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SYMMETRIC FORMULATIONS FOR MODAL ANALYSIS OF INTERNAL FLUID STRUCTURE SYSTEMS.

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ABSTRACT

This paper deals with appropriate computational methods for modal analysis of elastic structures containing an inviscid fluid (gas or liquid). These methods, based on Ritz-Galerkin projection using appropriate functional basis, allow us to construct *reduced models* expressed in terms of physical displacement vector field u in the structure, and generalized coordinates vector \mathbf{r} describing the behavior of the fluid. Those reduced models lead to symmetric generalized eigenvalue matrix system involving a reduced number of degrees of freedom for the fluid. More precisely, we construct symmetric matrix models of the fluid considered as a subsystem, by considering the response of the fluid to a prescribed normal displacement of the fluid-structure interface. Two distinct situations are analyzed, namely linear vibrations of an elastic structure completely filled with a compressible gas or liquid and linear vibrations of an elastic structure containing an incompressible liquid with free surface effects due to gravity. The first case is a *structural acoustic* problem with modal interaction between *structural modes* with *acoustic modes* in rigid motionless cavity. Wall impedance can also be easily introduced in order to take into account fluid-structure interface dissipation, for further forced response studies. The second case is a *hydroelastic-sloshing* problem with modal interaction between *incompressible hydroelastic structural*

modes with *incompressible liquid sloshing modes* in rigid motionless cavity, involving an *elastogravity operator* related to the wall normal displacement of the fluid-structure interface. For the construction of reduced models, the static behavior at zero frequency play an important role. This is why we start from “well-posed” variational formulations of the problem, in the sense that zero-frequency behavior must be well retrieved in the equations. It should be noted that the so-called “quasi-static correction” term plays a fundamental role in the Ritz-Galerkin procedure (error truncation). The general methodology corresponds to dynamic substructuring procedures adapted to fluid-structure modal analysis. For general presentations of computational methods using appropriate finite element and dynamic substructuring procedures applied to modal analysis of elastic structures containing inviscid fluids (sloshing, hydroelasticity and structural-acoustics), we refer the reader to Morand and Ohayon (1995).

1 STRUCTURAL ACOUSTIC PROBLEM

We consider here the harmonic vibrations of an elastic structure completely filled with an homogeneous, inviscid and compressible fluid considered as weightless (linear acoustic fluid). Among the applications, we find, for instance, the important problem of acoustic comfort of “passengers” related

to the vibroacoustic environment such in launcher payloads, aircraft, automobiles. The particular case of a compressible liquid with a free surface, neglecting gravity effects, is also considered (for applications involving compressibility effects in huge cryogenic tanks).

After an original derivation of the linearized equations of the fluid-structure coupled system, introducing an appropriate linear constraint in order to obtain a “well-posed” problem at zero frequency, we discuss the basic ideas leading to a reduced model of the “fluid subsystem”. For this purpose, acoustic modes in rigid motionless cavity are introduced as Ritz projection vector basis, including mandatory zero-frequency (static) solution of the coupled system. Let us recall that the internal fluid-structure system is characterized by a strong resonant behavior. Therefore, a direct finite element description can be costly. That is why, starting from one of the possible variational formulations of the problem, *reduced symmetric matrix models* are constructed.

2 STRUCTURAL ACOUSTIC EQUATIONS

2.1 Structure Subjected to a Fluid Pressure Loading

We consider an elastic structure occupying the domain Ω_S at equilibrium. The internal fluid-structure interface is denoted as Σ . The angular frequency is denoted as ω . The chosen unknown field in the structure domain Ω_S is the displacement field u . The linearized deformation tensor is denoted as $\epsilon_{ij}(u)$ and the corresponding stress tensor is denoted as $\sigma_{ij}(u)$. We denote by ρ_S the constant mass density at equilibrium and by n the unit normal, external to the structure domain Ω_S . Let δu be the test function, associated to u , belonging to the admissible space \mathcal{C}_u .

The weak variational formulation describing the response of the structure Ω_S to given harmonic forces of amplitude F^d on the external structure boundary $\partial\Omega_S \setminus \Sigma$, and to fluid pressure field p acting on the internal fluid-structure interface Σ is written as follows.

