}E^{\mathbb{P}}[\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2|\mathcal{G}_t] \geq r$, for two reasons: insiders are risk averse and insiders' portfolio is risky. Then, with $r > 0$, $E^{\mathbb{P}}[(r(u) + \frac{1}{2}\|\theta_a(u)\|^2)|\mathcal{F}_t] - \frac{1}{2}E^{\mathbb{P}}[\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2|\mathcal{F}_t] \geq 0$.

Setting $a(u) = 0$ and $v^{\mathbb{F}}(u)$ instead of $v^{\mathbb{G}}(u)$ for $u \in [t, T]$ and changing \mathcal{G}_t into \mathcal{F}_t in Equation (56), we get an outsider investor's derived utility:

$$(57) \quad J^{\mathbb{F}}(t, x, n, T) := \ln(x) + E^{\mathbb{P}}[\int_t^T (r(u) + \frac{1}{2}\|\theta(u)\|^2 - \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2)du|\mathcal{F}_t].$$

$$\text{So, } \Delta J(t, x, n, T) := J^{\mathbb{G}}(t, x, n, T) - J^{\mathbb{F}}(t, x, n, T) =$$

$$\Delta J^{asy} + \Delta J^{sym} - E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2du|\mathcal{G}_t] + E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2du|\mathcal{F}_t].$$

Setting $a = 0$, the assumption in Remark 4.2 holds for the outsider case, meaning with $E^{\mathbb{P}}[(r(u) + \frac{1}{2}\|\theta(u)\|^2)|\mathcal{F}_t] - \frac{1}{2}E^{\mathbb{P}}[\|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2|\mathcal{F}_t] \geq 0$.

To highlight the effect of insider information, let us ignore insider-executives' portfolio constraints;

that is, set $v^{\mathbb{G}}(u) = 0$ in Equation (56). Subtracting Equation (57) from Equation (56), we see that the substantial increment owing to insider information can be decomposed into a *symmetric impact* $E^{\mathbb{P}}(\Delta J^{sym}(t)|\mathcal{F}_t)$ and an *asymmetric impact* $E^{\mathbb{P}}(\Delta J^{asy}(t)|\mathcal{F}_t)$. The symmetric impact, determined by the squared norm of the compensating process, implies the same benefit to the insiders whether the (noisy) information indicates good news, i.e., $W_1(T^*) > W_1(t)$, or bad news, i.e., $W_1(T^*) < W_1(t)$. The asymmetric impact becomes zero, if, for any time, $u \in [t, T]$, $a(u)$ is independent of \mathcal{F}_u ; otherwise, good news and bad news can cause an unequal impact on insiders' derived utility.

Generally, whether $a(u)$ is independent of \mathcal{F}_u , is determined by the type of insider information. For example, if insiders know the (noisy) terminal stock price, then the substantial increment has only a symmetric impact; however, if they know the peak of the stock's return, within a term up to T^* , then good news or bad news causes an asymmetric impact on insiders' derived utilities.

5 Insiders' optimal exercise policy

In Section 4, we assume ESO are European. We now incorporate the American feature of ESO. Consider an insider-executive's portfolio that includes ESO with a continuous partial exercise feature. We define an exercise rate process \dot{n} as the number of ESO exercised per unit time elapsed; therefore, we have

$$(58) \quad dN(t) = -\dot{n}(t)dt, \quad N(0) \in \mathbb{N},$$

where $N(0)$ is the number of ESO initially granted to executives and

$$(59) \quad t_{vo} \text{ is the end of the option vesting period.}$$

Recall the portfolio constraint at time t (49):

$$(60) \quad K(t, \omega) = \left[\frac{n\Delta_R(t)S_1(t)}{X(t)}, \infty \right) \times (-\infty, \infty)^{d-1} \times [0, 0].$$

The optimal portfolio process $\pi^{\mathbb{G}^*}$ and exercise process $(\dot{n})^*$ jointly solve

$$(61) \quad \mathbb{J}^{\mathbb{G}} := \mathbb{J}^{\mathbb{G}}(X_*^{x,n}(t), S(t), t, T) := \operatorname{ess\,sup}_{(\pi^{\mathbb{G}}, \dot{n}) \in \mathcal{A}^{\mathbb{G}}(t, x, K, T) \times \mathcal{N}(t)} E^{\mathbb{P}}[U(X_{\pi^{\mathbb{G}}, \dot{n}}^{x,n}(T)) | \mathcal{G}_t],$$

where $X_*^{x,n}$ denotes the wealth with initial cash endowment x , n units of ESO granted, X is generated by the optimal portfolio process $\pi^{\mathbb{G}^*}$ and exercise process $(\dot{n})^*$. Recall that $\mathcal{A}^{\mathbb{G}}$ is defined in Equation (42) as the opportunity set of insiders' portfolio processes.

We need some notations and definitions. Recall that Δ_R is the hedge ratio used to replicate the executive's call option.

Some results from Section 4.3, equations (45) and (46), are useful below:

$$(62) \quad r_v^{\mathbb{G}}(t) = r(t) + \delta(v^{\mathbb{G}}(t))$$

$$(63) \quad b_{a,v}^{\mathbb{G}}(t) = b(t) + \sigma(t)a(t) + v^{\mathbb{G}}(t) + \delta(v^{\mathbb{G}}(t))\mathbf{1}$$

$$(64) \quad \theta_{a,v}(t) = \sigma^{-1}(t)[b_{a,v}^{\mathbb{G}}(t) - r_v^{\mathbb{G}}(t)\mathbf{1}]$$

$$(65) \quad \delta(v(t)) = -n\Delta_R(t)\frac{S_1(t)}{X(t)} \times v_1(t)$$

$$(66) \quad v_1^{\mathbb{G}}(t) = \frac{1}{h_{1,1}(t)} \max \left(\frac{n\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) - g_{11}a_1(t), 0 \right)$$

so that

$$(67) \quad b_{a,v}^{\mathbb{G}}(t) - r_v^{\mathbb{G}}(t)\mathbf{1} = b(t) + \sigma(t)a(t) + v^{\mathbb{G}}(t) - r(t)\mathbf{1}.$$

The subjective price can be written in terms of a change of measure. Let $Z_{a,v}$ be defined to be

the Doléan-Dade exponential,

$$Z_{a,v}(t) = \exp \left\{ - \int_0^t \theta_{a,v}^\top(s) d\tilde{W}(s) - \frac{1}{2} \int_0^t \|\theta_{a,v}(s)\|^2 ds \right\}.$$

If Z is a martingale, then it defines a new probability measure, which we denote by $P^{a,v}$. Next, define the process $\gamma_{a,v}$ to be

$$(68) \quad \gamma_{a,v}(t) = \exp \left\{ - \int_0^t r_v^\mathbb{G}(s) ds \right\}.$$

The drift rate vector for the risky assets under $P^{a,v}$ conditional on \mathcal{G}_t is, thus,

$$b(t) + \sigma(t)a(t) - [b(t) + \sigma(t)a(t) + v^\mathbb{G}(t) - r(t)\mathbf{1}] = r(t)\mathbf{1} - v^\mathbb{G}(t);$$

in particular, the drift rate for the executive's stock is the process $r - v_1^\mathbb{G}$, where $v_1^\mathbb{G} \geq 0$. Also, the risk-free rate under the probability measure used for the subjective price is equal to

$$(69) \quad r_v^\mathbb{G}(t) = r(t) - n\Delta_R(t) \frac{S_1(t)}{X(t)} \times v_1^\mathbb{G}(t) < r(t).$$

From Karatzas and Kou (1996) Theorem 7.4, the subjective price of the European call option is

$$(70) \quad \begin{aligned} \hat{c}(t) &= \frac{E[\gamma_{a,v}(T)Z_{a,v}(T)(S_1(T) - K)^+ | \mathcal{G}_t]}{\gamma_{a,v}(t)Z_{a,v}(t)} = \frac{E^{a,v}[\gamma_{a,v}(T)(S_1(T) - K)^+ | \mathcal{G}_t]}{\gamma_{a,v}(t)} \\ &= \frac{E^{a,v}[\gamma_{a,v}(T)S_1(T)\mathbf{1}_{S_1(T) > K} | \mathcal{G}_t]}{\gamma_{a,v}(t)} - K \frac{E^{a,v}[\gamma_{a,v}(T)\mathbf{1}_{S_1(T) > K} | \mathcal{G}_t]}{\gamma_{a,v}(t)}. \end{aligned}$$

Lemma 5.1 *In evaluating the subjective call price (whether American or European) under the probability measure $P^{a,v}$, we use the (stochastic) risk-free rate,*

$$(71) \quad r_v^\mathbb{G} = r(t) + \delta(v^\mathbb{G}(t)) = r(t) - n\Delta_R(t) \frac{S_1(t)}{X(t)} v_1^\mathbb{G}(t),$$

and the stock has a (stochastic) dividend yield that is equal to

$$(72) \quad q_1^{\mathbb{G}}(t) = v_1^{\mathbb{G}}(t) + \delta(v^{\mathbb{G}}(t)) = v_1^{\mathbb{G}}(t) - n\Delta_R(t) \frac{S_1(t)}{X(t)} \times v_1^{\mathbb{G}}(t).$$

As a consequence

$$(73) \quad r_v^{\mathbb{G}} - q_1^{\mathbb{G}}(t) := r(t) - v_1^{\mathbb{G}}(t) = r(t) - \frac{1}{h_{1,1}(t)} \max \left(\frac{n\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) - g_{11}a_1(t), 0 \right).$$

Proof: The risk-free rate $r_v^{\mathbb{G}}$ used here is given in Section 4.3 and equation (46).

Regarding the dividend yield, first recall (see paragraph above Proposition 3.1) that $\tilde{W}(t) := W(t) - \int_0^t a(s)ds$ is a $(d+1)$ -vector-valued (\mathbb{P}, \mathbb{G}) -standard Brownian motion. By Girsanov's theorem, and the definition of $Z_{a,v}$, the $(d+1)$ -vector process $W^{a,v}$ defined by

$$W^{a,v}(t) = \tilde{W}(t) + \int_0^t \theta_{a,v}(s)ds$$

is a $(\mathbb{P}^{a,v}, \mathbb{G})$ -standard Brownian motion. The $(d+1)$ -vector of stock prices S satisfies the stochastic differential equation

$$dS(t) = \text{diag}(S(t)) \left(b(t)dt + \sigma(t)a(t)dt + \sigma(t)d\tilde{W}(t) \right).$$

Using (47) in Section 4.3, $\theta_{a,v}(t) = \sigma^{-1}(t)[b_{a,v}^{\mathbb{G}}(t) - r_v^{\mathbb{G}}(t)\mathbf{1}]$, and (46) yield

$$\begin{aligned} dS(t) &= \text{diag}(S(t)) \left(b(t)dt + \sigma(t)a(t)dt + \sigma(t)(d\tilde{W}(t) + \theta_{a,v}(t)dt - \theta_{a,v}(t)dt) \right) \\ &= \text{diag}(S(t)) (b(t)dt + \sigma(t)a(t)dt + \sigma(t)(dW^{a,v}(t) - \theta_{a,v}(t)dt)) \\ &= \text{diag}(S(t)) (b(t)dt + \sigma(t)a(t)dt + r_v^{\mathbb{G}}(t)\mathbf{1}dt - b_{a,v}^{\mathbb{G}}(t)dt + \sigma(t)dW^{a,v}(t)) \\ &= \text{diag}(S(t)) (r_v^{\mathbb{G}}(t)\mathbf{1}dt - v^{\mathbb{G}}(t)dt - \delta(v^{\mathbb{G}}(t))dt + \sigma(t)dW^{a,v}(t)). \end{aligned}$$

So, the drift rate for S_1 is $r_v^G(t) - q_1^G(t)$, where $q_1^G(t) = v_1^G(t) + \delta(v^G(t))$. Equation (65) ends the proof. •

The above lemma allows us to find conditions under which the option may be exercised early.

Recall that the vector a is defined above, Section 3.2, Corollary 3.2:

$$a_1(t) = \frac{\lambda[\lambda(W_1(T^*) - W_1(t)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(t))]}{T^* - t}, \quad t \leq T < T^*.$$

And before presenting the theorem, let us make three brief comments.

- (i) Most executive stock options have a vesting period denoted as t_{vo} , meaning the length of time that an executive must wait before being allowed to exercise their ESO. It is introduced in (59).
- (ii) Note that $n\Delta_R(t)S_1(t) \leq X(t)$: This holds because the wealth process, X , includes the $n\Delta_R$ shares by construction.
- (iii) Regulations normally require that executives not trade after the start of a prescribed blackout trading period, which is defined in Section 7; thus $T < T^*$.

5.1 On the early block exercise of American call options.

We discuss the general principles for early block exercise of American call options, where block exercise means at any time t , the option holder decides whether to exercise all the options held or keep them all alive. Here, we do not consider the executives' portfolio but allow for a dividend yield distributed by holding a firm stock, as in Lemma 5.1, equation (72). We use the two lemmas proved in Section 5.1 to prove Theorem 5.4, in which we formulate executives' portfolio constraints into a stochastic dividend yield in the auxiliary market.

We assume that the risk-free rate, $r > 0$, is stochastic and the firm's stock S_1 has a stochastic dividend yield, q . So, under the risk-neutral measure, Q and with $E^Q(\cdot)$ as the expectation operator

on the filtered probability space $(\Omega, \mathcal{F}, Q, \mathbb{F})$, S_1 satisfies the stochastic differential equation,

$$dS_1(t) = S_1(t)(r(t) - q(t))dt + S_1(t)\sigma(t)dW(t).$$

Lemma 5.2 and Lemma 5.3 show that if $q \leq 0$, then the call holder would never exercise the option early; but if $q > 0$ on an open interval with positive probability, then the call holder may exercise early. Our proof involves finding a lower bound for the European call, and we establish that lower bound by considering put-call parity, with stochastic parameters.

Write $c(t)$ and $p(t)$ (or $c(t, S_1, q)$ and $p(t, S_1, q)$ if needed) for the price at date t of a European call option and put option, respectively, on S_1 , with maturity T and strike price K ; that is,

$$c(t) = E^Q[e^{\{-\int_t^T r(s)ds\}}(S_1(T) - K)^+ | \mathcal{F}_t]; \quad p(t) = E^Q[e^{\{-\int_t^T r(s)ds\}}(K - S_1(T))^+ | \mathcal{F}_t]$$

Actually, the regular function c exists because of the Markov property of process S_1 .

Now let C represent the price process of the corresponding American call, where the essential supremum is taken over the set of all \mathcal{F} -stopping times $\tau \leq T$:

$$(74) \quad C(t) = \operatorname{ess\,sup}_{t \leq \tau \leq T} E^Q[e^{\{-\int_t^\tau r(s)ds\}}(S_1(\tau) - K)^+ | \mathcal{F}_t].$$

Similarly, write $p(t)$ (resp. $P(t)$) for the price of a European (resp. American) put as of date t on S_1 with the same maturity and strike price. It is easy to see that, at maturity, $c(T) - p(T) = S_1(T) - K$. Then at time $t = 0$: $c(0) - p(0) = E^Q[e^{\{-\int_0^T r(s)ds\}}(S_1(T) - K)]$.

Below we need and use

$$(75) \quad c(t) - p(t) = E^Q[e^{\{-\int_t^T r(s)ds\}}(S_1(T) - K) | \mathcal{F}_t].$$

Let the equivalent probability measure Q^q defined by

$$\frac{dQ^q}{dQ} = e^{\{-\int_0^T (r(s)-q(s))ds\}} \frac{S_1(T)}{S_1(0)}$$

and let us define the following processes:

$$(76) \quad B(t, T) := E^Q[e^{\{-\int_t^T r(s)ds\}}|\mathcal{F}_t], \quad D(t, T) := E^{Q^q}[e^{\{-\int_t^T q(s)ds\}}|\mathcal{F}_t].$$

Here, $B(t, T)$ is the price of risk-free zero coupon bonds as of date t with maturity T , and $D(t, T)$ is a similar calculation using the dividend yield but under the equivalent probability measure Q^q .

This definition is viable: the process $Y_u := e^{\{-\int_0^u (r(s)-q(s))ds\}} S_1(u)$ is a (Q, \mathbb{F}) -martingale, since the Itô formula implies $dY_u = Y_u \sigma(u) dW(u)$. Thus, $E^Q[e^{\{-\int_0^T (r(s)-q(s))ds\}} \frac{S_1(T)}{S_1(0)}] = E^Q[\frac{Y_T}{Y_0}] = 1$.

