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Theoretical investigation on fractal-fractional nonlinear ordinary differential equations

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Abstract

With this work, we aim at establishing some of the prerequisites for the existence and uniqueness solution of nonlinear fractal-fractional differential equation. The cases of exponential decay, power law, generalized Mittag-Leffler kernels, and the Delta-Dirac case were among the four that were taken into consideration. In order to accomplish this, we first established a number of significant inequalities connected to the well-known Gronwall inequality. We explained in great detail how the maximal and minimal solutions for each scenario were derived. Finally, we reported the convergence of the four successive approximation scenarios for each instance.

Keywords: fractal-fractional differentiation and integration, maximal and minimal solutions, successive approximation, inequalities.

1 Introduction

From fractal derivative[1] and various fractional differential operators, fractal-fractional differential and integral operators were created[2]. Numerous theoretical investigations, such as inequality, functional analysis, and others, have been conducted using these operators[[3]-[5]]. New classes of ordinary and partial differential equations have resulted from them. For instance, the class of fractal differential equations with the Delta-Dirac kernel lacks fractional characteristics. This class, which combines fractal derivative and fractional derivative with an exponential decay kernel[6], should be used for systems with fading memory. Fractal-fractional differential and integral equations with power-law kernel is a combination of fractal derivative and fractional derivative with power-law kernel[7]. We can thus predict that this class might be employed

for processes with power law behavior since fractal-fractional with the power law kernel is a mix of fractal derivative[1] and fractional derivative with the power law kernel[7]. differential equations that combine fractal and fractional derivatives with the extended Mittag-Leffler kernel[8] are known as fractal-fractional differential equations with the kernel[9]. For processes that display characteristics of the generalized Mittag-Leffler function, such as the crossover from stretched exponential to power law, this class can be used. They are specifically used to model processes in accordance with the properties connected to the kernel for application reasons. However, unless their answers can be found, they cannot be applied. which, given the complexity of these operators, is not always the case. To at least achieve numerical solutions, a few restricted numerical approaches are offered. However, the validity of these numerical approaches' accuracy depends on the assurance of precise and distinctive solutions. Mathematicians have developed several methodologies during the past ten years to ensure that a particular differential equation admits a singular solution. This has attracted numerous academics over the past few decades and set up a new branch of pure mathematics. The lack of a generic formula for solving nonlinear ordinary differential equations (ODEs) made the study of existence and uniqueness of solutions crucial. Nearly 200 years after the theory of differential equations was developed, the compact form of existence and uniqueness theory emerged. However, since the notion is still relatively new and will require more consideration for theoretical, methodological, and practical frameworks, less results have been obtained in the case of fractal-fractional differential and integral equations. The goal of this study is to extend current existence and uniqueness conclusions discovered for classical ordinary differential equations[[10]-[14]] to the situation of fractal-fractional nonlinear differential equations. Existence and uniqueness of maximal and minimal solutions, existence, uniqueness, and convergence of successive approximation for four separate scenarios will be addressed[[15],[16],[17]]. However, the process will begin with the identification of certain significant inequalities, like the Gronwall[[18],[19]].

2 Some inequalities associated to nonlinear fractal-fractional differential equations

We shall present here some inequalities associated to these differential equations accordingly to the properties of the function $f(t, y(t))$. We assume that $(t, y(t)) \in R_0$ where

$$R_0 = \{(t, y) \mid t_0 \leq t \leq t_0 + a, |y - y_0| < b\} \quad (1)$$

$\exists c > 0$ such that

$$|f(t, y(t))| < c(1 + |y|). \quad (2)$$

Then, in the case of fractal derivative[1], we shall have

$$\begin{cases} {}^F_0 D_t^\beta y(t) = f(t, y(t)) & \text{if } t > t_0 \\ y(t_0) = y_0 & \text{if } t = t_0 \end{cases} . \quad (3)$$

Then, the above equation is converted to

$$y(t) = y_0 + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau. \quad (4)$$

We apply the norm on both side to obtain

$$|y(t)| \leq |y_0| + \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y(\tau))| d\tau. \quad (5)$$

Using the hypothesis leads to the following inequality

$$\begin{aligned} |y(t)| &\leq |y_0| + \beta \int_{t_0}^t \tau^{\beta-1} c(1 + |y(\tau)|) d\tau \\ &\leq |y_0| + \beta c \int_{t_0}^t \tau^{\beta-1} d\tau + \beta c \int_{t_0}^t \tau^{\beta-1} |y(\tau)| d\tau \\ &\leq |y_0| + c(t - t_0)^\beta + \beta c \int_{t_0}^t \tau^{\beta-1} |y(\tau)| d\tau. \end{aligned} \quad (6)$$

We let $m(t) = |y(t)|$ then, we assume that the function $m(t)$ satisfies the conditions under derivation of the Gronwall inequality, then, we have

$$m(t) \leq v(t) + \beta c \int_{t_0}^t \tau^{\beta-1} m(\tau) d\tau. \quad (7)$$

Therefore, by the Gronwall inequality, we get

$$\begin{aligned} m(t) &\leq v(t) \exp\left(c(t - t_0)^\beta\right) \\ &\leq \left[|y_0| + c(t - t_0)^\beta\right] \exp\left(c(t - t_0)^\beta\right). \end{aligned} \quad (8)$$

In the case of exponential decay, we have

$$\begin{cases} {}_0^{FFE} D_t^{\alpha, \beta} y(t) = f(t, y(t)) & \text{if } t > t_0 \\ y(t_0) = y_0 & \text{if } t = t_0 \end{cases}. \quad (9)$$

Applying the fractal-fractional integral[2] on both sides, we get

$$y(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau. \quad (10)$$

From the above, we have

$$|y(t)| \leq (1 - \alpha) \beta t^{\beta-1} |f(t, y(t))| + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y(\tau))| d\tau. \quad (11)$$

By the hypothesis, we have the following

$$\begin{aligned}
|y(t)| &\leq (1-\alpha)\beta a^{\beta-1}c(1+|y(t)|) + \alpha\beta \int_{t_0}^t \tau^{\beta-1}c(1+|y(\tau)|)d\tau \\
&\leq (1-\alpha)\beta a^{\beta-1}c + (1-\alpha)\beta a^{\beta-1}c|y| \\
&\quad + \alpha\beta c \int_{t_0}^t \tau^{\beta-1}d\tau + \alpha\beta c \int_{t_0}^t \tau^{\beta-1}|y(\tau)|d\tau.
\end{aligned} \tag{12}$$

We assume that

$$1 + (\alpha - 1)\beta a^{\beta-1}c > 0 \tag{13}$$

such that

$$\begin{aligned}
|y(t)| &\leq \frac{(1-\alpha)\beta a^{\beta-1}c}{1+(\alpha-1)\beta a^{\beta-1}c} + \frac{\alpha c(t-t_0)^\beta}{1+(\alpha-1)\beta a^{\beta-1}c} \\
&\quad + \frac{\alpha\beta c}{1+(\alpha-1)\beta a^{\beta-1}c} \int_{t_0}^t \tau^{\beta-1}|y(\tau)|d\tau \\
&\leq v(t) + \int_{t_0}^t \Omega(\tau)|y(\tau)|d\tau.
\end{aligned} \tag{14}$$

By the Gronwall inequality, we obtain

$$\begin{aligned}
|y(t)| &\leq v(t) \exp \int_{t_0}^t \Omega(\tau) d\tau \\
&\leq \left(\frac{(1-\alpha)\beta a^{\beta-1}c}{1+(\alpha-1)\beta a^{\beta-1}c} + \frac{\alpha c(t-t_0)^\beta}{1+(\alpha-1)\beta a^{\beta-1}c} \right) \exp \left(\frac{\alpha\beta c(t-t_0)^\beta}{1+(\alpha-1)\beta a^{\beta-1}c} \right).
\end{aligned} \tag{15}$$

In the case of power-law kernel, we have

$$\begin{cases} {}_0^{FFP}D_t^{\alpha,\beta}y(t) = f(t, y(t)) & \text{if } t > t_0 \\ y(t_0) = y_0 & \text{if } t = t_0 \end{cases} . \tag{16}$$

Applying the fractal-fractional integral[2] on both sides, we get

$$y(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{17}$$

Then, we have

$$\begin{aligned}
|y(t)| &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y(\tau))| d\tau \\
&\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} c(1+|y(\tau)|) d\tau \\
&\leq \frac{\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau + \frac{\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau \\
&\leq \frac{\beta c}{\Gamma(\alpha)} \left[\int_0^t \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau - \int_0^{t_0} \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau \right] \\
&\quad + \frac{\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau.
\end{aligned} \tag{18}$$