For all real ω and $\forall \delta u \in \mathcal{C}_u$, find $u \in \mathcal{C}_u$ such that

$$\begin{aligned} \tilde{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx \\ - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \end{aligned} \quad (1)$$

in which

$$\tilde{k}(u, \delta u) = k(u, \delta u) + k_G(u, \delta u) + k_{P_0}(u, \delta u) \quad (2)$$

and where $k(u, \delta u)$ is the mechanical elastic stiffness such that

$$k(u, \delta u) = \int_{\Omega_S} \sigma_{ij}(u) \epsilon_{ij}(\delta u) \, dx \quad (3)$$

and where $k_G(u, \delta u)$ and $k_{P_0}(u, \delta u)$ are such that

$$\begin{aligned} k_G(u, \delta u) = \int_{\Omega_S} \sigma_{ij}^0 u_{l,i} \delta u_{l,j} \, dx \\ k_{P_0} = \int_{\Sigma} P_0 n_1(u) \cdot \delta u \, d\sigma \end{aligned} \quad (4)$$

In equations (4) and (5), $k_G(u, \delta u)$ represents the classical structural prestress geometric symmetric bilinear form in which σ_{ij}^0 denotes the prestress tensor, and $k_{P_0}(u, \delta u)$ represents an additional prestress symmetric bilinear form due to rotation of normal n , in which P_0 denotes the initial pressure existing in the reference equilibrium configuration. Finally, $n_1(u)$ represents the variation of normal n between the reference configuration and the actual configuration.

2.2 Fluid Subjected to a Wall Normal Displacement

Since the fluid is inviscid, instead of describing the small motion of the fluid by a fluid displacement vector field u_F (see Bermudez and Rodriguez (1994)), which requires an appropriate discretization of the fluid irrotationality constraint $\text{curl } u_F = 0$, we will use the pressure scalar field p . It should be noted that, the small movements corresponding to $\omega \neq 0$ are obviously irrotational, but, in the static limit case, i.e. at zero frequency, we consider only fluids which exhibit a physical irrotational behavior.

Let us denote by c the (constant) sound speed in the fluid, and by ρ_F , the (constant) mass density of the fluid at rest. We denote as Ω_F the domain occupied by the fluid at rest (which is taken

as the equilibrium state). The local equations describing the harmonic response of the fluid to a prescribed arbitrary normal displacement $u.n$ of the fluid-structure interface Σ are such that

$$\nabla p - \rho_F \omega^2 u_F = 0 \quad |\Omega_F \quad (5)$$

$$p = -\rho_F c^2 \nabla \cdot u_f \quad |\Omega_F \quad (6)$$

$$u_F \cdot n = u.n \quad |\Sigma \quad (7)$$

$$\text{curl } u_F = 0 \quad |\Omega_F \quad (8)$$

Equation (5) corresponds to the linearized Euler equation in the fluid. Equation (6) corresponds to the constitutive equation of the barotropic fluid. Equation (7) corresponds to the wall slipping condition. Equation (8) corresponds to the irrotationality condition, only necessary in order to ensure that when $\omega \rightarrow 0$, u_F tends to static irrotational motion, which corresponds to the hypothesis that for $\omega = 0$, we only consider irrotational motions (for simply connected fluid domain).

2.3 Relation between static pressure p^s and $u.n$

For $\omega = 0$, equations (6) and (7) lead to the constant pressure static field which is related to the normal wall displacement by the relation (see Ohayon and Felippa, (1990), Morand and Ohayon, (1995))

$$p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} u.n \, d\sigma \quad (9)$$

in which $|\Omega_F|$ denotes the measure of the volume occupied by domain Ω_F .

2.4 Equations in terms of p and $u.n$

The elimination of u_F between equations (5), (6), (7) and (8) leads to

$$\Delta p + \frac{\omega^2}{c^2} p = 0 \quad |\Omega_F \quad (10)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 u.n \quad |\Sigma \quad (11)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u.n \, d\sigma = 0 \quad (12)$$

Equation (10) is the classical Helmholtz equation expressed in terms of p . Equation (11) corresponds to the kinematic condition defined by equation (7). The linear constraint defined by equation (12) corresponds to the global mass conservation which ensures that the boundary problem defined by equations (10) to (11) is equivalent to the problem defined by equations (5) to (8). In the absence of the condition defined by equation (12), we would obtain a boundary value problem in terms of p which is not valid for $\omega = 0$ and which does not allow us to retrieve the value of p^s given by equation (9). This condition is usually omitted in literature.