For the dividends, we have at time t , t being fixed

$$\begin{aligned} E^Q[e^{\{-\int_t^T r(s)ds\}} S_1(T)|\mathcal{F}_t] &= E^Q[e^{\{-\int_t^T (r(s)-q(s)+q(s))ds\}} S_1(T)|\mathcal{F}_t] \\ &= S_1(t) E^Q[e^{\{-\int_t^T (r(s)-q(s))ds\}} \frac{S_1(T)}{S_1(t)} e^{\{-\int_t^T q(s)ds\}}|\mathcal{F}_t] \end{aligned}$$

and using Bayes formula for the \mathcal{F}_t -conditional expectation yields

$$(77) \quad E^Q[e^{\{-\int_t^T r(s)ds\}} S_1(T)|\mathcal{F}_t] = S_1(t) E^{Q^q}[e^{\{-\int_t^T q(s)ds\}}|\mathcal{F}_t] = S_1(t) D(t, T),$$

Thus, put-call parity can be written, for any $t \leq T$, as follows: using (75), (76) and (2),

$$(78) \quad c(t) - p(t) = S_1(t) D(t, T) - KB(t, T),$$

and using the that $p \geq 0$ this implies that

$$(79) \quad c(t) \geq S_1(t) D(t, T) - KB(t, T).$$

Lemma 5.2 *On the event $\{(t, \omega) : q(t, \omega) \leq 0\}$, the call holder would never choose to exercise the call early.*

Proof: When $q \leq 0$, $D(t, T) \geq 1$ and because $B(t, T) < 1$, we have

$$c(t) \geq S_1(t)D(t, T) - KB(t, T) > S_1(t) - K.$$

Now suppose the owner of the call is considering exercising at date $t < T$. If they exercise, they receive $S_1(t) - K$; but if they sell the option instead, they receive $c(t)$, which is worth more. Therefore, the call holder would never choose to exercise the call early. •

Lemma 5.3 *Suppose $q \geq 0$ and $q > 0$ on some open time interval(s) with positive probability. Then $c(t) < C(t)$ is satisfied on the event*

$$A_t := \{S_1(t)D(t, T) - KB(t, T) < c(t) < S_1(t) - K\},$$

implying that an early exercise exists on this event A_t , the probability of which is positive.

Proof: Since $q \geq 0$ and $q > 0$ on some open time interval(s) with positive probability, then $D(t, T) < 1$ on this event. Define the functions $f : S_1 \mapsto S_1D(t, T) - KB(t, T)$ and $g : S_1 \mapsto S_1 - K$. Note that f and g define straight lines. Since they have different slopes, they must intersect at some point, which we denote by $S_1^\circ(t) = K \frac{1-B(t, T)}{1-D(t, T)}$. Moreover, f has a lower slope for $S_1 > S_1^\circ$, $f(S_1) < g(S_1)$; that is,

$$S_1(t)D(t, T) - KB(t, T) < S_1(t) - K \text{ on the event } \{(t, \omega) | S_1(t) > S_1^\circ(t)\}.$$

Using (78) we see that the following events are equal:

$$\{S_1(t)D(t, T) - KB(t, T) < c(t) < S_1(t) - K\} = \{0 < p(t) < S_1(t) - K - (S_1(t)D(t, T) - KB(t, T))\}.$$

Since $S_1(t) \rightarrow \infty$ yields $p(t) \equiv p(t, S_1(t)) \rightarrow 0$, this event has a strictly positive probability.

On the other hand, recalling the definition (74) and using the stopping time $\tau = t$, we immediately see that

$$C(t) \geq (S_1(t) - K)^+ \geq S_1(T) - K.$$

Thus, we have shown that $c(t) < S_1(t) - K \leq C(t)$ is satisfied on the event

$$A_t := \{(t, \omega) | S_1(t) > S_1^\circ(t)\} \cap \{S_1(t)D(t, T) - KB(t, T) < c(t) < S_1(t) - K\},$$

meaning that an early exercise exists on this event A_t . Finally note the equivalence

$$S_1 > S_1^\circ = K \frac{1 - B(t, T)}{1 - D(t, T)} \iff S_1(t)D(t, T) - KB(t, T) < S_1(t) - K$$

so actually

$$A_t = \{S_1(t)D(t, T) - KB(t, T) < c(t) < S_1(t) - K\}$$

and we prove above that $P(A_t) > 0$. •

5.2 Good news, bad news, and insider's exercise policy

Theorem 5.4 demonstrates that if an insider's information is good news, then they are more likely to hold onto their ESO rather than exercising them, and if their information is bad news, they are more likely to exercise the ESO sooner, effectively freeing up their portfolio constraints.

We define the upper bound of blackouts as the one making insider-executives' substantial increment $\Delta \mathcal{J}(x, t, T)$ (cf. Section 4.4) of the derived utility equal to zero, when outsiders can freely trade until T^* and the terminal trading time of insiders is just before the blackout starts, assuming they have same cash-only endowments.

Theorem 5.4 Denote n as the value of the process N at initial time t . The insider information, a_1 (cf. Corollary 3.2), influences the early exercise decision in the following ways.

1. First, if the insider has good news, satisfying

$$(80) \quad a_1(t) > \frac{1}{g_{11}} \left(\frac{n\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right) =: A(n),$$

then this insider will not exercise the call options possessed early but will hold onto them until closer to the time that the good news is revealed, which we assume happens at date T^* .

2. If the insider has (relatively) bad news satisfying

$$(81) \quad a_1(t) < \frac{1}{g_{11}} \left(\frac{n\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right) = A(n),$$

then, the insider will exercise at least one option early, before the bad news is revealed.

3. Moreover, on this event, the first element of vector $\pi^{\mathbb{G}}$ is given by

$$(82) \quad \pi_1^{\mathbb{G}}(t) = \frac{n\Delta_R(t)S_1(t)}{X(t)},$$

That is, the insider is forced to hold the granted ESO and is not willing to hold more than that if exercise is not allowed before the vesting ends.

4. If, at the end of the vesting period, $t = t_{vo}$, the insider has bad news satisfying

$$(83) \quad a_1(t) < \frac{1}{g_{11}} \left(\frac{\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right) =: A(1),$$

then the insider will exercise all of their options on the event A_t defined in Lemma 5.3, where $S_1(t)$ satisfies $\hat{c}(t) = \hat{c}(t, S_1, a, v) < S_1(t) - K$, i.e., $\hat{c}(t)$, the subjective price defined in equation (70), which

has been adjusted due to portfolio constraints, is less than the immediate payoff from exercise.

Proof: If the risk-free rate in the subjective option price is equal to $r_v^{\mathbb{G}}(t) = r(t) + \delta(v^{\mathbb{G}}(t))$ as in (46), then we can rewrite the risk-neutral drift rate with respect to the insider's filtration \mathbb{G} as,

$$\begin{aligned}
 r(t) - v_1^{\mathbb{G}} &= r(t) - n\Delta_R(t)\frac{S_1(t)}{X(t)} \times v_1^{\mathbb{G}}(t) + n\Delta_R(t)\frac{S_1(t)}{X(t)} \times v_1^{\mathbb{G}}(t) - v_1^{\mathbb{G}} \\
 (84) \qquad &= r_v^{\mathbb{G}} - v_1^{\mathbb{G}} \left(1 - n\Delta_R(t)\frac{S_1(t)}{X(t)} \right),
 \end{aligned}$$

where $v_1^{\mathbb{G}} \geq 0$ according to its definition (66). Recall $n\Delta_R(t)S_1(t) \leq X(t)$ and the dividend yield $q_1^{\mathbb{G}}$ are defined above in Lemma 5.1:

$$(85) \qquad q_1^{\mathbb{G}}(t) := v_1^{\mathbb{G}}(t) \left(1 - n\Delta_R(t)\frac{S_1(t)}{X(t)} \right).$$

Item 1. By Lemma 5.2, if the dividend yield $q_1^{\mathbb{G}}$ is less than or equal to zero, then the call holder will not exercise early. From the definition of $v_1^{\mathbb{G}}$, and noting that $h_{11} > 0$, $q_1^{\mathbb{G}}(t) = 0$,

$$\frac{n\Delta_R(t)S_1(t)}{X(t)} - g_{11}a_1(t) - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \leq 0.$$

Solving for a_1 and noting that g_{11} is greater than zero, Item 1 is proved.

Item 2. By Lemma 5.3, if $q_1^{\mathbb{G}} > 0$, then

$$\frac{n\Delta_R(t)S_1(t)}{X(t)} - g_{11}a_1(t) - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) > 0$$

and the options may be exercised early. This holds if $S_1(t)$ is sufficiently large, i.e., if

$$S_1(t) > \frac{X(t)}{n\Delta_R}(g_{11}a_1(t) + \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t)).$$

Solving for a_1 gives the assumption of Item 2.

Item 3. Note that from Equation (55) and the ensuing paragraph,

$$\begin{aligned} \pi^{\mathbb{G}}(t) &= [\sigma^\top(t)]^{-1} (\theta(t) + a(t) + \sigma^{-1}(t)v^{\mathbb{G}}(t)) \\ &= [\sigma^\top(t)]^{-1}\theta(t) + [\sigma^\top(t)]^{-1}a(t) + [\sigma^\top(t)]^{-1}\sigma^{-1}(t)v^{\mathbb{G}}(t) \\ &= h(t)[b(t) - r(t)\mathbf{1}] + g(t)a(t) + h(t)v^{\mathbb{G}}(t), \end{aligned}$$

where, θ is defined in Equation (4), and g , and h in (50). By Lemma 4.1,

$$v_1^{\mathbb{G}}(t) = \frac{1}{h_{1,1}(t)} \max \left(\frac{N(t)\Delta_R(t)S_1(t)}{X(t)} - g_{11}a_1(t) - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t), 0 \right).$$

So, if the insider information is bad news satisfying the condition of the proposition, then $v_1^{\mathbb{G}}(t) > 0$.

The first element of vector $\pi^{\mathbb{G}}$ becomes

$$\pi_1^{\mathbb{G}}(t) = (h(t)[b(t) - r(t)\mathbf{1}])_1 + g_{11}a_1(t) + h_{11}v_1^{\mathbb{G}}(t).$$

However,

$$(h(t)[b(t) - r(t)\mathbf{1}])_1 = \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t),$$

and the result quickly follows.

Item 4. The assumption

$$a_1(t) < \frac{1}{g_{11}} \left(\frac{\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right)$$

implies

$$a_1(t) < \frac{1}{g_{11}} \left(\frac{k\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right),$$

for all $k = 1, 2, \dots, n$ which means the inequality in Item 2 holds for any value of k . So, the event $\{a_1(t) < A(1)\}$ satisfies the condition A_t defined in Lemma 5.3, and the call holder may exercise all of the options. •

We can also apply these results to the case where there is no insider information, which corresponds to the assumption that $a_1 = 0$.

Corollary 5.5 1. *With no insider information, the executive will not exercise options early if*

$$\frac{n\Delta_R(t)S_1(t)}{X(t)} < \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t).$$

Intuitively, if the executive's options represent a relatively low percentage of their total wealth, they will not exercise early.

2. *With no insider information, the executive may exercise at least one option if*

$$\frac{n\Delta_R(t)S_1(t)}{X(t)} > \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t).$$

3. Moreover, on this event, the first element of vector $\pi^{\mathbb{F}}$ is given by

$$\pi_1^{\mathbb{F}}(t) = \frac{n\Delta_R(t)S_1(t)}{X(t)}.$$

That suggests that outside executives will not hold extra firm shares other than the non-transferable non-hedgeable ESO.

4. With no insider information, the executive may exercise all their options if

$$\frac{\Delta_R(t)S_1(t)}{X(t)} > \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t).$$

Proof: We apply Theorem 5.4 in case $a_1 = 0$.

1. is Theorem 5.4 Item 1.
2. is Theorem 5.4 Item 2.
3. is Theorem 5.4 Item 3.
4. is Theorem 5.4 Item 4. •

Theorem 5.4 proves that insiders' optimal exercise policies are different facing good news and bad news.

- First, if the insiders' information news is sufficiently good, they will not exercise options at all.
- Second, if insiders' information is bad news, they will exercise at least one option after the vesting end and before the option maturity.
- Third, if insiders' information is sufficiently bad, the executive's optimization problem has a *corner solution*; the optimal stock position is to hold non-transferable non-hedgeable ESO granted and not more than that. Short sales are not allowed, so the insider-executive cannot

achieve excessive expected utility by trading in S_1 alone when news is bad. However, insider-executives can still achieve excessive expected utility by trading in other assets correlated with the firm's stock.

- Fourth, if insiders' information is very bad, then they will exercise all options upon vesting end.

It should be emphasized that the terms "good" news and "bad" news are not permanent. To see this, first consider the following remark.

Remark 5.6 *Suppose σ_{11} is constant. Then, all else being equal, if S_1 increases then a_1 decreases.*

Proof: Because the matrix σ is lower triangular, S_1 is a function of W_1 only. In fact, if we assume that σ_{11} is constant, then

$$\frac{1}{\sigma_{11}} \left(\ln\left(\frac{S_1(t)}{S_1(0)}\right) - \int_0^t (b_1(u) - \frac{1}{2}\sigma_{11}^2) du \right) = W_1(t).$$

Also, recall Corollary 3.2:

$$a_1(t) = \frac{\lambda[\lambda(W_1(T^*) - W_1(t)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(t))]}{T^* - t}.$$

The result follows. •

For example, suppose $a_1(t) > \frac{1}{g_{11}} \left(\frac{N(t)\Delta_R(t)S_1(t)}{X(t)} - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right)$. In this case, $v^{\mathbb{G}} = 0$, and the insider would not exercise their option early. However, as S_1 increases, a_1 decreases, while the right-hand side of the inequality increases. Thus, the inequality can be reversed, $v^{\mathbb{G}}$ can become positive, and the insider may exercise the option early. This is analogous to the usual American call option problem for a stock paying a dividend yield: if S_1 is sufficiently high, the stock may be exercised early.

Note also that even if the insider is not able to achieve excessive expected utility by trading their own stock, they may be able to achieve excessive expected utility by trading in other correlated stocks, as we discuss in Section 4.4.

Let us briefly mention that if there is no insider information (i.e., $a_1(t) = 0$), then $\pi_1^G(t) = \frac{N(t)\Delta_R(t)S_1(t)}{X(t)}$ if

$$\frac{N(t)\Delta_R(t)S_1(t)}{X(t)} > \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t).$$

6 Policy recommendations on blackout trading regulation

In financial markets a blackout period is a period when insiders are prohibited from buying or selling shares that they have private information about or making changes to their investment plans. As explained earlier, regulators are concerned about executives using their information to gain an unfair advantage in the stock markets, so companies may enforce blackout periods at various times, such as before earnings announcements, before mergers or acquisitions, or before the release of a new product.

We discuss four practical questions.

- (1) Is a blackout still required when insider information is (very) noisy and insider-executives must obey non-transferable non-hedgeable constraints? See Proposition 6.1.
- (2) How long should a blackout be? See Section 6.2.
- (3) Who should take the role of the choice entity to enact a blackout, and who should mandate it? Look at the beginning of Section 6.3.
- (4) What assets should the blackout trade prohibitions include? See Proposition 6.4.

6.1 The necessity of blackout trading periods

Under the model in Section 3, we have the following proposition.