Noting that

$$\begin{aligned}
\int_0^t \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau &= t^{\alpha+\beta-1} B(\alpha, \beta) \\
\int_0^{t_0} \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau &= t^{\alpha+\beta-1} B\left(\frac{t_0}{t}, \alpha, \beta\right).
\end{aligned} \tag{19}$$

Therefore, we have

$$\begin{aligned}
|y(t)| &\leq \frac{\beta c}{\Gamma(\alpha)} t^{\alpha+\beta-1} \left[B(\alpha, \beta) - B\left(\frac{t_0}{t}, \alpha, \beta\right) \right] \\
&\quad + \frac{\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau \\
&\leq v(t) + \int_{t_0}^t \Omega(\tau) |y(\tau)| d\tau.
\end{aligned} \tag{20}$$

By the Gronwall inequality, we get

$$\begin{aligned}
|y(t)| &\leq v(t) \exp \int_{t_0}^t \Omega(\tau) d\tau \\
&\leq v(t) \exp \left(\frac{\beta c}{\Gamma(\alpha)} t^{\alpha+\beta-1} \left[B(\alpha, \beta) - B\left(\frac{t_0}{t}, \alpha, \beta\right) \right] \right) \\
&\leq \frac{\beta c}{\Gamma(\alpha)} t^{\alpha+\beta-1} \left[B(\alpha, \beta) - B\left(\frac{t_0}{t}, \alpha, \beta\right) \right] \\
&\quad \times \exp \left(\frac{\beta c}{\Gamma(\alpha)} t^{\alpha+\beta-1} \left[B(\alpha, \beta) - B\left(\frac{t_0}{t}, \alpha, \beta\right) \right] \right).
\end{aligned} \tag{21}$$

Noting that $B(z; \alpha, \beta)$ is the generalization of the incomplete beta function defined by

$$B(z; \alpha, \beta) = z^\alpha \sum_{n=0}^{\infty} \frac{(1-\beta)_n}{n! (\alpha+n)} z^n \tag{22}$$

where $(\cdot)_n$ is a Pochhammer symbol.

We shall consider now the case of the general Mittag-Leffler kernel

$$\begin{cases} {}_0^{FFM}D_t^{\alpha, \beta} y(t) = f(t, y(t)) & \text{if } t > t_0 \\ y(t_0) = y_0 & \text{if } t = t_0 \end{cases} \tag{23}$$

Applying the fractal-fractional integral[2] on both sides, we get

$$y(t) = (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{24}$$

Therefore, we write

$$|y(t)| \leq (1-\alpha)\beta t^{\beta-1} |f(t, y(t))| + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y(\tau))| d\tau. \tag{25}$$

By the hypothesis, we have

$$\begin{aligned}
|y(t)| &\leq (1-\alpha)\beta a^{\beta-1} c + (1-\alpha)\beta a^{\beta-1} c |y(t)| + \frac{\alpha\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau \\
&\quad + \frac{\alpha\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\leq \frac{\alpha\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau + \frac{\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau \\
&\leq \frac{\alpha\beta c}{\Gamma(\alpha)} \left[\int_0^t \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau - \int_0^{t_0} \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau \right] \\
&\quad + \frac{\beta c}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau.
\end{aligned}$$

Under the condition that

$$1 + (\alpha - 1) \beta a^{\beta-1} c > 0, \quad (27)$$

we shall have

$$\begin{aligned}
|y(t)| &\leq \frac{(1-\alpha)\beta a^{\beta-1}c}{1+(\alpha-1)\beta a^{\beta-1}c} + \frac{\alpha c t^{\alpha+\beta-1} [B(\alpha, \beta) - B(\frac{t_0}{t}, \alpha, \beta)]}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta-1}c)} \\
&\quad + \frac{\alpha\beta c}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta-1}c)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |y(\tau)| d\tau \\
&\leq v(t) + \int_{t_0}^t \Omega(\tau) |y(\tau)| d\tau.
\end{aligned} \quad (28)$$

Then,

$$\begin{aligned}
|y(t)| &\leq v(t) \exp \int_{t_0}^t \Omega(\tau) d\tau \\
&\leq v(t) \exp \left(\frac{\alpha c}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta-1}c)} t^{\alpha+\beta-1} \left[B(\alpha, \beta) - B\left(\frac{t_0}{t}, \alpha, \beta\right) \right] \right) \\
&\leq \left(\frac{(1-\alpha)a^{\beta-1}\beta c}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta-1}c)} \right. \\
&\quad \left. + \frac{\alpha c t^{\alpha+\beta-1}}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta-1}c)} [B(\alpha, \beta) - B(\frac{t_0}{t}, \alpha, \beta)] \right) \\
&\quad \times \exp \left(\frac{\alpha c t^{\alpha+\beta-1}}{\Gamma(\alpha)(1+(\alpha-1)\beta a^{\beta-1}c)} [B(\alpha, \beta) - B(\frac{t_0}{t}, \alpha, \beta)] \right).
\end{aligned} \quad (29)$$

3 Existence and uniqueness of the maximal and minimal solutions for fractal-fractional differential equations

We present first the convergence of successive approximation[10] for the four equations.

Theorem. 1. Assume that the function $H(t, u)$ is continuous and positive $\forall t \in [t_0, t_0 + a], 0 \leq u \leq 2b$ and $\forall t_1, t_0 < t_1 < t_0 + a, u(t) = 0$ is the only differentiable function on $t_0 \leq t < t_1$ satisfying

$$\begin{cases} {}^{FF}D_t^{\alpha, \beta} u(t) = H(t, u(t)), \\ y(t_0) = 0 \end{cases} \quad (30)$$

where ${}^{FF}D_t^\alpha$ is a fractal-fractional derivative[2].

For $t_0 \leq t \leq t_1, f \in C[R_0, R]$ where

$$R_0 = \{(t, y) \mid t_0 \leq t \leq t_0 + a, |y - y_0| < b\} \quad (31)$$

and $\forall (t, y), (t, \bar{y}) \in R_0$

$$|f(t, y) - f(t, \bar{y})| \leq H(t, |y - \bar{y}|). \quad (32)$$

Then, the fractal-fractional differential equation[2]

$$\begin{cases} {}^{FF}D_t^{\alpha, \beta} y(t) = f(t, y(t)), & \text{if } t > t_0 \\ y(t_0) = y_0, & \text{if } t = t_0 \end{cases} \quad (33)$$

has at most one solution on $t_0 \leq t \leq t_0 + a$.

We shall present the proof for the four cases starting with the classical fractal case[1]

$$\begin{cases} {}^F D_t^\beta y(t) = f(t, y(t)), & \text{if } t > t_0 \\ y(t_0) = y_0, & \text{if } t = t_0 \end{cases} \quad (34)$$

Applying the integral yields

$$y(t) = y(t_0) + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau. \quad (35)$$

We now assume that the existence of two solutions y_1 and y_2 . We call their difference

$$\Lambda(t) = |y_1(t) - y_2(t)|, \Lambda(t_0) = 0. \quad (36)$$

Then, we write

$$\begin{aligned} |y_1(t) - y_2(t)| &= \beta \left| \int_{t_0}^t \tau^{\beta-1} (f(\tau, y_1) - f(\tau, y_2)) d\tau \right| \\ &\leq \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_1) - f(\tau, y_2)| d\tau \\ &\leq \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, \Lambda(\tau))| d\tau \end{aligned} \quad (37)$$

which obtain using the theorem hypothesis. We also have that $\Lambda(t_0) = 0$. Let $t_1 \in (t_0, t_0 + a)$, we shall have that

$$\Lambda(t) \leq r(t), t_0 \leq t < t_1 \quad (38)$$

since maximal solution principle properties are verified here[[15],[16],[17]]. But the first hypothesis provides that

$$\Lambda(t) = 0 \text{ on } t_0 \leq t < t_1 \quad (39)$$

which completes the proof.

We shall now consider the case with exponential decay

$$y(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau. \quad (40)$$

We choose $y_1(t)$ and $y_2(t)$

$$\begin{aligned} \Lambda(t) &= |y_1(t) - y_2(t)| \\ &\leq (1 - \alpha) \beta t^{\beta-1} |f(t, y_1) - f(t, y_2)| \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_1) - f(\tau, y_2)| d\tau \\ &\leq (1 - \alpha) \beta t^{\beta-1} |H(t, \Lambda(t))| \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, \Lambda(\tau))| d\tau. \end{aligned} \quad (41)$$

We have that $\Lambda(t_0) = 0$. Let $t_1 \in (t_0, t_0 + a)$, we have

$$\Lambda(t) \leq r(t), t_0 \leq t < t_1 \quad (42)$$

and clearly then

$$\Lambda(t) = 0 \text{ on } t_0 \leq t < t_1 \quad (43)$$

which completes the proof.

