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Trajectory Tracking in Linear Complementarity Systems with and without State-Jumps: A Passivity Approach

working preprint

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Abstract

This work is largely concerned with trajectory tracking in linear complementarity systems (LCS). The main analytical tool for stability analysis and control design is passivity and the associated linear matrix inequalities (LMIs). Cases with and without state-jumps, with and without parametric uncertainties, are analysed. Theoretical findings are illustrated with many examples from circuits with set-valued, nonsmooth electronic components, and networks with unilateral interactions.

1 Introduction

Complementarity dynamical systems are a class of nonsmooth nonlinear systems, that has received attention in the Automatic Control and Applied Mathematics literature because of their applications in many fields like electrical circuits with nonsmooth components, mechanical systems with unilateral contact, networks with unilateral interactions, economics with projected dynamical systems, genetics, traffic flow and neural networks, *etc.*, see [23, 13, 55]. Trajectory tracking in fully-actuated complementarity Lagrangian systems has been tackled in [66, 65, 67, 48, 47, 22, 21, 11, 69, 68, 61, 53, 75, 74]. The stabilization is studied in [27, 73, 7], the output regulation is analyzed in [82]. The Lyapunov stability of linear complementarity systems (LCS) has been tackled in [30, 27, 19, 18, 49]. Tracking in complementarity mechanical systems with state jumps is studied in [45, 10]. The trajectory tracking problem is solved in [66, 65, 67, 48, 47, 22, 21, 11, 69, 68, 61, 53, 75, 74] using different strategies. It is noteworthy that in most of these articles, the desired trajectory is suitably modified in the neighborhood of state jumps (impacts), and/or the tracking error dynamics 'stability is tailored to the problem. The tracking of cyclic trajectories which undergo constrained/unconstrained/impacting phases of motion (including multiple constraints and multiple impacts phenomena) is studied in [22, 21, 11, 69, 68] for n -degree-of-freedom (dof) systems. Basically, the controllers are extended passivity-based algorithms which switch between three sub-controllers (persistent contact with constant number of activated constraints, mode obtained from the deactivation of one or several constraints, impacting transition phase between two persistent contact modes when the number of activated constraints increases). The authors introduce the notion of weak stability, which extends Lyapunov stability by disregarding the Lyapunov function variation during the transition phases (nonmonotonic Lyapunov functions are used). Trajectory tracking in planar billiards is studied in [66, 65, 67, 48, 47]. The objective is to control a two-dof particle inside a closed compact domain when trajectories collide with the domain's boundary and never undergo persistent contact with it (a kind of vibro-impact systems [15, section 1.3.2]).

This article is dedicated to trajectory tracking in LCS, which make a class of complementarity systems different from Lagrangian complementarity systems studied in the above references. The main differences between Lagrangian systems with unilateral constraints and impacts, and LCS as in this paper, are that impacts in mechanical systems are state-dependent, see (134), and passivity does not hold when the complementarity variables (*i.e.*, the contact force multiplier and the gap

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function which usually is a nonlinear function of the position) are used as the input and output variables to define the supply rate (see Remark 14.1). In passive LCS, state jumps are essentially exogenous (*i.e.*, they are triggered by the signal $Fu(t)$ in the variable w in (2) below), and passivity holds between the complementarity variables. Moreover the complementarity variables are not of the same form in mechanical systems and in LCS. In the latter they take the form of a linear function of the state, the multiplier and an exogenous signal (see Definition 4 below and w in (2)), while unilateral constraints in Lagrangian systems always have $D = 0$ and $F = 0$ (hence excluding the strong passivity of the closed-loop system), and usually are a nonlinear function of one part of the state only, *i.e.*, positions. However even linear unilaterally constrained Lagrangian systems do not fit within the class of LCS studied in this article, because of the different nature of state jumps as explained above. One major consequence is that state jumps can be triggered instantaneously by the exogenous signal, which is not possible in mechanical systems where exogenous signals (external forces and torques) do not act directly in the complementarity constraints. The way the desired trajectories are designed in the above works and in this article, are quite different as well. When the unilateral constraints are replaced by unilateral springs, things differ significantly as illustrated in section 14. Certainly, the frameworks that are the closest to what is presented in the sequel, are in [82] and [61] (though the class of studied systems is not the same, and the robustness analysis is not led in [61]). This is detailed below in Remark 3.7.

The main contributions of this article are to propose a solution for the trajectory tracking in LCS, in the case where all parameters are known, and when parameter uncertainties are taken into account. The case of absolutely continuous solutions is treated, as well as the case of solutions with state jumps (when parameters are known). Passivity and maximal monotonicity of suitable operators are the tools which allow to design stable closed-loop systems. Matrix inequalities stemming from passivity are central in the design process. The theoretical developments are illustrated by numerical examples on nonsmooth circuits, networks with unilateral interactions, and mechanics with unilateral springs.

This article is organised as follows: useful notations and definitions are given in the next paragraph; section 2 is dedicated to recall the main results about the well-posedness (existence and uniqueness of solutions) of LCS, as well as state jump mappings; section 3 presents in details the control design and the error dynamics analysis, in the known parameters case, with and without state jumps; section 4 is devoted to extend the results to the case when parametric uncertainties are considered; the particular case of FOSwP is analysed in section 5; a simple scalar example is studied in section 6, and many circuit examples are analysed in sections 7 through 13, a simple mechanical system with a unilateral spring is presented in section 14, and a network with unilateral interactions is presented in section 7; conclusions end the article in section 16, and several useful tools, results, code design details, are provided in Appendices A.1 through A.8 and in Appendix B.

Notation and Definitions The elements of a matrix $M \in \mathbb{R}^{n \times m}$ are denoted M_{ij} . $M_{\bullet,j}$ is its j th column, and $M_{i,\bullet}$ is its i th row. A matrix $M \in \mathbb{R}^{n \times n}$, possibly nonsymmetric, is said to be positive definite, $M \succ 0$ (resp. positive semi-definite, $M \succeq 0$) if $x^\top M x > 0$ for all $x \neq 0 \in \mathbb{R}^n$ (resp. $x^\top M x \geq 0$). M is a P-matrix if all its principal minors are positive; a positive definite matrix is a P-matrix. The minimum and the maximum eigenvalues of M are denoted $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$. The maximum singular value of $M \in \mathbb{R}^{n \times m}$ is denoted $\sigma_{\max}(M)$, $\text{Im}(M)$ is its image, $\text{Ker}(M)$ is its null space. The singular values are ordered as $\sigma_{\max}(M) = \sigma_1(M) \geq \sigma_2(M) > \dots > \sigma_r(M) > 0$, where $r = \text{rank}(M)$. Let $M = M^\top$, its eigenvalues are real and ordered as $\lambda_{\max}(M) = \lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_n(M) = \lambda_{\min}(M)$. M^\dagger denotes the Moore-Penrose pseudo-inverse of M . Let $1 \leq p < +\infty$, a Lebesgue integrable function $f : I \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to $\mathcal{L}_{loc}^p(I; \mathbb{R})$ if $(\int_I \|f(t)\|^p dt)^{\frac{1}{p}} < +\infty$ for any open set I (extended spaces). If $I = \mathbb{R}^n$ then $f \in \mathcal{L}^p(\mathbb{R}^n; \mathbb{R})$. Also $f \in \mathcal{L}^\infty(\mathbb{R}^n; \mathbb{R})$ if $f(\cdot)$ is Lebesgue integrable and $\|f(x)\| < +\infty$ almost everywhere on \mathbb{R}^n . We denote $\|f\|_{I,p} \triangleq (\int_I \|f(t)\|^p dt)^{\frac{1}{p}}$ for $1 \leq p < +\infty$, and $\|f\|_{I,\infty} \triangleq \text{esssup}_{t \in I} \|f(t)\|$ (when I is obvious from the context it may be avoided). The range of the function $f(\cdot)$ is $\text{Im}(f)$, its domain is $\text{dom}(f) = \{x \mid f(x) < +\infty\}$. Right and left limits of the function $f(\cdot)$ at t are denoted $f(t^+)$ and $f(t^-)$. Let S be a nonempty set, $\text{rint}(S)$ is its relative interior [76]. Let S and S' be closed sets, $d_H(S, S')$ denotes the Hausdorff distance between S and S' . AC is for absolute continuity, BV is for bounded variation, LBV is locally BV, and RCLBV is for right-continuous LBV. Let $S \subseteq \mathbb{R}^m$ be a closed convex set containing 0. Its polar cone $S^\circ = \{x \in \mathbb{R}^m \mid x^\top v \leq 0, \forall v \in S\} = -S^*$ where S^* is the dual cone. Let us recall some useful definitions [23, 76, 77, 44, 25].

Definition 1. Let $S \subset \mathbb{R}^n$ be a closed, non-empty and convex set. The normal cone to S at a

point $x \in S$ is given by

$$\mathcal{N}_S(x) = \{g \in \mathbb{R}^n \mid g^\top(z - x) \leq 0, \forall z \in S\}. \quad (1)$$

For $x \notin S$ one usually sets $\mathcal{N}_S(x) = \emptyset$. The tangent cone $\mathcal{T}_S(\cdot)$ is the polar to the normal cone.

The domain of a set-valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is $\text{Dom}(T) = \{x \in \mathbb{R}^n \mid T(x) \neq \emptyset\}$.

Definition 2. A multivalued mapping $T : \text{Dom}(T) \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is monotone if for all $u_1, u_2 \in \text{Dom}(T)$, for all $v_1 \in T(u_1), v_2 \in T(u_2)$, then

$$\langle u_1 - u_2, v_1 - v_2 \rangle \geq 0.$$

It is maximal if its graph cannot be enlarged without destroying monotonicity.

Definition 3. A multivalued mapping $T : \text{Dom}(T) \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is hypomonotone if there exists a real $k > 0$ such that for all $u_1, u_2 \in \text{Dom}(T)$, for all $v_1 \in T(u_1), v_2 \in T(u_2)$, we have:

$$\langle u_1 - u_2, v_1 - v_2 \rangle \geq -k\|u_1 - u_2\|^2$$

The normal cone to a convex closed nonempty set, defines a maximal monotone set-valued mapping. More generally, let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Its subdifferential at x is defined as the set $\partial\varphi(x) = \{g \in \mathbb{R}^n \mid \varphi(v) - \varphi(x) \geq g^\top(v - x), \forall v \in \mathbb{R}^n\}$, and it defines a maximal monotone mapping. The normal cone in (1) is the subdifferential of the indicator function defined as $\Psi_S(x) = 0$ if $x \in S$, $\Psi_S(x) = +\infty$ if $x \notin S$. Another function plays an important role: the support function of the set S , defined as $\sigma_S(y) = \sup_{v \in S} v^\top y$. It is related to the indicator function as: $\eta \in \mathcal{N}_S(v) = \partial\Psi_S(v) \Leftrightarrow v \in \partial\sigma_S(\eta) = \mathcal{N}_S^{-1}(\eta)$ (the indicator and the support functions are conjugate functions, and their subdifferentials are inverse mappings).

Definition 4. A linear complementarity system (LCS) with inputs is a nonsmooth dynamical system defined as:

$$\begin{cases} (a) & \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t) \quad (\text{almost everywhere}) \\ (b) & 0 \leq \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + Fu(t) \geq 0 \end{cases} \quad (2)$$

where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $w \in \mathbb{R}^m$, A, B, C, D, E and F are constant matrices of appropriate dimensions, $u : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ is an exogenous signal or a control input. The constraint in (2) (b) is a linear complementarity problem (LCP) in λ , denoted $\text{LCP}(D, Cx + Fu)$.

A mode of the LCP (2) (b) corresponds to a pair (λ, w) with $\lambda_i = 0$ and $w_i > 0$ (noncontact indices), $\lambda_j > 0$ and $w_j = 0$ (contact indices), $\lambda_k = w_k = 0$ (degenerate indices), $i \in \mathcal{I} \subseteq \{1, \dots, m\}$, $j \in \mathcal{J} \subseteq \{1, \dots, m\}$, $k \in \mathcal{K} \subseteq \{1, \dots, m\}$, $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K} = \{1, \dots, m\}$, $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset$, $\mathcal{I} \cap \mathcal{J} = \emptyset$, $\mathcal{I} \cap \mathcal{K} = \emptyset$, and $\mathcal{J} \cap \mathcal{K} = \emptyset$. The switching times between modes i and j correspond to junction times, or detachment times. It is noteworthy that most of the results in this article apply to linear cone complementarity systems (LCCS), with (b') $K \ni \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + Fu(t) \in K^*$, where $K \subseteq \mathbb{R}^m$ is a nonempty closed convex cone, $K^* = \{z \in \mathbb{R}^m \mid z^\top w \geq 0, \forall w \in K\}$ is its dual cone. Using tools from Convex Analysis it is possible to rewrite equivalently (2) as the differential inclusion (DI):

$$\dot{x}(t) \in Ax(t) + Eu(t) - B(D + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(Cx(t) + Fu(t)) \triangleq -H(t, x(t)), \quad (3)$$

where $\lambda(t) \in -(D + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(Cx(t) + Fu(t))$. We shall see later that the DI may sometimes be written in a slightly different way when the input is given a specific form, see (21). A state x^* is an equilibrium point of (2) if and only if there exist λ^* and $w^* \in \mathbb{R}^n$ such that the mixed LCP (MLCP)

$$\begin{cases} 0 = Ax^* + B\lambda^* \\ 0 \leq \lambda^* \perp w^* = Cx^* + D\lambda^* \geq 0 \end{cases} \quad (4)$$

holds. Passivity plays an important role in the sequel of this work. Let us define Moreau's first order sweeping process:

Definition 5. Moreau's first order sweeping process (FOSwP) is a differential inclusion of the form:

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + f(t, x) \quad (5)$$

where $\mathcal{N}_{S(t)}(x(t))$ is the normal cone to the closed convex non-empty set $S(t) \subseteq \mathbb{R}^n$ at $x(t)$, defined in (1), and $f(t, x)$ is a single-valued map such that $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 6. A quadruple (A, B, C, D) is said to be passive, or dissipative with respect to the supply rate $u^\top y$, if there exists a non-negative function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called a storage function, such that for all $t_0 \leq t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2([t_0, t_1]; \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m)$ such that $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t) dt \geq V(x(t_1)) \quad (6)$$

The inequality (6) is called the dissipation inequality. Equivalently, the following LMI

$$\begin{pmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -D - D^\top \end{pmatrix} \preceq 0 \quad (7)$$

has a solution $P = P^\top \succcurlyeq 0$. Then $V(x) = \frac{1}{2}x^\top Px$. The quadruple (A, B, C, D) is said strictly state passive if the LMI (7) holds with $P = P^\top \succ 0$ and $A^\top P + PA + \epsilon P \preceq 0$ for some $\epsilon > 0$. It is said strongly passive if the LMI (7) holds with strict inequality and $P = P^\top \succ 0$.

When LCS are considered, then the multiplier λ is chosen as the input u while w is chosen as the output y for the passivity property. Passivity is then to be seen as a structural property related to internal variables of the LCS, and motivated by the fact that it corresponds to a physical property in some applications like circuits.

Definition 7. [25, Definitions 2.34, 2.58, 2.78] The transfer function $H(s) = C(sI - A)^{-1}B + D \in \mathbb{C}^{m \times m}$ is positive real (PR) if: $H(s)$ has no pole in $\mathbf{Re}[s] > 0$, $H(s)$ is real for all positive real s , $H(s) + H^*(s) \succcurlyeq 0$ for all $\mathbf{Re}[s] > 0$. The transfer function $H(s) \in \mathbb{C}^{m \times m}$ is strictly positive real (SPR) if $H(s - \epsilon)$ is PR for some $\epsilon > 0$ [25, Definition 2.58] and it is strong SPR (SSPR) if $H(s)$ is analytic in $\mathbf{Re}[s] \geq 0$ and $\mathbf{Re}[H(j\omega)] \geq \delta > 0$, for all $\omega \in [-\infty, \infty]$ and some $\delta \in \mathbb{R}$ [25, Definition 2.78].

Strict state passivity is related to SPR transfer functions, while strong passivity is related to SSPR transfer functions [25, 62]. Using the Schur complement theorem (see e.g., [25, Theorem A.65]), strong passivity implies that $D \succ 0$ and $-A^\top P - PA \succ 0$.

2 Well-posedness of Time-Varying LCS

2.1 LCS with External Inputs

LCS as in (2) are nonlinear nonsmooth dynamical systems with inputs, whose well-posedness (existence, uniqueness, and continuous dependence of solutions) has been investigated. Let us provide a short summary of well-posedness results in [27, 30, 29, 28, 24, 26, 19, 82]. Most of them apply to larger classes of nonsmooth systems than LCS. Let us remind that a right-continuous function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ of locally bounded variation (RCLBV), has a countable set of discontinuities on I (allowing for left accumulations of jump instants), and possesses a differential measure df , see [64]. The conditions associated with the results recalled below, are sufficient. However they are well fitted with the conditions imposed in this article for stability purposes, i.e., passivity.

1. (Well-posedness based on reduction to an ODE)

- Assume that D is a P-matrix. Then from the fundamental theorem of Complementarity Theory [44], it follows that λ is a piecewise continuous, Lipschitz single-valued function of x and u . Hence the LCS in (2) is the ODE:

$$\dot{x}(t) = Ax(t) + B\lambda(x(t), u(t)) + Eu(t).$$

Provided $u(\cdot)$ satisfies Lebesgue integrability conditions, an AC solution exists on \mathbb{R}_+ for any bounded initial condition, with uniqueness and continuous dependence on initial data [36, 34].