Proposition 6.1 *Recall the definition $\Delta J(t, x, n, T) = J^{\mathbb{G}}(t, x, n, T) - J^{\mathbb{F}}(t, x, n, T)$.*

(i) *Under log-utility with portfolio constraints given by equation (49) $K(t, \omega) = [\frac{N(t)\Delta_R(t)S_1(t)}{X(t)}, \infty) \times (-\infty, \infty)^{d-1} \times [0, 0]$, we have*

$$E^{\mathbb{P}}[\Delta J(t, x, n, T)|\mathcal{F}_t] \geq \frac{1}{2} \int_t^T \frac{1}{T^* - u} du - \frac{1}{2} C_1(t),$$

where

$$C_1(t) = E^{\mathbb{P}}\left[\int_t^T (C_0(u))^2 h_{1,1}(u) du \middle| \mathcal{F}_t\right] + E^{\mathbb{P}}\left[\int_t^T r^2(u) du \middle| \mathcal{F}_t\right],$$

with

$$C_0(u) = \frac{1}{h_{1,1}(u)} \left(\frac{n\Delta_R(u)S_1(u)}{X(u)} - \sum_{i=1}^d (b_i(u) - r(u)) h_{i,1}(u) \right).$$

If $C_1(t) < \infty$, then the substantial increment defined in Section 4.4 (i),

$$\Delta \mathcal{J}(t, x, n, T) = E^{\mathbb{P}}[\Delta J(t, x, n, T)|\mathcal{F}_t] \rightarrow \infty \text{ as } T \rightarrow T^*.$$

(ii) *Moreover the map $\lambda \mapsto E^{\mathbb{P}}[\Delta J(t, x, n, T)|\mathcal{F}_t]$ is non-decreasing.*

Proof: (i) Recall that $a_1(u) = \lambda[\lambda(W_1(T^*) - W_1(u)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(u))]/(T^* - u)$ and $a_{d+1}(u) = \sqrt{1 - \lambda^2}[\lambda(W_1(T^*) - W_1(u)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(u))]/((T^* - u))$ are Gaussian random variables, independent of \mathcal{F}_t with Gaussian laws $\mathcal{N}(0, \frac{\lambda^2}{T^* - u})$ and $\mathcal{N}(0, \frac{1 - \lambda^2}{T^* - u})$, respectively, under \mathbb{P} . Thus

$$E^{\mathbb{P}}\left[\int_t^T \frac{1}{2} \|a(u)\|^2 du \middle| \mathcal{F}_t\right] = \frac{1}{2} E^{\mathbb{P}}\left[\int_t^T [(a_1(u))^2 + (a_{d+1}(u))^2] du \middle| \mathcal{F}_t\right] = \frac{1}{2} \ln \frac{T^* - t}{T^* - T}.$$

Moreover, because $a_i(u)$, $i = 1, d + 1$ have mean zero and are independent of \mathcal{F}_u , for any sufficiently integrable \mathbb{F} -adapted process \hat{b} ,

$$E^{\mathbb{P}}[a_i(u)\hat{b}(u)|\mathcal{F}_t] = E^{\mathbb{P}}[a_i(u)]E^{\mathbb{P}}[\hat{b}(u)|\mathcal{F}_t] = 0.$$

From Equations (56) and (57), since $E^{\mathbb{P}}[\int_t^T (\frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2) du] \geq 0$,

$$\begin{aligned} E^{\mathbb{P}}[\Delta J(t, x, n, T)|\mathcal{F}_t] &\geq E^{\mathbb{P}}(\Delta J^{asy}(t)|\mathcal{F}_t) + E^{\mathbb{P}}(\Delta J^{sym}(t)|\mathcal{F}_t) - E^{\mathbb{P}}[\int_t^T (\frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2) du|\mathcal{F}_t] \\ &= E^{\mathbb{P}}[\int_t^T \theta^{\top}(u)a(u)du|\mathcal{F}_t] + E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|a(u)\|^2 du|\mathcal{F}_t] - E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2 du|\mathcal{F}_t] \\ (86) \quad &= 0 + \frac{1}{2} \ln \frac{T^* - t}{T^* - T} - E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2 du|\mathcal{F}_t]. \end{aligned}$$

Next, recall that we are writing the $(i, j)^{th}$ element of the matrix $\sigma^{-1}(u)$ as g_{ij} . Because $\sigma_{i, d+1} = \sigma_{d+1, i} = 0$ for $i \neq d + 1$, it follows that $g_{i, d+1} = g_{d+1, i} = 0$ for $i \neq d + 1$, as well. Thus, using $g_{d+1, d+1} = 1$, by assumption:

$$(\sigma^{-1}(u)v^{\mathbb{G}}(u))^T = (g_{1,1}v_1^{\mathbb{G}} + g_{1,d+1}v_{d+1}^{\mathbb{G}}, \dots, g_{d,1}v_1^{\mathbb{G}} + g_{d,d+1}v_{d+1}^{\mathbb{G}}, g_{1,d+1}v_1^{\mathbb{G}} + g_{d+1,d+1}v_{d+1}^{\mathbb{G}}) = (g_{1,1}v_1^{\mathbb{G}}, \dots, g_{d,1}v_1^{\mathbb{G}}, v_{d+1}^{\mathbb{G}}).$$

Note that $h = g^{\top}g$, and so $h_{1,1} = \sum_{i=1}^{d+1}(g_{i,1})^2 = \sum_{i=1}^d(g_{i,1})^2$, which implies that $\frac{(g_{1,1}(u))^2}{h_{1,1}(u)} \leq 1$. Now,

$$\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2 = (v_1^{\mathbb{G}}(u))^2 \sum_{i=1}^d (g_{i,1}(u))^2 + (v_{d+1}^{\mathbb{G}}(u))^2 = (v_1^{\mathbb{G}}(u))^2 h_{1,1}(u) + (v_{d+1}^{\mathbb{G}}(u))^2.$$

Recall Lemma 4.1: $v_1^{\mathbb{G}}(u) = (C_0(u) - \frac{g_{1,1}(u)}{h_{1,1}(u)}a_1(u))^+$, $v_{d+1}^{\mathbb{G}}(t) = r(t) - a_{d+1}(t)$. Then,

$$E^{\mathbb{P}}[(v_1^{\mathbb{G}}(u))^2 h_{1,1}(u)|\mathcal{F}_t] = E^{\mathbb{P}}[(C_0(u) - \frac{g_{1,1}(u)}{h_{1,1}(u)}a_1(u))^2 \mathbf{1}_{(v_1^{\mathbb{G}}(u) > 0)} h_{1,1}(u)|\mathcal{F}_t]$$

$$\leq E^{\mathbb{P}}[(C_0(u) - \frac{g_{1,1}(u)}{h_{1,1}(u)}a_1(u))^2 h_{1,1}(u) | \mathcal{F}_t].$$

Using the independence between \mathcal{F}_t and $a_1(u)$, $u \geq t$ centered with variance $\frac{\lambda^2}{T^* - u}$, and $\frac{(g_{1,1}(u))^2}{h_{1,1}(u)} \leq 1$, this bound is

$$E^{\mathbb{P}}[(C_0(u))^2 h_{1,1}(u) + \frac{(g_{1,1}(u))^2}{h_{1,1}(u)} a_1^2(u) | \mathcal{F}_t] \leq E^{\mathbb{P}}[(C_0(u))^2 h_{1,1}(u) | \mathcal{F}_t] + \frac{\lambda^2}{T^* - u}.$$

Once again we use the fact that $a_{d+1}(u)$ has mean zero and is independent of \mathcal{F}_t with variance $\frac{1 - \lambda^2}{T^* - u}$:

$$E^{\mathbb{P}}[(v_{d+1}^{\mathbb{G}}(u))^2 | \mathcal{F}_t] = E^{\mathbb{P}}[(r(u) - a_{d+1}(u))^2 | \mathcal{F}_t] = E^{\mathbb{P}}[r^2(u) | \mathcal{F}_t] + \frac{1 - \lambda^2}{T^* - u}.$$

So,

$$E^{\mathbb{P}}[\int_t^T (\frac{1}{2} \|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2) du | \mathcal{F}_t] \leq$$

$$\frac{1}{2} E^{\mathbb{P}} \left[\int_t^T C_0^2(u) h_{1,1}(u) du + \int_t^T r^2(u) du | \mathcal{F}_t \right] + \frac{1}{2} \int_t^T \frac{du}{T^* - u}.$$

Therefore, using (86)

$$\begin{aligned} E^{\mathbb{P}}[\Delta J(t, x, n, T) | \mathcal{F}_t] &\geq \frac{1}{2} \int_t^T \frac{du}{T^* - u} - \frac{1}{2} E^{\mathbb{P}}[\int_t^T (C_0(u))^2 h_{1,1}(u) du | \mathcal{F}_t] - \frac{1}{2} E^{\mathbb{P}}[\int_t^T r^2(u) du | \mathcal{F}_t] \\ &= \frac{1}{2} \int_t^T \frac{du}{T^* - u} - \frac{1}{2} C_1(t), \end{aligned}$$

as desired.

(ii) We first recall (86) $E^{\mathbb{P}}[\Delta J(t, x, n, T) | \mathcal{F}_t] =$

$$E^{\mathbb{P}}(\Delta J^{asy}(t) + \Delta J^{sym}(t) | \mathcal{F}_t) - E^{\mathbb{P}}[\int_t^T (\frac{1}{2} \|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2) du | \mathcal{F}_t] + E^{\mathbb{P}}[\int_t^T (\frac{1}{2} \|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2) du | \mathcal{F}_t],$$

where $\Delta J^{asy} := E^{\mathbb{P}}[\int_t^T \theta^\top(u)a(u)du|\mathcal{G}_t]$ and $\Delta J^{sym} := E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|a(u)\|^2 du|\mathcal{G}_t]$. We have above proven that $E^{\mathbb{P}}[\int_t^T \theta^\top(u)a(u)du|\mathcal{F}_t] = 0$ and $E^{\mathbb{P}}[\int_t^T \frac{1}{2}\|a(u)\|^2 du|\mathcal{F}_t] = \frac{1}{2} \ln \frac{T^*-t}{T^*-T}$. The last term $E^{\mathbb{P}}[\int_t^T (\frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2) du|\mathcal{F}_t]$ obviously does not depend on λ , so we have to look only at the term

$$\lambda \mapsto -E^{\mathbb{P}}[\int_t^T (\frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2) du|\mathcal{F}_t].$$

Recall Lemma 4.1,

$$v_1^{\mathbb{G}}(t) = \frac{1}{h_{1,1}(t)} \left(\frac{n\Delta_R(t)S_1(t)}{X(t)} - g_{11}a_1(t) - \sum_{i=1}^d (b_i(t) - r(t))h_{i,1}(t) \right)^+, \quad v_{d+1}^{\mathbb{G}}(t) = r(t) - a_{d+1}(t).$$

So $\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2 = |g_{11}(u)v_1^{\mathbb{G}}(u)|^2 + |v_{d+1}^{\mathbb{G}}(u)|^2$ since the matrix σ^{-1} is lower triangular with the last term $\sigma_{d+1,d+1}^{-1} = 1$.

Tonelli Theorem allows us to commute the conditional expectation and the integral in time.

$$E^{\mathbb{P}}[\int_t^T (|\sigma^{-1}(u)v_{d+1}^{\mathbb{G}}(u)|^2) du|\mathcal{F}_t] = \int_t^T E^{\mathbb{P}}[|\sigma^{-1}(u)v_{d+1}^{\mathbb{G}}(u)|^2|\mathcal{F}_t] du = \int_t^T E^{\mathbb{P}}[(r(u)-a_{d+1}(u))^2|\mathcal{F}_t] du.$$

Corollary 3.2 gives

$$\begin{aligned} a_1(t) &= \frac{\lambda[\lambda(W_1(T^*) - W_1(t)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(t))]}{T^* - t} \\ a_i(t) &= 0, \quad i = 2, 3, \dots, d, \\ a_{d+1}(t) &= \frac{\sqrt{1 - \lambda^2}[\lambda(W_1(T^*) - W_1(t)) + \sqrt{1 - \lambda^2}(W_{d+1}(T^*) - W_{d+1}(t))]}{T^* - t}. \end{aligned}$$

Once again we use that $a(u)$ is conditionally independent of \mathcal{F}_t for $u > t$, so

$$E^{\mathbb{P}}[(r(u) - a_{d+1}(u))^2|\mathcal{F}_t] = E^{\mathbb{P}}[r(u)^2|\mathcal{F}_t] + E^{\mathbb{P}}[a_{d+1}(u)^2] = E^{\mathbb{P}}[r(u)^2|\mathcal{F}_t] + \frac{1 - \lambda^2}{T^* - u}.$$

We now look at (using $h_{1,1} = g_{1,1}^2$)

$$g_{11}(u)v_1^{\mathbb{G}}(u) = \left(\frac{g_{11}(u)}{h_{1,1}(u)} \left(\frac{n\Delta_R(u)S_1(u)}{X(u)} - \sum_{i=1}^d (b_i(u) - r(u))h_{i,1}(u) \right) - a_1(u) \right)^+.$$

This is the positive part of a Gaussian random variable under the \mathcal{F}_t conditional expectation

$\mathcal{N}\left(\frac{g_{11}(u)}{h_{1,1}(u)} \frac{n\Delta_R(u)S_1(u)}{X(u)} - \sum_{i=1}^d (b_i(u) - r(u))h_{i,1}(u), \frac{\lambda^2}{T^* - u}\right)$, so we use this lemma:

Lemma 6.2 *Let X a Gaussian random variable $\mathcal{N}(m, \sigma^2)$. Then*

$$E[(X^+)^2] = \int_{y \leq \frac{m}{\sigma}} (m - y\sigma)^2 \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

and $\sigma \mapsto E[(X^+)^2]$ is non increasing.

As a consequence, denote $m(u) = \frac{g_{11}(u)}{h_{1,1}(u)} \left(\frac{n\Delta_R(u)S_1(u)}{X(u)} - \sum_{i=1}^d (b_i(u) - r(u))h_{i,1}(u) \right)$,

$$E^{\mathbb{P}}[|\sigma^{-1}(u)v_1^{\mathbb{G}}(u)|^2 | \mathcal{F}_t] = \int_{y \leq \frac{m(u)\sqrt{T^* - u}}{\lambda}} \left(m(u) - \frac{y\lambda}{\sqrt{T^* - u}} \right)^2 \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

Gathering both expressions, we can conclude that the following is non-increasing:

$$\lambda \mapsto E^{\mathbb{P}}\left[\int_t^T \frac{1}{2} \|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2 du | \mathcal{F}_t\right] = E^{\mathbb{P}}\left[\int_t^T \frac{1}{2} (g_{11}(u)v_1^{\mathbb{G}}(u))^2 + (v_{d+1,d+1}^{\mathbb{G}}(u))^2 du | \mathcal{F}_t\right],$$

which concludes the proof of point (ii). •

Proposition 6.1 sheds light on two issues.

- First, it supports the legal basis for implementing blackout trading periods, without which noisy insider information (at any level of quality except pure noise) invalidates firms' incentives for executives in the same way as accurate insider information does, even with a binding non-transferable

non-hedgeable portfolio constraint.

- Second, insiders' substantial increment of derived utility (compared to outsiders') increases with respect to a portfolio holding period T at a speed proportional to information quality, namely λ , which indicates that the better the information quality insiders hold, the longer the blackout must be to effectively prevent insiders from obtaining derived utility above a certain level.

Given executive attributes, we should identify blackout trading period regulatory schemes that prevent the harmful effects of insider trading (particularly, nullifying aligning incentives).

6.2 Inadequacy and excessiveness of blackout trading period

The start of a blackout is the end of the trading period, T . Blackouts cannot start too late (early); equivalently, the trading period end T cannot be too close (far) to the information disclosure time T^* . We denote the earliest time that a blackout can start as T^E and the latest time that blackout can start as T^L . If the trading end falls in (T^E, T^L) , the blackout regulation is efficient.

Earliest start to a blackout period. We define T^E as $T^E := \max(T^{E_1}, T^{E_2})$, where a blackout starting earlier than T^{E_1} nullifies the incentive of ESO and starting earlier than T^{E_2} is unfair to insiders, as we now explain.

Noticing that the subjective price $\hat{C}(t, T) = \hat{C}(t)$ satisfies $\hat{C}(t, t) = 0$ if at-the-money or out-of-the-money, and $T \rightarrow \hat{C}(t, T)$ is increasing, we can define T^{E_1} as

$$(87) \quad T^{E_1} = \min(\inf\{T | \hat{C}(t, T) > S_1(t) - K\}, T^*).$$

The T^{E_1} is defined based on the following facts.

- (i) Recall from Theorem 5.4 item 4, that the insider will exercise all of their options if insider

news is sufficiently bad.

(ii) As $T \rightarrow \hat{C}(t, T)$ is increasing, the options with short maturities are more likely to be exercised altogether soon, which quickly nullifies the incentive of ESO.

(iii) For options with maturity T^* , owing to insider trading restriction on exercising options, the executives cannot sell the resulting shares during the blackout period. That makes exercising an option meaningless when insider news is bad. In that case, the blackout effectively shortens the option maturity.

(iv) To keep the incentive sustainable, the blackout shouldn't start earlier than T^{E_1} .

We define T^{E_2} as the unique solution in Proposition 6.3.