In the case of power-law, we have

$$\begin{cases} {}_0^{FFP} D_t^{\alpha, \beta} y(t) = f(t, y(t)) & \text{if } t > t_0 \\ y(t_0) = y_0 & \text{if } t = t_0 \end{cases}. \quad (44)$$

Applying the fractal-fractional integral[2] on both sides, we obtain

$$y(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (45)$$

We consider $y_1(t)$ and $y_2(t)$ again, here we have

$$\begin{aligned}\Lambda(t) &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y_1) - f(\tau, y_2)| d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |H(\tau, \Lambda(\tau))| d\tau.\end{aligned}\quad (46)$$

Then

$$\Lambda(t) \leq r(t), \forall t \in [t_0, t_1] \quad (47)$$

where $r(t)$ is the maximal solution of

$${}^{\text{FFP}}_0 D_t^{\alpha, \beta} u(t) = H(t, u(t)). \quad (48)$$

Again with the assumption

$$\Lambda(t) = 0 \text{ on } t_0 \leq t < t_1 \quad (49)$$

which also concludes the proof.

In the case of generalized Mittag-Leffler kernel, we shall have

$$\begin{cases} {}^{\text{FFM}}_0 D_t^{\alpha, \beta} y(t) = f(t, y(t)) & \text{if } t > t_0 \\ y(t_0) = y_0 & \text{if } t = t_0 \end{cases} . \quad (50)$$

Applying the fractal-fractional integral[2] on both sides, we obtain

$$y(t) = (1-\alpha)\beta t^{\beta-1} f(t, y(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (51)$$

Considering $y_1(t)$ and $y_2(t)$, we have

$$\begin{aligned}\Lambda(t) &\leq (1-\alpha)\beta t^{\beta-1} |f(t, y_1) - f(t, y_2)| \\ &\quad + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y_1) - f(\tau, y_2)| d\tau \\ &\leq (1-\alpha)\beta t^{\beta-1} |H(t, \Lambda(t))| \\ &\quad + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |H(\tau, \Lambda(\tau))| d\tau \\ &\leq r(t)\end{aligned}\quad (52)$$

where $r(t)$ is the maximal solution of

$${}_0^{FFM}D_t^{\alpha,\beta}u(t) = H(t, u(t)). \quad (53)$$

In the above, we have used several times the maximal solution principle, in the next theorem, we shall present a detailed proof of this[[15],[16],[17]].

Theorem 2. Let $f \in C[R_0, R]$, R_0 is defined as before, and

$$|f(t, y)| < M_f \quad (54)$$

on R_0 . Then, there exists a maximal solution and a minimal of

$$\begin{cases} {}_{t_0}^{FF}D_t^{\alpha,\beta}y(t) = f(t, y(t)), & \text{if } t > t_0 \\ y(t_0) = y_0, & \text{if } t = t_0 \end{cases} \quad (55)$$

on $[t_0, t_0 + \lambda]$ where λ depends on the chosen fractal-fractional derivative[2].

Proof. We let $0 < \xi \leq \frac{b}{2}$, we consider in the case of the classical version with initial condition

$${}_t^FD_t^\beta y(t) = f(t, y(t)) + \xi, \quad y(t_0) = y_0 + \xi. \quad (56)$$

For simplicity, we let $f_\xi(t, y(t))$ to be $f(t, y(t)) + \xi$. We can now define

$$R_0^\xi = \left\{ (t, y) \mid t \in [t_0, t_0 + a], |y - (y_0 + \xi)| < \frac{b}{2} \right\}. \quad (57)$$

Clearly, $R_0^\xi \subset R_0$ and

$$|f_\xi(t, y)| \leq M + \frac{b}{2} \quad \text{on } R_0^\xi. \quad (58)$$

We can conclude that

$${}_t^FD_t^\beta y(t) = f_\xi(t, y(t)) \quad (59)$$

has a solution $y(t, \xi)$ on the interval $[t_0, t_0 + \lambda]$ where λ is obtained as follows:

$$\lambda = |y(t) - (y_0 + \xi)| < \frac{b}{2}, \quad (60)$$

then

$$\begin{aligned} \int_{t_0}^t \left(M + \frac{b}{2}\right) \beta \tau^{\beta-1} d\tau &< \frac{b}{2} \Rightarrow \left(M + \frac{b}{2}\right) (t^\beta - t_0^\beta) < \frac{b}{2} \\ (2M + b)t^\beta &< b \Rightarrow (2M + b)a^\beta < b \\ \Rightarrow a &< \left(\frac{b}{2M + b}\right)^{\frac{1}{\beta}} \end{aligned} \quad (61)$$

where $\lambda = \min \left\{ a, \left(\frac{b}{2M+b} \right)^{\frac{1}{\beta}} \right\}$. Therefore, for $0 < \xi_2 < \xi_1 \leq \xi$, we have that

$$y(t_0, \xi_2) < u(t_0, \xi_1). \quad (62)$$

We then have that

$${}^F D_t^\beta y(t, \xi_2) \leq f(t, y(t, \xi_2)) + \xi_2 \quad (63)$$

Then

$$\begin{aligned} y(t, \xi_2) &\leq y(t_0, \xi_2) + \beta \int_{t_0}^t \tau^{\beta-1} (f(\tau, y(\tau, \xi_2)) + \xi_2) d\tau \\ &\leq y(t_0, \xi_2) + \xi_2 (t^\beta - t_0^\beta) + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau, \xi_2)) d\tau. \end{aligned} \quad (64)$$

But also

$${}^F D_t^\beta y(t, \xi_1) > f(t, y(t, \xi_2)) + \xi_2, \forall t \in [t_0, t_0 + \lambda]. \quad (65)$$

This clearly leads us to

$$y(t, \xi_2) < y(t, \xi_1), \forall t \in [t_0, t_0 + \lambda]. \quad (66)$$

Indeed, the family of functions is equicontinuous and uniformly bounded on $[t_0, t_0 + \lambda]$, it therefore follows that $\exists \{\xi_n\}$ which decreases such that $\xi_n \rightarrow \infty, n \rightarrow \infty$ and that

$$z(t) = \lim_{n \rightarrow \infty} y(t, \xi_n) \quad (67)$$

exists on $[t_0, t_0 + \lambda]$. We have that $z(t_0) = y_0$. Having f uniformly continuous, then

$$\lim_{n \rightarrow \infty} f(t, y(t, \xi_n)) = f(t, y(t)). \quad (68)$$

Therefore

$$y(t, \xi_n) = y_0 + \xi_n + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau, \xi_n)) d\tau \quad (69)$$

and

$$\lim_{n \rightarrow \infty} y(t, \xi_n) = z(t) = y_0 + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau \quad (70)$$

which is the solution of our equation. We shall now show in addition that the obtained $r(t)$ is indeed the maximal solution on $[t_0, t_0 + \lambda]$ satisfying all the requirements of the maximal solution[[15],[16],[17]]. To do this, we assume $z(t)$ to be another maximal solution existing on $[t_0, t_0 + \lambda]$. Therefore

$$z(t_0) = z_0 < z_0 + \xi = y(t_0, \xi). \quad (71)$$

$${}^F D_t^\beta z(t) < H(t, z(t)) + \xi_1. \quad (72)$$

$$\begin{aligned} z(t) &< z(t_0) + \beta \int_{t_0}^t \tau^{\beta-1} (\xi + H(\tau, z(\tau))) d\tau \\ &< z(t_0) + \xi (t^\beta - t_0^\beta) + \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, z(\tau)) d\tau \\ &< z(t_0) + \xi (t^\beta - t_0^\beta) + \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, z(\tau)) d\tau \end{aligned} \quad (73)$$

and then

$${}^F D_t^\beta z(t, \xi) \geq H(t, u(t, \xi)) + \xi \quad (74)$$

for $t \in [t_0, t_0 + \lambda]$ and $\xi \leq \frac{b}{2}$, we have that

$$z(t) < u(t, \xi), \forall t \in [t_0, t_0 + \lambda]. \quad (75)$$

We have from the uniqueness of the maximal solution that $z(t, \xi)$ converges uniformly to $r(t)$ on $[t_0, t_0 + \lambda]$ as $\xi \rightarrow 0$ which completes the proof in the case of classical fractal derivative[1].