- Assume that $D \succ 0$ (not necessarily symmetric), then using the formalism in (3), the result can be extended to systems with operators $(D + \mathcal{M}_t)^{-1}(\cdot)$ where $\mathcal{M}_t(\cdot)$ is a maximal monotone operator for each fixed t . Using [19, Proposition 1] or [3], the mapping $(D + \mathcal{M}_t)^{-1}(\cdot)$ is then single-valued, defined everywhere, and Lipschitz continuous with constant $\frac{1}{\lambda_{\min}(D + D^\top)}$ for each fixed t . Therefore the mapping $H(t, x)$ in (3) is single-valued Lipschitz continuous in x and classical results on ODE's well-posedness apply.

2. (Well-posedness based on passivity and relaxations of passivity) Let us summarize the results presented in [27, 82, 29, 31, 19, 28], which deal with the case with external inputs.

(a) [27, Theorem 7.5] Assume that:

- i) $(B^\top, D + D^\top)^\top$ has full column rank,
- ii) (A, B, C, D) is passive with $P = P^\top \succ 0$, and minimal,
- iii) $u(\cdot)$ is piecewise continuous with rational Laplace transform.

Then for any $x(0) = x_0$, there exists a unique global solution to (2) such that $(\lambda, x, w) \in \mathcal{L}_\delta^2(\mathbb{R}_+; \mathbb{R}^{m+n+m})$, the space of Schwartz' distributions, with regular parts in $\mathcal{L}_{loc}^2 \subseteq \mathcal{L}_{loc}^1$, and atomic part with isolated atoms (instants of state jumps) that is represented by sums of Dirac measures (higher degree distributions do not occur).

(b) [29, Theorems 7, 8, 9] Assume that:

- i) (A, B, C, D) is passive with $P = P^\top \succ 0$,
- ii) $Fu(t) \in Q_D^* + \text{Im}(C)$ for all $t \geq [0, T)$, $T > 0$ (see section 2.1 for the definition of Q_D and of its dual cone),
- iii) $u(\cdot)$ is a Bohl function (*i.e.*, $u(t) = M \exp(Nt)R$ for some constant matrices M, N, R).

Then $x(0^+)$ is calculated as in (9), there exists a unique AC solution $x : (0, T) \rightarrow \mathbb{R}^n$, and $\lambda(\cdot)$ is locally Lebesgue integrable. The solution is a forward solution, *i.e.*, it is the concatenation of Bohl functions defined between state jumps. Left accumulations of state jumps are allowed.

(c) [82, Corollary 2] [23, Corollary 5.9] Assume that:

- i) $D \succcurlyeq 0$, $\text{Ker}(D + D^\top) \subseteq \text{Ker}(PB - C^\top)$ for some $P = P^\top \succ 0$,
- ii) $\text{rint}(\mathbb{R}_+^m - Fu(t)) \subseteq \text{rint}(\text{Im}(\partial\sigma_{\mathbb{R}_+^m - Fu(t)} + D))$,
- iii) $D\mathbb{R}_+^m \subseteq \text{Im}(C)$,
- iv) $\text{Im}(C) - \mathbb{R}_+^m = \mathbb{R}^m$,
- v) for each $x \in \mathbb{R}^n$ and each $t \geq 0$, if the set $\Lambda = \{\lambda \in \mathbb{R}_+^m \mid v = Cx + D\lambda + Fu(t) \geq 0, \lambda^\top v = 0\}$ has a nonzero element, and $\Lambda \cap \text{Im}(D + D^\top) \neq \emptyset$.

Then if $Fu(\cdot)$ is AC (resp. RCLBV), there exists a unique AC (resp. RCLBV) solution to (2) for any $x(0)$ such that $Cx(0) \in \text{Im}(\partial\sigma_{\mathbb{R}_+^m - Fu(0)} + D)$.

(d) [28, Theorems 11, 26] Let:

- i) (A, B, C, D) be passive,
- ii) $\text{Im}(C) \cap \text{rint}(\text{Im}(D + \mathcal{N}_{\mathbb{R}_+^m}) - Fu(t)) \neq \emptyset$,
- iii) $Fu(\cdot)$ be AC.

Then there exists a unique AC solution for any $x(0) = x_0 \in \text{Dom}(H(0, \cdot))$.

(e) [19, Theorem 1, Corollaries 2, 3] Assume that:

- i) $D = \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix} \in \mathbb{R}^{m_2 \times m_2}$, $m_2 \leq m$
- ii) $F = 0$,
- iii) $u(\cdot)$ is continuous with $\dot{u} \in \mathcal{L}_{loc}^1(\mathbb{R}_+; \mathbb{R}^p)$,
- iv) there exists a full-rank $R = R^\top \in \mathbb{R}^{n \times n}$ such that $R^2 B_1 = C_1^\top$, $B = (B_1 \ B_2)$, $C = (C_1^\top \ C_2^\top)^\top$,
- v) there exists $w_0 \in \mathbb{R}^{m_1}$ at which the operator $w \mapsto \partial\Psi_{\mathbb{R}_+^{m_1}}(C_1 R^{-1}w)$ is continuous,
- vi) the operator $w \mapsto RB_2(D_2 + \partial\Psi_{\mathbb{R}_+^{m_2}})^{-1}(-C_2 R^{-1}w)$ is well-defined, single-valued and Lipschitz continuous.

Then there exists a unique continuous right-differentiable solution for any $x(0) \in \text{Dom}(\partial\Psi_{\mathbb{R}_+^{m_1}}(C_1 R^{-1}\cdot)) = \{w \in \mathbb{R}^n \mid C_1 R^{-1}w \geq 0\}$, with $\dot{x} \in \mathcal{L}^\infty(\mathbb{R}_+; \mathbb{R}^n)$. The result holds if $m_2 = 0$ and i-v) are verified, or if $m_2 = m$ and vi) is verified (then item 1 applies). Condition vi) holds if D_2 is a P-matrix [19, Proposition 1, Corollary 1].

(f) [60, Theorems 5.3, 5.4] and [59, Theorems 1, 2] deal with DIs of the form:

$$\dot{x}(t) \in g(t, x(t)) - B(D + F_{t,x}^{-1})^{-1}(Cx(t)),$$

where $F_{t,x} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is maximal monotone for each (t, x) . For briefness we focus on $F_{t,x}(\cdot) = \mathcal{N}_{K(t,x)}(\cdot)$ [59]. Assume that:

- i) $D \succcurlyeq 0$, $\text{Im}(D) \subset \text{Im}(C)$, $\text{Ker}(D + D^\top) \subseteq \text{Ker}(PB - C^\top)$ for some $P = P^\top \succ 0$,
- ii) $K(t, x)$ has closed convex values, $K(t, x) \cap \text{Im}(C) \neq \emptyset$, $d_H(K(t, x) \cap \text{Im}(C), K(s, y) \cap \text{Im}(C)) \leq k_1|t - s| + l_2\|x - y\|$ for all t, s, x, y , $l_1 \geq 0$, $0 \leq l_2 \leq \frac{\lambda_2}{\|C\|}$, λ_2 the smallest positive eigenvalue of CC^\top ,
- iii) $(\mathcal{N}_{K(t,x)}^{-1} + D)^{-1}(Cx) \neq \emptyset \Rightarrow \text{Im}(D + D^\top) \cap (\mathcal{N}_{K(t,x)}^{-1} + D)^{-1}(Cx) \neq \emptyset$,
- iv) $\text{Im}(C) \cap \text{rint}(\text{Im}(\mathcal{N}_{K(t,x)}^{-1} + D)) \neq \emptyset$,
- v) $g(\cdot, \cdot)$ is continuous in t and Lipschitz continuous in x .

Then existence and uniqueness of Lipschitz continuous solutions is guaranteed for each initial condition such that $(\mathcal{N}_{K(0,x_0)}^{-1} + D)^{-1}(Cx_0) \neq \emptyset$. Moreover, $\|\dot{x}(t)\| \leq \alpha + \beta\|x(0)\|$ for some $\alpha > 0$ and $\beta > 0$.

3. (Well-posedness by transformation to a sweeping process [24, 23, 12]) Assume that $D = 0$, and that:

- i) $PB = C^\top$ for some $P = P^\top \succ 0$,
- ii) $u \in \mathcal{L}_{loc}^1(\mathbb{R}_+; \mathbb{R}^p)$,
- iii) $\text{Im}(C) - \mathbb{R}_+^m = \mathbb{R}^m$.

Then the LCS in (2) can be rewritten equivalently as a FOSwP (see Definition 5, and [24, 23] for details on the transformation, and [64, 23] for details on FOSwP). The basic condition i) is implied by, but does not imply, the passivity of $(A, B, C, 0)$. Depending on the signal $u(\cdot)$ being AC (resp. RCLBV), the solution is shown to be AC (resp. RCLBV), defined on \mathbb{R}_+ , and uniqueness holds. For AC to hold it is needed that $x(0) \in S(0)$ (no initial jump).

The various assumptions made in items 2 and 3 (which may be thought as constraint qualifications in many instances, a concept which is familiar in Optimization) have different meanings:

- property of the plant's model: item 2 (a) ii), (b) i), (c) i), (d) i), (e) i) ii) iv), item 3 i),
- constraints qualifications: item 2 (a) i), (b) ii), (c) ii), iii), iv), v), (e) v), (f) ii), iii), iv), item 3 iii),
- exogenous signals properties: item 2 (a) iii), (b) iii), conclusion of item 2 (c), item 2 (e) iii), (f) v), conclusion of item 3.

The condition in item 2 (b) ii) stems from a fundamental result in Complementarity Theory [44, Theorem 3.8.6, Corollary 3.8.10], which guarantees the existence of solutions to the LCP in (2). This assumption thus secures that there exists a bounded multiplier $\lambda(t)$ which solves the LCP (notice that at the end time T , there may be a state jump if this assumption does not hold). This is an assumption quite similar to item 2 (c) ii), and item 2 (d) ii). Roughly speaking, they all guarantee that the various operators have nonempty domains so that the problem makes sense. They stem from [77, Theorem 12.43] to guarantee maximal monotonicity (see [19, section 3.2.1]). Such CQ may not be easy to check and may require some developments [17]. Condition in item 2 (c) iii) and (f) i) ($\text{Im}(D) \subset \text{Im}(C)$) are used to compute Vladimirov's pseudo distance (see [84] for the definition, which is beyond the scope of this article) between time-varying and/or state-dependent sets.

Multipliers boundedness (AC solutions)

Well-posedness analyses usually focus solely on the state. However, the boundedness of the multiplier is an important feature, since it may be used for feedback in this article. The condition in item 2 (b) ii) guarantees the existence of a solution to the LCP in (2). If the conditions in item 3 are verified, then explicit upper-bounds $\|\dot{x}(t)\| \leq \dot{v}(t) + \alpha(t)\|x(t)\|$ can be obtained for some integrable $\alpha(\cdot)$ and AC $v(\cdot)$ [83, Theorem 4.3] [64, section 5.2, Theorem 2.1]. In our case, these upper-bounds depend on the properties of the set-valued map $t \mapsto S(t)$ (which in turn depend on $Fu(\cdot)$ being AC, or Lipschitz continuous, or LBV [24, Proposition 3.2]). Following the developments in [82] (see item 2. (c) in section 2), $\lambda = \lambda^{\text{im}}(t, x) + \lambda^{\text{ker}}$, where $x \mapsto \lambda^{\text{im}}(t, x) \in \text{Im}(D + D^\top)$ is Lipschitz continuous, while $\lambda^{\text{ker}} \in \text{Ker}(D + D^\top)$. Moreover, λ^{im} is the least norm element in the (possibly) set-valued right-hand side of the DI, see [82, Equ. (15), Lemma 3].

It is inferred that in case the existence of AC solutions is proved, then $B\lambda(\cdot)$ is a bounded selection of the (possibly set-valued) right-hand side in (2). The part of λ inside $\text{Ker}(B)$ does not play any role in $x(\cdot)$, but we may need the whole multiplier vector for feedback purpose. It is therefore

reasonable to assume that the minimum norm element is available and that it satisfies a linear growth condition in $\|x\|$. Finally, it is worth recalling that even if $x(\cdot)$ is AC, the multiplier $\lambda(\cdot)$ may have discontinuities at junction times when $w(\cdot)$ reaches the boundary (*i.e.*, $w = 0$), when D is not a P-matrix. In the case of item 1, the multipliers are AC when $Fu(t)$ is AC since they are a Lipschitz continuous function of AC functions.

From a more practical point of view, multipliers in LCS can be calculated by constructing the contact LCP [23, section 2.4.1]. Thus they appear to be a nonsmooth function of both state and input. The uniqueness of λ can be deduced from the conditions in item 2 (a) and (b), where passivity is the main property, plus some constraints qualification.

State jumps

Let the solution $x(\cdot)$ of (2) be RCLBV. Then $x(\cdot)$ may undergo discontinuities and the LCS has to be interpreted in the Measure Differential Inclusion (MDI) formalism [64, 46, 23]. In other words, the DI (3) has to be embedded into an MDI: $dx \in (Ax(t) + Eu(t))dt - B(D + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(Cx(t^+) + Fu(t))d\nu$, where dx is the differential measure associated with the RCLBV function $x(\cdot)$ [64], dt is the Lebesgue measure, and $d\nu$ is a specific Radon measure, see [82, Definition 4] for a rigorous introduction. The dissipation inequality extends to MDI, see [15, section 5.4.4.5] [25, section 7.2.4.1].

Let the set of time instants at which the state undergoes a jump, be denoted as \mathcal{J}_x , and the set of time instants at which $Fu(\cdot)$ is discontinuous be denoted as \mathcal{J}_{Fu} . Then if conditions of item 2 (a) hold, $\mathcal{J}_x \subseteq \{0\} \cup \mathcal{J}_{Fu}$ [27, 24]. In other words, the state can jump only initially, or at times of discontinuity in $Fu(\cdot)$ (or in the set $S(t)$ in the FOSWP formalism). *This means that state jumps can be chosen and dwell times can be imposed (contrarily to complementarity mechanical systems), by suitably choosing $Fu(t)$.*

In the case of BV solutions, and under the conditions stated above, the term $B\lambda$ is a Dirac measure at times of state jumps, and we may denote $\lambda = \lambda^{imp} + \lambda^{reg}$. The input in (15) below, may be defined with the total λ ("impulsive" input), or just with its function part λ^{reg} . In both cases, the stability analysis has to incorporate the jumps. At an instant of jump t , using (3) and [82, section 3.4], we have [82, section 3.4]:

$$x(t^+) - x(t^-) \in -B(D + \partial\sigma_{S(t^+)})^{-1}(Cx(t^+)) \quad (8)$$

with $S(t^+) = \mathbb{R}_+^m - Fu(t^+)$. State jump rules formulations are given in [23, Lemma 2.3], compiling results in [27, 24, 29, 50]. They are based on a maximum dissipation principle (a kind of plastic impact). Let us introduce such rules briefly. If (A, B, C, D) is passive with $P = P^\top \succ 0$ and if $0 \in \mathcal{K} \triangleq \{z \in \mathbb{R}^n \mid Cz + Fu(t^+) \in Q_D^*\}$ (notice that the set $\mathcal{K} \neq \emptyset$ by the condition in item 2 (b) ii)), $Q_D = \{z \in \mathbb{R}^m \mid 0 \leq z \perp Dz \geq 0\}$, then the jumps dissipate "energy", *i.e.*, $V(x(t^+)) \leq V(x(t^-))$ [50, Lemma 3], and the state jump rule is given by:

$$x(t^+) = \operatorname{argmin}_{x \in \mathcal{K}} \frac{1}{2}(x - x(t^-))^\top P(x - x(t^-)). \quad (9)$$

Notice that Q_D is a polyhedral set, and so is its dual cone [77, Lemma 6.45, Theorem 3.52]. If $D \succ 0$ then $Q_D = \{0\}$, $Q_D^* = \mathbb{R}^m$, $\mathcal{K} = \mathbb{R}^n$ and using (9), $x(t^+) = x(t^-)$. If $D = 0$ then $Q_D = \mathbb{R}_+^m = Q_D^*$, $\mathcal{K} = \{z \in \mathbb{R}^n \mid Cz + Fu(t^+) \geq 0\}$, hence $0 \in \mathcal{K}$ if and only if $Fu(t) \geq 0$. Thus strongly passive systems have continuous-time solutions, but strictly state passive systems may have state jumps. Notice that $x(t^+) - x(t^-) = B\sigma$, with $\lambda^{imp} = \sigma\delta_t$, and $0 \leq \sigma \perp D\sigma \geq 0$. As shown in [27, 23, 50] the state jump does not depend on a particular choice of the storage function matrix P . The meaning of the complementarity conditions at discontinuity times is explained in [27, Theorems 6.1, 9.1], see also [23, Lemma 2.3].

Zeno behavior

LCS undergo two classes of events, *i.e.*, switching between modes of the LCP (2) (b) (AC solutions), and state discontinuities (BV solutions). Zenoness with AC solutions is tackled in [81, 30, 80] when $Fu(\cdot) = 0$. It states that if $BSOL(D, Cx)$ is a singleton, then the LCS (2) is Zeno-free. This holds if D is a P-matrix. Thus junction and detachment times are separated (dwell time). The conditions stated in [29, Theorem 7] also prevent Zeno behaviour in LCS with external inputs and state jumps, by imposing condition ii) in item 2 (b) between jumps.