Proposition 6.3 *There exists a unique solution, T^{E_2} , to $E^{\mathbb{P}}[J^{\mathbb{G}}(t, x, n, T^{E_2}) - J^{\mathbb{F}}(t, x, n, T^*) | \mathcal{F}_t] = 0$.*

Proof: Using Definitions (56) and (57), the mapping $f : T \rightarrow$

$$\begin{aligned} E^{\mathbb{P}}[J^{\mathbb{G}}(t, x, n, T) - J^{\mathbb{F}}(t, x, n, T^*) | \mathcal{F}_t] &= E^{\mathbb{P}}\left[\int_t^T (r(u) + \frac{1}{2}\|\theta_a(u)\|^2)du | \mathcal{F}_t\right] - E^{\mathbb{P}}\left[\int_t^T (\frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2)du | \mathcal{F}_t\right] \\ &- E^{\mathbb{P}}\left[\int_t^{T^*} (r(u) + \frac{1}{2}\|\theta(u)\|^2 - \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{G}}(u)\|^2)du | \mathcal{F}_t\right] \end{aligned}$$

is increasing according to Remark 4.2 and $\Delta\mathcal{J}(t, x, n, T) \rightarrow \infty$ as $T \rightarrow T^*$ using Proposition 6.1.

On the other hand, $f(t) = -E^{\mathbb{P}}[\int_t^{T^*} (r(u) + \frac{1}{2}\|\theta(u)\|^2 - \frac{1}{2}\|\sigma^{-1}(u)v^{\mathbb{F}}(u)\|^2)du | \mathcal{F}_t] < 0$ according to Remark 4.2 applied to the non-insider. As f is a continuous function, a unique solution for $f(T) = 0$ exists. •

The definition of T^{E_2} is based on the fact that if the blackout starts too early, the insider's expected utility is lower than the outsider's utility, and that is unfair to insiders. Thus, T^E , represents the earliest appropriate start of a blackout.

Latest start to a blackout period. If the blackout period starts too soon, it can give the insider

an unfair disadvantage; but if the blackout period starts too late, it may give the insider an unfair advantage, which would concern regulators. Therefore, one possible definition for the latest advisable time to begin the blackout period, denote T^{L_1} , is

$$(88) \quad T^{L_1} = \inf\{T \leq T^* \mid E^{\mathbb{P}}[J^{\mathbb{G}}(t, x, n, T) \geq k_1 J^{\mathbb{F}}(t, x, n, T^*) \mid \mathcal{F}_t]\},$$

for some $k_1 \geq 1$. The choice of k_1 is a judgement call, and so is somewhat subjective. For example, if $k_1 = 1.5$, then when $T > T^{L_1}$ the insider achieves over 50% more expected utility than the outsider, which may be considered excessive.

On the other hand, the firm may worry about the incentivizing effects of their executive options. So, they could define the latest time to begin a blackout period as T^{L_2} , given by

$$(89) \quad T^{L_2} = \inf\{T \leq T^* \mid E^{\mathbb{P}}[J^{\mathbb{G}}(t, x, n, T) \geq k_2 J^{\mathbb{G}}(t, x, 0, T) \mid \mathcal{F}_t]\},$$

for some $k_2 \geq 1$. Here we are comparing insiders with and without the n options. The reasoning here is that the firm would like to know that the n options will continue to incentivize the insiders, so that they will continue to work to increase the value of their options; if the options have a negligible effect on their expected utility, we assume they are not sufficiently incentivized.

Given these considerations, a firm would choose $T^L = \min\{T^{L_1}, T^{L_2}\}$ for the latest recommended start to the blackout period.

6.3 Optimal regulations

The analysis in the previous section demonstrates that a blackout is considered to be effective if the blackout starts at some date $t_{\mathcal{B}} \in [T^E, T^L]$. However, T^E and T^L are conditional on executives' specific attributes, including the nature of their information (the type, quality, and dates of future events that give them improved predictive power), total wealth, wealth composition, and portfolio constraints. As firms have incentives to establish effective blackouts, it becomes apparent that firms'

superior relevant information on these attributes makes them the best entity, rather than the SEC or other regulators, to mandate blackout-trading-period boundaries.

6.4 Blackout trading prohibited asset list

Another relevant question is, what assets should blackout trading prohibitions include? Should they include firms' securities (i.e., stock and ESO) only or additional components of executives' other wealth? The following proposition demonstrates that the asset list depends on whether the risk that comprises the insider information is idiosyncratic or systematic.

We consider a scenario in which, within the firm, some executives have insider information and others do not, and they hold the same portfolio process $\pi = (\pi_1, \dots, \pi_{(d+1)})$ when the blackout starts. Because it is hard in practice to screen insiders from outsiders and almost impossible to implement a blackout trading prohibition rule on insiders and outsiders differently, we seek an asset list that applies equally to all executives during a blackout.

In the following proposition, let $[t_{\mathcal{B}}, T^*]$ represent a general blackout period, which might be given by a legal requirement or something else.

During this period, there are some restrictions on the executives' trading. In the following proposition, we suppose that they are forced to keep the number of shares of stock S_1 fixed during this period. In this case, for $u \in [t_{\mathcal{B}}, T]$, the portfolio constraint is of the form $K(u, \omega) = [\pi_1(u), \pi_1(u)] \times (-\infty, \infty)^{d-1} \times [0, 0]$ instead of (49).

Here, π is the portfolio process. Recall $\Delta_R(u)$, which represents the *hedge ratio* for the ESO; that is, $\Delta_R(u)S_1(u)$ would be the position in the stock required to replicate the ESO if trading were allowed. So, while the number of shares (and number of ESO) remains constant, at levels, $N^S(t_{\mathcal{B}})$

and $N(t_{\mathcal{B}})$, respectively:

$$(90) \quad \pi_1(u) := \frac{[N^S(t_{\mathcal{B}}) + N(t_{\mathcal{B}})\Delta_R(u)]S_1(u)}{X(u)},$$

with a similar definition for other stocks if there is a blackout for them.

We would like to see whether the insider can still achieve abnormally high utility by trading in other firms' shares. For purposes of comparison, we assume that the individual who does not have insider information has the same portfolio constraint as the insider who does.

Proposition 6.4 *Once again, log-utility is assumed. Given $T \in [t_{\mathcal{B}}, T^*)$, we assume $\pi_v^G(t_{\mathcal{B}}) = \pi_v^{\mathbb{F}}(t_{\mathcal{B}})$: Insider and non-insider start with the same portfolio, and then they face the constraints given by $K(u, \omega)$, for $u \in [t_{\mathcal{B}}, T]$. For G given by (5) and its corresponding $a(t) = \alpha(t, G)$ defined in Corollary 3.2,*

- if $\forall i = 2, \dots, d, \sigma_{i,1}(u) = 0$, and $K(u, \omega) = [\pi_1(u), \pi_1(u)] \times (-\infty, \infty)^{d-1} \times [0, 0]$, then $\Delta\mathcal{J}(t_{\mathcal{B}}, x, n, T) = 0$;
- if $\exists i = 2, \dots, d$, s.t. $\sigma_{i,1}(u) \neq 0$ and $K(u, \omega) = [\pi_1(u), \pi_1(u)] \times (-\infty, \infty)^{d-1} \times [0, 0]$, then $\Delta\mathcal{J}(t_{\mathcal{B}}, x, n, T) \rightarrow \infty$, as $T \rightarrow T^*$;
- if $\exists i = 2, \dots, d$, s.t. $\sigma_{i,1}(u) \neq 0$ and $K(u, \omega) = [\pi_1(u), \pi_1(u)] \times \dots \times [\pi_d(u), \pi_d(u)] \times [0, 0]$, then $\Delta\mathcal{J}(t_{\mathcal{B}}, x, n, T) = 0$.

Proof:

(i) If the firm's stock S_1 is the only non-tradable asset in the blackout period, then the constraint on the executive's portfolio is $K(u, \omega) = [\pi_1(u), \pi_1(u)] \times (-\infty, \infty)^{d-1} \times [0, 0]$, for all $u \in [t_{\mathcal{B}}, T]$, where $\pi_1(u)$ is defined in (90) above. In this case, the effective domain of the support function of constraint $K(u, \omega)$ [generally defined in (40), and the specified $K(u, \omega)$ here means executives cannot trade on

firm stock to change its position but can trade on other assets during the blackout] is derived as $\tilde{K} = (-\infty, \infty) \times [0, 0]^{d-1} \times (-\infty, \infty)$ and the scalar support function $\delta(v)$ determining $v^{\mathbb{G}}(u)$ and $v^{\mathbb{F}}(u)$ is $\delta(v) = -\pi_1(u) \times v_1(u)$. Then, thanks to Equation (11.4) in Cvitanić and Karatzas (1992, p. 790),

$$v^{\mathbb{G}}(u) := \arg \min_{v \in \tilde{K}} [2\delta(v) + \|\sigma^{-1}(u)[b(u) + v(u) - r(u)\mathbf{1}] + a(u)\|^2];$$

$$v^{\mathbb{F}}(u) := \arg \min_{v \in \tilde{K}} [2\delta(v) + \|\sigma^{-1}(u)[b(u) + v(u) - r(u)\mathbf{1}]\|^2].$$

From \tilde{K} , we see that $v_i^{\mathbb{G}}(t) = v_i^{\mathbb{F}}(t) = 0$ for $i = 2, \dots, d$, but that $v_1^{\mathbb{G}}(t)$ and $v_1^{\mathbb{F}}(t)$ can take any value from $(-\infty, \infty)$; in particular, $v_1^{\mathbb{G}}(t) < 0$ and $v_1^{\mathbb{F}}(t) < 0$ are allowed. From Equations (47) and (48), in Lemma 4.1 it is not hard to see that

$$v_1^{\mathbb{G}}(u) = \frac{1}{h_{1,1}(u)} (\pi_1(u) - \sum_{i=1}^d (b_i(u) - r(u))h_{i,1}(u) - g_{1,1}(u)a_1(u)),$$

$$v_1^{\mathbb{F}}(u) = \frac{1}{h_{1,1}(u)} (\pi_1(u) - \sum_{i=1}^d (b_i(u) - r(u))h_{i,1}(u)).$$

This implies that $v_1^{\mathbb{F}}(u) - v_1^{\mathbb{G}}(u) = \frac{g_{1,1}(u)}{h_{1,1}(u)} a_1(u)$. Since log utility is assumed, recall the insiders' optimal constrained portfolio process (55):

$$\pi_v^{\mathbb{G}}(u) = [\sigma^{\top}(u)]^{-1}(\theta(u) + \sigma^{-1}(u)v^{\mathbb{G}}(u) + a(u)) = [\sigma^{\top}(u)]^{-1}(\theta(u) + a(u)) + h(u)v^{\mathbb{G}}(u),$$

while with $a = 0$ for an outsider,

$$\pi_v^{\mathbb{F}}(u) = [\sigma^{\top}(u)]^{-1}(\theta(u) + \sigma^{-1}(u)v^{\mathbb{F}}(u)) = [\sigma^{\top}(u)]^{-1}\theta(u) + h(u)v^{\mathbb{F}}(u).$$

By Equation (48) and assumption, $\sigma_{i,1}(u) = 0$, $i = 2, \dots, d$, $[\sigma^{\top}(u)]^{-1}a(u) = (g_{1,1}(u)a_1(u), 0, \dots, 0)^{\top}$.

But given the assumption that $\pi_{v,1}^{\mathbb{G}}(t_{\mathcal{B}}) = \pi_{v,1}^{\mathbb{F}}(t_{\mathcal{B}})$, together with the constraint $K(u, \omega)$, it follows

that $\pi_{v,1}^{\mathbb{G}} = \pi_{v,1}^{\mathbb{F}}$ on $[t_{\mathcal{B}}, T^*]$. Hence $\pi^{\mathbb{G}}(u) = \pi^{\mathbb{F}}(u)$, and thus, $\Delta\mathcal{J}(t_{\mathcal{B}}, x, n, T) = 0$.

(ii) If $\exists i = 2, \dots, d$, s.t. $\sigma_{i,1}(u) \neq 0$, then $\pi_v^{\mathbb{G}}(u)$ is still a function of $a(u)$, and the rest of the proof is similar to Proposition 6.1. So, since we assume log-utility, the same calculations yield $\Delta\mathcal{J}(t_{\mathcal{B}}, x, n, T) \rightarrow \infty$, as $T \rightarrow T^*$ even if the first component in $K(t, \omega)$ is a singleton.

(iii) If $K(u, \omega) = [\pi_1(t, u), \pi_1(t, u)] \times \dots \times [\pi_d(t, u), \pi_d(t, u)]$, then the constraint is a singleton and necessarily $\pi_v^{\mathbb{G}}(u) = \pi_v^{\mathbb{F}}(u)$, and $\Delta\mathcal{J}(t_{\mathcal{B}}, x, n, T) = 0$, as required. •

Although insider and outsider executives are facing the same portfolio constraints, the strength of the constraints could be different. For example, if insiders know the firm's stock will rise or decline for sure and they cannot vary their position during a blackout, then the insiders' constraint is effectively stricter than that of outsiders. Two facts reflect this point: $v^{\mathbb{G}}$ generally is a function of a , and there is a value difference between $v^{\mathbb{G}}$ and $v^{\mathbb{F}}$.

The financial intuition of Proposition 6.4 is that to prevent insiders from getting extra substantial utility after a blackout trading period starts.

(i) If the insider information is purely idiosyncratic, then the firm should list only the firm's stock on the blackout trading prohibition list. The disadvantage of having stricter portfolio constraints caused by blackout trading periods offsets the advantage of possessing insider information. Consequently, there is no substantial increment of insiders' derived utility compared to that of outsider executives.

(ii) If the insider information is not purely idiosyncratic, insider-executives can acquire infinite derived utility by trading other firms' shares, even when restricted from trading their own firms' shares. Thus, the SEC or other regulators should restrict executives from trading in all assets, including firm shares during blackouts.

Please note that although our model assumes log utility and a specific insider information type, Proposition 6.4 is derived without needing those specifications. Hence, they are safe to use as a legal basis generally for enacting blackout trading period regulations.

7 Policy recommendations on firm incentives

Finding a fixed blackout window that works across all firm insiders is impossible. Furthermore, as we discuss in Section 7.2, the ESO incentive has tolerance effects in which allocating additional ESO might render the blackout too short. Therefore, it is critical for firms to develop effective incentivizing schemes. We suggest re-examining reload stock options.

7.1 Effective blackout trading periods, too good to be true

The following are our concerns regarding effective blackouts (we use blackout to mean blackout trading period).

- First, an effective blackout, i.e. $t_B \in (T^E, T^U]$, might not always exist as it is possible to have $T^E > T^U$.
- Second, the boundaries of effective blackouts vary across individuals; hence, there is no uniform effective blackout.
- Third, even for a particular executive, the effective blackout is not static. Because an executive's total wealth and portfolio constraints change dynamically, a fixed blackout window for a particular insider might switch among different states (inadequate, effective, and excessive) from time to time.
- Fourth, job termination could reduce the portfolio holding period and, equivalently, extend the blackout trading period from inadequate to effective. Thus, ESO provide different incen-

tives for different insider-executives, depending jointly on the adequacy of blackouts and the foreseeability of job termination.

For these four reasons and because, in practice, firms can mandate only a single predetermined blackout to all corporate insiders, developing alternative incentives for ESO is critical.

7.2 Tolerance effect of executive stock options

In previous sections, we assumed that executives have insights but are price-takers not affecting the future stock price. In that case, granting ESO can motivate insider-executives to boost the current spot price to achieve a higher derived utility, considered a short-term incentive. We have shown that an inadequate blackout invalidates the incentives of firm-granted ESO.

However, in reality, executives' insider information could be due to their efforts. Hence, in this section, we assume that executives can determine or at least influence the future stock price. The firm then has the motivation to grant more ESO to better align the interests of executives and shareholders in the long run. We show that, even if the blackout as initially set is effective, it can become excessively long as the firm grants more ESO. The reasoning is that the subjective option price $\hat{C}(t)$ is the objective option price $C(t)$ minus the devaluation due to the portfolio constraints; the more non-transferable non-hedgeable options granted, the greater the devaluation. Hence, the exercising condition, $\hat{C}(t) < S(t) - K$, is more likely to be satisfied. To prevent $\hat{C}(t) < S(t) - K$ from happening so that the incentive can be maintained, the trading horizon T needs to be extended, as subjective option price is an increasing function of option maturity T . If the predetermined blackout does not allow a sufficiently long trading period, then insiders will exercise all options (See the definition of T^{E_1} in Section 6.2), which invalidates both the long-term and short-term incentives of ESO.⁷ We call this the *tolerance effect* of the ESO and offer a scheme for granting Reload Stock Options (RSO) written on the firm's stock as an alternative long-term incentive.