2.5 Variational formulation in terms of (u, p)

Let δp be the test function, associated to p , belonging to the admissible space \mathcal{C}_p . The weak variational formulation corresponding to equations (10) to (12) is obtained by the usual test-function method using Green's formula. The weak variational formulation corresponding to the modal analysis of the structural acoustic problem is then obtained by setting $F^d = 0$ in equation (1). The eigenvalue structural acoustic variational formulation is then stated as follows. Find ω^2 and $u \in \mathcal{C}_u$ and $p \in \mathcal{C}_p$, such that for all $\delta u \in \mathcal{C}_u$ and $\delta p \in \mathcal{C}_p$, we have

$$\tilde{k}(u, \delta u) - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx \quad (13)$$

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx + \omega^2 \int_{\Sigma} u.n \delta p \, d\sigma \quad (14)$$

with the constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, dx + \int_{\Sigma} u.n \, d\sigma = 0 \quad (15)$$

The variational formulation defined by equations (13), (14) and (15) is written here for the first time (to author knowledge). In effect, usually, only equations (13) and (14) are written, and as pointed

out above, are not valid for $\omega = 0$. In the case of a direct finite element discretization of equations (13), (14) and (15), we obtain a matrix eigenvalue system of the type $\mathbf{A} \mathbf{Y} = \omega^2 \mathbf{B} \mathbf{Y}$, in which \mathbf{A} and \mathbf{B} are not symmetric. That is why various symmetric formulations using for the fluid pressure field p and displacement potential φ , defined up to an additive constant and such that $u_F = \nabla \varphi$, have been derived. The resulting *symmetric* formulations are then obtained by elimination of p or φ (see, Felippa and Ohayon (1990), Liu and Uras (1988), or for detailed complete analysis, Morand and Ohayon (chapters 8 and 9, 1995)). In the present case, we are not considering a direct finite element approach of the variational formulation defined by equations (13), (14) and (15).

2.6 Symmetric Reduced Model

Let us consider the following two basic problems. The first one corresponds to the acoustic modes in rigid motionless cavity and is obtained by setting $u = 0$ into equations (14) and (15). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace \mathcal{C}_p^* of \mathcal{C}_p

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Omega_F} p \, dx = 0 \right\} \quad , \quad (16)$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathcal{C}_p^*$ such that, for all $\delta p \in \mathcal{C}_p^*$, we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, dx = \omega^2 \frac{1}{\rho_F c^2} \int_{\Omega_F} p \delta p \, dx \quad (17)$$

with the constraint

$$\int_{\Omega_F} p \, dx = 0 \quad (18)$$

It should be noted that, in practice, if the constraint condition (18) is “omitted”, we only add a first *non physical* zero frequency constant pressure mode, the other modes corresponding to $\omega \neq 0$ remaining the same as those defined by equations (17) and (18). This zero frequency mode must not be retained in any Ritz-Galerkin projection analysis. In addition, we have the following orthogonality conditions

$$\begin{aligned} \frac{1}{\rho_F c^2} \int_{\Omega_F} p_\alpha p_\beta \, dx &= \mu_\alpha \delta_{\alpha\beta} \quad , \\ \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_\alpha \cdot \nabla p_\beta \, dx &= \mu_\alpha \omega_\alpha^2 \delta_{\alpha\beta} \end{aligned} \quad (19)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u \cdot n$. The solution, denoted as $p^s(u \cdot n)$, is given by equation (9). For any deformation $u \cdot n$ of the fluid-structure interface, $p^s(u \cdot n)$ belongs to a subset of \mathcal{C}_p , denoted as $\mathcal{C}^{u \cdot n}$

$$\mathcal{C}^{u \cdot n} = \left\{ p^s \in \mathcal{C}_p; p^s = -\frac{\rho_F c^2}{|\Omega_F|} \int_{\Sigma} u \cdot n \, d\sigma \right\} \quad (20)$$

In the variational formulation defined by equations (13), (14) and (15), p is searched under the form

$$p = p^s(u \cdot n) + \sum_{\alpha=1}^{N_p} r_\alpha p_\alpha \quad (21)$$

in which N_p denotes the number of retained acoustic modes. The decomposition (21) is unique. In addition, it should be noted that, since each eigenvector p_α corresponding to $\omega_\alpha \neq 0$, verifies the constraint defined by equation (18). Consequently, using equation (9), we deduce that p and $u \cdot n$ satisfy the constraint defined by equation (18). The decomposition defined by equation (21) corresponds to a decomposition of the admissible class \mathcal{C}_p into the direct sum of the admissible classes defined respectively by equations (20) and (16)

$$\mathcal{C}_p = \mathcal{C}^{u \cdot n} \oplus \mathcal{C}_p^* \quad (22)$$

Following equation (21), the test function δp is then searched under the following form

$$\delta p = p^s(\delta u \cdot n) + \sum_{\alpha=1}^{N_p} \delta r_\alpha p_\alpha \quad (23)$$