2.2 Time-Varying LCS (TVLCS)

We may also encounter TVLCS of the form:

$$\begin{cases} (a) & \dot{x}(t) = A(t)x(t) + B(t)\lambda(t) + E(t)u(t) \quad (\text{almost everywhere}) \\ (b) & 0 \leq \lambda(t) \perp w(t) = C(t)x(t) + D(t)\lambda(t) + F(t)u(t) \geq 0 \end{cases} \quad (10)$$

when we deal with uncertainties which may be time-varying. The TVLCS is rewritten equivalently as:

$$\begin{aligned} \dot{x}(t) \in & A(t)x(t) + E(t)u(t) - B(t)(D(t) + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(C(t)x(t) + F(t)u(t)) \\ & = A(t)x(t) + E(t)u(t) - B(t)(D(t) + \mathcal{N}_{\mathbb{R}_+^m - F(t)u(t)}^{-1})^{-1}(C(t)x(t)) \end{aligned} \quad (11)$$

Clearly, some of the results described above can be used to analyse (11), in particular cases. One such case is as follows. Assume that $A = A(t)$, $E = E(t)$, $F = F(t)$, $D = 0$, C and B are constant. Assume that there exists $P = P^\top \succ 0$ such that $PB = C^\top$. Then (11) is rewritten equivalently as:

$$\dot{x}(t) \in A(t)x(t) + E(t)u(t) - B\mathcal{N}_{\mathbb{R}_+^m - F(t)u(t)}(Cx(t)) \quad (12)$$

After the classical state change $z = Rx$, $R^2 = P$, $R = R^\top \succ 0$ [23], (12) is rewritten equivalently as:

$$\dot{z}(t) \in RA(t)R^{-1}z(t) + RE(t)u(t) - \mathcal{N}_{\mathbb{R}_+^m - F(t)u(t)}(z(t)), \quad (13)$$

which is a FOSwP with affine perturbation. Then the material in item 3 section 2.1 can be used to analyse the well-posedness with both AC and RCLBV solutions. The general problem of well-posedness of (11) is not tackled in this work.

3 Trajectory Tracking in the Nominal Case

In this section, it is assumed that the plant model and parameters have no uncertainties. That is, the matrices A , B , C , D , E , F in (2) are known.

3.1 Controller Design

Let us consider the LCS with external input in (2). Let us define the system which generates the desired trajectory to be tracked, as follows:

$$\begin{cases} \dot{x}_d(t) = Ax_d(t) + B\lambda_d(t) + Eu_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = Cx_d(t) + D\lambda_d(t) + Fu_d(t) \geq 0 \end{cases} \quad (14)$$

where $u_d(\cdot)$ is an input. Following the LCS of the desired system in (14), the designer has to perform a preliminary analysis of the desired dynamics to determine a suitable trajectory for tracking. This analysis may involve numerical methods or the approaches presented in [79, 54]. The aim is to design a feedback controller such that the error dynamics with state vector $e \triangleq x - x_d$ possesses some stability property, to be defined later. Let us state the following that will be used later:

Assumption 3.1. *The solution $x_d : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ of the LCS (14) is AC and uniformly bounded, and the multiplier vector $\lambda_d(u_d, x_d)$ is a bounded function of time.*

Conditions such that this holds can be obtained from the results stated in section 2.

Assumption 3.2. *The state $x(\cdot)$ and the multiplier $\lambda(\cdot)$ are available for measurement.*

In practice, the multiplier may be a physical quantity (like voltages and currents in circuits, contact force in mechanics), and Assumption 3.2 is reasonable (and studying state observers in the loop is outside the scope of this article). Thus the feedback controller in the plant (2) is chosen generically as:

$$u(x, \lambda, t) = K[x - x_d(t)] + G[\lambda - \lambda_d(t)] + u_d(t). \quad (15)$$

where the example in Appendix A.3 plays the role of a motivating example for introducing feedback from λ in the controller (15). Inserting (15) into (2) for some feedback gains $K \in \mathbb{R}^{p \times n}$ and $G \in \mathbb{R}^{p \times m}$ gives rise to the closed-loop LCS:

$$\begin{cases} \dot{x}(t) = (A + EK)x(t) + (B + EG)\lambda(t) - EKx_d(t) - EG\lambda_d(t) + Eu_d(t) \quad (\text{a.e.}) \\ 0 \leq \lambda(t) \perp w(t) = (C + FK)x(t) + (D + FG)\lambda(t) - FKx_d(t) - FG\lambda_d(t) + Fu_d(t) \geq 0 \end{cases} \quad (16)$$

It is important to note that in general both $\lambda_d(\cdot)$ and $u_d(\cdot)$ in (14) may be discontinuous time-functions, hence (16) is in general an LCS as (2) with a potentially discontinuous term $-FKx_d(t) - FG\lambda_d(t) + Fu_d(t)$ inside the complementarity constraints. As a consequence jumps in $x(\cdot)$ may occur unless conditions hold (see the end of section 3.2). We shall come back on state jumps in section 3.3. Rewriting (14) equivalently as

$$\begin{cases} \dot{x}_d(t) = (A + EK)x_d(t) + (B + EG)\lambda_d(t) + Eu_d(t) - EKx_d(t) - EG\lambda_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = (C + FK)x_d(t) + D\lambda_d(t) + Fu_d(t) - FKx_d(t) \geq 0 \end{cases} \quad (17)$$

gives rise to the error dynamics:

$$\begin{aligned} \dot{e}(t) &= (A + EK)e(t) + (B + EG)(\lambda(t) - \lambda_d(t)) \\ 0 \leq \begin{pmatrix} w(t) \\ w_d(t) \end{pmatrix} &= \begin{pmatrix} C + FK \\ -C - FK \end{pmatrix} e(t) + \begin{pmatrix} D + FG & -FG \\ 0 & D \end{pmatrix} \begin{pmatrix} \lambda(t) \\ \lambda_d(t) \end{pmatrix} + \begin{pmatrix} 0 & C \\ C + FK & -FK \end{pmatrix} \begin{pmatrix} x(t) \\ x_d(t) \end{pmatrix} \\ &+ \begin{pmatrix} Fu_d(t) \\ Fu_d(t) \end{pmatrix} \perp \begin{pmatrix} \lambda(t) \\ \lambda_d(t) \end{pmatrix} \geq 0 \end{aligned} \quad (18)$$

Clearly, the LCS in (18) is well-posed if both (14) and (16) are. However, the LCS in (18) cannot be used for well-posedness directly, because $x(\cdot)$ acts as an exogenous signal in complementarity constraints, whose properties have to be proved. The closed-loop' well-posedness has to be tackled with (16) and (17) (or (14)). In view of the structure of the closed-loop LCS in (16), let us state the following assumptions, which will be used in the sequel for both well-posedness and stability purposes.

Assumption 3.3. *There exist matrices K and G such that the plant's closed-loop quadruple $(A + EK, B + EG, C + FK, D + FG)$ is strictly state passive.*

From Assumption 3.3, there exist gain matrices K and G such that the following nonlinear matrix inequality with unknowns P , K and G :

$$M \triangleq \begin{pmatrix} (A + EK)^\top P + P(A + EK) & P(B + EG) - (C + FK)^\top \\ (B + EG)^\top P - (C + FK) & -(D + FG) - (D + FG)^\top \end{pmatrix} \preceq \begin{pmatrix} -\epsilon P & 0 \\ 0 & 0 \end{pmatrix}, \quad (19)$$

has a solution $P = P^\top \succ 0$. This may be replaced by the more stringent assumptions:

Assumption 3.4. *There exist matrices K and G such that the plant's closed-loop quadruple $(A + EK, B + EG, C + FK, D + FG)$ is strongly passive.*

From Assumption 3.4, there exist gain matrices K and G such that the nonlinear matrix inequality:

$$M \prec 0 \quad (20)$$

has a solution $P = P^\top \succ 0$. For numerical purposes, it is important to note that the BMIs in (19) and (20) are transformed into LMIs, as detailed in A.2. Using Convex Analysis, the closed-loop system (16) is rewritten equivalently as the DI:

$$\dot{x}(t) \in (A + EK)x(t) - (B + EG)((D + FG) + \partial\sigma_{S(t)})^{-1}((C + FK)x(t) - EKx_d(t) - EG\lambda_d(t) + Eu_d(t)). \quad (21)$$

where $S(t) = \{v \in \mathbb{R}_+^m \mid v + FKx_d(t) + FG\lambda_d(t) - Fu_d(t) \geq 0\}$ is closed nonempty convex for each t , and $\sigma_{S(t)}(\cdot)$ is the support function of $S(t)$. The DI in (21) has the form

$$\dot{x}(t) \in -\mathcal{M}(t, x(t)) + f(t), \quad (22)$$

with $f(t) = -EKx_d(t) - EG\lambda_d(t) + Eu_d(t)$ and $\mathcal{M}(\cdot, \cdot)$ is a set-valued operator. In view of (14) and Assumption 3.1, depending on the system's parameters, the multiplier $\lambda_d(\cdot)$ may be discontinuous at some instants, hence the set $S(t)$ may also be discontinuous at those instants ($D \succ 0$ prevents such jumps). The LCS in (14) is also equivalently rewritten as the DI:

$$\dot{x}_d(t) \in Ax_d(t) - B(D + \partial\sigma_{S_d(t)})^{-1}(Cx_d(t)), \quad (23)$$

with $S_d(t) = \{v \in \mathbb{R}^m \mid v - Fu_d(t) \geq 0\}$, and the set $S_d(t)$ is continuous as long as $u_d(t)$ is. Therefore the DI (23) also fits with (22). Let us now examine (18). Using that $x(t) = e(t) + x_d(t)$, the error dynamics may be rewritten as:

$$\dot{e}(t) \in \tilde{A}e(t) - \tilde{B}(\tilde{D} + \mathcal{N}_{\tilde{S}(t, e(t))}^{-1})^{-1}(\tilde{C}e(t)) \quad (24)$$

with $\tilde{S}(t, e) = \{z \in \mathbb{R}^{2m} \mid z + \tilde{E}e + \tilde{F}(t) \geq 0\}$, $\tilde{E} = \begin{pmatrix} 0 \\ C + FK \end{pmatrix}$, $\tilde{F}(t) = \begin{pmatrix} Cx_d(t) + Fu_d(t) \\ Cx_d(t) + Fu_d(t) \end{pmatrix}$, $\tilde{D} = \begin{pmatrix} D + FG & -FG \\ 0 & D \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} C + FK \\ -C - FK \end{pmatrix}$, $\tilde{B} = (B + EG \quad -B - EG)$, $\tilde{A} = A + EK$. The DI in (24) has a state and time-dependent polyhedral set $\tilde{S}(\cdot, \cdot)$ which renders its study more complex. For each t and e the normal cone $\mathcal{N}_{\tilde{S}(t, e)}$ defines a maximal monotone mapping, which may allow to recast (24) in the framework of item 2 (f) in section 2, and deduce conditions such that (24) has a Lipschitz solution. A second equivalent way to write the LCS (18) is:

$$\dot{e}(t) \in \bar{A}e(t) - \bar{B}(\bar{D} + \mathcal{N}_{\bar{S}(t)}^{-1})(\bar{C}e(t)) \Leftrightarrow \begin{cases} \dot{e}(t) = \bar{A}e(t) + \bar{B}\tilde{\lambda}(t) \\ 0 \leq \tilde{\lambda}(t) \perp \bar{C}e(t) + \bar{D}\tilde{\lambda}(t) + \bar{F}(t) \geq 0 \end{cases}, \quad (25)$$

with $\bar{A} = \tilde{A}$, $\bar{B} = \tilde{B}$, $\bar{C} = \begin{pmatrix} C + FK \\ 0 \end{pmatrix}$, $\bar{D} = \tilde{D}$, $\bar{S}(t) = \{z \in \mathbb{R}^{2m} \mid z + \bar{F}(t) \in \mathbb{R}_+^{2m}\}$, $\bar{F}(t) = \begin{pmatrix} Cx_d(t) + Fu_d(t) \\ Cx_d(t) + Fu_d(t) \end{pmatrix}$. It is noteworthy that some conditions have to be imposed so that the construction of the DIs in (24) and (25) is possible, see section 3.3 for more details. Now we have at our disposal several (equivalent) formalisms for the closed-loop plant dynamics (16) and (21), the desired trajectory generator (14) and (23), and the error dynamics (18) and (24) and (25). This is useful for the well-posedness analyses relying on the various results recalled in section 2. A difficulty is to determine under which conditions the passivity in Assumptions 3.3 or 3.4, implies the passivity of the quadruples $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, or $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, see (30).

Remark 3.5 (Closed-loop system's well-posedness). *In the framework of this article, the well-posedness of the plant dynamics is not a fundamental issue, since it is only the closed-loop system which is used in the analysis. In a similar way as feedback can be used to make an unstable system, stable in closed-loop, it may be used to render an ill-posed plant's model, well-posed in closed loop. The well-posedness of the closed-loop plant LCS (16) (equivalently the DI in (21)) can be inferred from [24, 28, 82, 60, 19], assuming that the desired signals $x_d(\cdot)$, $\lambda_d(\cdot)$ and $u_d(\cdot)$ are AC or $\mathcal{L}_{loc}^2(\subseteq \mathcal{L}_{loc}^1)$, and that some basic constraint qualification conditions hold. Clearly in this setting Assumption 3.1 is important. When $D + FG > 0$ (this is the case if Assumption 3.4 holds) then the results reported in item 1 in section 2.1 apply. If $D + FG = 0$ (which may be the case if Assumption 3.3 holds) then the results in item 3 in section 2.1 apply. When $D + FG \succcurlyeq 0$ but not null, then the various results in item 2 in section 2.1 can be used. This requires to check the constraint qualifications listed in item 2 (a) through (f). This is not tackled in this article whose primary goal is tracking control.*

Remark 3.6. *The trajectory tracking problem as tackled in this work, can be interpreted as a synchronization problem between the master system (14), and the plant (2).*

Remark 3.7. *The output regulation problem tackled in [82] differs from the problem tackled in this work: First the desired systems are different: The desired system to be tracked is defined by the dynamics of the quadruple (A_r, G_r, H_r, J_r) as mentioned in [82] which is different from the real system defined. It is noticeable that the controller u in [82] is introduced only in the ordinary differential equation (ODE) of the real system, but not in the variational inequality (VI). Besides, there is no presence of a desired controller u_d in the desired system, which is autonomous. Second the objectives are different: In the presence of uncertainties, the real and the desired systems have different dynamics. Hence, in this case, the aim of the regulation problem in [82] is to design a controller using internal model principle that achieves a zero steady-state regulation error: $e \triangleq x - \Pi x_r$ for some matrix Π . But, the aim of the tracking problem tackled in this work, in the presence of uncertainties, is to ultimately bound the tracking error: $e \triangleq x - x_d$, where $x_d(\cdot)$ is generated by desired dynamics. Third the controller in [82] is state feedback, while state and multiplier feedback are considered in this article.*

The authors in [61] study tracking for Measure Differential Inclusions (MDIs), which differ from LCS investigated in this paper. They give sufficient conditions for the uniform convergence of MDIs with maximal monotonicity properties. In the tracking control of MDIs, the authors considered any desired trajectory x_d of locally bounded variation, and control inputs with an impulsive part (which we avoid, see section 3.3 and condition (C1)). In our approach, the desired trajectory x_d is designed independently by a LCS as shown in (14). Another difference with [61] is that the authors are interested in designing a controller such that the MDI of the closed-loop system is uniformly convergent with zero tracking error, while this paper is primarily concerned with studying the stability of the error dynamics without explicitly considering the convergence of the closed-loop system.

3.2 Error Dynamics Stability Analysis (no state jumps)

This section is dedicated to the stability analysis of the error dynamics (18), and it is assumed that all trajectories are at least absolutely continuous. The arguments used for the proof of Proposition 3.8 are similar to those employed in [20, 4, 19], and are given for completeness.

Proposition 3.8. *Suppose that Assumptions 3.1, 3.2 and 3.3 hold, and that the solution $x(\cdot)$ of the closed-loop LCS (16) is AC on \mathbb{R}_+ . Then, the error dynamics in (18) has a globally exponentially stable equilibrium point $e^* = 0$.*

Proof. Let $\Delta\lambda(t) \triangleq \lambda(t) - \lambda_d(t)$ and $\Delta w(t) \triangleq w(t) - w_d(t) = (C + FK)e(t) + (D + FG)\Delta\lambda(t)$. Consider the Lyapunov function candidate $V(e) = e^\top P e$. From the assumptions $e(\cdot)$ is absolutely continuous and thus it has a derivative almost everywhere. Along the error dynamics trajectories (18) it holds that:

$$\dot{V}(t) = e^\top [(A + EK)^\top P + P(A + EK)]e + 2e^\top P(B + EG)\Delta\lambda(t)$$

From the complementarity conditions in (16) and (17) we obtain equivalently:

$$\lambda(t) \in -\mathcal{N}_{S(t)}((C + FK)x(t) + (D + FG)\lambda(t))$$

and

$$\lambda_d(t) \in -\mathcal{N}_{S(t)}((C + FK)x_d(t) + (D + FG)\lambda_d(t))$$

with $S(t)$ defined after (21). It follows from the monotonicity of the normal cone mapping that:

$$[w(t) - w_d(t)]^\top [\lambda(t) - \lambda_d(t)] = \Delta w(t)^\top \Delta\lambda(t) \leq 0. \quad (26)$$

In matrix form (the time argument is dropped on the right-hand side):

$$\begin{aligned} \dot{V}(t) &= \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} 0 & (C + FK)^\top \\ C + FK & D + FG + (D + FG)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \\ &= \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + 2\Delta\lambda^\top \Delta w \leq \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \leq 0, \end{aligned} \quad (27)$$

where $(C + FK)e = \Delta w - (D + FG)\Delta\lambda$. So, for all e and $\Delta\lambda$ and using the strict state passivity, it is inferred:

$$\dot{V}(t) \leq -\epsilon e(t)^\top P e(t) = -\epsilon V(t).$$

Classical arguments yield $V(t) \leq V(0) \exp(-\epsilon t)$. Using the inequality: $\lambda_{\min}(P) \|e\|^2 \leq e^\top P e$, the following is obtained

$$\|e(t)\|^2 \leq \frac{V(t)}{\lambda_{\min}(P)} \leq \frac{V(0)}{\lambda_{\min}(P)} \exp(-\epsilon t)$$

Therefore, the equilibrium point of the error dynamics is globally exponentially stable. \square

Notice that using (18) it follows that $\lambda(t) - \lambda_d(t)$ converges exponentially fast to $\text{Ker}(B - EG)$.