In particular, using (45), (46), (47), an insider's optimal constrained portfolio process is

$$(91) \quad \begin{aligned} \pi_v^{\mathbb{G}}(t) &= [\sigma^\top(t)]^{-1} \theta_{a,v}(t) = [\sigma(t) \sigma^\top(t)]^{-1} [b_{a,v}^{\mathbb{G}}(t) - r_v^{\mathbb{G}}(t) \mathbf{1}] \\ &= [\sigma(t) \sigma^\top(t)]^{-1} [b(t) - r(t) + v^{\mathbb{G}}(t)] + [\sigma^\top(t)]^{-1} a(t). \end{aligned}$$

By construction, the noisy information is not a traded asset; that is, $\pi_{v,d+1}^{\mathbb{G}} = 0$ and only $\pi_{v,1}^{\mathbb{G}}$ and the proportion assigned to the risk-free asset $(1 - \sum_{i=1}^d \pi_{v,i}^{\mathbb{G}})$ are affected by $a_1(t)$, which is an increasing function of $W_1(T^*)$. (Cf. Corollary 3.2.) By Equation (91), $\pi_{v,1}^{\mathbb{G}}$ is a linear function of $\sigma_{1,1}^{-1}(t) a_1(t)$. Hence, the higher the value of $a_1(t)$, the greater the proportion of the firm's stock the executives should optimally hold. If, on the other hand, the value of $a_1(t)$ is low enough to make $\pi_{v,1}^{\mathbb{G}} < \frac{n\Delta_R(t)S_1(t)}{X(t)}$, then the non-transferable and non-hedgeable (i.e., no short selling) constraint imposed by the executive stock option prevents the executives from trading optimally. As a result, the executives have the motivation to boost the future terminal value of their stock, i.e., $S_1(T^*)$ [or, equivalently, $W_1(T^*)$] to make the optimal proportion of the firm's stock $\pi_1^{\mathbb{G}} \geq \frac{n\Delta_R(t)S_1(t)}{X(t)}$ positive to rid themselves of the constraint, to get more wealth, and to improve the expected derived utility. Moreover, by Equation (66), if $a_1(t)$ is positive and high enough, $v_1^{\mathbb{G}} = 0$, which directly renders the escape from constraints.

Building on our analysis, we further claim that firms have the motivation to provide stronger incentives to tighten the non-transferable non-hedgeable constraints. If the constraint is stricter, e.g., a large grant of non-transferable non-hedgeable ESO, then the opportunity set of $\pi_1^{\mathbb{G}}$ is $[\zeta, \infty)$ where ζ is a positive constant. Then, the insider-executives are motivated to boost $S_1(T^*)$ until the optimal portfolio satisfies, $\pi_1^{\mathbb{G}^*} \geq \zeta$, which enables them to escape from the non-transferable non-hedgeable constraints. In other words, a stronger incentive comes from stricter non-transferable non-hedgeable constraints by granting more ESO.

7.3 Reload stock options (RSO) incentives

Here we study the exercise policy and pricing of RSO. We find that the exercise of American ESO (RSO with infinite reloads) is determined backwardly (forwardly) and is affected by (robust to) insider trading and portfolio constraints. Granting ESO successively induces executives' successive short-term performance, which can be weakened because of the tolerance effect caused by the insider trading. Granting long-term RSO with infinite reloads incentivizes insider-executives' long-term performance. We recommend that firms reconsider using RSO, which have fallen out of favor in recent years.

Definition 7.1 *An RSO, invented by Frederic W. Cook and Co. in 1987, is a non-transferable non-hedgeable American call option that grants additional at-the-money options upon exercising the initial one. The option holder pays the strike price in stock already possessed instead of paying in cash (stock-for-stock). Meanwhile, a new strike is set to be the market value of the underlying stock at the time the option is exercised.*

To elaborate this definition, we consider an example. Suppose an executive owns one share and one reload option. As explained in Dybvig and Loewenstein (2003), if $S_1(t) > K$ at some date $t \leq T$, and if the executives exercise one reload option, then they receive a payoff of $1 - K/S_1(t)$ shares plus $K/S_1(t)$ new reload options with new strike price $S_1(t)$. So, the executives end up with less than one share (that is the payment), but they receive more reload options that are currently “at-the-money”. If only one reload is allowed, then these new options are “plain vanilla” call options.

In the case where infinite reloads are possible, Dybvig and Loewenstein (2003) made a breakthrough contribution on RSO pricing. Saly et al. (1999), Brenner et al. (2000), Dai and Kwok (2005), Ingersoll (2007), Bélanger and Forsyth (2008), and Dai and Kwok (2008) have also contributed to the RSO pricing literature. They showed that the optimal exercise policy of an RSO with infinite

reloads is to exercise the options whenever the stock price reaches a historical record new high, and the value of the reload option always lies between the value of an American call and the stock price, irrespective of the number of reloads and the maturity.

We claim that if we further consider executives' insider trading as well as non-transferable non-hedgeable portfolio constraints, the optimal exercise policy stated in Dybvig and Loewenstein (2003) still holds. The logic is as follows. Cvitanić and Karatzas (1992) and Karatzas and Kou (1996) elegantly transform a constrained portfolio optimization problem into an unconstrained one with adjusted drift rates. Pikovsky and Karatzas (1996) endow the portfolio optimization framework the flexibility to incorporate the anticipative feature of insider trading, again through drift rate adjustment. However, whether the results in Dybvig and Loewenstein (2003) hold does not depend on the value of drifts as inputs; therefore, considering executives' insider trading and portfolio constraints, the optimal exercise policy for RSO with infinite reloads holds.

We now assume that

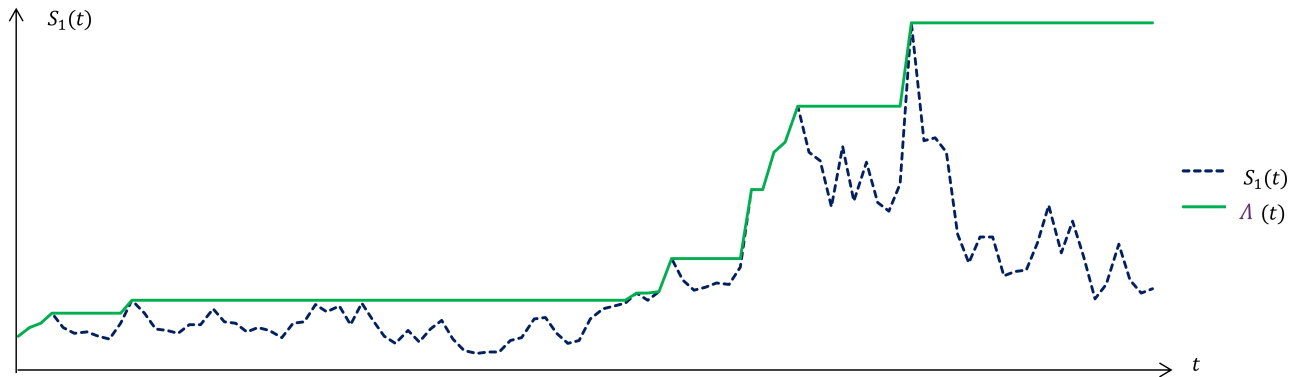
- (i) executives have insider information regarding the terminal values of a driving Brownian motion;
- (ii) executives pay the strike price in mature stock already in their possession rather than in cash;
- (iii) executives always hold enough mature shares to pay for the exercise price (we do not assume the executive can borrow the necessary shares);
- (iv) an RSO can be exercised only after its predetermined vesting period;
- (v) executives are prohibited from short-selling the firm's stock and transferring the options;
- (vi) executives are not allowed to sell the shares they own during the blackout trading period.

Note that exercising an RSO through "stock-for-stock" belongs to the intra-company approach; it does not involve contemporaneous sales into the market and, hence, is allowed during a blackout (See Nathan and Hoffman 2013).

Holding RSO and reloading them with an optimal reload policy is equivalent to accumulating the

instant payoff at the time of each exercise, realized at any appropriate reloading time. Therefore, without considering the present value discount, the realized cash pay-off at time t is the difference between the current value and the previous historical record highs. Let $\Lambda(u) := \max(S_1(s), 0 \leq s \leq u)$ be the historical high, which is a non-decreasing process. See Figure 0.

Fig 0. RSO payoff envelop under optimal exercise



We employ the common practice of setting the number of additional non-transferable non-hedgeable American at-the-money call options, granted upon exercising the initial one, equal to the pre- and post-exercise strike price ratio. At any time $u \in [t, T]$, the number of RSO converted from one option at time t^0 is $\frac{S_1(t^0)}{S_1(u)}$, where t^0 is the RSO granting time at which the strike price of the RSO was initially set as $S_1(t^0)$.

To price RSO, we refer to a result of Dybvig and Loewenstein (2003, Lemma 1, p. 9). However, we use a different stochastic discount factor, namely, an insider's: \mathbb{G} -progressively measurable subjective stochastic discount factor accounting for non-transferable non-hedgeable constraints as follows. For

any time $t \in [t^0, T]$,

$$H_v^{\mathbb{G}}(t) := \exp\left(-\int_0^t r_v^{\mathbb{G}}(s) ds\right) Z_v^{\mathbb{G}}(t), \quad (23)$$

where

$$(92) \quad Z_v^{\mathbb{G}}(t) := \exp\left(-\int_0^t [\theta_{a,v}(s)]^T d\tilde{W}(s) - \frac{1}{2} \int_0^t \|\theta_{a,v}(s)\|^2 ds\right),$$

processes $r_v^{\mathbb{G}}$ and $\theta_{a,v}$ are defined in Section 3.6 in (46) and (47).

An insider's subjective price of at-the-money RSO, expiring at T , with infinite reloads, which was converted from one option granted at time t^0 , is

$$\hat{R}(S_1(t), t^0, t, T) = E^{\mathbb{P}}\left[\int_t^T \frac{H_v^{\mathbb{G}}(u) S_1(t^0)}{H_v^{\mathbb{G}}(t) S_1(u)} d\Lambda(u) \mid \mathcal{F}_t\right], \quad (25)$$

recalling that $\Lambda(u) := \max(S_1(s), 0 \leq s \leq u)$ is a non-decreasing process, the firm's stock price's envelope.

The subjective stochastic discount factor process for insiders, $H_v^{\mathbb{G}}$, is determined by a , the information compensating process, as well as $v^{\mathbb{G}}$, the drift rate adjustment reflecting insiders' portfolio constraints. Setting $v^{\mathbb{G}} \equiv 0$, we obtain the objective stochastic discount factor, which in turn determines the firm's cost of RSO granted to insiders, denoted as $R(S_1(t), t^0, t, T)$.

According to the Law of One Price, using a different discount factor does not vary the upper bound of options. Therefore, after we consider insider information and portfolio constraints, the result in Dybvig and Loewenstein (2003) still holds, that the upper bound, at any time t of the firm's cost (objective price) of granting an at-the-money RSO with infinite reloads is the spot stock price at the initial granting time.

RSO fell out of favor around 2006, and firms gradually stopped granting them thereafter for two

reasons: the pricing difficulty and the claim that RSO bestow too many lucrative shares to executives.

First, in 2004, the Financial Accounting Standard Board made the RSO optional reporting mandatory [FAS123(R)]. It reacted to the extensive use of share-based compensation, asking for reports of fair value, reflecting grant-date share price and other pertinent factors, including volatility, restrictions, and inherent conditions.

Although much progress has been made on ESO pricing, considering non-transferable non-hedgeable constraints and adding the reload feature escalates the pricing difficulty, the Board continues to believe that the reload term makes it impossible to estimate a reasonably fair value of options at the grant date. It states that subsequent granting of reload options should be accounted for as a separate award when the reload options are granted [See FAS123(R) paragraphs 24 to 26; see also Saly *et al.* 1999]. However, the objective price, $R(S_1(t), t^0, t, T)$ (the case where we set $v^G \equiv 0$), gives the firm's cost of RSO, taking all the aforementioned factors into account.

Second, RSO have been blamed for bestowing too many lucrative shares to the executives. Our work endeavors to test the truth or falsehood of that claim from a new perspective by taking executives' insider trading into account. The objective price of the RSO, $R(S_1(t), t^0, t, T)$, is certainly less than $S_1(t)$, showing that the firm's cost of granting one at-the-money RSO with infinite reloads is no more than granting one share of firm stock. Hence, the claim that RSO are money pumps for executives is groundless.

8 Simulation and sensitivity analysis

We use Monte Carlo simulation to demonstrate how executives' insider information changes the incentivizing mechanism of their ESO. For simplicity, we consider European ESO. We distinguish between low- and high-volatility regimes; see Table 1, Panel A and Panel B. For each volatility regime, we study two types of insider information: good news and bad news, and compare the results with

those of an outsider executive, who has no information.

Table 1: Sensitivity analysis and determinants of ESO efficiency

Panel A: Low-volatility regime							
	BS	Obj	Sub	Inc_U(e-03)	Inc_P	DWC	Eff(e-03)
Outsider	2.522	18.350	18.578	0.0480	0.143	-0.228	0.049
Insider (good news)	2.522	2.522	2.522	0.0087	0.553	55.e-05	0.0087
Insider (bad news)	2.522	4.471	5.152	7.5	0.279	-0.681	8.8
Panel B: High-volatility regime							
	BS	Obj	Sub	Inc_U(e-03)	Inc_P	DWC	Eff(e-03)
Outsider	4.662	4.662	4.114	11.0	0.329	0.55	9.6
Insider (good news)	4.662	24.227	24.848	2.5	0.139	-0.621	2.6
Insider (bad news)	4.662	0.427	0.566	18.0	0.368	-0.139	24.0

For each panel, we report three ESO prices: “plain vanilla” Black-Scholes prices (BS), ESO objective prices (Obj), and executives’ subjective prices (Sub). The objective price (also termed as the firm cost in the literature) is the price without considering portfolio constraints, while the subjective price is the one taking non-transferable non-hedgeable constraints into account. For outsider executives, the objective price is the Black-Scholes price. However, for insider-executives, they are not equivalent.

We distinguish between and report utility incentives and price incentives. We define *utility incentive* (Inc_U) as the change in executive’s derived utility per unit of stock price change, i.e., $\frac{\partial J^E(t,x,n,T)}{\partial S_1(t)}$ for executives uninformed and $\frac{\partial J^G(t,x,n,T)}{\partial S_1(t)}$ for insiders.

We define *price incentives* (Inc_P) as the delta of the subjective price $\hat{c}(t)$, i.e., the change in the logarithm of the subjective price of an ESO per unit of share price change, $\frac{\partial \ln(\hat{c}(t))}{\partial S_1(t)}$.

We also present the deadweight cost (DWC) of granting ESO, which is an ESO’s objective price net of the subjective price; that is, $c(t) - \hat{c}(t)$. A positive (negative) value of deadweight cost indicates an ESO granting efficiency loss (gain).

We measure and report the overall ESO granting efficiency (Eff) as the utility incentives adjusted by multiplying a deadweight cost discount (premium), $e^{\{-\frac{c(t)-\hat{c}(t)}{c}\}}$, if the deadweight cost is positive (negative); that is, we define the granting efficiency by $e^{\{-\frac{c(t)-\hat{c}(t)}{c}\}} \partial J^G(t,x,n,T) / \partial S_1(t)$. We argue that the utility incentives do not take into account the fact that the deadweight cost is not always positive, and if it is negative, it is a good thing. This is why we multiply the utility incentive by this exponential term, to discount the utility incentive if the deadweight cost ratio is positive and to increase it if the deadweight cost ratio is negative, much like taking a present value.

The benchmark setting of the parameters determining ESO efficiency are Low-volatility regime (stock volatility $[\sigma_{11} = 20\%]$, index volatility $[\sigma_{22} = 10\%]$) and High-Volatility regime (stock volatility $[\sigma_{11} = 50\%]$, index volatility $[\sigma_{22} = 30\%]$), and stock price $[S_1 = 10 \text{ dollars}]$, index price $[S_2 = 6 \text{ dollars}]$, correlation $\rho_{12} = 60\%$, stock drift $[b_1 = 15\%]$, index drift $[b_2 = 8\%]$, risk-free rate $[r = 4\%]$, investment horizon $[T = 5 \text{ year}]$, strike $[K = 10 \text{ dollars}]$, stock vesting period $[t_{vs} = 1 \text{ year}]$, option granted $[N = 200 \text{ shares}]$, stock granted $[N^S = 200 \text{ shares}]$, cash endowment $[x = 1000 \text{ dollars}]$, info disclosure time $[T^* = 6 \text{ years}]$, information $[W_1(T^*) = 1]$, noise $[W_3(T^*) = -0.5]$, info quality $[\lambda = 0.6]$.