We now examine the case with the exponential decay kernel.

$$\begin{cases} {}^{FFE} D_t^{\alpha, \beta} y(t) = f(t, y(t)), & \text{if } t > t_0 \\ y(t_0) = y_0, & \text{if } t = t_0 \end{cases}. \quad (76)$$

In this case, we first want to find λ .

$$y(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau. \quad (77)$$

Applying absolute value on both sides to obtain

$$\begin{aligned} |y(t)| &\leq (1 - \alpha) \beta t^{\beta-1} |f_\xi(t, y(t))| \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |f_\xi(\tau, y(\tau))| d\tau \\ &\leq (1 - \alpha) \beta a^{\beta-1} \left(M_f + \frac{b}{2} \right) \\ &\quad + \alpha \beta \left(M_f + \frac{b}{2} \right) \int_{t_0}^t \tau^{\beta-1} d\tau \end{aligned} \quad (78)$$

$$\begin{aligned}
&\leq (1-\alpha)\beta a^{\beta-1}\left(M_f + \frac{b}{2}\right) \\
&\quad + \alpha\left(M_f + \frac{b}{2}\right)\left(t^\beta - t_0^\beta\right) \\
&\leq (1-\alpha)\beta a^\beta\left(M_f + \frac{b}{2}\right) + \alpha\left(M_f + \frac{b}{2}\right)t^\beta \\
&\leq (1-\alpha)\beta a^\beta\left(M_f + \frac{b}{2}\right) + \alpha\left(M_f + \frac{b}{2}\right)a^\beta < \frac{b}{2}.
\end{aligned}$$

Then, we get

$$a^\beta < \frac{b}{(2M_f + b)\{(1-\alpha)\beta + \alpha\}}. \quad (79)$$

Therefore, in this case

$$\lambda = \min \left\{ a, \left(\frac{b}{(2M_f + b)\{(1-\alpha)\beta + \alpha\}} \right)^{\frac{1}{\beta}} \right\}. \quad (80)$$

Again for $0 < \xi_2 < \xi_1 \leq \xi$, we will have

$$y(t_0, \xi_2) < y(t_0, \xi_1). \quad (81)$$

since

$$y(t_0, \xi_1) = y(t_0) + \xi_1 \text{ and } y(t_0, \xi_2) = y(t_0) + \xi_2. \quad (82)$$

Then

$${}^{FFE}D_t^\beta y(t, \xi_2) \leq f(t, y(t, \xi_2)) + \xi_2, \forall t \in [t_0, t_0 + \lambda]. \quad (83)$$

This clearly leads us to

$$\begin{aligned}
|y(t)| &\leq (1-\alpha)\beta t^{\beta-1}(f(t, y(t, \xi_2)) + \xi_2) \\
&\quad + \alpha\beta \int_{t_0}^t \tau^{\beta-1}(f(\tau, y(\tau, \xi_2)) + \xi_2) d\tau \\
&\leq (1-\alpha)\beta t^{\beta-1}f(t, y(t, \xi_2)) + (1-\alpha)\beta t^{\beta-1}\xi_2 \\
&\quad + \alpha\beta \int_{t_0}^t \tau^{\beta-1}f(\tau, y(\tau, \xi_2)) d\tau + \alpha\beta \int_{t_0}^t \tau^{\beta-1}\xi_2 d\tau \\
&\leq (1-\alpha)\beta a^{\beta-1}f(t, y(t, \xi_2)) + (1-\alpha)\beta t^{\beta-1}\xi_2 \\
&\quad + \alpha\beta\left(t^\beta - t_0^\beta\right) + \alpha\beta \int_{t_0}^t \tau^{\beta-1}f(\tau, y(\tau, \xi_2)) d\tau.
\end{aligned} \quad (84)$$

On the other hand, we have

$${}^{FFE}D_t^\beta y(t, \xi_1) \leq f(t, y(t, \xi_2)) + \xi_2, \forall t \in [t_0, t_0 + \lambda]. \quad (85)$$

This shows that

$$y(t, \xi_2) < y(t, \xi_n), \forall t \in [t_0, t_0 + \lambda]. \quad (86)$$

We go through the same routine to arrive

$$\begin{aligned} y(t, \xi_n) &= y_0 + \xi_n + (1 - \alpha) \beta t^{\beta-1} f(t, y(t, \xi_n)) \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau, \xi_n)) d\tau \end{aligned} \quad (87)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} y(t, \xi_n) &= r(t) = y_0 + (1 - \alpha) \beta t^{\beta-1} f(t, r(t)) \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, r(\tau)) d\tau. \end{aligned} \quad (88)$$

The next can be concluded by the same way as before.

We now examine the case of power-law.

$$\begin{cases} {}^{FFP}D_t^{\alpha, \beta} y(t) = f(t, y(t)), & \text{if } t > t_0 \\ y(t_0) = y_0, & \text{if } t = t_0 \end{cases}. \quad (89)$$

We first find λ .

$$y(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (90)$$

Applying absolute value on both sides to obtain

$$\begin{aligned} |y(t)| &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} |f_\xi(\tau, y(\tau))| d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)} \left(M_f + \frac{b}{2} \right) \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)} \left(M_f + \frac{b}{2} \right) t^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \\ &\leq \frac{\beta}{2\Gamma(\alpha)} (2M_f + b) a^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \\ &< \frac{b}{2}. \end{aligned} \quad (91)$$

Then, we get

$$a < \max_{t \in [t_0, t_0 + \lambda]} \left(\frac{b\Gamma(\alpha)}{\beta(2M_f + b)(B(\beta, \alpha) - B(\frac{t_0}{t}, \beta, \alpha))} \right)^{\frac{1}{\alpha + \beta - 1}}. \quad (92)$$

Therefore, in this case

$$\lambda = \min \left\{ a, \max_{t \in [t_0, t_0 + \lambda]} \left(\frac{b\Gamma(\alpha)}{\beta(2M_f + b)(B(\beta, \alpha) - B(\frac{t_0}{t}, \beta, \alpha))} \right)^{\frac{1}{\alpha + \beta - 1}} \right\}. \quad (93)$$

We follow the same routine and reach

$$y(t, \xi_n) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau, \xi_n)) (t - \tau)^{\alpha-1} d\tau \quad (94)$$

and

$$\lim_{n \rightarrow \infty} y(t, \xi_n) = r(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} f(\tau, r(\tau)) (t - \tau)^{\alpha-1} d\tau. \quad (95)$$

We consider now the case with the generalized Mittag-Leffler kernel

$$\begin{cases} {}_{t_0}^{FFM} D_t^{\alpha, \beta} y(t) = f(t, y(t)), & \text{if } t > t_0 \\ y(t_0) = y_0, & \text{if } t = t_0 \end{cases}. \quad (96)$$

We shall find λ in this case.

$$y(t) = (1 - \alpha) \beta t^\beta f(t, y(t)) + \frac{\alpha \beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (97)$$

Then, we get

$$\begin{aligned} |y(t)| &\leq (1 - \alpha) \beta t^\beta |f_\xi(t, y(t))| + \frac{\alpha \beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} |f_\xi(\tau, y(\tau))| d\tau \\ &\leq (1 - \alpha) \beta t^\beta \left(M_f + \frac{b}{2} \right) + \frac{\alpha \beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t - \tau)^{\alpha-1} \left(M_f + \frac{b}{2} \right) d\tau \\ &\leq (1 - \alpha) \beta a^\beta \left(\frac{2M_f + b}{2} \right) + \frac{\alpha \beta}{\Gamma(\alpha)} \left(\frac{2M_f + b}{2} \right) t^{\alpha + \beta - 1} \left(\begin{array}{c} B(\beta, \alpha) \\ -B(\frac{t_0}{t}, \beta, \alpha) \end{array} \right) \\ &< \frac{b}{2}. \end{aligned} \quad (98)$$

We let

$$a^\gamma = \max(a^\beta, a^{\alpha+\beta-1}). \quad (99)$$

Then, we have

$$\begin{aligned} |y(t)| &\leq (1-\alpha)\beta a^\gamma (2M_f + b) + \frac{\alpha\beta}{\Gamma(\alpha)} (2M_f + b) a^\gamma \\ &\quad \times \max_{t \in [t_0, t_0 + \lambda]} \left(\begin{array}{c} B(\beta, \alpha) \\ -B\left(\frac{t_0}{t}, \beta, \alpha\right) \end{array} \right) \\ &< b. \end{aligned} \quad (100)$$

Therefore here

$$\lambda = \min \left\{ a, \left(\frac{b\Gamma(\alpha)}{(2M_f + b)\Sigma} \right)^{\frac{1}{\gamma}} \right\}. \quad (101)$$

where

$$\Sigma = \left\{ \Gamma(\alpha)(1-\alpha)\beta + \alpha\beta \max_{t \in [t_0, t_0 + \lambda]} \left(\begin{array}{c} B(\beta, \alpha) \\ -B\left(\frac{t_0}{t}, \beta, \alpha\right) \end{array} \right) \right\}. \quad (102)$$

Following the same routine, we arrive at

$$y(t, \xi_n) = (1-\alpha)\beta t^{\beta-1} f(t, y(t, \xi_n)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau, \xi_n)) (t-\tau)^{\alpha-1} d\tau \quad (103)$$

and

$$\lim_{n \rightarrow \infty} y(t, \xi_n) = r(t) = (1-\alpha)\beta t^{\beta-1} f(t, r(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} f(\tau, r(\tau)) (t-\tau)^{\alpha-1} d\tau. \quad (104)$$

The rest can be deduced as before. Note that for the minimal solution, we have to set

$$\begin{cases} {}^F D_t^\alpha u(t) = f(t, u) - \xi, \\ u(t_0) = u_0 - \xi \end{cases}. \quad (105)$$

For simplicity, we put

$$f_\xi(t, u) = f(t, u) + \xi \quad (106)$$

We shall now the convergence of the successive approximations for the four cases.