Comments on passivity Following on from the proof of Proposition 3.8, the supply rate in (26) indicates that the passivity, in this case, is studied between the input $\lambda - \lambda_d$ and the output $w - w_d$. The variation of the storage function in (27) is written as $\dot{V}(t) \leq 2\Delta\lambda^\top \Delta w$. This shows the incremental passivity [72, 25] of the LCS defined in (16).

Continuity of the solutions A sufficient condition for no state-jump is both matrices D and $D + FG$ are P-matrices, and $u_d(\cdot)$ is continuous. Then (14) has continuously differentiable solutions $x_d(\cdot)$ with uniqueness for any initial data, and $\lambda_d(\cdot)$ is time-continuous (being a Lipschitz continuous function of x_d and u_d). Thus the trajectories of the closed-loop system (16) (or (21)) are also state-jump free. Let now $G = 0$ in (15). Then $\lambda_d(\cdot)$ does not appear in (16). Thus provided both $x_d(\cdot)$ and $u_d(\cdot)$ are continuous, no state jump occurs, except possibly at the initial time. These results follow by applying, *e.g.*, items 1 or 2 (a) in section 2. When $D = 0$ and/or $D + FG = 0$ (which cannot be excluded by Assumption 3.3), item 3 in section 2 can be used. We may also rely on item 2 (f) to analyse (24) and guarantee it has Lipschitz continuous solutions for the admissible initial state. Let us check conditions i) of item 2 (f) 2 under Assumption 3.3 ($\Rightarrow D + FG \succcurlyeq 0$) and $D \succcurlyeq 0$ (most of the conditions assuring Assumption 3.1 imply it):

(a) $\tilde{D} \succcurlyeq 0 \Leftrightarrow \tilde{D} + \tilde{D}^\top \succcurlyeq 0$. Using Lemma A.3, equivalently we have:

1. $\text{Im}(G^\top F^\top) \subseteq \text{Im}(D + D^\top)$,
2. $D + FG + (D + FG)^\top \succcurlyeq FG(D + D^\top)^\dagger G^\top F^\top$. A sufficient condition for this to hold, is $D + FG \succ 0$, and $\sigma_{\max}(FG(D + D^\top)^\dagger G^\top F^\top) < \sigma_{\max}(D + FG + (D + FG)^\top) = \lambda_{\max}(D + FG + (D + FG)^\top)$. From [8, Corollary 9.6.5] we have: $\sigma_{\max}(FG(D + D^\top)^\dagger G^\top F^\top) \leq \sigma_{\max}^2(FG)\sigma_{\max}((D + D^\top)^\dagger)$. If $D + D^\top = 0$ the matrix inequality is satisfied. If $D + D^\top \neq 0$, $\text{rank}(D + D^\top) = d$, then $\sigma_{\max}((D + D^\top)^\dagger) = \sigma_d^{-1}(D + D^\top)$ [8, Fact 6.3.28], and $\sigma_d(D + D^\top) = \lambda_d(D + D^\top)$ since $D + D^\top \succcurlyeq 0$ is symmetric. Thus a sufficient condition is $\sigma_{\max}^2(FG) < \lambda_d(D + D^\top)\lambda_{\max}(D + FG + (D + FG)^\top)$. Another sufficient condition, following similar steps, is $D + D^\top \succ 0$, $\sigma_{\max}(FG) < \lambda_d(D + FG + (D + FG)^\top)\lambda_{\max}(D + D^\top)$. We see that in both cases we allow for $D + D^\top$ of $D + FG + (D + FG)^\top$ to be low rank, which hampers $\tilde{D} \succ 0$. Finally, we may use Lemmas A.3, A.5, as well as the matrix decompositions described after Lemma A.5, to get necessary and sufficient conditions.

(b) $\text{Im}(\tilde{D}) \subseteq \text{Im}(\tilde{C}) \Leftrightarrow \text{Im}(D + FG) + \text{Im}(FG) \subseteq \text{Im}(C + FK)$ and $\text{Im}(D) \subseteq \text{Im}(C + FK)$.

(c) $\text{Ker}(\tilde{D} + \tilde{D}^\top) \subseteq \text{Ker}(P\tilde{B} - \tilde{C}^\top)$ for some $\mathbb{R}^{n \times n} \ni P = P^\top \succ 0 \Leftrightarrow \{(x, y) \in \mathbb{R}^{m \times m} \mid ((D + FG) + (D + FG)^\top)x - FGy = 0, -G^\top F^\top x + (D + D^\top)y = 0\} \subseteq \{(x, y) \in \mathbb{R}^{m \times m} \mid x - y \in \text{Ker}(P(B + EG) - (C + FK)^\top)\}$. This is verified if $FG = 0$ and Assumption 3.3 holds, since in this case $\text{Ker}(D + D^\top) \subseteq \text{Ker}(P(B + EG) - (C + FK)^\top)$ [29] [25, section 3.8]. If D and $D + FG$ are P-matrices, and if $FG(D + D^\top)^{-1}G^\top F^\top \neq D + FG + (D + FG)^\top$, then $\text{Ker}(\tilde{D} + \tilde{D}^\top) = \{0\}$, hence the inclusion holds. If $FG(D + D^\top)^{-1}G^\top F^\top = D + FG + (D + FG)^\top$, then $\text{Ker}(\tilde{D} + \tilde{D}^\top) = \mathbb{R}^m \times \mathbb{R}^m$, hence the inclusion holds only if $P(B + EG) = (C + FK)^\top$.

The various conditions stated in section 2 are sufficient only, so it may be that some of them are unnecessary in some cases (*e.g.*, condition b) above is needed for continuity arguments using the Vladimirov's pseudo-distance defined in [84] as done in [82, 60, 59], but it is not necessary at all [23, section 5.3] [28]). Condition b) is not necessary for the well-posedness of (18) when both D and $D + FG$ are P-matrices. The point is also that the error system (18) or (24), is a specific interconnection of both subsystems, which does not necessarily inherits good properties of the subsystems (*e.g.*, $D \succcurlyeq 0$ and $D + FG \succcurlyeq 0$ may not imply $\tilde{D} \succcurlyeq 0$). Conditions in a) and c) are necessary for the passivity of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ [29] [25, section 3.8].

The stability proof shows that under the proposition's assumptions (like the continuity of the solutions) the generalized equation $0 \in \tilde{A}e^* - \tilde{B}(\tilde{D} + \mathcal{N}_{\tilde{S}(t)}^{-1})^{-1}(\tilde{C}e^*)$ has the unique solution $e^* = 0$ for all $t \geq 0$.

3.3 Error Dynamics Stability Analysis including State Jumps

An important question is whether or not the tracking control framework developed above is suitable for trajectories with discontinuities. Let us study how to relax Assumption 3.1 and the continuity of the closed-loop state $x(\cdot)$. State jumps are known to add difficulty to the trajectory tracking problem, especially when the jump times are unknown. The so-called peaking phenomenon, due to non-synchronized jumps in the plant and the desired trajectories, has long been known to be one of the obstacles. The jumps in $x(\cdot)$ and $x_d(\cdot)$ may arise from different reasons:

1. Let the conditions in section 2.1 item 2 (a) or item 3 hold for (A, B, C, D) . Jumps at t in $x_d(\cdot)$ occur only if Fu_d is discontinuous at t and $\text{rank}(D) < m$.
2. Let the conditions in section 2.1 item 2 (a) or item 3 hold for $(A + EK, B + EG, C + FK, D + FG)$. Jumps at t in the closed-loop state occur only if $\text{rank}(D + FG) < m$ and $-FKx_d - FG\lambda_d + Fu_d$ is discontinuous at t .

The following is deduced, where \mathcal{T}_f denotes the set of discontinuity times of the function $f(\cdot)$:

Lemma 3.9. *Let $FG\lambda_d(\cdot)$ and $u_d(\cdot)$ be bounded functions of time. (a) $\mathcal{T}_{x_d} \subseteq \{t_0\} \cup \mathcal{T}_{Fu_d}$ and (b) $\mathcal{T}_x \subseteq \{t_0\} \cup \mathcal{T}_{Fu_d} \cup \mathcal{T}_{FG\lambda_d}$.*

Proof. (a) is obvious. (b): $\mathcal{T}_x \subseteq \{t_0\} \cup \mathcal{T}_{-FKx_d - FG\lambda_d + Fu_d} \subseteq \{t_0\} \cup \mathcal{T}_{FKx_d} \cup \mathcal{T}_{FG\lambda_d} \cup \mathcal{T}_{Fu_d} \subseteq \{t_0\} \cup \mathcal{T}_{FG\lambda_d} \cup \mathcal{T}_{Fu_d}$ since $\mathcal{T}_{FKx_d} \subseteq \mathcal{T}_{Fu_d}$. \square

Lemma 3.9 indicates that when $FG\lambda_d$ is continuous, then either $x_d(\cdot)$ jumps while $x(\cdot)$ is continuous, or the inverse, or they jump simultaneously. This will be illustrated on examples. Notice that when $x_d(\cdot)$ jumps, then λ_d is a Dirac measure, and the meaning of the feedback in (15) has to be carefully studied, as well as the mere meaning of the complementarity constraints in (16) which contain the term $FG\lambda_d$. Also depending on FG , the sum of two Dirac measures, one stemming from λ_d (due to a jump in $u_d(t)$) and one stemming from λ (due to a jump in $-FKx_d - FG\lambda_d + Fu_d$)

could occur in the differential part of the closed-loop dynamics (see section 6): we may call this a forbidden situation (this is not to be confused with the case of Measure Differential Inclusions, or with passive LCS with state jumps [23, 24, 27]). It is inferred that:

(C1) If $x_d(\cdot)$ jumps, we impose $FG = 0$. A sufficient condition to prevent forbidden situations, is that $FG = 0$ if $u_d(\cdot)$ is discontinuous. In other words, only continuous $u_d(\cdot)$ is allowed if $FG \neq 0$.

In view of condition **(C1)**, we can refine item 2: Jumps in the closed-loop state can occur at t if:

1. $FG = 0$ and $-FKx_d + Fu_d$ is discontinuous at t ,
2. or $FG \neq 0$, x_d is continuous at t and λ_d jumps at t (which can occur if u_d is continuous at t at a junction time with the constraint boundary $(Cx_d + D\lambda_d + Fu_d)_i = 0$ for some $1 \leq i \leq m$).

Conditions in item 2 are not straightforward, because we want that $FG \neq 0$, but FG also multiplies λ , see (16), and passivity implies $D + FG \succcurlyeq 0$. The scalar case $m = 1$ is analysed in section 6.

The first step is to characterize the jumps in (18), then to study the variation $\Delta V(e(t)) = V(e(t^+)) - V(e(t^-)) = (e(t^+) - e(t^-))^\top P(e(t^+) - e(t^-))$.

Case $G = 0$ and $D = 0$

In this case λ_d does not enter (16), and $\tilde{D} = 0$. Let Assumptions 3.1 and 3.3 hold, and let us study the case when $x(\cdot)$ jumps while $x_d(\cdot)$ is continuous. Then $(A + EK, B, C + FK, 0)$ is strictly state passive, and $PB = (C + FK)^\top$ for some $P = P^\top \succ 0$. The closed-loop system in (21) becomes:

$$\dot{x}(t) \in (A + EK)x(t) - BN_{S(t)}((C + FK)x(t)) - EKx_d(t) + Eu_d(t). \quad (28)$$

where $S(t) = \{v \in \mathbb{R}_+^m \mid v - FKx_d(t) + Fu_d(t) \geq 0\}$. This DI can be rewritten equivalently as a FOSWP, see item 3 in section 2 and [23, 24, 12], as:

$$\dot{\zeta}(t) \in R(A + EK)R^{-1}\zeta(t) + RE(-Kx_d(t) + u_d(t)) - \mathcal{N}_{\Phi(t)}(\zeta(t))$$

where $R^2 = P$, $R = R^\top \succ 0$, $\zeta = Rx$, $\Phi(t) = \{Rx \mid (C + FK)x - FKx_d(t) + Fu_d(t) \in S(t)\}$. We see that $\Phi(t) \neq \emptyset$ for each t if and only if there exists x such that $(C + FK)x - FKx_d(t) + Fu_d(t) \geq 0$. This is guaranteed if a condition as in item 3 iii) holds, i.e., $\text{Im}(C + FK) - \mathbb{R}_+^m = \mathbb{R}^m$ (which is a constraint qualification). Assume in addition that $u_d(\cdot)$ is continuous on \mathbb{R}_+ . Suppose that $Cx_d(0) + Fu_d(0) \geq 0$, hence $x_d(\cdot)$ is continuous at $t = 0$. Then a jump in $x(\cdot)$ (thus in $e(\cdot)$) can occur only at the initial time and $V(0^+) - V(0^-) = e(0^+)^\top Pe(0^+) - e(0^-)^\top Pe(0^-) = (e(0^+) + e(0^-))^\top P(e(0^+) - e(0^-)) = (x(0^+) + x(0^-) - 2x_d(0))^\top P(x(0^+) - x(0^-)) = x(0^+)^\top Px(0^+) - x(0^-)^\top Px(0^-) - 2x_d(0)^\top P(x(0^+) - x(0^-))$. Now using the passivity of $(A + EK, B, C + FK, 0)$, we have (see the paragraph State Jumps at the end of section 2): $x(0^+)^\top Px(0^+) - x(0^-)^\top Px(0^-) \leq 0$, provided that $0 \in \{z \in \mathbb{R}^n \mid (C + FK)z - FKx_d(0) + Fu_d(0) \geq 0\}$, equivalently $0 \in \Phi(0)$, equivalently $-FKx_d(0) + Fu_d(0) \geq 0$. In this case, $V(0^+) - V(0^-) \leq -2x_d(0)^\top P(x(0^+) - x(0^-))$. We have $P(x(0^+) - x(0^-)) \in -\mathcal{N}_{\mathcal{K}}(x(0^+))$ [23, Lemma 2.3]. Thus it is necessary and sufficient that $x_d(0) \in (\mathcal{N}_{\mathcal{K}}(x(0^+)))^\circ = \mathcal{T}_{\mathcal{K}}(x(0^+))$ to guarantee that the right-hand side is nonpositive. Therefore, the following has been proved:

Lemma 3.10. *Assume that $u_d(\cdot)$ is time-continuous, $G = 0$, $D = 0$, $-FKx_d(0) + Fu_d(0) \geq 0$, and $x_d(0) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_P[\mathcal{K}; x(t_0^-)])$. Then at an initial state jump, we have $V(0^+) - V(0^-) \leq 0$, where $P = P^\top \succ 0$ is a solution of the passivity LMI associated with the triple $(A + EK, B, C + FK)$ and $V(t) = V(e(t))$.*

The condition on $x_d(0)$ is certainly not easy to check in general (but $x_d(0) = 0$ is always suitable). The interest of Lemma 3.10 is that it allows for a jump in $x(0)$ while $x_d(0)$ does not jump. But applying it at any time of state jump $t_k > 0$ implies to impose a suitable desired state, which may not be possible in our framework where x_d is generated by (14). Under the same conditions, assume that $Fu_d(\cdot)$ has a discontinuity at time t_c . From (14) and (16), both $x(\cdot)$ and $x_d(\cdot)$ may jump at t_c , so λ and λ_d are Dirac measures at t_c . From (16) the discontinuity may act in both terms FKx_d and Fu_d . Using (21) and (23), this implies that the jumps' magnitudes in the sets $S(t)$ and $S_d(t)$, may not be equal. The post-jump states are computed using (9), where $\mathcal{K} = \{z \in \mathbb{R}^n \mid (C + FK)z - FKx_d(t) + Fu_d(t) \geq 0\}$ for the closed-loop plant, and $\mathcal{K}_d = \{z \in \mathbb{R}^n \mid Cz + Fu_d(t) \geq 0\}$ for the desired system.

Extension for further jumps at $t > 0$

Let us now place ourselves in another perspective. Until now we have assumed that $x_d(\cdot)$ is time-continuous. First notice that condition **(C1)** can be relaxed, if we admit that the state $x_d(\cdot)$ of the dynamics in (14) can be arbitrarily reset to some value at arbitrary times, without considering λ_d as a Dirac measure. This has important consequences, because as we shall see this means that the desired dynamics is no longer autonomous (the desired state has to be modified online, a common feature in trajectory tracking for systems undergoing state jumps [69, 68, 74]). Consider (16), with Assumption 3.3.