Table 2 lists all the determinants of (European) ESO efficiency. We set each determinant at two levels - high and low, deviating from the benchmark level. Therefore, 33 scenarios are considered in the analysis of insiders' ESO granting efficiency. For each scenario, we further distinguish high and low volatility regimes, as well as good news and bad news, overall there are 116 cases (33 scenarios by 2 volatility regimes by 2 news, good and bad), among which, 29 scenarios apply to both outsiders and insiders, hence, we study these scenarios to compare between insiders and outsiders. The rest of the scenarios about information disclosure time and information quality apply to insiders only.

Table 2: List of scenarios considered in insiders' ESO granting efficiency analysis

1	Benchmark
2	stock price = 12
3	stock price = 6
4	index price = 10
5	index price = 2
6	correlation = 0.9
7	correlation = 0.3
8	stock drift = 0.2
9	stock drift = 0.12
10	index drift = 0.12
11	index drift = 0.06
12	risk-free rate = 0.06
13	risk-free rate = 0.02
14	investment horizon = 7
15	investment horizon = 3
16	stock volatility = 0.5
17	stock volatility = 0.13
18	index volatility = 0.15
19	index volatility = 0.05
20	strike = 12
21	strike = 8
22	stock vesting period = 1.5
23	stock vesting period = 0.5
24	option granted = 2000 shares
25	option granted = 20 shares
26	stock granted = 2000 shares
27	Stock granted = 20 shares
28	cash endowment = 10000
29	cash endowment = 500
30	Information disclosure time=6.4
31	Information disclosure time=5.6
32	Information quality =0.9
33	Information quality =0.2

Our simulated results reconcile with our theoretical results. Based on the benchmark parameter setting, the subjective prices of ESO perceived by outsider executives are less than or equal to the firms' granting costs, which implies positive deadweight costs of ESO granted to outsider executives and which, in turn, reduces ESO incentivizing and results in a lower granting efficiency. (Note that the occasional negative deadweight cost observed is due to rounding errors.) By contrast, the subjective price perceived by insider-executives is usually greater than firms' granting costs, resulting in a negative deadweight cost for ESO granted to insider-executives and increasing the ESO incentivizing. However, when executives have insider information, their utility incentives could become weaker; therefore, the overall granting efficiency of ESO to insider-executives becomes lower than to outsider executives. This is the case in the simulations of high-volatility regimes when insiders' information is good news.

Table 3 Panel A (Panel B) summarizes the number of cases considered in the low-volatility (high-volatility) regime, and under the regime, the total number of counts if the deadweight cost of granting ESO to insiders is lower than the one to outsiders, the total number of counts if the deadweight cost of granting ESO to insiders is negative, the total number of counts if the ESO granting efficiency to insiders is lower than the one to outsiders, the total number of counts if the ESO granting efficiency to insiders is negative. Panel C presents the total sum of both regimes.

Table 3: List of scenarios considered in insiders' ESO granting efficiency analysis

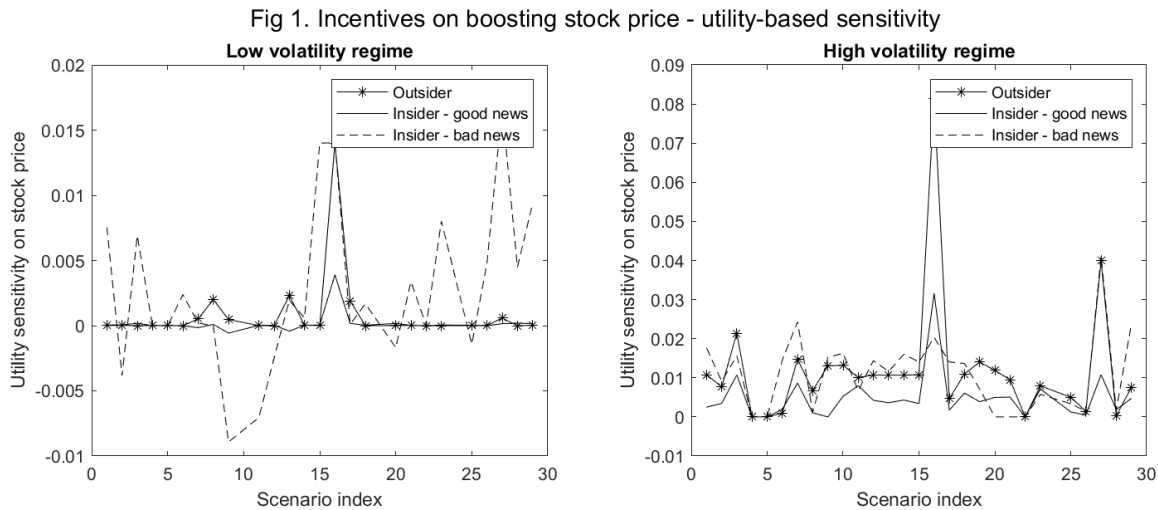
Panel A: Low-volatility regime					
	DWC		Eff		Total
	News (good)	News (bad)	News (good)	News (bad)	
Count # of scenarios	29	29	29	29	58
Count if DWC insider < outsider	29	28			57
Count if DWC insider < 0	29	28			57
Count if Eff insider < outsider			14	12	26
Count if Eff insider < outsider			4	8	12
Panel B: High-volatility regime					
	DWC		Eff		Total
	News (good)	News (bad)	News (good)	News (bad)	
Count # of scenarios	29	29	29	29	58
Count if DWC insider < outsider	26	29			55
Count if DWC insider < 0	24	29			53
Count if Eff insider < outsider			17	5	22
Count if Eff insider < outsider			2	1	3
Panel C: Pooled volatility regime					
	DWC		Eff		Total
	News (good)	News (bad)	News (good)	News (bad)	
Count # of scenarios	58	58	58	58	116
Count if DWC insider < outsider	55	57			112
Count if DWC insider < 0	53	57			110
Count if Eff insider < outsider			31	17	48
Count if Eff insider < outsider			6	9	15

See Panel C of Table 3. Although 112 cases show a reduction in deadweight cost, and among them, 110 cases turn outsiders' positive deadweight cost into a negative, 48 cases show weakened granting efficiency of ESO compared to outsiders' due to insider-executives' ability to profit from outsider wealth owing to their predictive information. Remarkably, the weakening of the granting efficiency is more significant when insiders have good news regardless of the volatility regime. Note

that 31 of 58 good news cases *vs.* 17 of 58 bad news cases show that ESO granting efficiency to insiders is lower than to outsiders. However, under high-volatility regimes when insider news is bad, the granting efficiency to insider-executives is typically higher than to outsider executives (See Panel B). What's more, we observe overall 15 cases of weakened granting efficiencies (i.e., the granting efficiency for insiders is lower than that of outsiders) in Panel C, 12 of which appear in low-volatility regime (See Panel A).

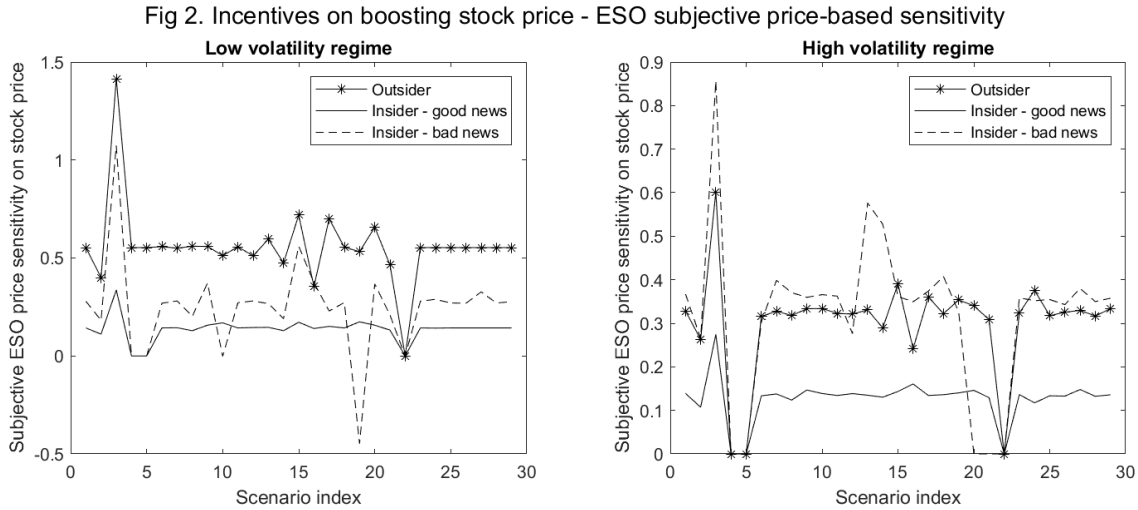
For each scenario, we study how parameter changes affect ESO incentives and report the results below. The following comparative statics are of high interest to executives, firms, and the public.

1. Across 29 scenarios, ESO incentivize executives to boost firm stock prices. See Figure 1; in most cases, these incentives, as measured by utility sensitivities to stock price changes, are stronger to the insider-executives with bad news information than to outsider executives; and the incentives to insider-executives with good news information are weaker than the incentives to outsider executives.



It highlights the distinction between utility incentive and subjective price incentive when insider information is considered, and insiders' objective changes to maximizing utility generated by

total wealth including outsider wealth, i.e., the asset other than firm stocks or options. See Figure 2; subjective ESO price sensitivity with respect to the stock price shows a different pattern rendering the ranking order from utility incentivize executives in the low regime.



2. The impact of insider trading on insider’s objective and subjective prices are similar with a volatility regime given. See Figure 3 and Figure 4. For the low-volatility regime, symmetric impacts dominate: insider information, whether good or bad news, increases insiders’ objective prices and subjective prices of their ESO, compared to outsiders’. However, prices increase more to insider-executives with good news. For high- volatility regimes, asymmetric impacts dominate, meaning that if insiders’ information is good (bad) news, it increases (decreases) insider-executives’ objective and subjective prices of their ESO, compared with outsider executives’ prices.
3. We compare objective prices and subjective prices, and find that outsider executives’ subjective and objective prices almost coincide in a low-volatility regime (see Figure 5), but subjective is lower than objective prices across all cases in a high-volatility regime (see Figure 6), implying that the non-transferable non-hedgeable constraint takes effect when the market is more

Fig 3. Impacts of insider information on objective price

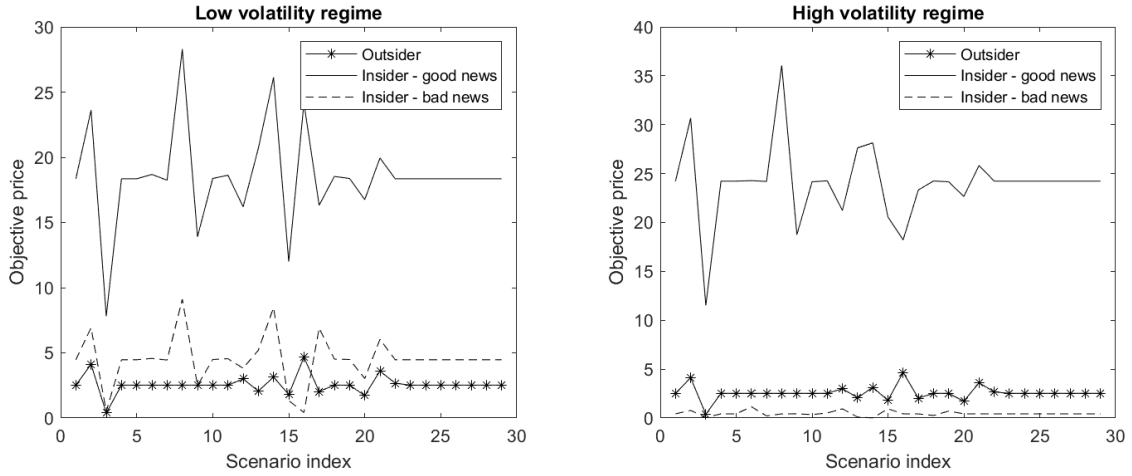
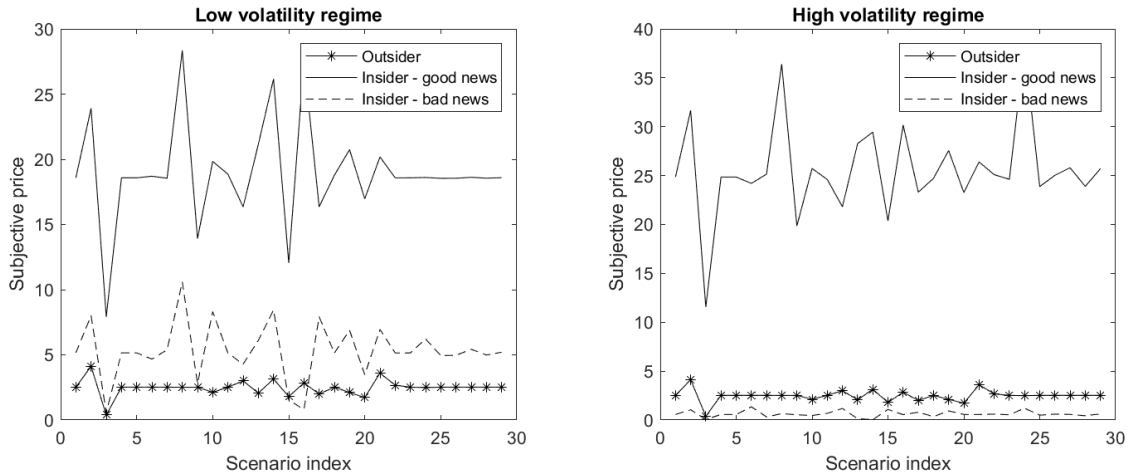


Fig 4. Impacts of insider information on subjective price



volatile. Usually, insiders' objective and subjective prices move in tandem when parameters change. Insiders' subjective price is usually higher than the objective one regardless of the volatility regime or the news implying that the information advantage mitigates and dominates the portfolio constraint disadvantage.

4. It is well known that option price and underlying volatility have a positive relation. However, we find that when insider-executives trade with bad news, higher firm stock volatility devalues ESO subjective price, especially in a low-volatility regime. See Figure 7.