4 Convergence of successive approximation for fractal-fractional ordinary differential equations

We will discuss the convergence of consecutive approximation solutions for nonlinear differential equations with fractal-fractional derivatives[2] in this section. Four unique cases will be presented.

4.1 Convergence of successive approximation for fractal derivative

Theorem 3. We shall now assume that the conditions of theorem hold. We assume in addition that, $H(t, u)$ is nondecreasing in a $\forall t \in [t_0, t_0 + a]$ that $|H(t, y)| < M_f$ on R_0 and $\lambda = \min\{a, \gamma\}$ where γ will be obtained with respect to the chosen fractal-fractional differential operator[2]. Then in the case of classical fractal derivative, the successive approximation defined by

$$y_{n+1}(t) = y(t_0) + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) d\tau. \quad (107)$$

In the case of exponential kernel, we write

$$y_{n+1}(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y_n(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) d\tau. \quad (108)$$

In the case of power-law, we write

$$y_{n+1}(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) (t - \tau)^{\alpha-1} d\tau. \quad (109)$$

In the case of generalized Mittag-Leffler kernel, we have

$$y_{n+1}(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y_n(t)) + \frac{\alpha \beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) (t - \tau)^{\alpha-1} d\tau. \quad (110)$$

In all cases, $y_{n+1}(t)$ exists on it respective interval $t_0 \leq t \leq t_0 + \lambda$ as a continuous function and converges uniformly on this interval to the exact solution $y(t)$.

Proof. In the case of classical, we suppose that $y_k(t)$ is defined and continuous on $t_0 \leq t \leq t_0 + \lambda$, $\lambda = \min\left\{a, \left(\frac{b}{M_f}\right)^{\frac{1}{\beta}}\right\}$ and satisfied $|y_k(t) - y_0| \leq b, \forall k = 0, 1, 2, \dots, n$

$$y_{n+1}(t) = y_0 + \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) d\tau. \quad (111)$$

The continuity of $y_{n+1}(t)$ follows since $f(t, y_n(t))$ is defined and continuous on $t_0 \leq t \leq t_0 + \lambda$. Also, we

have that

$$\begin{aligned}
|y_{n+1}(t) - y_0| &\leq \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) d\tau \\
&\leq \beta M_f \int_{t_0}^t \tau^{\beta-1} d\tau \\
&\leq M_f (t^\beta - t_0^\beta) \\
&\leq a^\beta M_f < M_f \lambda < b.
\end{aligned} \tag{112}$$

Indeed, using the successive approximation, we have that $y_{n+1}(t)$ are defined and continuous on $t_0 \leq t \leq t_0 + \lambda$ with

$$|y_{n+1}(t) - y_0| \leq b, \forall n = 0, 1, 2, \dots \tag{113}$$

For the function $u(t)$, we define approximations recursively

$$u_0(t) = M_f (t^\beta - t_0^\beta) \tag{114}$$

and

$$u_{n+1}(t) = \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, u_n(\tau)) d\tau, t_0 \leq t \leq t_0 + \lambda. \tag{115}$$

Then,

$$\begin{aligned}
|y_1(t) - y_0| &\leq \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_0(\tau))| d\tau \\
&\leq M_f (t^\beta - t_0^\beta) \\
&= u_0(t).
\end{aligned} \tag{116}$$

We shall now assume that for any fixed k , we have that

$$|y_k(t) - y_{k-1}(t)| \leq u_{k-1}(t). \tag{117}$$

But

$$|y_{k+1}(t) - y_k(t)| \leq \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_k(\tau)) - f(\tau, y_{k-1}(\tau))| d\tau. \tag{118}$$

The theorem hypothesis together with the monotonic property of H in u provide us with

$$\begin{aligned}
|y_{k+1}(t) - y_k(t)| &\leq \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_k(\tau)) - f(\tau, y_{k-1}(\tau))| d\tau \\
&\leq \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, y_k(\tau) - y_{k-1}(\tau))| d\tau \\
&\leq \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{k-1}(\tau))| d\tau \\
&= u_k(t).
\end{aligned} \tag{119}$$

We then shown by induction

$$|y_{n+1}(t) - y_n(t)| \leq u_n(t). \tag{120}$$

Then,

$$\begin{aligned}
|y_{n+1}(t) - y_n(t)| &\leq \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_n(\tau)) - f(\tau, y_{n-1}(\tau))| d\tau \\
&\leq \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, y_n(\tau) - y_{n-1}(\tau))| d\tau \\
&\leq \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{n-1}(\tau))| d\tau \\
&= u_n(t).
\end{aligned} \tag{121}$$

We consider $n \leq k \leq m$. We use the above inequality and the relation between H and f to have

$$\begin{aligned}
|y_n(t) - y_m(t)| &\leq |y_n(t) - y_{n+1}(t)| + |y_{m+1}(t) - y_m(t)| + |y_{n+1}(t) - y_{m+1}(t)| \\
&\leq \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_{n-1}(\tau)) - f(\tau, y_n(\tau))| d\tau \\
&\quad + \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_m(\tau)) - f(\tau, y_{m-1}(\tau))| d\tau \\
&\quad + \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_n(\tau)) - f(\tau, y_m(\tau))| d\tau
\end{aligned} \tag{122}$$

$$\begin{aligned}
&\leq \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{n-1}(\tau))| d\tau + \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{m-1}(\tau))| d\tau \\
&\quad + \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, |y_n - y_m|) d\tau.
\end{aligned}$$

Our next issue is to evaluate $|y_n(t) - y_m(t)|$. We know that $u_{n+1}(t) < u_n(t), \forall n$ thus

$$|y_n(t) - y_m(t)| \leq 2\beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, y_{n-1}(\tau))| d\tau + \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, |y_n - y_m|) d\tau. \quad (123)$$

Nevertheless, due to monotonicity of $H(t, u)$ with respect to u . We have that

$$|y_n(t) - y_m(t)| \leq r_n(t), t_0 \leq t \leq t_0 + \lambda \quad (124)$$

where indeed $r_n(t)$ is the maximal solution of

$$y_n(t) \leq 2\beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{n-1}(\tau))| d\tau + \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, y_n(\tau)) d\tau \quad (125)$$

for each n . We have shown however that $r_n(t) \rightarrow 0$ on $t_0 \leq t \leq t_0 + \lambda$, $y_n(t)$ converges uniformly on $t_0 \leq t \leq t_0 + \lambda$ as $n \rightarrow \infty$. Noting that the solution is unique. On the other hand, however, the defined sequence is bounded and equicontinuous. We shall show the proof. If we assume that

$$\lim_{n \rightarrow \infty} |y_n(t) - y_{n-1}(t)| \rightarrow 0. \quad (126)$$

Then, $(y_n(t))$ converges toward $y(t)$ which is the unique solution. We now let

$$\Lambda(t) = \lim_{n \rightarrow \infty} \sup |y_n(t) - y_{n-1}(t)|. \quad (127)$$

We aim to show that $\Lambda(t)$ is continuous $\forall t \in [t_0, t_0 + \lambda]$. We have that on R_0 , $|f(t, y)| < M_f$, we have that t_1 and t_2 such that $t_1 > t_2 \in [t_0, t_0 + \lambda]$, we have by integration

$$\begin{aligned}
|y_n(t) - y_{n-1}(t)| &\leq \beta \int_{t_0}^{t_1} \tau^{\beta-1} |f(\tau, y_{n-1}(\tau))| d\tau + \beta \int_{t_0}^{t_1} \tau^{\beta-1} |f(\tau, y_{n-2}(\tau))| d\tau \\
&\leq M_f |t_1^\beta - t_0^\beta| + M_f |t_1^\beta - t_0^\beta| \\
&\leq 2M_f |t_1^\beta - t_0^\beta|.
\end{aligned} \quad (128)$$

But, on the other hand, we have that the function t^β is differentiable on $[t_0, t_0 + \lambda]$, $\exists \xi \in [t_0, t]$ such that

$$\left| t_1^\beta - t_0^\beta \right| = \beta \xi^{\beta-1} |t_1 - t_0|. \quad (129)$$