Proposition 3.11. *Let $V(e) = e^\top P e$, with P a solution of the closed-loop passivity LMI, and let $\{0\} \in \mathcal{K} = \{z \in \mathbb{R}^n \mid (C + FK)z - FKx_d - FG\lambda_d(t^+) + Fu_d(t) \in Q_{D+FG}^*\}$. Assume that $x_d(t^+) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_P[\mathcal{K}; x(t^-)])$ and $x_d(t^+)^\top P x(t^-) \geq x_d(t^-)^\top P x(t^-)$, then $\Delta V(e(t)) \leq (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-))$.*

Proof. At a state jump time:

$$\begin{aligned}
\Delta V(e(t)) &= (x(t^+) - x(t^-))^\top P (x(t^+) + x(t^-)) - (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&\quad - (x_d(t^+) - x_d(t^-))^\top P (x(t^+) + x(t^-)) + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&\leq -(x(t^+) - x(t^-))^\top P (x_d(t^+) + x_d(t^-)) - (x_d(t^+) - x_d(t^-))^\top P (x(t^+) + x(t^-)) \\
&\quad + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&= -(x_d(t^+) + x_d(t^-))^\top P (x(t^+) - x(t^-)) - (x_d(t^+) - x_d(t^-))^\top P (x(t^+) - x(t^-)) \\
&\quad - 2(x_d(t^+) - x_d(t^-))^\top P x(t^-) + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&= -2x_d(t^+)^\top P (x(t^+) - x(t^-)) - 2(x_d(t^+) - x_d(t^-))^\top P x(t^-) \\
&\quad + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-))
\end{aligned} \tag{29}$$

The first term in the first equality in (29) is nonpositive from the passivity. It is noteworthy that we cannot infer the same conclusions about the last term in (29) from (17) because these dynamics are equivalent to that in (14). Let us consider the last equality in (29). We know that $P(x(t^+) - x(t^-)) \in -\mathcal{N}_{\mathcal{K}}(x(t^+))$ [23, Lemma 2.3]. Thus the nonpositivity of the first term is equivalent to $x_d(t^+) \in (\mathcal{N}_{\mathcal{K}}(x(t^+)))^\circ = \mathcal{T}_{\mathcal{K}}(x(t^+))$. Using (9) the first condition follows. The second condition is obvious. \square

Thus under the conditions of Proposition 3.11, $\Delta V(e(t)) \leq 0$ if and only if $x_d(t^+)^\top P x_d(t^+) \leq x_d(t^-)^\top P x_d(t^-)$, which means that the desired state jump is dissipative with respect to the closed-loop storage function (in general there is no reason that it should satisfy this property). This makes a set of constraints that the reset desired state $x_d(t^+)$ has to satisfy. It is noteworthy that the conditions of Proposition 3.11 are sufficient only. Examples show that they may not be satisfied, while $\Delta V(e(t)) \leq 0$, see section 6.4. Also, we note that the reset mechanism is not needed when $G = 0$ (then condition **(C1)** holds true), while the characterization of $\Delta V(e(t))$ as in (29) remains valid.

Using the DI (25)

The formalism in (24) is not convenient for the state jumps analysis, because it involves a state-dependent set. Let us recall the error dynamics in (25) to analyze the state discontinuities.

$$\dot{e}(t) \in \bar{A}e(t) - \bar{B}(\bar{D} + \mathcal{N}_{\bar{S}(t)}^{-1})(\bar{C}e(t)) \Leftrightarrow \begin{cases} \dot{e}(t) = \bar{A}e(t) + \bar{B}\lambda(t) \\ 0 \leq \lambda(t) \perp \bar{C}e(t) + \bar{D}\lambda(t) + \bar{F}(t) \geq 0 \end{cases} ,$$

with $\bar{A} = \tilde{A}$, $\bar{B} = \tilde{B}$, $\bar{C} = \begin{pmatrix} C + FK \\ 0 \end{pmatrix}$, $\bar{D} = \tilde{D}$, $\bar{S}(t) = \{z \in \mathbb{R}^{2m} \mid z + \bar{F}(t) \in \mathbb{R}_+^{2m}\}$, $\bar{F}(t) = \begin{pmatrix} Cx_d(t) + Fu_d(t) \\ Cx_d(t) + Fu_d(t) \end{pmatrix}$. At a state jump time t , the DI in (25) may be written as:

$$e(t^+) - e(t^-) \in -\bar{B}(\bar{D} + \mathcal{N}_{\bar{S}(t^+)}^{-1})(\bar{C}e(t^+)).$$

which has a solution (possibly with jumps) if the quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is passive with positive definite storage function. Then (9) holds. Thus it is inferred that $\Delta V(e(t)) \leq 0$ provided that $0 \in \mathcal{K} = \{e \in \mathbb{R}^n \mid \bar{C}e + \bar{F}(t) \in Q_{\bar{D}}^*\}$, and:

$$\begin{pmatrix} -P(A + EK) - (A + EK)^\top P & -P(B + EG) + (C + FK)^\top & P(B + EG) \\ -(B + EG)^\top P + C + FK & D + FG + (D + FG)^\top & FG \\ (B + EG)^\top P & (FG)^\top & D + D^\top \end{pmatrix} \succcurlyeq 0, \tag{30}$$

where $P = P^\top \succ 0$ is a solution to $-M \succcurlyeq 0$ with M in (20). Notice that (30) holds only if $M \preceq 0$, hence only if P is also a solution of the plant's closed-loop LMI. Thus, the solution P of (30) must be the solution of NLMI in (20). From Assumption 3.3 we have $-M \succcurlyeq 0$. Using Lemma A.3, we infer that (30) holds if and only if:

1. $\bar{D} + \bar{D}^\top \succcurlyeq 0$ (see item a) at the end of section 3.2 for sufficient conditions),
2. $\text{Im} \begin{pmatrix} -(B + EG)^\top P + C + FK \\ (B + EG)^\top P \end{pmatrix} \subseteq \text{Im}(\bar{D} + \bar{D}^\top)$,
3.
$$-M \succcurlyeq \begin{pmatrix} -P(B + EG) + (C + FK)^\top & P(B + EG) \end{pmatrix} (\bar{D} + \bar{D}^\top)^\dagger \begin{pmatrix} -(B + EG)^\top P + C + FK \\ (B + EG)^\top P \end{pmatrix} \quad (31)$$

Some comments arise:

- A sufficient condition for (31) to hold is $\bar{D} = -\bar{D}^\top \iff D + FG = -(D + FG)^\top$, $D = -D^\top$, $FG = 0$. This is allowed by strict state passivity in Assumption 3.3. However item 2 then implies that $-(B + EG)^\top P + C + FK = 0$ and $(B + EG)^\top P = 0$, hence $C + FK = 0$ and $B + EG = 0$ since P is full-rank. Thus in (16) the ODE part and the complementarity part are decoupled.
- Assume that $D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_1 \succ 0$. Item 1 implies that $FG = \begin{pmatrix} (FG)_1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\bar{D} + \bar{D}^\top = \begin{pmatrix} \bar{D}_1 + \bar{D}_1^\top & 0 \\ 0 & 0 \end{pmatrix}$, and $(\bar{D} + \bar{D}^\top)^\dagger = \begin{pmatrix} (\bar{D}_1 + \bar{D}_1^\top)^\dagger & 0 \\ 0 & 0 \end{pmatrix}$, with $\bar{D}_1 + \bar{D}_1^\top = \begin{pmatrix} D_1 + (FG)_1 + D_1 + (FG)_1^\top & 0 & -(FG)_1 \\ 0 & 0 & 0 \\ -(FG)_1^\top & 0 & D_1 + D_1^\top \end{pmatrix}$. Similar calculations can be done for M in (20), and also for the ranges inclusion in item 2. Pursuing the calculations and matrices partitions allows to simplify (31).
- Assume that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is passive. A jump in $e(\cdot)$ occurs at t only if $\bar{F}(\cdot)$ jumps at t . In turn $\bar{F}(\cdot)$ jumps at t if and only if $Cx_d + Fu_d$ jumps at t , while $x_d(\cdot)$ jumps only if $u_d(\cdot)$ does. Hence $e(\cdot)$ jumps at t only if $u_d(\cdot)$ does. How is this related to the jumps in $x(\cdot)$ in (16) and in $x_d(\cdot)$ in (14)? As we saw above, if $x_d(\cdot)$ jumps at t_c then λ_d is a Dirac measure at t_c and the complementarity constraints in (16) are meaningless at t_c . This means that applying an impulsive feedback control to an LCS (2) with $FGu \neq 0$, has to be avoided as pointed out in condition (C1). In fact, the DIs in (24), (25), (21), and (22) are constructed with the underlying assumption that the complementarity conditions can be rewritten equivalently as inclusion into a normal cone (see, e.g., [23, Equ. (B.1)] [76, Corollary 23.5.4]). For instance, $f(t)$ in (22) is a Dirac measure if $EG\lambda_d$ is. This might let one think that in this case, $x(\cdot)$ jumps at t_c . We set that this is true if $F = 0$ only, in which case $x_d(\cdot)$ can jump only initially, see (14). Obviously, if λ_d is a Dirac measure, both formalisms (complementarity and inclusion) require further analysis for their understanding if $FG \neq 0$.
- The conditions in this paragraph are different from those of the foregoing paragraph, because now we consider both the desired system and the closed-loop plant simultaneously, instead of looking at (28) only. However, jumps in both $x(\cdot)$ and $x_d(\cdot)$ are permitted under some conditions as stated above. A quick examination of the LMI in (30) shows that the occurrence of state jumps is quite restricted in this context. Indeed, if $D + D^\top = 0$, then (30) implies that $B + EG = 0$ and $FG = 0$. In fact, the LMI in (30) shows that studying state jumps from the passivity error dynamics in (25) is almost impossible.
- The controller may be impulsive if $G \neq 0$ and a jump occurs in $x(\cdot)$. Condition (C1) still applies.

Analysis of jump sets \mathcal{K} and \mathcal{K}_d

These sets are crucial in the state jump characterization and computation, see (8). We have:

$$\mathcal{K} = \{z \in \mathbb{R}^n \mid Cz + Fu_d(t^+) + FK(z - x_d(t^+)) \in Q_{D+FG}^*\} \quad (32)$$

and

$$\mathcal{K}_d = \{z \in \mathbb{R}^n \mid Cz + Fu_d(t^+) \in Q_D^*\} \quad (33)$$

In general both sets are different. However if $FG = 0$ then they differ only by the term $FK(z - x_d(t^+))$. Since $x(t^+)$ belongs to \mathcal{K} , both sets are almost equal if the tracking error $x(t^-) - x_d(t^-)$ and FK are very small. So if in addition both storage functions matrices P (calculated from the closed-loop system LMI) and P_d (computed from the desired dynamics LMI when this dynamics is passive) are closed one to each other, both state jumps are almost the same as well. On the other hand, the state jump as computed in (8) is independent on P (or P_d , respectively), provided it is a solution of the passivity LMI. Thus, if $EG = 0$, which implies that the "input" matrix of the closed-loop system is equal to the input matrix of the desired system, the forms of \mathcal{K} and \mathcal{K}_d in (32) and (33) will imply that both jumps in x and x_d are close one to each other.

The jump-mismatch (peaking) phenomenon

This is a well-known phenomenon in trajectory tracking when state jumps are present, as recalled in the introduction. The problem that is faced in the following is twofold: 1) is the mismatch issue present? If it is, how can it be coped with? It will appear that if the plant's parameters are known, the peaking is not present in our systems, because passivity constraints prevent it from occurring: [did we prove this really?](#) passivity forces the jumps (in either x or x_d or both) to occur in such a way that the Lyapunov function of the error system decreases. Things are different when parameter uncertainty is considered. In the scalar case treated in section 6, it will be shown that if both states jump then they jump at the same time. However, it is also possible that x jumps while x_d does not, and *vice versa*.

4 Robustness Analysis: Parametric Uncertainties

It is of interest to analyse the tracking problem when the plant's dynamics have uncertainties. In this case, the desired dynamics' matrices in (14), and the plant's model matrices, differ. The desired dynamics (14) has to be designed using a nominal plant model.

4.1 Controller Design

The controller $u(t) = K_0[x(t) - x_d(t)] + G_0[\lambda(t) - \lambda_d(t)] + u_d(t)$ is designed from the plant's nominal quadruple (A_0, B_0, C_0, D_0) , along the same procedure as in the foregoing section. Therefore the desired system (14) is represented by the following LCS:

$$\begin{cases} \dot{x}_d(t) = A_0 x_d(t) + B_0 \lambda_d(t) + E_0 u_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = C_0 x_d(t) + D_0 \lambda_d(t) + F_0 u_d(t) \geq 0. \end{cases} \quad (34)$$

If strong passivity is used (Assumption 3.4) instead of strict state passivity as in the foregoing section (Assumption 3.3), the controller gains are computed assuming there exist matrices K_0 and G_0 such that the inequality (20) is satisfied for the nominal plant, *i.e.*,

$$M_0 \triangleq \begin{pmatrix} (A_0 + E_0 K_0)^\top P_0 + P_0 (A_0 + E_0 K_0) & P_0 (B_0 + E_0 G_0) - (C_0 + F_0 K_0)^\top \\ (B_0 + E_0 G_0)^\top P_0 - (C_0 + F_0 K_0) & -D_0 - F_0 G_0 - (D_0 + F_0 G_0)^\top \end{pmatrix} \prec 0 \quad (35)$$

has a solution $P_0 = P_0^\top \succ 0$. The plant dynamics is represented as follows

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A)x(t) + (B_0 + \Delta B)\lambda(t) + (E_0 + \Delta E)u(t) \\ 0 \leq \lambda(t) \perp w(t) = (C_0 + \Delta C)x(t) + (D_0 + \Delta D)\lambda(t) + (F_0 + \Delta F)u(t) \geq 0 \end{cases} \quad (36)$$

where $\Delta A = A - A_0$, ΔB , ΔC , ΔD , ΔE and ΔF represent additive uncertainties. Sufficient conditions on the uncertainties upperbounds are calculated in the next section so that some stability is guaranteed. The closed-loop system is given by:

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A + (E_0 + \Delta E)K_0)x(t) + (B_0 + \Delta B + (E_0 + \Delta E)G_0)\lambda(t) \\ \quad - (E_0 + \Delta E)(K_0 x_d(t) + G_0 \lambda_d(t) - u_d(t)) \\ 0 \leq \lambda(t) \perp w(t) = (C_0 + \Delta C + (F_0 + \Delta F)K_0)x(t) + (D_0 + \Delta D + (F_0 + \Delta F)G_0)\lambda(t) \\ \quad - (F_0 + \Delta F)(K_0 x_d(t) + G_0 \lambda_d(t) - u_d(t)) \geq 0 \end{cases} \quad (37)$$

It is noteworthy that the well-posedness of (37) may not be guaranteed for any uncertainties. Some of the results in section 2 can be used. The following assumption is supposed to hold in this section:

Assumption 4.1. The closed-loop system (37) is well-posed, i.e., it has unique AC solutions for any initial condition $x(t_0) = x_0$ satisfying $w(t_0, x_0) \geq 0$, equivalently:

$$x_0 \in \text{Dom} \left((D_0 + \Delta D + (F_0 + \Delta F)G_0 + \partial\sigma_{S(t)})^{-1} \right) = \text{Im} \left((D_0 + \Delta D + (F_0 + \Delta F)G_0 + \partial\sigma_{S(t)}) \right),$$

with $S(t) = \{\nu \in \mathbb{R}^m \mid \nu + (F_0 + \Delta F)(K_0 x_d + G_0 \lambda_d - u_d) \geq 0\}$.

Remark 4.2. It is clear that time-varying uncertainties give rise to a time-varying LCS in (37). As pointed out in section 2.2, it is only in few particular cases that the well-posedness of time-varying LCS has been studied. In this section stability relies on the fact that the closed-loop nominal system is strongly passive, and that the closed-loop plant feedthrough matrix $(D_0 + \Delta D + (F_0 + \Delta F)G_0) \succ 0$. Thus item 1 in 2 applies if this matrix is constant.

4.2 Error Dynamics Stability Analysis

The error dynamics is given by:

$$\begin{cases} \dot{e}(t) = (A_0 + E_0 K_0)e(t) + (B_0 + E_0 G_0)\Delta\lambda(t) + \Delta A x(t) + \Delta B \lambda(t) + \Delta E K_0 e(t) + \Delta E G_0 \Delta\lambda(t) \\ \quad + \Delta E u_d(t) \\ \Delta w(t) = w(t) - w_d(t) = (C_0 + F_0 K_0)e(t) + (D_0 + F_0 G_0)\Delta\lambda(t) + \Delta C x(t) + \Delta D \lambda(t) + \Delta F K_0 e(t) \\ \quad + \Delta F G_0 \Delta\lambda(t) + \Delta F u_d(t) \\ 0 \leq w(t) \perp \lambda(t) \geq 0 \quad \text{and} \quad 0 \leq w_d(t) \perp \lambda_d(t) \geq 0 \end{cases} \quad (38)$$

with $\Delta\lambda(t) = \lambda(t) - \lambda_d(t)$. Let:

$$p(x, t, \lambda) \triangleq \Delta A x(t) + \Delta B \lambda(t) + \Delta E K_0 e(t) + \Delta E G_0 \Delta\lambda(t) + \Delta E u_d(t) \quad (39)$$

and

$$q(x, t, \lambda) \triangleq \Delta C x(t) + \Delta D \lambda(t) + \Delta F K_0 e(t) + \Delta F G_0 \Delta\lambda(t) + \Delta F u_d(t). \quad (40)$$

Let us now state the stability result.