Fig 5. Subjective vs. Objective price comparison - low volatility regime

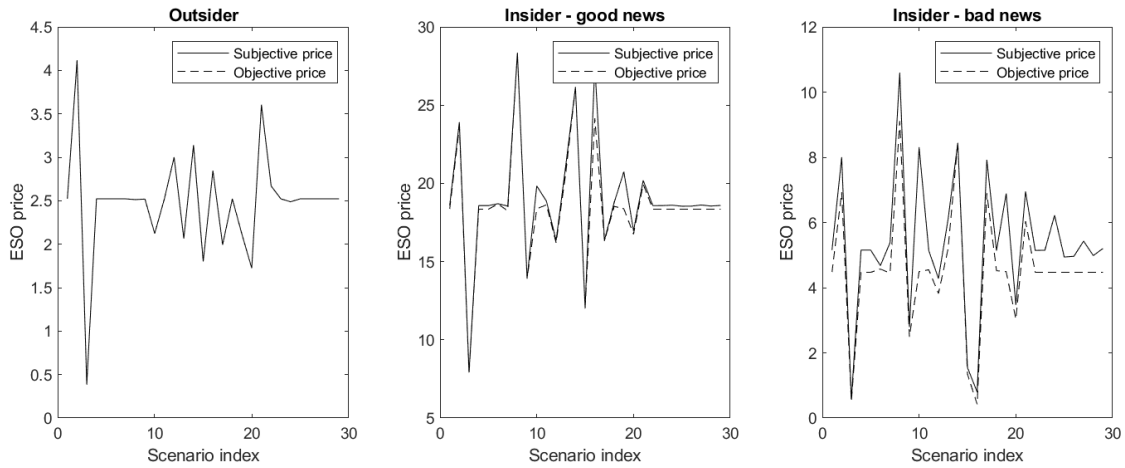
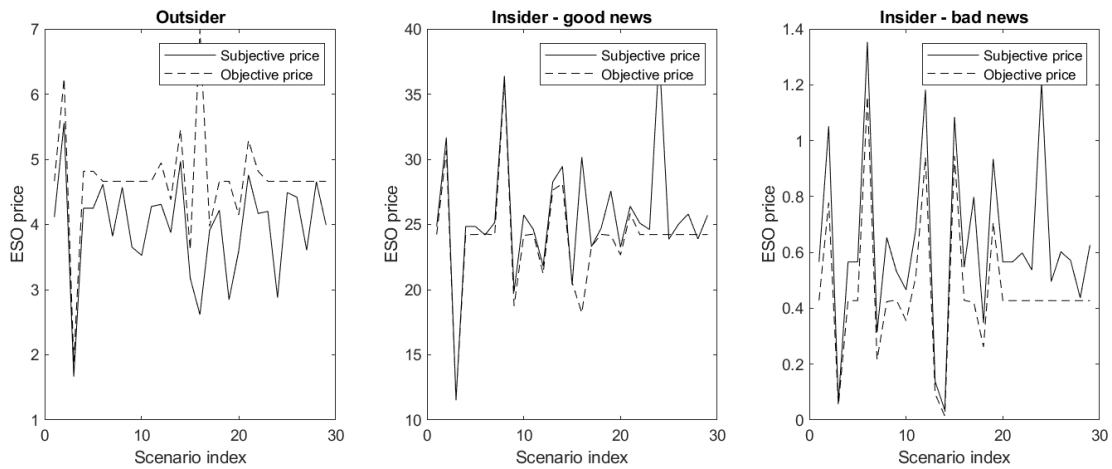


Fig 6. Subjective vs. Objective price comparison - high volatility regime



5. A lower positive correlation between firm stock returns and index returns results in higher positive granting efficiencies to insider-executives, see Figure 8. Under a high-volatility regime regardless of news, the amplification of bad news is even greater than good news. However, that feature does hold for a low-volatility regime. This implies that when the market is more volatile, risk diversification will remove insider's information advantage; hence, insider-executives stay more with firm equity-based compensation, such as ESO, and the granting efficiency is enhanced.

6. We assume that ESO expire at the end of the investment horizon, and find that ESO granting

Fig 7. Incentives on boosting risk taking

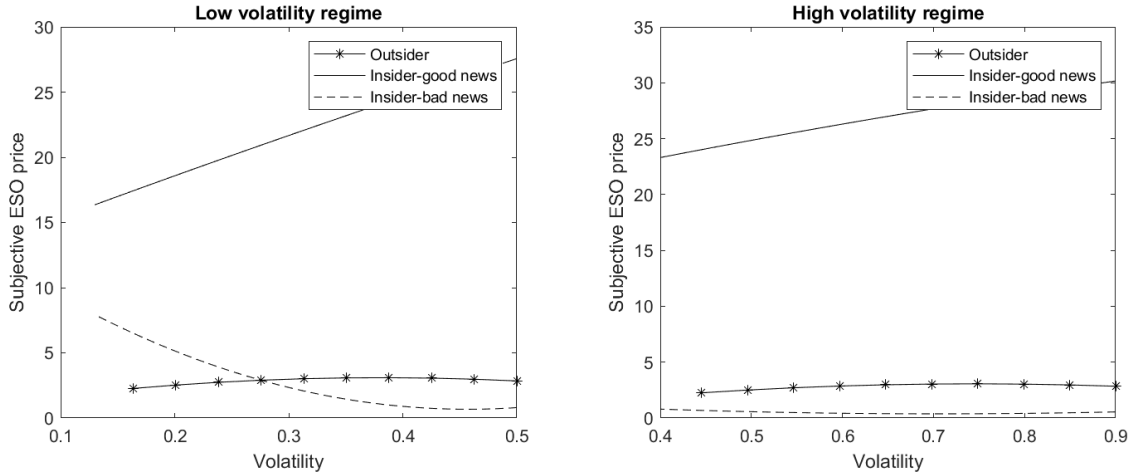
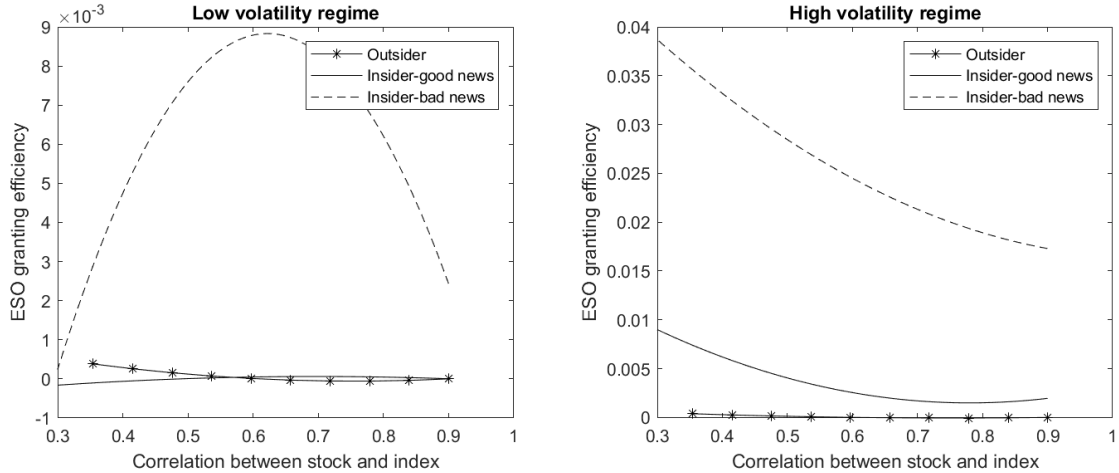


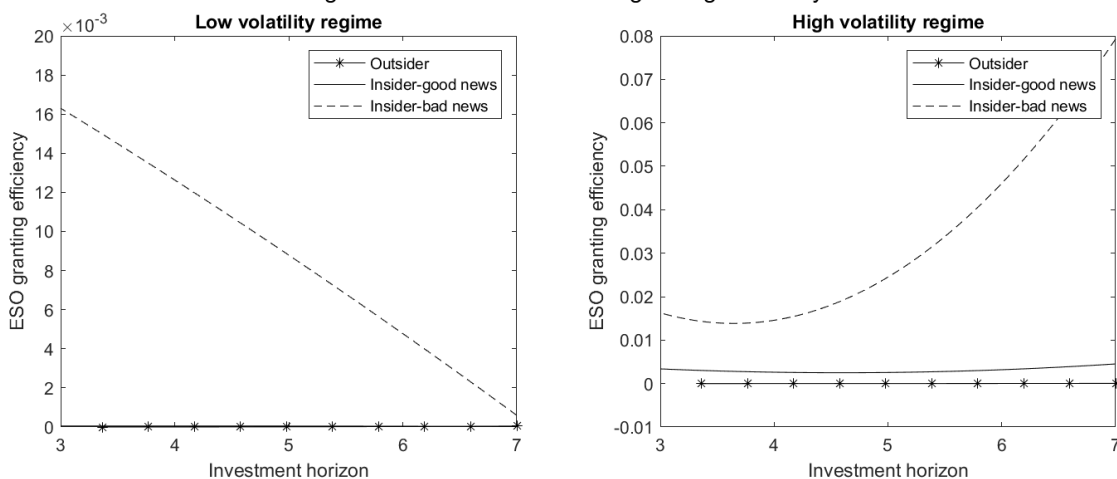
Fig 8. Stock and market index Correlation and granting efficiency



efficiency is irrelevant to the investment horizon for outsiders and insiders with good news but is largely determined by the investment horizon for insider-executives when news is bad. See Figure 9. Longer option maturities decrease granting efficiencies in a low-volatility regime and improves granting efficiency in the a high-volatility regime. This comparison suggests that in a stable market, a firm granting ESO should focus on incentivizing short-term goal and that in a more volatile market, a firm granting ESO should focus on incentivizing long-term goals.

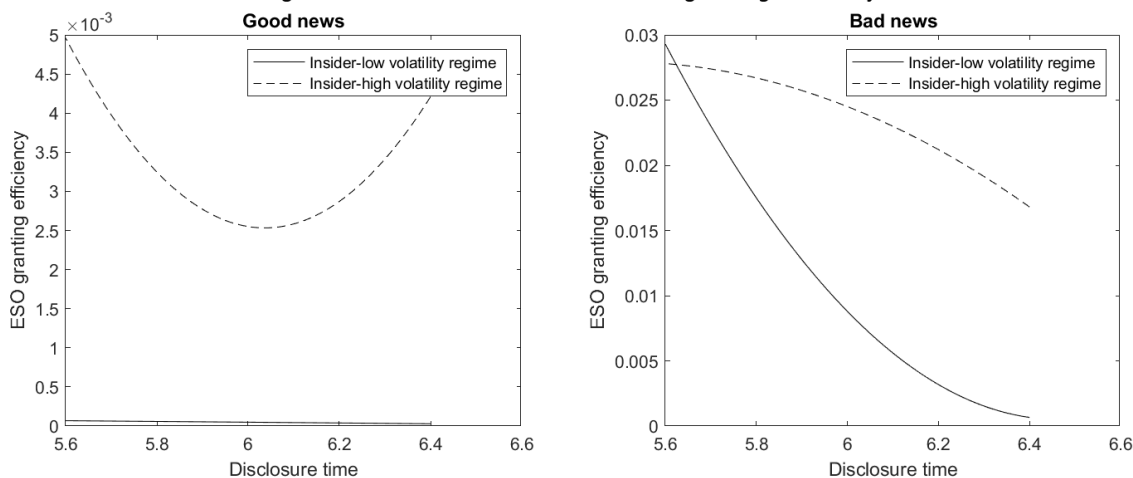
7. We fixed the investment horizon at five years and simulated different information disclosure times. See Figure 10. We find that the closer investment horizons are to information disclo-

Fig 9. Investment horizon and granting efficiency



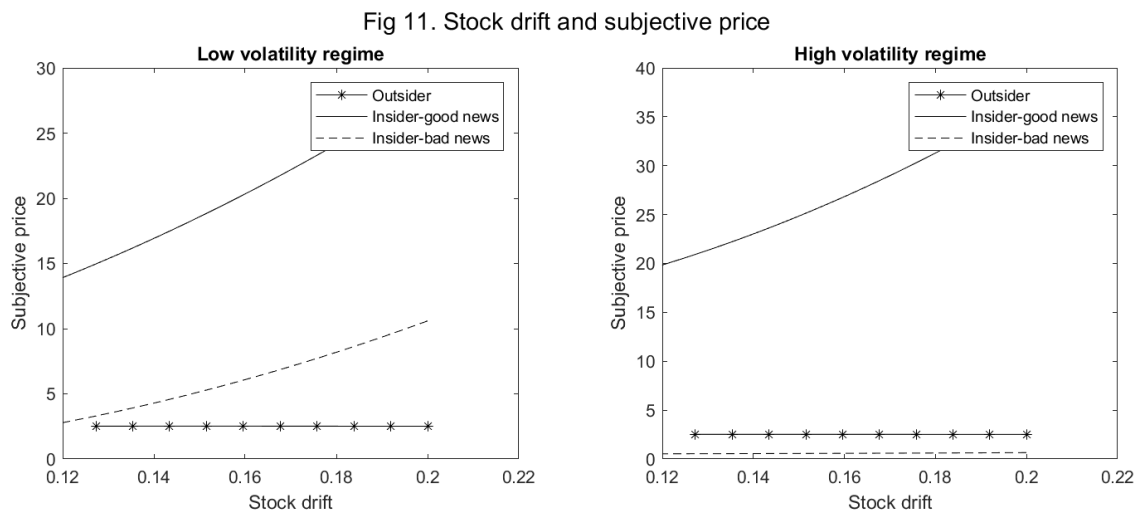
sure times, the higher are incentivizing efficiencies, suggesting that although blackouts are of critical importance, blackouts should not be too long when news is bad. As a regulator has no information to judge the news type, incentive innovation becomes important.

Fig 10. Information disclosure time and granting efficiency



8. While stocks' drift rates are not part of the Black-Scholes pricing formula and, thus, do not affect the objective prices of outsider executives' European ESO, they affect outsider executives' subjective prices. See Figure 11. Higher stock drift rates do not necessarily result in higher outsiders' subjective prices. However, higher stock drift rates generally result in greater

insiders' subjective prices, except the case of bad news under high-volatility regime, as insider-executives know that eventually, their stock price goes down. Putting effort to boost stock price temporarily is not as attractive as to utilize the information to trade using outsider wealth when the market is volatile (active) enough.



9. Subjective ESO prices are affected also by the drift rate and volatility of the market index. See Figure 12 and Figure 13. As the index becomes more attractive, i.e., as its drift rate increases or its volatility decreases (all else being equal), the outsider's ESO position becomes relatively less attractive. However, if executives have insider information on the firm's future stock price, the relationship between stocks and the market index changes from substitutes to complements; that is, insiders' subjective price of ESO increases as the index's drift increases and as the index's volatility decreases.

10. The impact of ESO moneyness on granting efficiency depends on the news. See Figure 14. If the news is bad, at-the-money ESO have the highest granting efficiency. Firms should grant at-the-money ESO to executives with bad news insider information and stop granting them out-of-the-money ESO due to negative granting efficiency in the low-volatility regime. The

Fig 12. Index drift and ESO subjective price

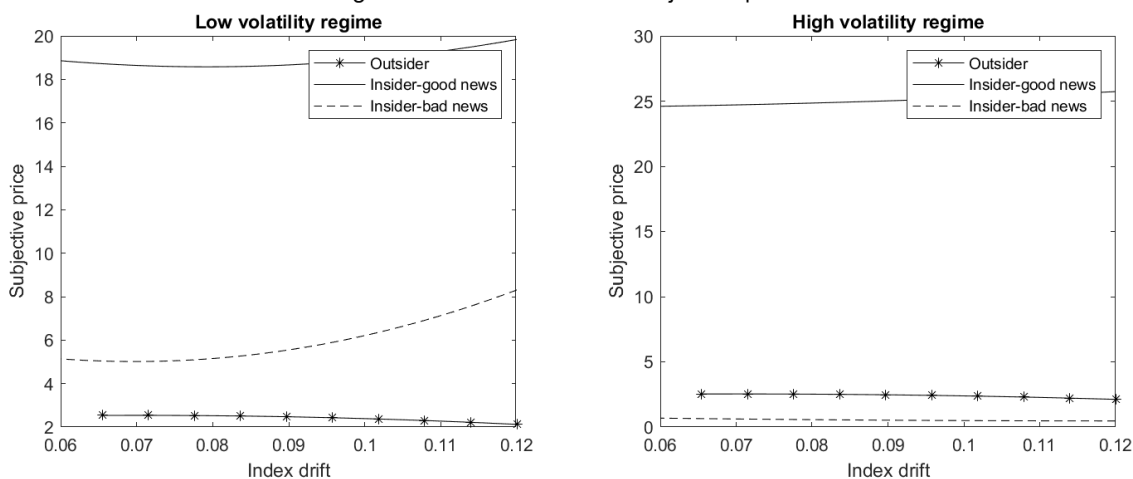
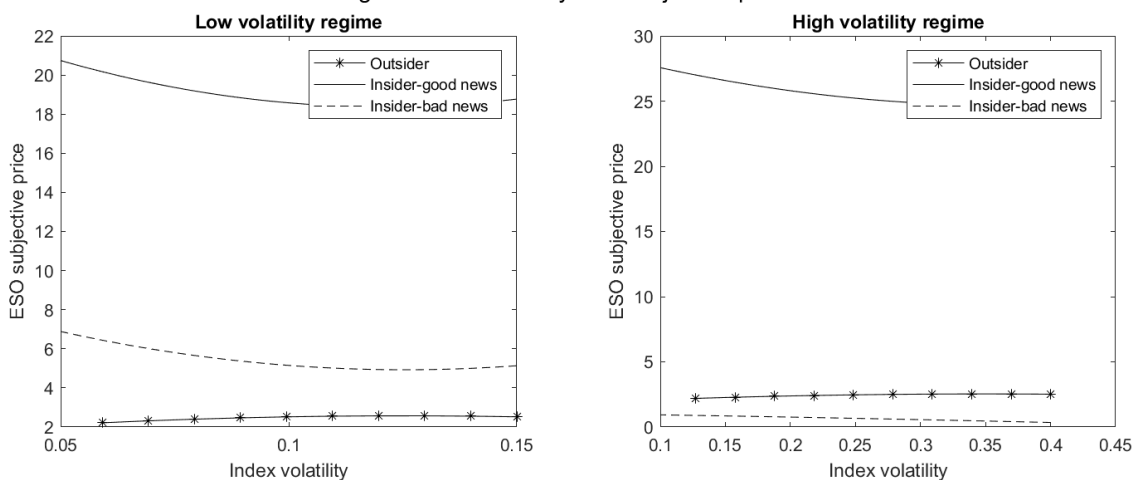


Fig 13. Index volatility and subjective price



impact of ESO moneyness on granting efficiency is not significant to insiders if the news is good, or to outsiders.