Therefore, we have

$$|y_n(t) - y_{n-1}(t)| < 2M_f \beta \xi^{\beta-1} |t_1 - t_0|. \quad (130)$$

Therefore, by using t_1 and t_2 , we have

$$\begin{aligned} |y_n(t_1) - y_{n-1}(t_1)| &\leq |y_n(t_2) - y_{n-1}(t_2)| + 2M_f \beta \xi^{\beta-1} |t_1 - t_2| \\ &\leq \Lambda(t_2) + 2M_f \beta \xi^{\beta-1} |t_1 - t_2| + \varepsilon. \end{aligned} \quad (131)$$

For large n , if $n > 0$. Hence, we have

$$|\Lambda(t_1) - \Lambda(t_2)| < 2\beta M_f \xi^{\beta-1} |t_1 - t_2|. \quad (132)$$

This leads to owning the fact that t_1 and t_2 could be interchangeable

$$|\Lambda(t_1) - \Lambda(t_2)| \leq \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, |y_n(\tau) - y_{n-1}(\tau)|) d\tau \quad (133)$$

which proves the continuity property of $\Lambda(t)$. We have by assumption that

$$|y_{n+1}(t) - y_n(t)| \leq \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, |y_n(\tau) - y_{n-1}(\tau)|) d\tau. \quad (134)$$

For any fixed t on $[t_0, t_0 + \lambda]$, we have a sequence of integers $n_1 < n_2 < n_3 \dots$ such that

$$|y_{n+1}(t) - y_n(t)| \rightarrow \Lambda(t) \text{ as } n = n_k \rightarrow \infty \quad (135)$$

and that

$$\Lambda^*(t) = \lim_{n=n_k \rightarrow \infty} |y_n(t) - y_{n-1}(t)| \quad (136)$$

exists uniformly on $t_0 \leq \tau \leq t_0 + \lambda$. Then

$$m(t) \leq \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, \Lambda^*(\tau)) d\tau. \quad (137)$$

Noting, however, that H is nondecreasing according to the theorem hypothesis, with respect to u and $\Lambda^*(\tau) \leq \Lambda(\tau)$, we can obtain from the above equality that

$$\Lambda(t) \leq \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, \Lambda(\tau)) d\tau. \quad (138)$$

On the other hand, $r(t)$ is the maximal solution,

$$\Lambda(t) \leq r(t) \quad (139)$$

where

$$r(t) = 0, \forall t \in [t_0, t_0 + \lambda]. \quad (140)$$

Thus, we can conclude that

$$\Lambda(t) = 0, \forall t \in [t_0, t_0 + \lambda] \quad (141)$$

which completes the proof.

4.2 Convergence of successive approximation for fractal-fractional derivative with exponential decay kernel

In the case of exponential decay kernel, we have

$$\begin{aligned} |y(t)| &\leq (1 - \alpha) \beta t^{\beta-1} |f(t, y(t))| + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y(\tau))| d\tau \\ &\leq (1 - \alpha) \beta t^{\beta-1} M_f + \alpha \beta M_f \int_{t_0}^t \tau^{\beta-1} d\tau \\ &\leq (1 - \alpha) \beta a^{\beta-1} M_f + \alpha M_f (t^\beta - t_0^\beta) \\ &\leq (1 - \alpha) \beta a^\beta M_f + \alpha M_f a^\beta < b. \end{aligned} \quad (142)$$

Then, we have

$$\lambda = \min \left\{ a, \left(\frac{b}{((1 - \alpha) \beta + \alpha) M_f} \right)^{\frac{1}{\beta}} \right\}. \quad (143)$$

The following successive approximation is defined

$$y_{n+1}(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y_n(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) d\tau \quad (144)$$

and exists on $t_0 \leq t \leq t_0 + \lambda$ as continuous functions, which also converges uniformly to the solution

$$y(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y(\tau)) d\tau. \quad (145)$$

Again, we assume that $y_k(t)$ is defined and continuous on $t_0 \leq t \leq t_0 + \lambda$ with the property that

$$|y_k(t)| \leq b, \forall k = 0, 1, 2, \dots \quad (146)$$

We have by definition that $f(t, y_n(t))$ is continuous and defined on $[t_0, t_0 + \lambda]$ therefore

$$y_{n+1}(t) = (1 - \alpha) \beta t^{\beta-1} f(t, y_n(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} f(\tau, y_n(\tau)) d\tau \quad (147)$$

is also defined and continuous on $[t_0, t_0 + \lambda]$. Additionally, with the definition of λ , we shall have that

$$|y_{n+1}(t)| \leq \{(1 - \alpha) \beta + \alpha\} M_f a^\beta < b. \quad (148)$$

The above insures that $\forall n$ fixed the successive approximation is defined and continuous on $[t_0, t_0 + \lambda]$.

We now build the successive approximation of (u_n) . We choose

$$u_0(t) = M_f \left[(1 - \alpha) \beta t^{\beta-1} + \alpha (t^\beta - t_0^\beta) \right] \quad (149)$$

and

$$u_{n+1}(t) = (1 - \alpha) \beta t^{\beta-1} H(t, u_n(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, u_n(\tau)) d\tau \quad (150)$$

with $t \in [t_0, t_0 + \lambda]$. We shall show that $t \in [t_0, t_0 + \lambda]$ and n fixed

$$|y_n(t) - y_{n-1}(t)| \leq u_{n-1}(t). \quad (151)$$

By the induction, we have when $n = 1$

$$|y_1(t)| \leq M_f \left[(1 - \alpha) \beta t^{\beta-1} + \alpha (t^\beta - t_0^\beta) \right] = u_0(t). \quad (152)$$

We assume for a given n fixed

$$|y_n(t) - y_{n-1}(t)| \leq u_{n-1}(t). \quad (153)$$

Now

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq (1 - \alpha) \beta t^{\beta-1} |f(t, y_n(t)) - f(t, y_{n-1}(t))| \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |f(\tau, y_n(\tau)) - f(\tau, y_{n-1}(\tau))| d\tau \\ &\leq (1 - \alpha) \beta t^{\beta-1} H(t, |y_n(t) - y_{n-1}(t)|) \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, |y_n(\tau) - y_{n-1}(\tau)|) d\tau. \end{aligned} \quad (154)$$

By induction hypothesis and the monotonic property of $H(t, u)$ with respect to u , we obtain

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq (1 - \alpha) \beta t^{\beta-1} H(t, u_{n-1}(t)) \\ &\quad + \alpha \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, u_{n-1}(\tau)) d\tau = u_n(t). \end{aligned} \quad (155)$$

Let then $1 \leq n \leq m$. We will have that

$$\begin{aligned} |y_n(t) - y_m(t)| &= |y_n(t) - y_{n+1}(t) + y_{m+1}(t) - y_m(t) + y_{n+1}(t) - y_{m+1}(t)| \\ &\leq |y_n(t) - y_{n+1}(t)| + |y_{m+1}(t) - y_m(t)| + |y_{n+1}(t) - y_{m+1}(t)|. \end{aligned} \quad (156)$$

From the previous inequality, we have that

$$|y_{n+1}(t) - y_n(t)| \leq (1 - \alpha) \beta t^{\beta-1} |H(t, u_{n-1}(t))| + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{n-1}(\tau))| d\tau \quad (157)$$

$$|y_{m+1}(t) - y_m(t)| \leq (1 - \alpha) \beta t^{\beta-1} |H(t, u_{m-1}(t))| + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{m-1}(\tau))| d\tau. \quad (158)$$

m and n being interchangeable, we set

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq 2 \left\{ (1 - \alpha) \beta t^{\beta-1} |H(t, u_{n-1}(t))| + \alpha \beta \int_{t_0}^t \tau^{\beta-1} |H(\tau, u_{n-1}(\tau))| d\tau \right\} \\ &\quad + (1 - \alpha) \beta t^{\beta-1} H(t, |y_n(t) - y_m(t)|) + \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, |y_n(\tau) - y_m(\tau)|) d\tau. \end{aligned} \quad (159)$$

But, we know that

$$|y_n(t) - y_m(t)| \leq r_n(t), t \in [t_0, t_0 + \lambda] \quad (160)$$

where $r_n(t)$ is the maximal solution of

$${}^{FFE}D_t^{\alpha, \beta} y(t) = H(t, y) + 2H(t, y_{n-1}(t)) \quad (161)$$

with $y_n(t) = 0$ for any fixed n . The conclusion is reached.