Proposition 4.3. Let Assumptions 3.1, 3.2, 3.4 hold for the nominal system. Let

$$\begin{aligned} \Delta A^\top \Lambda_A^{-1} \Delta A \preceq I_n, \quad \Delta B^\top \Lambda_B^{-1} \Delta B \preceq I_m, \quad \Delta C^\top \Lambda_C^{-1} \Delta C \preceq I_n \\ \Delta D^\top \Lambda_D^{-1} \Delta D \preceq I_m, \quad \Delta E^\top \Lambda_E^{-1} \Delta E \preceq I_p, \quad \Delta F \Lambda_F^{-1} \Delta F^\top \preceq I_m \end{aligned} \quad (41)$$

hold for any $\Lambda_k = \Lambda_k^\top \succ 0$, $k \in \{A, B, C, D, E, F\}$, and assume there exist P_0 , K_0 and G_0 such that the matrix inequality

$$\left(\begin{array}{cc|ccccccc} -(M_0)_{11} & -(M_0)_{12} & P_0 & P_0 & K_0^\top & I_n & 0 & 0 & 0 \\ -(M_0)_{12} & -(M_0)_{22} & 0 & 0 & 0 & 0 & G_0^\top & I_m & I_m \\ \hline P_0 & 0 & \Lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ P_0 & 0 & 0 & \tilde{\Lambda}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ K_0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & \frac{1}{2}I_n & 0 & 0 & 0 \\ 0 & G_0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & \Lambda_2^{-1} & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_2^{-1} \end{array} \right) \succ 0 \quad (42)$$

and $P_0 = P_0^\top \succ 0$, with $\tilde{\Lambda}_F \triangleq I_p + \Lambda_F$, $\tilde{\Lambda}_1 \triangleq \Lambda_A + 2\Lambda_E + \Lambda_B$, $\tilde{\Lambda}_2 \triangleq \Lambda_C + \Lambda_D + 4I_m$. Then, the solution of the error dynamics in (38) is globally uniformly ultimately bounded. Note that $(M_0)_{ij}$ denotes the element of the matrix M_0 in (35) located at the i th row and j th column where $i, j \in \{1, 2\}$.

Proof. The derivative of the Lyapunov function candidate $V(t) = e^\top P_0 e$ along the closed-loop trajectories is calculated as follows:

$$\dot{V} = e^\top \left[(A_0 + E_0 K_0)^\top P_0 + P_0 (A_0 + E_0 K_0) \right] e + 2e^\top P_0 (B_0 + E_0 G_0) \Delta\lambda + 2e^\top P_0 p(x, t, \lambda)$$

In matrix form

$$\begin{aligned} \dot{V}(t) = & \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} (A_0 + E_0 K_0)^\top P_0 + P_0 (A_0 + E_0 K_0) & P_0 (B_0 + E_0 G_0) \\ (B_0 + E_0 G_0)^\top P_0 & 0 \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \\ & + 2e^\top P_0 p(x, t, \lambda) \\ & \pm \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} 0 & (C_0 + F_0 K_0)^\top \\ C_0 + F_0 K_0 & D_0 + F_0 G_0 + (D_0 + F_0 G_0)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \end{aligned}$$

The purpose of the \pm term is to obtain the matrix in (35) and

$$\begin{aligned} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} 0 & (C_0 + F_0K_0)^\top \\ C_0 + F_0K_0 & D_0 + F_0G_0 + (D_0 + F_0G_0)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} &= 2\Delta\lambda^\top [(C_0 + F_0K_0)e + (D_0 + F_0G_0)\Delta\lambda] \\ &= 2\Delta\lambda^\top (\Delta w - q(x, t, \lambda)) \end{aligned}$$

Then,

$$\dot{V}(t) = \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M_0 \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + 2e^\top P_0 p(x, t, \lambda) + 2\Delta\lambda^\top (\Delta w - q(x, \lambda)).$$

Let us substitute the values of p and q and write explicitly the following

$$\begin{aligned} 2e^\top P_0 p(x, t, \lambda) - 2\Delta\lambda^\top q(x, \lambda) &= 2e^\top P_0 [\Delta Ax + \Delta B\lambda + \Delta EK_0 e + \Delta EG_0 \Delta\lambda + \Delta E u_d \\ &\quad \pm \Delta Ax_d \pm \Delta B\lambda_d] \\ &\quad - 2\Delta\lambda^\top [\Delta Cx + \Delta D\lambda + \Delta FK_0 e + \Delta FG_0 \Delta\lambda \\ &\quad + \Delta F u_d \pm \Delta Cx_d \pm \Delta D\lambda_d]. \end{aligned}$$

Hence,

$$\begin{aligned} 2e^\top P_0 p(x, t, \lambda) - 2\Delta\lambda^\top q(x, \lambda) &= \\ \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} (\Delta A + \Delta EK_0)^\top P_0 + P_0(\Delta A + \Delta EK_0) & P_0(\Delta B + \Delta EG_0) - (\Delta C + \Delta FK_0)^\top \\ (\Delta B + \Delta EG_0)^\top P_0 - (\Delta C + \Delta FK_0) & -\Delta D - \Delta FG_0 - (\Delta D + \Delta FG_0)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \\ + 2e^\top P_0 (\Delta Ax_d + \Delta B\lambda_d + \Delta E u_d) - 2\Delta\lambda^\top (\Delta Cx_d + \Delta D\lambda_d + \Delta F u_d) \end{aligned}$$

Let

$$\Delta M_0 \triangleq \begin{pmatrix} -(\Delta A + \Delta EK_0)^\top P_0 - P_0(\Delta A + \Delta EK_0) & -P_0(\Delta B + \Delta EG_0) + (\Delta C + \Delta FK_0)^\top \\ -(\Delta B + \Delta EG_0)^\top P_0 + (\Delta C + \Delta FK_0) & \Delta D + \Delta FG_0 + (\Delta D + \Delta FG_0)^\top \end{pmatrix}$$

Thus,

$$\dot{V} \leq - \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top (-M_0 + \Delta M_0) \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + 2e^\top P_0 (\Delta Ax_d + \Delta B\lambda_d + \Delta E u_d) - 2\Delta\lambda^\top (\Delta Cx_d + \Delta D\lambda_d + \Delta F u_d)$$

Let $a(t) \triangleq \Delta Ax_d + \Delta B\lambda_d + \Delta E u_d$ and $b(t) \triangleq \Delta Cx_d + \Delta D\lambda_d + \Delta F u_d$. For any $\Lambda_1^\top = \Lambda_1 \succ 0$ and $\Lambda_2^\top = \Lambda_2 \succ 0$, it holds that:

$$\begin{aligned} |2e^\top P_0 a(t)| &\leq e^\top P_0 \Lambda_1 P_0 e + a^\top(t) \Lambda_1^{-1} a(t) \\ |2\Delta\lambda^\top b(t)| &\leq \Delta\lambda^\top \Lambda_2 \Delta\lambda + b^\top(t) \Lambda_2^{-1} b(t) \end{aligned}$$

So,

$$\begin{aligned} \dot{V} &\leq -z^\top \left[-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] z + a^\top(t) \Lambda_1^{-1} a(t) + b^\top(t) \Lambda_2^{-1} b(t) \\ &\leq -\lambda_{\min} \left[-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] \|z\|^2 + \lambda_{\min}^{-1}(\Lambda_1) \|a(t)\|^2 + \lambda_{\min}^{-1}(\Lambda_2) \|b(t)\|^2 \end{aligned}$$

where $z(t) \triangleq \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}$. Let us prove that if the conditions, (41) and (42) hold, then the matrix $-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$ is positive definite. For this purpose, the upper-bounding of ΔM_0 can be done term by term as:

$$\begin{aligned} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \Delta M_0 \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} &= -2e^\top P_0 (\Delta A + \Delta EK_0) e - 2e^\top P_0 (\Delta B + \Delta EG_0) \Delta\lambda \\ &\quad + 2e^\top (\Delta C + \Delta FK_0)^\top \Delta\lambda + 2\Delta\lambda^\top (\Delta D + \Delta FG_0) \Delta\lambda \end{aligned}$$

Thus, for any $\Lambda_k = \Lambda_k^\top \succ 0$, $k \in \{A, B, C, D, E, F\}$

$$2e^\top P_0 (\Delta A + \Delta EK_0) e \geq -e^\top [P_0 (\Lambda_A + \Lambda_E) P_0 + \Delta A^\top \Lambda_A^{-1} \Delta A + K_0^\top \Delta E^\top \Lambda_E^{-1} \Delta E K_0] e$$

$$2\Delta\lambda^\top(\Delta B + \Delta E G_0)^\top P_0 e \geq -e^\top [P_0(\Lambda_B + \Lambda_E)P_0]e - \Delta\lambda^\top [\Delta B^\top \Lambda_B^{-1} \Delta B + G_0^\top \Delta E^\top \Lambda_E^{-1} \Delta E G_0] \Delta\lambda$$

$$2\Delta\lambda^\top(\Delta C + \Delta F K_0)e \geq -\Delta\lambda^\top [\Lambda_C + \Delta F \Lambda_F^{-1} \Delta F^\top] \Delta\lambda - e^\top [\Delta C^\top \Lambda_C^{-1} \Delta C + K_0^\top \Lambda_F K_0] e$$

$$2\Delta\lambda^\top(\Delta D + \Delta F G_0)\Delta\lambda \geq -\Delta\lambda^\top [\Lambda_D + \Delta D^\top \Lambda_D^{-1} \Delta D + G_0^\top \Lambda_F G_0 + \Delta F \Lambda_F^{-1} \Delta F^\top] \Delta\lambda$$

Assume that the constraints on uncertainties in (41) stated in Proposition 4.3 are satisfied, then

$$\begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \Delta M_0 \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \geq - \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} P_0 \tilde{\Lambda}_1 P_0 + K_0^\top \tilde{\Lambda}_F K_0 + 2I_n & 0 \\ 0 & \tilde{\Lambda}_2 + G_0^\top \tilde{\Lambda}_F G_0 \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}$$

Thus:

$$\begin{aligned} z^\top & \left[-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] z \geq \\ z^\top & \begin{pmatrix} (-M_0)_{11} - P_0 \Lambda_1 P_0 - P_0 \tilde{\Lambda}_1 P_0 - K_0^\top \tilde{\Lambda}_F K_0 - 2I_n & -(M_0)_{12} \\ -(M_0)_{12} & -(M_0)_{22} - \tilde{\Lambda}_2 - \Lambda_2 - G_0^\top \tilde{\Lambda}_F G_0 \end{pmatrix} z \end{aligned} \quad (43)$$

Applying the Schur complement Lemma A.3 for positive definiteness, it is inferred that the matrix obtained in (43) is positive definite if and only if the matrix inequality (42) in Proposition 4.3 holds. The inequality obtained in (42) is a nonlinear matrix inequality due to the upper-left sub-matrix M_0 . Then, in order to solve this inequality, it must be transformed into LMI, according

$$\text{to Appendix A.2, by multiplying the left and right-hand sides with the matrix } \begin{pmatrix} Q_0 & 0 & \dots & 0 \\ 0 & I & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & I \end{pmatrix}$$

where $Q_0 = P_0^{-1}$ and by defining a new variable $N_0 = K_0 Q_0$. It follows that

$$M_{0_{in}} \triangleq - \begin{pmatrix} -Q_0 A_0^\top - Q_0 A_0 - N_0^\top E_0^\top - E_0 N_0 & -B_0 - E_0 G_0 + Q_0 C_0^\top + N_0^\top F_0^\top \\ -B_0^\top - G_0^\top E_0^\top + C_0 Q_0 + F_0 N_0 & D_0 + F_0 G_0 + (D_0 + F_0 G_0)^\top \end{pmatrix} \quad (44)$$

So, the inequality in (42) is written as;

$$\begin{pmatrix} \begin{array}{cc|ccccccc} -(M_{0_{in}})_{11} & -(M_{0_{in}})_{12} & I_n & I_n & N_0^\top & Q_0 & 0 & 0 & 0 \\ -(M_{0_{in}})_{21} & -(M_{0_{in}})_{22} & 0 & 0 & 0 & 0 & G_0^\top & I_m & I_m \end{array} \\ \hline I_n & 0 & \Lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & \tilde{\Lambda}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ N_0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 & 0 & 0 \\ Q_0 & 0 & 0 & 0 & 0 & \frac{1}{2}I_n & 0 & 0 & 0 \\ 0 & G_0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & \Lambda_2^{-1} & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_2^{-1} \end{array} \succ 0 \quad (45)$$

where $(M_{0_{in}})_{ij}$ denotes the element of the matrix $M_{0_{in}}$ in (44) located at the i th row and j th column with $i, j \in \{1, 2\}$. The LMI in (45) can be solved, under some conditions, in the new variables $Q_0 = Q_0^\top \succ 0$, N_0 and G_0 . Thus, it is possible to say that the inequality:

$z^\top \left[-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] z \succ 0$ holds. As a consequence of the result obtained about positive definiteness of the matrix obtained in (43) and due to symmetry, there exists $\mu > 0$ such that $-\lambda_{\min} \left[M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] = -\mu < 0$. According to Assumption 3.1 where u_d , x_d and λ_d are bounded, the terms $\|a(t)\|^2$ and $\|b(t)\|^2$ are bounded such that $\|a(t)\|^2 < \beta_1$ and $\|b(t)\|^2 < \beta_2$ for some $\beta_1, \beta_2 > 0$. Then, it is inferred that:

$$\begin{aligned} \dot{V} & \leq -\mu \|z\|^2 + \lambda_{\min}^{-1}(\Lambda_1) \beta_1 + \lambda_{\min}^{-1}(\Lambda_2) \beta_2 \\ & \leq -\mu \|e\|^2 - \mu \|\Delta\lambda\|^2 + \lambda_{\min}^{-1}(\Lambda_1) \beta_1 + \lambda_{\min}^{-1}(\Lambda_2) \beta_2 \\ & \leq -\mu \|e\|^2 + \lambda_{\min}^{-1}(\Lambda_1) \beta_1 + \lambda_{\min}^{-1}(\Lambda_2) \beta_2 \end{aligned}$$

It shows that $\dot{V} < 0$ outside the ball $B_r(0) \subset \mathbb{R}^n$, with $r \triangleq \sqrt{\frac{\lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2}{\mu}}$. Then, the solution of (38) is GUUB [57, 43]. According to Theorem A.2, the ultimate bound with $\alpha_1(\|e\|) = \lambda_{\min}(P_0)\|e\|^2$ and $\alpha_2(\|e\|) = \lambda_{\max}(P_0)\|e\|^2$ is given by

$$\|e\| \leq \alpha_1^{-1}(\alpha_2(r)) = \sqrt{\frac{\lambda_{\max}(P_0)r^2}{\lambda_{\min}(P_0)}} = \sqrt{\frac{\lambda_{\max}(P_0)(\lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2)}{\lambda_{\min}(P_0)\mu}}$$

□

Notice that under the conditions of the proposition, then $M_0 + \Delta M_0 \succ 0$. This implies that the closed-loop system's quadruple in (37) is strongly passive (in other words the uncertainties do not destroy the strong passivity of the nominal closed-loop system). In particular this implies that $D_0 + \Delta D + (F_0 + \Delta F)G_0 \succ 0$. Consequently, Assumption 3.1 and item 1 in section 2 guarantee that the solutions of (37) are AC as long as uncertainties are constant matrices. Time-varying uncertainties yield a closed-loop system which can be analyzed along the lines of section 2.2.

4.3 Relaxing Strong Passivity to Strict State Passivity

It is of interest to relax the strong passivity condition (Assumption 3.4) of Proposition 4.3, with strict state passivity (Assumption 3.3). The strict state passivity BMI in (35) modified to $M_0 \preceq \begin{pmatrix} -\epsilon P_0 & 0 \\ 0 & 0 \end{pmatrix}$. This means that the nominal closed-loop system may have $D_0 + F_0G_0 \succ 0$, hence $M_0 \preceq 0$. Therefore we have to find conditions which guarantee

$$-M_0 + \Delta M_0 - \begin{pmatrix} \epsilon' P_0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \succcurlyeq 0 \quad (46)$$

for some $0 < \epsilon' < \epsilon$, where M_0 corresponds to the strict state passivity LMI. To this end, we may rely on Lemma A.3 in Appendix A.5.

Proposition 4.4. *Let Assumptions 3.1, 3.2, 3.3 hold for the nominal system, and Assumption 4.1 holds for the closed-loop system. Then the matrix inequality in (46) holds if and only if:*

1. $R \triangleq (D_0 + F_0G_0) + (D_0 + F_0G_0)^\top + (\Delta D + \Delta FG_0) + (\Delta D + \Delta FG_0)^\top - \Lambda_2 \succcurlyeq 0$,
2. $Q \triangleq -(A_0 + E_0K_0 + \frac{\epsilon'}{2}I_n)^\top P_0 - P_0(A_0 + E_0K_0 + \frac{\epsilon'}{2}I_n) - (\Delta A + \Delta EK_0)^\top P_0 - P_0(\Delta A + \Delta EK_0) - P_0\Lambda_1P_0 \succcurlyeq 0$, with $-(A_0 + E_0K_0)^\top P_0 - P_0(A_0 + E_0K_0) \succcurlyeq \epsilon P_0$, $\epsilon > \epsilon' > 0$,
3. $\text{Im}(S^\top) \subseteq \text{Im}(R)$, with $S \triangleq -P_0(B_0 + E_0G_0) + (C_0 + F_0K_0)^\top - P_0(\Delta B + \Delta EG_0) + (\Delta C + \Delta FK_0)^\top$,
4. $Q \succcurlyeq SR^\dagger S^\top$.