Variational formulation in δu defined by equation (13) becomes

$$\begin{aligned} \tilde{k}(u, \delta u) + k^s(u, \delta u) - \sum_{\alpha=1}^{N_p} r_\alpha \int_{\Sigma} p_\alpha n \cdot \delta u \, d\sigma = \\ \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx \end{aligned} \quad (24)$$

in which $\tilde{k}(u, \delta u)$ is defined by equation (2) and $k^s(u, \delta u)$ is such that

$$k^s(u, \delta u) = \frac{\rho_F c^2}{|\Omega_F|} \left(\int_{\Sigma} u \cdot n \, d\sigma \right) \left(\int_{\Sigma} \delta u \cdot n \, d\sigma \right) \quad (25)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of equation (24) can be written as

$$[\tilde{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_{\alpha} r_{\alpha} = \omega^2 \mathbf{M} \mathbf{U} \quad (26)$$

in which symmetric matrices $\tilde{\mathbf{K}}$ and \mathbf{K}^s correspond to finite element discretization of stiffness symmetric bilinear forms defined by equations (2), (3), (4) and (25) respectively. In equation (26), \mathbf{M} denotes the structural symmetric mass matrix and rectangular coupling matrix \mathbf{C}_{α} corresponds to the discretization of the coupling fluid-structure contribution $\int_{\sigma} p \delta u \cdot n \, d\sigma$. The discretized form of equation (14) in δp can then be written as

$$\omega_{\alpha}^2 \mu_{\alpha} r_{\alpha} = \omega^2 \mu_{\alpha} r_{\alpha} + \omega^2 \mathbf{C}_{\alpha}^T \mathbf{U} \quad (27)$$

From equations (26) and (27), we obtain the following *symmetric matrix reduced model*

$$\begin{pmatrix} \mathbf{K}^{\text{tot}} & \mathbf{0} \\ \mathbf{0} & \text{Diag} \mu_{\alpha} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} = \omega^2 \begin{pmatrix} \mathbf{M}^{\text{tot}} & \mathbf{D} \\ \mathbf{D}^T & \text{Diag} (\mu_{\alpha} / \omega_{\alpha}^2) \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{r} \end{pmatrix} \quad (28)$$

in which \mathbf{r} denotes the vector of N generalized coordinates r_{α} , with $1 \leq \alpha \leq N_p$, and

$$\mathbf{K}^{\text{tot}} = \tilde{\mathbf{K}} + \mathbf{K}^s \quad (29)$$

$$\mathbf{M}^{\text{tot}} = \mathbf{M} + \sum_{\alpha=1}^{N_p} \frac{1}{\omega_{\alpha}^2 \mu_{\alpha}} \mathbf{C}_{\alpha} \mathbf{C}_{\alpha}^T \quad (30)$$

$$\mathbf{D}_{\alpha} = \sum_{\alpha=1}^{N_p} \frac{1}{\omega_{\alpha}^2} \mathbf{C}_{\alpha} \quad (31)$$

Further diagonalization of equation (28) implies a projection of \mathbf{U} on the solutions of the following eigenvalue problem

$$\mathbf{K}^{\text{tot}} \mathbf{U}_{\beta} = \lambda_{\beta} \mathbf{M}^{\text{tot}} \mathbf{U}_{\beta} \quad (32)$$

Setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_{\beta} \mathbf{U}_{\beta} \quad (33)$$

in which q_{β} are the generalized coordinates describing the structure. Using the orthogonality conditions associated with the solutions of equation (32), i.e. $\mathbf{U}_{\beta'}^T \mathbf{M}^{\text{tot}} \mathbf{U}_{\beta} = \mu_{\beta}^s \delta_{\beta\beta'}$ and $\mathbf{U}_{\beta'}^T \mathbf{K}^{\text{tot}} \mathbf{U}_{\beta} = \mu_{\beta}^s \lambda_{\beta} \delta_{\beta\beta'}$, equation (28) becomes

$$\begin{pmatrix} \text{Diag} \lambda_{\beta} & \mathbf{0} \\ \mathbf{0} & \text{Diag} \omega_{\alpha}^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \omega^2 \begin{pmatrix} I_{N_u} & [\mathbf{C}_{\beta\alpha}] \\ [\mathbf{C}_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} \quad (34)$$

It should be noted that *in vacuo structural modes* can also be used (instead of those defined by equation (32)), but the resulting matrix system would not be diagonal with respect to \mathbf{U} .