Since

$$\Lambda_*^{\alpha, \beta}(t) = \lim_{n=n_k \rightarrow \infty} |y_n(t) - y_m(t)| \quad (162)$$

exists uniformly on $t_0 \leq \tau \leq t_0 + \lambda$. Thus,

$$\Lambda^{\alpha, \beta}(t) \leq (1 - \alpha) \beta t^{\beta-1} H(t, \Lambda^{\alpha, \beta}(t)) + \alpha \beta \int_{t_0}^t \tau^{\beta-1} H(\tau, \Lambda^{\alpha, \beta}(\tau)) d\tau \quad (163)$$

with the properties of H with respect to u and $\Lambda_*^{\alpha,\beta}(t) = \Lambda^{\alpha,\beta}(t)$. Then, we have

$$\begin{aligned}\Lambda^{\alpha,\beta}(t) &\leq (1-\alpha)\beta t^{\beta-1}H(t, \Lambda^{\alpha,\beta}(t)) + \alpha\beta \int_{t_0}^t \tau^{\beta-1}H(\tau, \Lambda^{\alpha,\beta}(\tau))d\tau \\ &\leq (1-\alpha)\beta t^{\beta-1}H(t, r(t)) + \alpha\beta \int_{t_0}^t \tau^{\beta-1}H(\tau, r(\tau))d\tau \\ &\leq r(t)\end{aligned}\tag{164}$$

But $r(t) = 0$ on $[t_0, t_0 + \lambda]$, therefore

$$\Lambda^{\alpha,\beta}(t) = 0, \forall t \in [t_0, t_0 + \lambda].\tag{165}$$

We write

$$\begin{aligned}|\Lambda^{\alpha,\beta}(t_1) - \Lambda^{\alpha,\beta}(t_2)| &\leq 2(1-\alpha)\beta |t_1^\beta - t_2^\beta| M_f + 2\alpha |t_1^\beta - t_2^\beta| M_f \\ &\leq 2M_f((1-\alpha)\beta + \alpha) |t_1^\beta - t_2^\beta|.\end{aligned}\tag{166}$$

We have that the function t^β is differentiable on $[t_0, t_0 + \lambda]$, therefore, by the mean value theorem, $\xi \in [t_1, t_2]$ such that

$$\beta\xi^{\beta-1} |t_1 - t_2| = |t_1^\beta - t_2^\beta|,\tag{167}$$

therefore

$$|\Lambda^{\alpha,\beta}(t_1) - \Lambda^{\alpha,\beta}(t_2)| \leq 2M_f((1-\alpha)\beta + \alpha)\beta\xi^{\beta-1} |t_1 - t_2|.\tag{168}$$

We write

$$|\Lambda^{\alpha,\beta}(t_1) - \Lambda^{\alpha,\beta}(t_2)| < \varepsilon, \delta < \frac{\varepsilon}{2M_f((1-\alpha)\beta + \alpha)\beta\xi^{\beta-1}}\tag{169}$$

which shows that $\Lambda^{\alpha,\beta}$ is continuous.

By the theorem hypothesis, we have that

$$\begin{aligned}|y_{n+1}(t) - y_n(t)| &\leq (1-\alpha)\beta t^{\beta-1}H(t, |y_n(t) - y_{n-1}(t)|) \\ &\quad + \alpha\beta \int_{t_0}^t \tau^{\beta-1}H(\tau, |y_n(\tau) - y_{n-1}(\tau)|)d\tau.\end{aligned}\tag{170}$$

We reach the conclusion following the routine presented earlier.

4.3 Convergence of successive approximation for fractal-fractional derivative with power-law kernel

We consider now the case with power-law kernel. In this case, we have

$$\lambda = \min \{a, \gamma\} \quad (171)$$

where γ is obtained as

$$\begin{aligned} |y(t)| &< \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y(\tau))| d\tau \\ &< \frac{\beta}{\Gamma(\alpha)} M_f t^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \\ &< \frac{\beta}{\Gamma(\alpha)} M_f a^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \\ &< \frac{\beta}{\Gamma(\alpha)} M_f a^{\alpha+\beta-1} \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) < b. \end{aligned} \quad (172)$$

Then, we get

$$a < \left(\frac{\Gamma(\alpha) b}{\beta M_f \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right)} \right)^{\frac{1}{\alpha+\beta-1}}. \quad (173)$$

Then, we get

$$\lambda = \min \left\{ a, \left(\frac{\Gamma(\alpha) b}{\beta M_f \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right)} \right)^{\frac{1}{\alpha+\beta-1}} \right\}. \quad (174)$$

Then, we put

$$y_{n+1}(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} f(\tau, y_n(\tau)) d\tau. \quad (175)$$

$y_{n+1}(t)$ exists on $[t_0, t_0 + \lambda]$, since $f(t, y(t))$ is continuous on $[t_0, t_0 + \lambda]$. Also, $y_{n+1}(t)$ converges uniformly due to the properties of $f(t, y(t))$.

Proof. We write

$$|y_{n+1}(t)| \leq \frac{\beta}{\Gamma(\alpha)} M_f a^{\alpha+\beta-1} \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) < b. \quad (176)$$

Therefore $\forall n = 0, 1, 2, 3, \dots$

$$|y_{n+1}(t)| < b. \quad (177)$$

The next step is to define $u_n(t)$ as we did before. In this case, we put

$$u_0(t) = \frac{\beta}{\Gamma(\alpha)} M_f t^{\alpha+\beta-1} \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \quad (178)$$

and

$$u_{n+1}(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} H(\tau, u_n(\tau)) d\tau, \quad t_0 \leq t \leq t_0 + \lambda. \quad (179)$$

For $n = 0$, we have

$$\begin{aligned} |y_1(t)| &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y_0(\tau))| d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)} M_f t^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \\ &= u_0(t). \end{aligned} \quad (180)$$

We assume that $\forall n$ fixed

$$|y_n(t) - y_{n-1}(t)| \leq u_{n-1}(t). \quad (181)$$

We shall prove that this is also true for $n+1$

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |f(\tau, y_n(\tau)) - f(\tau, y_{n-1}(\tau))| d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |H(\tau, u_{n-1}(\tau))| d\tau \\ &\leq u_n(t). \end{aligned} \quad (182)$$

We let $1 \leq n \leq m$, using previous results, we have that

$$\begin{aligned} |y_n(t) - y_m(t)| &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |H(\tau, u_{n-1}(\tau))| d\tau \\ &\quad + \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |H(\tau, u_{m-1}(\tau))| d\tau \\ &\quad + \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} H(\tau, |u_n(\tau) - u_m(\tau)|) d\tau. \end{aligned} \quad (183)$$

We have that

$$u_{n+1}(t) < u_n(t), \forall n. \quad (184)$$

Therefore

$$\begin{aligned} |y_n(t) - y_m(t)| &\leq \frac{2\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} |H(\tau, u_{n-1}(\tau))| d\tau \\ &\quad + \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} H(\tau, |y_n(\tau) - y_m(\tau)|) d\tau. \end{aligned} \quad (185)$$

Again, we have

$$|y_n(t) - y_m(t)| \leq r_n(t), \quad t_0 \leq t \leq t_0 + \lambda. \quad (186)$$

The conclusion is reached following the routine presented before.

Alternatively, we have

$$\Lambda_1^{\alpha, \beta}(t) = \limsup_{n \rightarrow \infty} |y_n(t) - y_{n-1}(t)|. \quad (187)$$

Then, we write

$$\begin{aligned} |y_{n+1}(t_1) - y_n(t_1)| &\leq \limsup_{n \rightarrow \infty} |y_n(t) - y_{n-1}(t)| \\ &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t_1 - \tau)^{\alpha-1} |f(\tau, y_n(\tau)) - f(\tau, y_{n-1}(\tau))| d\tau \\ &\leq \frac{2\beta}{\Gamma(\alpha)} M_f t_1^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right) \end{aligned} \quad (188)$$

and

$$|y_n(t_2) - y_{n-1}(t_2)| \leq \frac{2\beta}{\Gamma(\alpha)} M_f t_2^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t_2}, \beta, \alpha\right) \right). \quad (189)$$

Therefore,

$$\begin{aligned} \Lambda_1^{\alpha, \beta}(t_1) &\leq \frac{2\beta}{\Gamma(\alpha)} M_f t_1^{\alpha+\beta} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right), \\ \Lambda_1^{\alpha, \beta}(t_2) &\leq \frac{2\beta}{\Gamma(\alpha)} M_f t_2^{\alpha+\beta} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t_2}, \beta, \alpha\right) \right). \end{aligned} \quad (190)$$

Substracting $\Lambda_1^{\alpha, \beta}(t_1)$ and $\Lambda_1^{\alpha, \beta}(t_2)$ yields

$$\begin{aligned} \left| \Lambda_1^{\alpha, \beta}(t_1) - \Lambda_1^{\alpha, \beta}(t_2) \right| &\leq \frac{2\beta}{\Gamma(\alpha)} M_f \left| t_1^{\alpha+\beta} - t_2^{\alpha+\beta} \right| \left(B\left(\frac{t_0}{t_2}, \beta, \alpha\right) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right) \\ &\leq \frac{2\beta}{\Gamma(\alpha)} M_f \left| t_1^{\alpha+\beta} - t_2^{\alpha+\beta} \right| M. \end{aligned} \quad (191)$$