Proof. Notice that Q and R are symmetric by construction. The proof follows from Lemma A.3. □

The tracking error ultimate boundedness can be shown as follows.

Proposition 4.5. *Assume that Proposition 4.4 holds, then the tracking error e is globally ultimately bounded as:*

$$\|e\| \leq \sqrt{\frac{\lambda_{\max}(P_0)\eta_r}{\lambda_{\min}^2(P_0)\epsilon'}} \quad (47)$$

Proof. Under the proposition's assumptions, the rate of change of the Lyapunov function $V(e) = e^\top P_0 e$ where $P_0 = Q_0^{-1}$ is the solution of the strict state passivity LMI $M_{0_{lin}} \preceq \begin{pmatrix} -\epsilon Q_0 & 0 \\ 0 & 0 \end{pmatrix}$ where $M_{0_{lin}}$ is defined in (44) is:

$$\begin{aligned} \dot{V} &\leq - \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \left[-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} + \underbrace{a^\top(t)\Lambda_1^{-1}a(t) + b^\top(t)\Lambda_2^{-1}b(t)}_{r(t)} \\ &\leq -\epsilon' e^\top P_0 e + r(t) \end{aligned}$$

where $\epsilon' > 0$ and $r(t)$ is a positive bounded term determined by the parametric uncertainties ΔA , ΔB , ΔC , ΔD , ΔE , and ΔF such that $\|r(t)\| \leq \eta_r$. So,

$$\dot{V} \leq -\epsilon' \lambda_{\min}(P_0) \|e\|^2 + \eta_r$$

Hence, the rate of change of the storage function $\dot{V} < 0$ outside the ball $B_r(0) \subset \mathbb{R}^n$, with $r \triangleq \sqrt{\frac{\eta_r}{\epsilon' \lambda_{\min}(P_0)}}$. Therefore, the tracking error is GUUB [57, 43] and the expression of the ultimate bound is in (47). \square

The difference between the case with strong passivity and the case with strict state passivity, is mainly that there is no $-||\Delta\lambda||^2$ that helps to accelerate the convergence in the second case (see (4.2)). Moreover the strong passivity allows us to dispense with strict conditions on the structure of the uncertainties. In a sense, strict state passivity allows us to obtain a more fragile robustness.

It is of interest to analyze under which conditions on the plant's nominal model and on the uncertainties, the conditions in items 1–4 in Proposition 4.4 hold. For item 1, assume that $\Delta D = 0$, $\Delta F = 0$, $\Delta C = 0$ (the complementarity constraint has no uncertainty) then we may take $\Lambda_2 = 0$ (since $b(t) = 0$), so that $R \succcurlyeq 0$ by closed-loop strict-state passivity. For item 2, using the fact that $Q_0 \succ 0$, Corollary A.6 can be used to derive sufficient condition such that $Q \succ 0$: $\sigma_{\max}(Q_0) > \epsilon \lambda_{\max}(P_0) > 2\sigma_{\max}(P_0)(\sigma_{\max}(\Delta A) + \sigma_{\max}(\Delta EK_0)) + \sigma_{\max}^2(P_0)\sigma_{\max}(\Lambda_1)$, where we used [8, Fact 9.14.15, Corollary 9.6.5]. Let us focus on items 3 and 4. We have $-M_0 = \begin{pmatrix} Q_0 & S_0 \\ S_0^\top & R_0 \end{pmatrix}$,

and $-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} Q_0 + \Delta Q & S_0 + \Delta S \\ S_0^\top + \Delta S^\top & R_0 + \Delta R \end{pmatrix}$, where the matrices stem from (35) and items 1, 2 and 3 in Proposition 4.4. Using Lemmata A.3 and A.5, the strict-state passivity LMI for the nominal system implies that $Q_0 \succ Q_0 - \epsilon P_0 \succcurlyeq S_0 R_0^\dagger S_0^\top$, and $\text{Im}(S_0 R_0^\dagger S_0^\top) \subseteq \text{Im}(Q_0 - \epsilon P_0)$ (we also have $\text{Im}(S_0 R_0^\dagger S_0^\top) \subseteq \text{Im}(Q_0) = \mathbb{R}^n$ since strict state passivity implies that $Q_0 \succ 0$), and $\text{Im}(S_0^\top) \subseteq \text{Im}(R_0) = \text{Im}(R_0^\dagger)$ [9, Proposition 8.1.7]. Assume that $\text{Im}(\Delta S^\top) \subseteq \text{Im}(S_0^\top)$. Then $\text{Im}(S^\top) = \text{Im}(S_0^\top + \Delta S^\top) \subseteq \text{Im}(S_0^\top)$. Thus item 3 is satisfied if $\text{Im}(R) = \text{Im}(R_0)$ (which holds if $\Delta D = 0$, $\Delta F = 0$, $\Delta C = 0$, since in this case $\Delta R = 0$). We can therefore state the following:

Lemma 4.6. *Let Assumption 3.3 hold for the nominal system. Assume that $\text{Im}(\Delta S^\top) \subseteq \text{Im}(S_0^\top)$, and that $\text{Im}(R) = \text{Im}(R_0)$, then $\text{Im}(S^\top) \subseteq \text{Im}(R)$.*

Notice that item 4 is $Q_0 + \Delta Q \succcurlyeq (S_0 + \Delta S)(R_0 + \Delta R)^\dagger (S_0^\top + \Delta S^\top)$. In the next proposition we derive conditions on the uncertainty matrices, that guarantee that the inequality in item 4 holds true when the nominal system is strictly state passive.

Proposition 4.7. *Let Assumption 3.3 hold for the quadruple (A_0, B_0, C_0, D_0) . Assume that: (i) $R \succcurlyeq 0$, (ii) $\text{Im}(\Delta R) \subseteq \text{Im}(R_0)$, (iii) $\text{rank}(R_0) = r$, $R_0 \neq 0$, $\sigma_{\max}(\Delta R) < \sigma_r(R_0)$, (iv) $\sigma_{\max}(OT) + \sigma_{\max}(\mathcal{O}((R_0^\dagger \Delta R)^2)) + \sigma_{\max}(\Delta Q) < \epsilon \lambda_{\max}(P_0)$, where OT is in (48). Then $Q \succcurlyeq SR^\dagger S^\top$.*

Proof. Since $\Delta R = \Delta R^\top$, using [9, Fact 8.4.3] and [9, Fact 8.4.38] and (ii), we have $(R_0 + \Delta R)^\dagger = (I_m + R_0^\dagger \Delta R)^\dagger (R_0^\dagger + R_0^\dagger \Delta R R_0^\dagger)^\dagger (I_m + \Delta R R_0^\dagger)^\dagger$. Using Corollary A.6, [9, Fact 8.3.33], [9, Corollary 11.6.5], it follows from (iii) that $I_m + R_0^\dagger \Delta R \succ 0$. Thus $(I_m + R_0^\dagger \Delta R)^\dagger = (I_m + R_0^\dagger \Delta R)^{-1} = \sum_{k=0}^{\infty} (-R_0^\dagger \Delta R)^k = I_m - R_0^\dagger \Delta R + \mathcal{O}((R_0^\dagger \Delta R)^2)$ [9, Proposition 11.3.10]. Therefore $(R_0 + \Delta R)^\dagger = (I_m - R_0^\dagger \Delta R)(R_0^\dagger + R_0^\dagger \Delta R R_0^\dagger)^\dagger (I_m - \Delta R R_0^\dagger) + \mathcal{O}((R_0^\dagger \Delta R)^2) = R_0^\dagger - R_0^\dagger \Delta R R_0^\dagger + \mathcal{O}((R_0^\dagger \Delta R)^2)$. Consequently, $(S_0 + \Delta S)(R_0 + \Delta R)^\dagger (S_0^\top + \Delta S^\top) = (S_0 + \Delta S)(R_0^\dagger - R_0^\dagger \Delta R R_0^\dagger + \mathcal{O}((R_0^\dagger \Delta R)^2))(S_0^\top + \Delta S^\top) = S_0 R_0^\dagger S_0^\top + OT + \mathcal{O}((R_0^\dagger \Delta R)^2)$, where OT stands for other terms defined as:

$$OT = S_0 R_0^\dagger \Delta S^\top - S_0 R_0^\dagger \Delta R R_0^\dagger S_0^\top - S_0 R_0^\dagger \Delta R R_0^\dagger \Delta S^\top + \Delta S R_0^\dagger S_0^\top + \Delta S R_0^\dagger \Delta S^\top - \Delta S R_0^\dagger \Delta R R_0^\dagger S_0^\top - \Delta S R_0^\dagger \Delta R R_0^\dagger \Delta S^\top. \quad (48)$$

The result follows from $Q_0 - S_0 R_0^\dagger S_0^\top \succcurlyeq \epsilon P_0$, (iv), Corollary A.6 and [9, Fact 11.16.18], which guarantee that $\epsilon P_0 \succ OT - \Delta Q + \mathcal{O}((R_0^\dagger \Delta R)^2)$, so that $Q_0 + \Delta Q \succcurlyeq (S_0 + \Delta S)(R_0 + \Delta R)^\dagger (S_0^\top + \Delta S^\top)$. \square

Item (iv) in Proposition 4.7 means that a class of nonzero, sufficiently small uncertainties are allowed. The matrix Λ_1 which appears in ΔQ can be chosen small if ΔA , ΔB , ΔE are small. As said above, ΔR can be made small if the uncertainties inside the complementarity constraints are small. In some cases there are no uncertainties in the complementarity constraints.

5 Tracking Control for First-order Sweeping Process (FOSwP)

Under some conditions (see, *e.g.*, section A.8), LCS can be equivalently rewritten as a FOSwP. Let us first rewrite both the closed-loop plant and the desired dynamics under the perturbed FOSwP format, when $D = 0$ in (14) and $D + FG = 0$ in (16). Here we are interested to see how the material in sections 3.2, 3.3 and 4.1, 4.2 and 4.3, adapt to trajectory tracking applied to FOSwP. Indeed the control of FOSwP has recently received much attention, especially its optimal control, stabilization, and the existence of periodic orbits [52, 37, 51, 39, 6, 38, 40, 42, 41, 32, 33, 70, 85, 56]. Thus it is of interest to investigate the trajectory tracking issue for such DIs.

The LCS of the desired system in (14) can be represented equivalently as a FOSwP when $D = 0$ if the conditions in section 2, item 3 are satisfied. These conditions require the system represented by the quadruple $(A, B, C, 0)$ to be passive which implies the condition $PB = C^\top$ where $P = P^\top \succ 0$ is the solution of the matrix inequality of strict passivity in (19), the controller $u \in \mathcal{L}_{loc}^1(\mathbb{R}_+; \mathbb{R}^p)$ and the constraint qualification $\text{Im}(C) - \mathbb{R}_+^m = \mathbb{R}^m$ holds. The FOSwP is represented as follows (see section A.8)

$$\dot{\zeta}_d(t) \in R_d A R_d^{-1} \zeta_d(t) - \mathcal{N}_{\phi_d(t)}(\zeta_d(t)) \quad (49)$$

where $R_d^2 = P_d$, $R_d = R_d^\top$, $\zeta_d = R_d x_d$ and $\phi_d(t) = \{R_d x_d \mid C x_d \in S_d(t)\}$ with $S_d(t) = \{v \in \mathbb{R}^m \mid v + F u_d(t) \geq 0\}$ and $x_d(t) = R_d^{-1} \zeta_d(t)$. Thus, $\phi_d(t) = \{\zeta_d \in \mathbb{R}^m \mid C R_d^{-1} \zeta_d + F u_d(t) \geq 0\}$.

Recall that the closed-loop system in (3.3) when $D + FG = 0$ is the following FOSwP:

$$\dot{\zeta}(t) \in R(A + EK)R^{-1}\zeta(t) + RE(-KR_d^{-1}\zeta_d(t) + u_d(t)) - \mathcal{N}_{\Phi(t)}(\zeta(t)) \quad (50)$$

where $\phi(t) = \{Rx \mid (C + FK)x \in S(t)\}$ with $S(t) = \{v \in \mathbb{R}^m \mid v - FKx_d(t^+) + F u_d(t) \geq 0\}$ and $\zeta(t) = R^{-1}x(t)$. It is noteworthy that the set $\phi(t)$ depends on the post-jump desired state x_d^+ . This means that it can be characterized only once the desired state jump has been calculated. Thus, $\phi(t) = \{\zeta \in \mathbb{R}^m \mid (C + FK)R^{-1}\zeta - FK R_d^{-1}\zeta_d(t^+) + F u_d(t) \geq 0\}$.

Recall that $f(t^+) = \lim_{s \rightarrow t^+} f(s)$ and $f(t^-) = \lim_{s \rightarrow t^-} f(s)$. It is noteworthy that $x \in \mathcal{K}$ if and only if $\zeta \in \phi(t)$ and $x_d \in \mathcal{K}_d$ if and only if $\zeta_d \in \phi_d(t)$.

Taking advantage of the FOSwP formalism, the aim in the following is to show under which conditions $x_d^+ = x^+$ holds. As a first step, let us check if $x_d^+ \in \partial\mathcal{K}$, it is required to show that $x_d^+ \in \mathcal{K}$ (the set \mathcal{K} is defined for (16)). Knowing that $x_d^+ \in \mathcal{K}_d$, then

$$C x_d^+ + F u_d \geq 0 \Leftrightarrow (C + FK)x_d^+ - FK x_d^+ + F u_d \geq 0 \quad (51)$$

which means that x_d^+ satisfies the condition of the set \mathcal{K} . Hence $x_d^+ \in \mathcal{K}$. In addition, given that $x_d^+ \in \partial\mathcal{K}_d$, it follows that

$$[C x_d^+ + F u_d]_k = 0 \Leftrightarrow [(C + FK)x_d^+ - FK x_d^+ + F u_d]_k = 0 \quad \text{for some } k \in \{1, \dots, m\} \quad (52)$$

Thus, the following is proved:

Lemma 5.1. *The post-jump desired state satisfies: $x_d^+ \in \partial\mathcal{K} \cap \partial\mathcal{K}_d$. So a necessary condition to have $x^+ = x_d^+$ is that $x^+ \in \partial\mathcal{K}$, which is equivalent to $x^- \notin \text{Int}(\mathcal{K})$.*

Assume that $P(x^- - x_d^+) \in \mathcal{N}_{\mathcal{K}}(x_d^+)$, equivalently $x_d^+ = \text{Proj}_P[\mathcal{K}; x^-] = x^+$ (the last equality holds from (9)). The question that arises is: what are the conditions such that the first inclusion holds? Let us present the given information in the following. The optimization problem in (9) for the desired system leads to the following:

$$x_d^+ = \text{Proj}_{P_d}[\mathcal{K}_d; x_d^-] \Leftrightarrow P_d(x_d^- - x_d^+) \in \mathcal{N}_{\mathcal{K}_d}(x_d^+) \quad (53)$$

Similarly, according to the optimization problem in (9) for the closed-loop system, the following is derived:

$$P(x^- - x^+) \in \mathcal{N}_{\mathcal{K}}(x^+) \quad (54)$$

Using the definition of the normal cone in (1), the equations in (53) and (54) are written as follows, for the desired and closed-loop systems, respectively:

$$(x_d^- - x_d^+)^\top P_d(x_d^- - x_d^+) \leq 0 \quad \forall x_d \in \mathcal{K}_d \quad (55)$$

$$(x^- - x^+)^\top P(x^- - x^+) \leq 0 \quad \forall x \in \mathcal{K} \quad (56)$$

And the third inclusion for $x_d^+ = x^+$ is equivalent to the VI:

$$(x^- - x_d^+)^\top P(x - x_d^+) \leq 0 \quad \forall x \in \mathcal{K} \quad (57)$$

Let $\zeta_d = R_d x_d$ and $\zeta = Rx$. Note that $x_d^+ \in \partial\mathcal{K}$ if and only if $Rx_d^+ \in \partial\phi(t) \Leftrightarrow RR_d^{-1}\zeta_d^+ \in \partial\phi(t)$, then the state $RR_d^{-1}\zeta_d$ becomes relevant. Hence, the set $RR_d^{-1}\phi_d(t)$ is considered in the following since $RR_d^{-1}\zeta_d \in RR_d^{-1}\phi_d(t)$. Let $\zeta'_d = RR_d^{-1}\zeta_d = Rx_d$ and $\phi'_d(t) = RR_d^{-1}\phi_d(t) = RK_d$ which is defined as:

$$\phi'_d(t) = \{\zeta'_d \in \mathbb{R}^2 \mid CR^{-1}\zeta'_d + Fu_d \geq 0\} \quad (58)$$

The desired system in (49) is written, in the terms of the new variable $\zeta'_d = RR_d^{-1}\zeta_d$, as follows:

$$\dot{\zeta}'_d \in RAR^{-1}\zeta'_d - RP_d^{-1}RN_{\phi'_d(t)}(\zeta'_d) \quad (59)$$

Similarly, $x^+ \in \partial\mathcal{K}$ if and only if $\zeta^+ \in \partial\phi(t)$. Hence, equivalently the three VIs are obtained:

$$(\zeta_d^- - \zeta_d^+)^\top (\zeta_d - \zeta_d^+) \leq 0 \quad \forall \zeta_d \in \phi'_d(t) \quad (60)$$

$$(\zeta^- - \zeta^+)^\top (\zeta - \zeta^+) \leq 0 \quad \forall \zeta \in \phi(t) \quad (61)$$

$$(\zeta^- - \zeta_d^+)^\top (\zeta - \zeta_d^+) \leq 0 \quad \forall \zeta \in \phi(t) \quad (62)$$

The optimization problem in (54) for the closed-loop FOSwP in (50) is written as follows:

$$\zeta^+ = \text{Proj}[\phi(t), \zeta^-] \Leftrightarrow \zeta^- - \zeta^+ \in \mathcal{N}_{\phi(t)}(\zeta^+) \quad (63)$$

Then, it is required to prove under which conditions $\zeta^- - \zeta_d^+ \in \mathcal{N}_{\phi(t)}(\zeta_d^+)$ (which is the VI (62)) holds, equivalently $\zeta_d^+ = \zeta^+$ (i.e., $R^{-1}\zeta^+ = R_d^{-1}\zeta^+ \Leftrightarrow x_d^+ = x^+$). Let us study this on a particular example.