2.7 Wall Impedance Condition

Referring to Kehr-Candille and Ohayon (1992), wall impedance condition corresponds to a particular fluid-structure interface modeling. This interface is considered as a third medium with infinitesimal thickness, without mass, and with the following constitutive equation

$$p = j\omega Z(\omega)(u \cdot n - u_F \cdot n) \quad (35)$$

in which $Z(\omega)$ denotes a complex impedance. Equations (7) and (11) must be replaced by equation (35), using $\partial p / \partial n = \rho_F \omega^2 u_F \cdot n$.

2.8 Case of a liquid with a free surface

Let us consider a liquid with a free surface at rest denoted as Γ , If we neglect gravity effects, the boundary condition on Γ is such that

$$p = 0 \mid_{\Gamma} \quad (36)$$

In this case, constraint condition (12) (or (15)) is replaced by equation (36). Equation (9) is replaced by $p^s = 0$. Admissible space defined by equation (16) becomes $\mathcal{C}_p^* = \{p \in \mathcal{C}_p; p = 0\}$.

In this case, the static problem defined in Section 2.3 leads to a zero pressure field.

3 HYDROELASTIC-SLOSHING PROBLEM

We consider the harmonic vibrations of an elastic structure partially filled with an homogeneous, inviscid and incompressible liquid, taking into account gravity effects on the free surface Γ . After an original derivation of the linearized equations of the fluid-structure coupled problem, introducing an appropriate linear constraint in order to obtain a “well-posed” problem at zero frequency, we discuss the main ideas leading to a reduced model of the “liquid subsystem”. For this purpose, sloshing modes in rigid motionless cavity are introduced as Ritz projection vector basis, including mandatory zero-frequency (static) solution of the coupled system. Let us recall that the internal fluid-structure system is characterized by a strong resonant behavior. Therefore, a direct finite element description can be costly. That is why, starting from one of the possible variational formulations of the problem, reduced symmetric matrix models of this system are constructed.

4 HYDROELASTIC-SLOSHING EQUATIONS

4.1 Structure Subjected to a Fluid Pressure Loading

The notations are the same that those defined in Section 2.1 The weak variational formulation describing the response of the structure Ω_S to given variation F^d of the applied forces with respect to the equilibrium state on the external structure boundary $\partial\Omega_S \setminus \Sigma$, and to fluid pressure field p acting on the internal fluid-structure interface Σ is written as follows.

For all real ω and $\forall \delta u \in \mathcal{C}_u$, find $u \in \mathcal{C}_u$ such that

$$\widehat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx - \int_{\Sigma} p n \cdot \delta u \, d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d \cdot \delta u \, d\sigma \quad (37)$$

in which

$$\widehat{k} = \widetilde{k} + k_{\Sigma} \quad (38)$$

In equation (38), $\widetilde{k}(u, \delta u)$ is defined by equation (2), and k_{Σ} is the *elastogravity* symmetric bilinear form such that (see Morand and Ohayon (1995))

$$k_{\Sigma}(u, \delta u) = -\frac{1}{2} \rho_F g \left\{ \int_{\Sigma} [zn_1(u) \cdot \delta u + u_z \delta u \cdot n] \, d\sigma + \int_{\Sigma} [zn_1(\delta u) \cdot u + \delta u_z u \cdot n] \, d\sigma \right\} \quad (39)$$

4.2 Fluid Subjected to a Wall Normal Displacement

We assume that the liquid is homogeneous, inviscid and incompressible. Free surface Γ is horizontal at equilibrium. We denote by z the external unit normal to Γ , and by g the gravity. The notations are similar to those of Section 2.2. The local equations describing the response of the fluid to a prescribed arbitrary normal displacement $u \cdot n$ of the fluid-structure interface Σ are such that

$$\nabla p - \rho_F \omega^2 u_F = 0 \quad |\Omega_F \quad (40)$$

$$\nabla \cdot u_f = 0 \quad |\Omega_F \quad (41)$$

$$u_F \cdot n = u \cdot n \quad |\Sigma \quad (42)$$

$$p = \rho_F g u_F \cdot n \quad |\Gamma \quad (43)$$

$$\text{curl } u_F = 0 \quad |\Omega_F \quad (44)$$

Equation (41) corresponds to the incompressibility condition. Equation (43) is the constitutive equation on the free surface Γ due to gravity effects.