11. For insiders, excessively long stock vesting periods usually induce a low granting efficiency because insiders cannot utilize the insider information. See Figure 15.
12. The greater the ratios of option endowment values over total initial wealth (options, stocks, and cash), the greater are granting efficiencies and the stronger is ESO alignment of executive and shareholders' interests. See Figure 16. However, granting an excessive number of ESO to

Fig 14. Strike price and granting efficiency

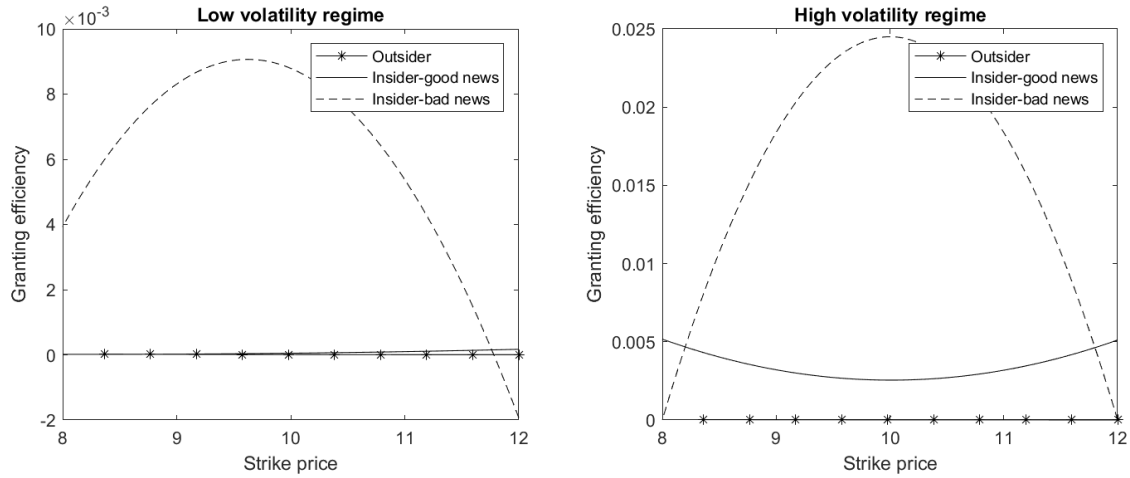
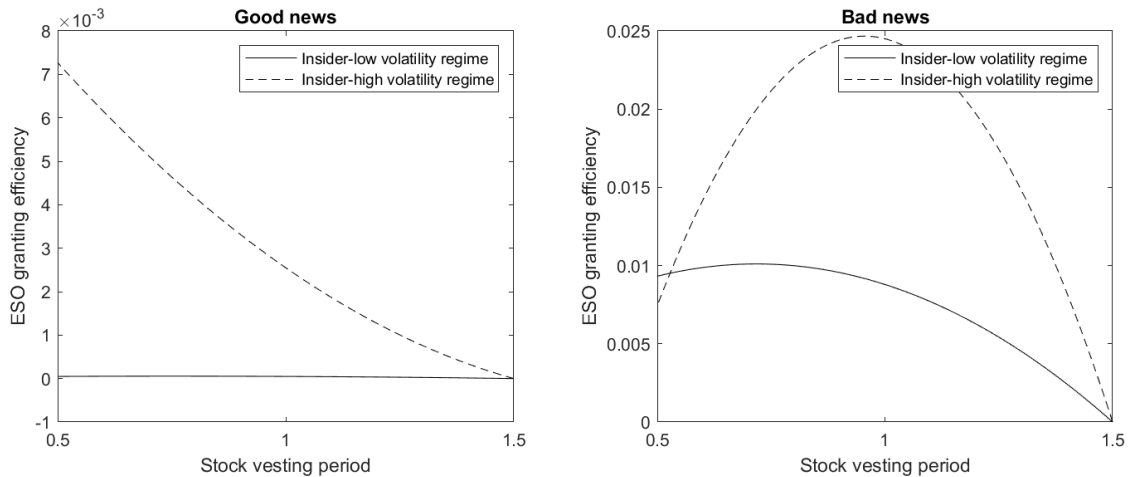


Fig 15. Stock vesting period and granting efficiency



insider-executives harms incentivization and results in a negative granting efficiency. This is especially significant when insiders' news is bad.

13. Granting efficiency, at any level of initial restricted stock endowment, is unconditionally positive. See Figure 17. However, we observe that the lowest level of initial restricted stock endowment induces the highest granting efficiency when news is bad. See Figure 18. The same pattern was observed in the study of granting efficiency with respect to cash endowment.
14. Information quality is important to incentivize insider-executives. With good (bad) news, the

Fig 16. ESO endowment and granting efficiency

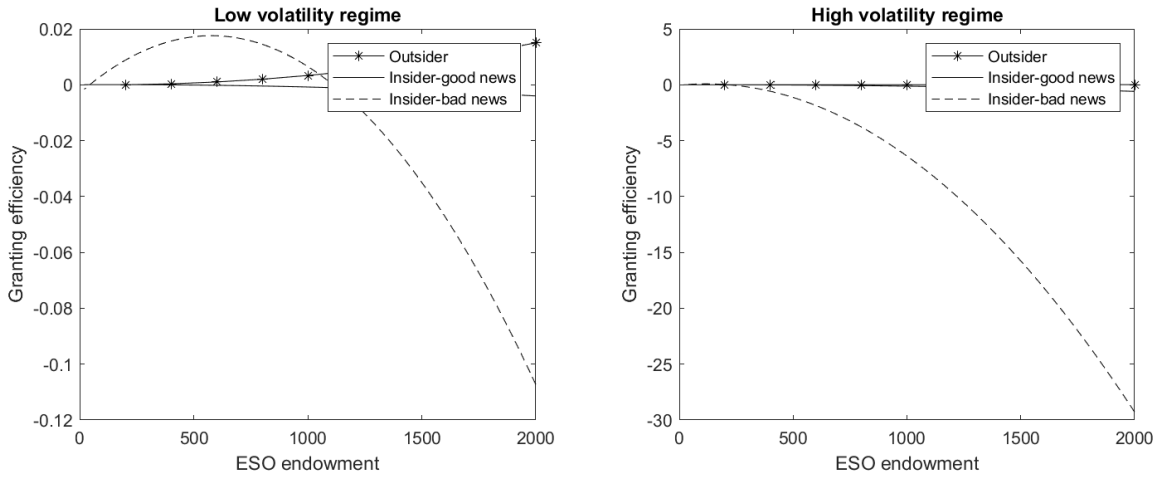
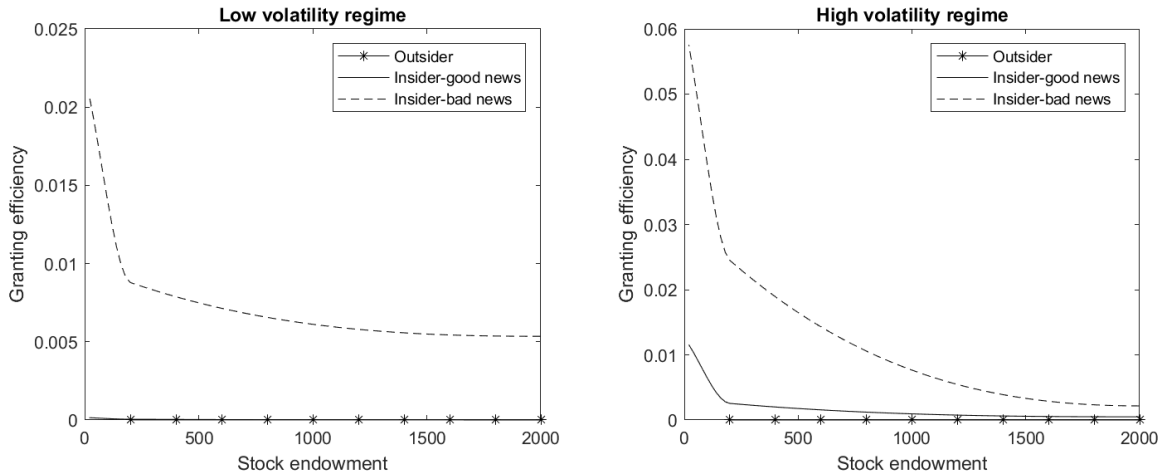


Fig 17. Stock endowment and granting efficiency



ESO granting efficiency shows a negative (positive) curvature with respect to the information quality. See Figure 19.

9 Conclusion

Properly incentivizing executives is essential for firms' performance, economic growth, and societal welfare. The predominant instrument for incentivizing executives has been non-transferable non-hedgeable American executive stock options. Colwell, Feldman, and Hu (2015) were the first to price the general case of such options analytically. This study demonstrates that insider-executives' use of

Fig 18. Cash endowment and granting efficiency

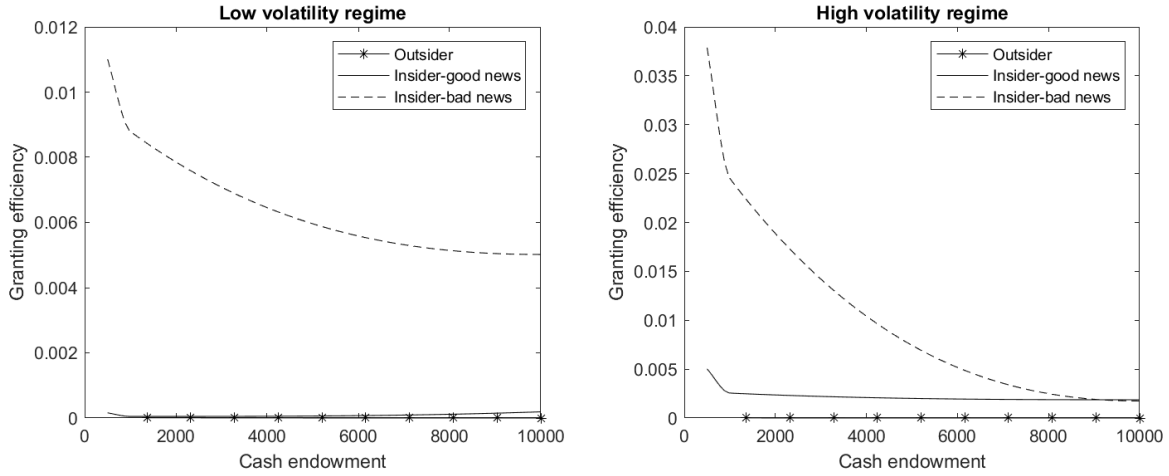
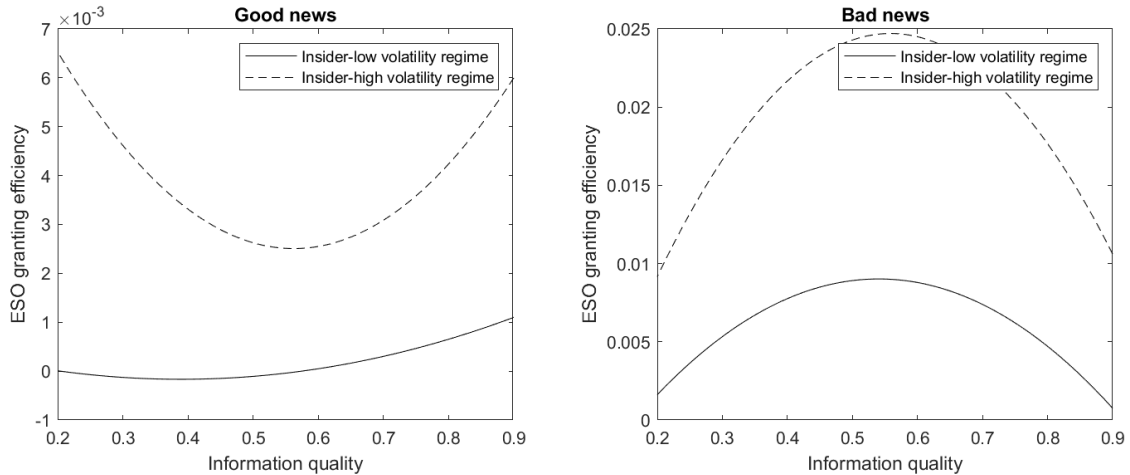


Fig 19. Information quality and granting efficiency



insider information nullifies ESO incentives. Despite non-transferable non-hedgeable restrictions and insider trading restrictions imposed in the Securities Exchange Act of 1934, Rule 10b5, executives might use portfolio optimization-style trading, rather than arbitrage style, in their outside wealth portfolios. The reason is that the Securities Exchange Act of 1934, Rule 10b5 Rule 10b5-1, which allows trading according to premeditated plans before the arrival of insider information, cannot be generally enforced.

We show how granting insider-executives with infinite reload of non-transferable non-hedgeable American ESO combined with blackout trading periods may realign executives' and stockholders' in-

terests. Analytically, we price RSO for insider-executives and identify their optimal exercise policies. We identify the lower and upper bounds of blackout trading periods. A blackout starting too early results in either a complete liquidation of ESO or unfairness to insiders, i.e., reducing executives' derived utility to below that of corresponding outsiders. A blackout starting too late gives insiders an excessive information advantage, which concerns regulators.

We adopt constrained primary asset portfolio optimization techniques and combine them with enlarged filtration techniques, which we further develop to allow for insiders' noisy information. To facilitate the pricing of insider information, we introduce imaginary non-tradable assets.

Our Monte Carlo simulation confirms that insider information could weaken ESO incentivizing power. The extent of the weakening depends on volatility regimes and insider news type (good or bad). There is greater weakening under low-volatility regimes and bad news. Sensitivity analyses agree with our theoretical results.

Policy implications suggest the reintroduction of the out-of-favor RSO combined with firm-imposed and SEC-regulated blackout trading periods of firms' issued securities. When insider information is idiosyncratic (systematic), a blackout trading prohibition includes the firm's stock only (all assets). Future empirical research will test the implications of this paper, and future theoretical research will address the issues here with the added possibility of executives' random job termination.

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Notes

¹Achieving infinite derived utility requires insider-executives to rebalance their portfolio at infinitely high frequency as the holding period approaches the public announcement of the insider information they have. Because continuous rebalancing is unrealistic, the statement can be understood as a sufficiently high derived utility from wealth using insider information makes insider-executives lose their interest in following the ESO incentives.

²A typical blackout begins a certain period prior to the public announcement of an event and ends upon the completion of one full trading day after the announcement.

³Effective blackout lower and upper bounds are both functions of executives' total wealth levels and ratios of nonvested compensation values to total wealth, which are dynamic.

⁴The granting efficiency is the utility incentives (i.e., the utility increment per unit of stock price increment) adjusted by a deadweight cost discount (premium), if the deadweight cost is positive (negative); see more details in Section 6.

⁵Insiders must initially file with the SEC Form 3 stating ownership of firm securities, report ownership changes on Form 4, and deferred such reporting on Form 5 so that outsiders can benefit from the information disclosed.

⁶The imaginary asset $d+1$ is not traded. We solve v so that it is not traded. Simplifying these parameters only simplifies v . It does not affect the value of the portfolio process $\pi_v^{\mathbb{G}^*}$, or the value of wealth $X(t)$. For the same reason, we do not need to estimate the parameters of the imaginary asset that is not traded when calibrating the model. Hence, setting $b_{d+1} = 0$ and $\sigma_{d+1,d+1} = 1$, is without loss of generality.

⁷We measure the short-term incentive of an ESO using the first-order derivative of $\mathbb{J}^{\mathbb{G}}$ with respect to $S_1(t)$, and measure the long-term incentive of an ESO using the first-order derivative of $\mathbb{J}^{\mathbb{G}}$ with respect to $S_1(T^*)$. We do not use the ESO's subjective price sensitivity because, with non-transferable non-hedgeable constraints, the objective of the optimal exercise policy and the portfolio optimization problem is to maximize the expected utility generated by terminal total wealth rather than to maximize the subjective price of the ESO. The two coincide only when the portfolio is unconstrained.