We know that $t^{\alpha+\beta}$ is differentiable on $[t_0, t_0 + \lambda]$, $\exists \xi \in [t_1, t_2]$ such that by the mean value theorem, we have

$$(\alpha + \beta) \xi^{\alpha+\beta-1} |t_1 - t_2| = \left| t_1^{\alpha+\beta} - t_2^{\alpha+\beta} \right|. \quad (192)$$

Therefore, we have

$$\left| \Lambda^{\alpha,\beta}(t_1) - \Lambda^{\alpha,\beta}(t_2) \right| \leq \frac{2\beta}{\Gamma(\alpha)} M_f (\alpha + \beta) \xi^{\alpha+\beta-1} |t_1 - t_2|. \quad (193)$$

Thus

$$\left| \Lambda^{\alpha,\beta}(t_1) - \Lambda^{\alpha,\beta}(t_2) \right| < \varepsilon, \quad \delta < \frac{\Gamma(\alpha)}{2\beta M_f (\alpha + \beta) \xi^{\alpha+\beta-1}} \quad (194)$$

which shows that $\Lambda^{\alpha,\beta}(t)$ is equicontinuous, therefore continuous. Following the routine presented before, we have that

$$\begin{aligned} \Lambda^{\alpha,\beta}(t) &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} H(\tau, \Lambda^{\alpha,\beta}(\tau)) (t - \tau)^{\alpha-1} d\tau \\ &\leq \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} H(\tau, r(\tau)) (t - \tau)^{\alpha-1} d\tau \\ &\leq r(t). \end{aligned} \quad (195)$$

But $r(t) = 0, \forall t \in [t_0, t_0 + \lambda]$, thus

$$\Lambda^{\alpha,\beta}(t) = 0, \forall t \in [t_0, t_0 + \lambda]. \quad (196)$$

4.4 Convergence of successive approximation for fractal-fractional derivative with Mittag-Leffler kernel

We now present the same procedure when the fractal-fractional differential equation is with the generalized Mittag-Leffler kernel. In this case, we first find λ as follows

$$\begin{aligned} |y(t)| &\leq (1 - \alpha) \beta t^{\beta-1} M_f + \frac{\alpha\beta}{\Gamma(\alpha)} M_f t^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \\ &\leq (1 - \alpha) \beta a^\alpha M_f + \frac{\alpha\beta}{\Gamma(\alpha)} M_f a^{\alpha+\beta-1} \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right). \end{aligned} \quad (197)$$

Take

$$\bar{a} = \max \{ a^\alpha, a^{\alpha+\beta-1} \}. \quad (198)$$

Then, we write

$$|y(t)| \leq \left\{ (1 - \alpha) \beta M_f + \frac{\alpha\beta}{\Gamma(\alpha)} M_f \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) \right\} \bar{a}. \quad (199)$$

Therefore

$$\lambda = \min \left\{ a, \left(\frac{b\Gamma(\alpha)}{\Gamma(\alpha)(1-\alpha)\beta M_f + \alpha\beta M_f \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right)} \right)^{\frac{1}{\gamma}} \right\}. \quad (200)$$

As presented before, we shall have due to the property of f on R that

$$|y_{n+1}(t)| \leq (1-\alpha)\beta a^\gamma M_f + \frac{\alpha\beta}{\Gamma(\alpha)} a^\gamma M_f \max_{t \in [t_0, t_0+a]} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right) < b. \quad (201)$$

In this case, u_n is constructed as follows

$$u_0(t) = (1-\alpha)\beta t^{\beta-1} M_f + \frac{\alpha\beta}{\Gamma(\alpha)} M_f t^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t}, \beta, \alpha\right) \right). \quad (202)$$

Then

$$u_{n+1}(t) = (1-\alpha)\beta t^{\beta-1} H(t, u_{n+1}(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} H(\tau, u_n(\tau)) d\tau. \quad (203)$$

We can easily establish that

$$|y_{n+1}(t) - y_n(t)| \leq u_n(t). \quad (204)$$

The rest of the proof will be similar to that of exponential decay. For the alternative proof, we only show that

$$\begin{aligned} |y_n(t_1) - y_{n-1}(t_1)| &\leq (1-\alpha)\beta t_1^{\beta-1} |f(t_1, y_n(t_1)) - f(t_1, y_{n-1}(t_1))| \\ &\quad + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^{t_1} \tau^{\beta-1} (t_1 - \tau)^{\alpha-1} |f(\tau, y_{n-1}(\tau)) - f(\tau, y_{n-2}(\tau))| d\tau \\ &\leq 2\beta(1-\alpha) M_f t_1^{\beta-1} + \frac{2\alpha\beta}{\Gamma(\alpha)} M_f t_1^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right) \end{aligned} \quad (205)$$

and

$$|y_n(t_2) - y_{n-1}(t_2)| \leq 2\beta(1-\alpha) M_f t_2^{\beta-1} + \frac{2\alpha\beta}{\Gamma(\alpha)} M_f t_2^{\alpha+\beta-1} \left(B(\beta, \alpha) - B\left(\frac{t_0}{t_2}, \beta, \alpha\right) \right). \quad (206)$$

Thus, we write

$$\begin{aligned} \left| \Lambda_2^{\alpha, \beta}(t_1) - \Lambda_2^{\alpha, \beta}(t_2) \right| &\leq 2M_f(1-\alpha)\beta \left| t_1^{\beta-1} - t_2^{\beta-1} \right| \\ &\quad + \frac{2\alpha\beta}{\Gamma(\alpha)} M_f \left| t_1^{\alpha+\beta-1} - t_2^{\alpha+\beta-1} \right| \left(B\left(\frac{t_0}{t_2}, \beta, \alpha\right) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right) \\ &\leq 2M_f(1-\alpha)\beta \left| t_1^{\alpha+\beta-1} - t_2^{\alpha+\beta-1} \right| \\ &\quad + \frac{2\alpha\beta}{\Gamma(\alpha)} M_f \left| t_1^{\alpha+\beta-1} - t_2^{\alpha+\beta-1} \right| \left(B\left(\frac{t_0}{t_2}, \beta, \alpha\right) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right). \end{aligned} \quad (207)$$

By the mean value theorem, $\exists \xi \in [t_1, t_2]$ such that

$$(\alpha + \beta - 1) \xi^{\alpha+\beta-2} |t_1 - t_2| = \left| t_1^{\alpha+\beta} - t_2^{\alpha+\beta} \right|. \quad (208)$$

Therefore

$$\begin{aligned} \left| \Lambda_2^{\alpha,\beta}(t_1) - \Lambda_2^{\alpha,\beta}(t_2) \right| &\leq \left\{ 2M_f(1-\alpha)\beta + \frac{2\alpha\beta}{\Gamma(\alpha)} M_f \left(B\left(\frac{t_0}{t_2}, \beta, \alpha\right) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right) \right\} \\ &\quad \times (\alpha + \beta - 1) \xi^{\alpha+\beta-2} |t_1 - t_2| < \varepsilon. \end{aligned} \quad (209)$$

We write

$$\delta < \frac{\varepsilon \Gamma(\alpha)}{2M_f(\Gamma(\alpha)(1-\alpha)\beta + \alpha\beta) \left(B\left(\frac{t_0}{t_2}, \beta, \alpha\right) - B\left(\frac{t_0}{t_1}, \beta, \alpha\right) \right)} \quad (210)$$

which shows that is equicontinuous thus continuous on $[t_0, t_0 + \lambda]$.

Finally, we can establish that

$$\Lambda_2^{\alpha,\beta}(t) \leq (1-\alpha)\beta t^{\beta-1} H(t, \Lambda_2^{\alpha,\beta}(t)) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_{t_0}^t \tau^{\beta-1} (t-\tau)^{\alpha-1} H(\tau, \Lambda_2^{\alpha,\beta}(\tau)) d\tau. \quad (211)$$

Thus, we have

$$\Lambda_2^{\alpha,\beta}(t) \leq r_2^{\alpha,\beta}(t) = 0, \forall t \in [t_0, t_0 + \lambda] \quad (212)$$

which completes the proof.

5 Conclusion

In the last five years, fractal-fractional differential and integral operators have provided new avenues for theoretical, methodological, and application-based research. They have added new classes for differential and integral equations, in particular. In all areas of science, technology, and engineering, complicated processes can be modeled using these equations, which can also be explored theoretically. Analyses and predictions demand modeling and solutions. However, these equations are frequently impossible to solve with the current analytical methods. We have shown many strategies to guarantee existence and unique solutions to these equations in order to enable the use of numerical technology. Some significant inequalities were also established.

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