4.3 Relation between static pressure p^s and $u \cdot n$

For $\omega = 0$, equations (41), (42) and (43) lead to the constant pressure static field which is related to the normal wall displacement by the relation (see Ohayon and Felippa (1990), Morand and Ohayon, (1995))

$$p^s = -\frac{\rho_F g}{|\Gamma|} \int_{\Sigma} u \cdot n \, d\sigma \quad (45)$$

in which $|\Gamma|$ denotes the measure of the area of free surface Γ .

4.4 Equations in terms of p and $u.n$

The elimination of u_F between equations (40) to (44) leads to

$$\Delta p = 0 \quad |\Omega_F \quad (46)$$

$$\frac{\partial p}{\partial n} = \rho_F \omega^2 u.n \quad |\Sigma \quad (47)$$

$$\frac{\partial p}{\partial z} = \frac{\omega^2}{g} p \quad |\Gamma \quad (48)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p d\sigma + \int_{\Sigma} u.n d\sigma = 0 \quad (49)$$

The linear constraint defined by equation (49) ensures that the boundary problem defined by equations (46) to (49) is equivalent to the problem defined by equations (40) to (44). This condition is usually omitted in literature.

4.5 Variational formulation in terms of (u, p)

Let δp be the test function, associated to p , belonging to the admissible space \mathcal{C}_p . The weak variational formulation corresponding to equations (46) to (49) is obtained by the usual test-function method using Green's formula. Recalling equation (37), the variational formulation of the *hydroelastic-sloshing* problem is then stated as follows. Find $u \in \mathcal{C}_u$ and $p \in \mathcal{C}_p$, such that for all $\delta u \in \mathcal{C}_u$ and $\delta p \in \mathcal{C}_p$, we have

$$\begin{aligned} \widehat{k}(u, \delta u) - \omega^2 \int_{\Omega_S} \rho_S u.\delta u dx \\ - \int_{\Sigma} p n.\delta u d\sigma = \int_{\partial\Omega_S \setminus \Sigma} F^d.\delta u d\sigma \end{aligned} \quad (50)$$

$$\begin{aligned} \frac{1}{\rho_F} \int_{\Omega_F} \nabla p.\nabla \delta p dx = \\ \frac{\omega^2}{\rho_F g} \int_{\Gamma} p \delta p dx + \omega^2 \int_{\Sigma} u.n \delta p d\sigma \end{aligned} \quad (51)$$

with the constraint

$$\frac{1}{\rho_F g} \int_{\Gamma} p d\sigma + \int_{\Sigma} u.n d\sigma = 0 \quad (52)$$

The variational formulation defined by equations (50), (51) and (52) is written here for the first time (to author knowledge).

4.6 Symmetric Reduced Model

Let us consider the following two basic problems. The first one corresponds to the sloshing modes in rigid motionless cavity and is obtained by setting $u = 0$ into equations (47) and (49). The calculation of these acoustic modes is generally done by using a finite element procedure. If we introduce the admissible subspace \mathcal{C}_p^* of \mathcal{C}_p

$$\mathcal{C}_p^* = \left\{ p \in \mathcal{C}_p; \int_{\Gamma} p d\sigma = 0 \right\} \quad , \quad (53)$$

the variational formulation of acoustic modes is stated as follows: find $\omega^2 > 0$ and $p \in \mathcal{C}_p^*$ such that, for all $\delta p \in \mathcal{C}_p^*$, we have

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p.\nabla \delta p dx = \omega^2 \frac{1}{\rho_F g} \int_{\Gamma} p \delta p d\sigma \quad (54)$$

with the constraint

$$\int_{\Gamma} p d\sigma = 0 \quad (55)$$

$$\begin{aligned} \frac{1}{\rho_F g} \int_{\Gamma} p_{\alpha} p_{\beta} d\sigma = \mu_{\alpha} \delta_{\alpha\beta} \quad , \\ \frac{1}{\rho_F} \int_{\Omega_F} \nabla p_{\alpha}.\nabla p_{\beta} dx = \mu_{\alpha} \omega^2 \delta_{\alpha\beta} \end{aligned} \quad (56)$$

The second basic problem corresponds to the *static* response of the fluid to a prescribed wall normal displacement $u.n$. The solution, denoted as $p^s(u.n)$, is given by equation (45). For any deformation $u.n$ of the fluid-structure interface, $p^s(u.n)$ belongs to a subset of \mathcal{C}_p , denoted as $\mathcal{C}^{u.n}$

$$\mathcal{C}^{u.n} = \left\{ p^s \in \mathcal{C}_p; p^s = -\frac{\rho_F g}{|\Gamma|} \int_{\Sigma} u.n d\sigma \right\} \quad (57)$$

In the variational formulation defined by equations (50) to (52), p is searched under the form

$$p = p^s(u.n) + \sum_{\alpha=1}^{N_p} r_{\alpha} p_{\alpha} \quad (58)$$

in which N_p denotes the number of retained sloshing modes. Fields p and $u.n$ satisfy the constraint defined by equation (52).

Variational formulation in δu defined by equation (50) becomes

$$\widehat{k}(u, \delta u) + k^s(u, \delta u) - \sum_{\alpha=1}^{N_p} r_\alpha \int_{\Sigma} p_\alpha n \cdot \delta u \, d\sigma = \omega^2 \int_{\Omega_S} \rho_S u \cdot \delta u \, dx \quad (59)$$

in which $\widehat{k}(u, \delta u)$ is defined by equation (38) and $k^s(u, \delta u)$ is such that

$$k^s(u, \delta u) = \frac{\rho_F g}{|\Gamma|} \left(\int_{\Sigma} u \cdot n \, d\sigma \right) \left(\int_{\Sigma} \delta u \cdot n \, d\sigma \right) \quad (60)$$

If we consider a finite element discretization of the structure, the corresponding discretized form of equation (60) can be written as

$$[\widehat{\mathbf{K}} + \mathbf{K}^s] \mathbf{U} - \sum_{\alpha=1}^n \mathbf{C}_\alpha r_\alpha - \omega^2 \mathbf{M} \mathbf{U} = \mathbf{F}^d \quad (61)$$

in which symmetric matrices $\widehat{\mathbf{K}}$ and \mathbf{K}^s correspond to finite element discretization of stiffness symmetric bilinear forms defined by equations (38) and (60) respectively. The discretized form of equation (51) in δp can then be written as

$$\omega_\alpha^2 \mu_\alpha r_\alpha = \omega^2 \mu_\alpha r_\alpha + \omega^2 \mathbf{C}_\alpha^T \mathbf{U} \quad (62)$$

From equations (61) and (62), we obtain a *symmetric matrix reduced model* whose expression is similar to the one given by expression (28). Similarly to Section 2.6, further diagonalization can be obtained by setting

$$\mathbf{U} = \sum_{\beta=1}^{N_u} q_\beta \mathbf{U}_\beta \quad (63)$$

in which q_β are the generalized coordinates describing the structure and \mathbf{U}_β are the eigenmodes of an eigenvalue problem similar to the one described by equation (32). We then obtain a similar matrix system than the one described by equation (34)

$$-\omega^2 \begin{pmatrix} \text{Diag} \lambda_\beta & 0 \\ 0 & \text{Diag} \omega_\alpha^2 \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} - \omega^2 \begin{pmatrix} I_{N_u} & [C_{\beta\alpha}] \\ [C_{\beta\alpha}]^T & I_{N_p} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathcal{F}^d \\ \mathbf{0} \end{pmatrix} \quad (64)$$

It should be noted that we can also use the *incompressible hydroelastic modes*, i.e. the modes of the coupled system constituted by the elastic structure containing an incompressible liquid, with $p = 0$ on Γ (through an *added mass operator*). In this case, the resulting matrix system is not completely diagonal with respect to \mathbf{U} variables (see Morand and Ohayon (1995)).

CONCLUSION

Using a new approach, we have presented convenient and versatile tools for low frequency modal analysis of elastic structures containing linear inviscid homogeneous fluids for *structural-acoustics* and *hydroelastic-sloshing* problems. Those formulations, together with dynamic substructuring techniques, lead to *symmetric reduced matrix systems* expressed in terms of generalized coordinates for the fluid (and if necessary for the structure).

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