Example 1. Consider the desired FOSwP in (49). Take $A = -I$, $B = I$, and $C = I$ where I is the 2×2 identity matrix. Let $E = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$ and $F = \begin{pmatrix} 2 & 1 \end{pmatrix}^\top$. The LMI in (19) for the desired system's quadruple $(A, B, C, 0)$ has a solution $P_d = P_d^\top = I \succ 0$. Thus, the desired system is strictly state passive. Given that $R_d^2 = P_d$, then $R_d = I$. Let us check if there exist K and P such that the closed-loop system's quadruple $(A + EK, B, C + FK, 0)$ is strictly passive. This means that it is required to check if the matrix inequality in (19) has a solution. The solution of (19) is given by

$$K = \begin{pmatrix} 0.07 & 0.035 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1.14 & 0.07 \\ 0.07 & 1.035 \end{pmatrix} \quad (64)$$

Given that $R^2 = P$, then $R = \begin{pmatrix} 1.07 & 0.034 \\ 0.034 & 1.017 \end{pmatrix}$. Thus the FOSwP associated with the closed-loop system in (50) is written as follows:

$$\begin{pmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} \in \begin{pmatrix} -0.93 & 0.034 \\ 0.034 & -1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} - \begin{pmatrix} 0.14 & 0.07 \\ 0.07 & 0.034 \end{pmatrix} \begin{pmatrix} \zeta_{1d}^+ \\ \zeta_{2d}^+ \end{pmatrix} + \begin{pmatrix} 1.07 \\ 0.034 \end{pmatrix} u_d - \mathcal{N}_{\phi(t)}(\zeta) \quad (65)$$

where $\mathcal{N}_{\phi(t)}(\zeta) = \{v \in \mathbb{R}^2 \mid v^\top (\zeta - \eta) \leq 0 \quad \forall \eta \in \phi(t)\}$ and $\phi(t)$ is given by:

$$\begin{aligned} \phi(t) &= \{\zeta \in \mathbb{R}^2 \mid (C + FK)R^{-1}\zeta - FKR^{-1}\zeta_d^+(t^+) + Fu_d(t) \geq 0\} \\ &= \left\{ \zeta \in \mathbb{R}^2 \mid \begin{pmatrix} 1.065 & 0.034 \\ 0.034 & 1.016 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} - \begin{pmatrix} 0.129 & 0.065 \\ 0.064 & 0.032 \end{pmatrix} \begin{pmatrix} \zeta_{1d}^+ \\ \zeta_{2d}^+ \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_d \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \end{aligned} \quad (66)$$

The desired FOSwP in (59) is written as follows:

$$\begin{pmatrix} \dot{\zeta}'_{1d} \\ \dot{\zeta}'_{2d} \end{pmatrix} \in \begin{pmatrix} \zeta'_{1d} \\ \zeta'_{2d} \end{pmatrix} - \begin{pmatrix} 1.14 & 0.07 \\ 0.07 & 1.035 \end{pmatrix} \mathcal{N}_{\phi'_d(t)}(\zeta'_d) \quad (67)$$

where $\mathcal{N}_{\phi'_d(t)}(\zeta'_d) = \{v \in \mathbb{R}^2 \mid v^\top (\zeta'_d - \eta_d) \leq 0 \quad \forall \eta_d \in \phi'_d(t)\}$ and $\phi'_d(t)$ is given by:

$$\phi'_d(t) = \{\zeta'_d \in \mathbb{R}^2 \mid CR^{-1}\zeta'_d + Fu_d \geq 0\} = \left\{ \zeta'_d \in \mathbb{R}^2 \mid \begin{pmatrix} 0.936 & -0.031 \\ -0.031 & 0.984 \end{pmatrix} \begin{pmatrix} \zeta'_{1d} \\ \zeta'_{2d} \end{pmatrix} + \begin{pmatrix} 2u_d \\ u_d \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \quad (68)$$

Let us define the set of active constraints for the desired system in (67) as:

$$A_d(\phi'_d(t)) = \{i \in \{1, 2\} \mid (CR^{-1})_{i\bullet} \zeta_d^{i+} + F_{i\bullet} u_d(t) = 0 \text{ and } \zeta_d^{i+} \in \phi'_d(t)\} \quad (69)$$

and we define $\Omega_{id} \triangleq \left\{ \zeta_d^{i+} \in \mathbb{R}^2 \mid \begin{pmatrix} 0.936 & -0.031 \\ -0.031 & 0.984 \end{pmatrix}_{i\bullet} \zeta_d^{i+} + \begin{pmatrix} 2u_d \\ u_d \end{pmatrix} = 0 \text{ and } \zeta_d^{i+} \in \phi'_d(t) \right\}$, for $i \in \{1, 2\}$. Let us draw the set $\phi'_d(t)$ in (68) as shown below in Fig. 1:

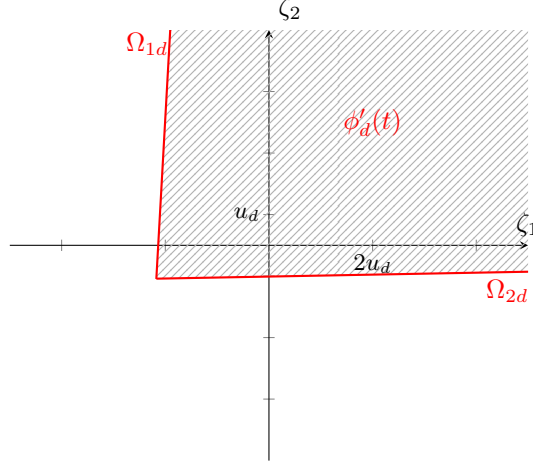


Figure 1

It is noticeable that the set $\phi(t)$ in (66) depends on the value of the state ζ_d^{i+} . This implies that there exists a unique set $\phi(t)$ associated with each ζ_d^{i+} which is the solution to the optimization problem in (53) with a specific set of active constraints applied. Let us consider the following cases based on the different values of ζ_d^{i+} , resulting in different sets of $\phi(t)$.

5.1 Case 1: the state $\zeta_d^{i+} \in \Omega_{1d} \cap \Omega_{2d}$

In this case, both constraints Ω_1 and Ω_2 in (69) for the desired system are considered active. According to the set of active constraints $A_d(\phi'_d(t))$ in (69) and given that the matrix CR^{-1} is invertible in this example, it follows that the state variable ζ_d^{i+} is expressed as $\zeta_d^{i+} = -RC^{-1}Fu_d \in \Omega_{1d} \cap \Omega_{2d}$. By substituting the value of ζ_d^{i+} in the set $\phi(t)$ in (66), it gives the following expression of $\phi(t)$:

$$\phi(t) = \{\zeta \in \mathbb{R}^2 \mid ((C + FK)R^{-1})\zeta + (FKC^{-1}F)u_d(t) + Fu_d(t) \geq 0\} \quad (70)$$

Figure 2 below shows both sets $\phi(t)$ and $\phi_d(t)$.

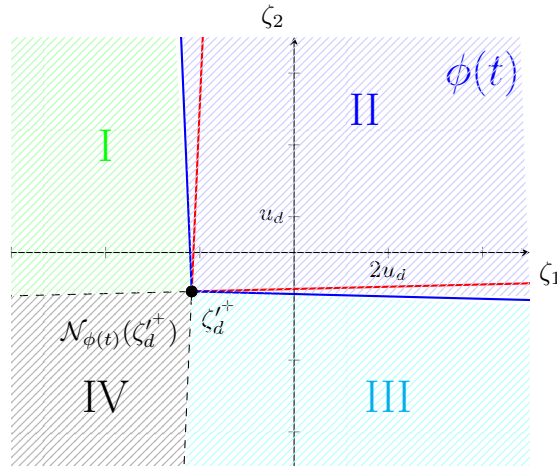


Figure 2

By observing Figure 2, it is noticed that the state ζ_d^{i+} is located at the corner of the set $\phi'_d(t)$ (i.e., $\zeta_d^{i+} \in \Omega_{1d} \cap \Omega_{2d}$ defined in (69)), and this corner represented by ζ_d^{i+} is shared by both sets $\phi'_d(t)$

and $\phi(t)$ such that $\zeta_d^{r+} \in \partial\phi(t) \cap \partial\phi'_d(t)$. Let us define the active constraints for the closed-loop system in (65) as follows:

$$A(\phi(t)) = \{i \in \{1, 2\} \mid ((C + FK)R^{-1})_{i\bullet}\zeta^+ + (FKC^{-1}F)_{i\bullet}u_d(t) + F_{i\bullet}u_d(t) = 0 \text{ for } \zeta^+ \in \phi(t)\} \quad (71)$$

and we define $\Omega_i \triangleq \left\{ \zeta^+ \in \mathbb{R}^2 \mid \begin{pmatrix} 1.065 & 0.034 \\ 0.034 & 1.016 \end{pmatrix}_{i\bullet} \zeta^+ + \begin{pmatrix} 2.35u_d \\ 1.75u_d \end{pmatrix}_{i\bullet} = 0 \text{ for } \zeta^+ \in \partial\phi(t) \right\}$, for $i \in \{1, 2\}$.

- If $\zeta^- \in \text{Region I}$ (i.e., $\zeta^- \notin \phi(t)$) and the active constraint is Ω_1 defined in (71), then the post-jump state $\zeta^+ \in \Omega_1$ and is defined as $\zeta^+ = \text{Proj}[\Omega_1; \zeta^-]$.
- If $\zeta^- \in \text{Region II}$ (i.e., $\zeta^- \in \text{int } \phi(t)$), then $\zeta^+ = \zeta^- \in \text{int } \phi(t)$.
- If $\zeta^- \in \text{Region III}$ (i.e., $\zeta^- \notin \phi(t)$) and the active constraint is Ω_2 as defined in (71), then $\zeta^+ \in \Omega_2$.
- If $\zeta^- \in \text{Region IV}$ (i.e., $\zeta^- - \zeta_d^{r+} \in \mathcal{N}_{\phi(t)}(\zeta_d^{r+})$) and both constraints Ω_1 and Ω_2 are active as defined in (71), then $\zeta^+ = \zeta_d^{r+}$.

Thus, regardless of the active constraint, $\zeta^+ = \zeta_d^{r+}$ if and only if $\zeta^- - \zeta_d^{r+} \in \mathcal{N}_{\phi(t)}(\zeta_d^{r+})$.

5.2 Case 2: the state $\zeta_d^{r+} \in \Omega_{1d}$

In this section, only one of the constraints for the desired system is considered active which is Ω_{1d} as defined in (69). Let $\zeta_d^{r+} = (-2.104u_d, u_d)^\top \in \Omega_{1d}$ and let us substitute the value of ζ_d^{r+} in the set $\phi(t)$ in (66), then:

$$\phi(t) = \left\{ \zeta \in \mathbb{R}^2 \mid \begin{pmatrix} 1.065 & 0.034 \\ 0.034 & 1.016 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} - \begin{pmatrix} 0.129 & 0.065 \\ 0.064 & 0.032 \end{pmatrix} \begin{pmatrix} -2.1036u_d \\ u_d \end{pmatrix} + \begin{pmatrix} 2u_d \\ u_d \end{pmatrix} \geq 0 \right\} \quad (72)$$

and it is represented graphically in Figure 3 below showing the different regions to be studied.

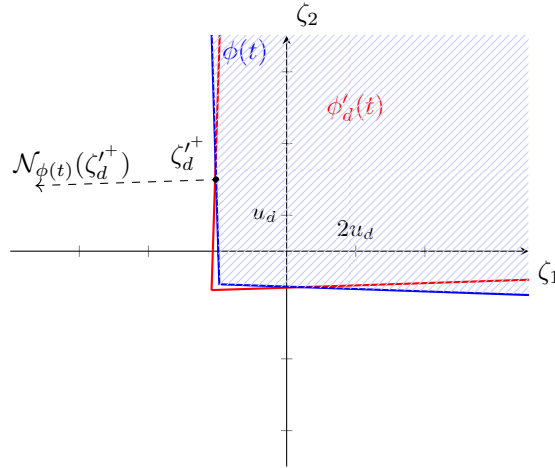


Figure 3

By observing Figure 3, the state ζ_d^{r+} is shared between the active constraint of the desired system Ω_{1d} and the set $\phi(t)$. It can be noticed that the set $\phi(t)$ in Figure 3 is different from that in Figure 1 due to the new selection of ζ_d^{r+} . Let us define the active constraints for the closed-loop system in (65) as follows:

$$A(\phi(t)) = \{i \in \{1, 2\} \mid ((C + FK)R^{-1})_{i\bullet}\zeta^+ + (FKR^{-1})_{i\bullet}\zeta_d^{r+}(t) + F_{i\bullet}u_d(t) = 0 \text{ for } \zeta^+ \in \phi(t)\} \quad (73)$$

and we define $\Omega_i = \left\{ \zeta^+ \in \mathbb{R}^2 \mid \Omega_i \triangleq \begin{pmatrix} 1.065 & 0.034 \\ 0.034 & 1.016 \end{pmatrix}_{i\bullet} \zeta^+ + \begin{pmatrix} 2.206u_d \\ 0.815u_d \end{pmatrix}_{i\bullet} = 0 \text{ for } \zeta^+ \in \phi(t) \right\}$, for $i \in \{1, 2\}$. The same reasoning as before is applicable in this case. It is worth noting that the equality $\zeta^+ = \zeta_d^{r+}$ holds if and only if ζ^- is selected such that $\zeta^- - \zeta_d^{r+} \in \mathcal{N}_{\phi(t)}(\zeta_d^{r+})$. Otherwise, the resulting feasible solution $\zeta^+ \neq \zeta_d^{r+}$, but it belongs to the boundary of the set $\phi(t)$, that is, $\zeta^+ \in \partial\phi(t)$, which can be any of the three possible domains: Ω_1 , Ω_2 , or $\Omega_1 \cap \Omega_2$ as defined in (73).

5.3 Case 3: the state $\zeta_d^{r+} \in \Omega_{2d}$

In this section, only one of the constraints for the desired system is considered active which is Ω_{2d} defined in (69). Let $\zeta_d^{r+} = (4u_d, -0.89u_d)^\top \in \Omega_{2d}$ and let us substitute the value of ζ_d^{r+} in the set $\phi(t)$ in (66), then:

$$\phi(t) = \left\{ \zeta \in \mathbb{R}^2 \mid \begin{pmatrix} 1.065 & 0.034 \\ 0.034 & 1.016 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} - \begin{pmatrix} 0.129 & 0.065 \\ 0.064 & 0.032 \end{pmatrix} \begin{pmatrix} 4u_d \\ -0.89u_d \end{pmatrix} + \begin{pmatrix} 2u_d \\ u_d \end{pmatrix} \geq 0 \right\} \quad (74)$$

which is represented in Figure 4

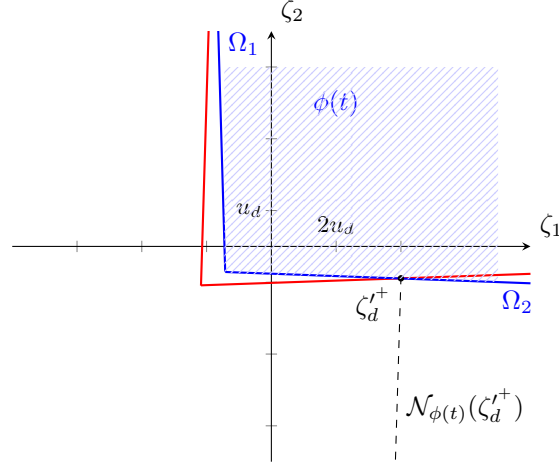


Figure 4

The state ζ_d^{r+} is shared by both sets $\phi(t)$ and $\phi_d(t)$, as shown in Figure 4. More precisely, $\zeta_d^{r+} \in \Omega_{2d} \cap \phi(t)$. Let us define the active constraints for the closed-loop system in (65) as follows:

$$A(\phi(t)) = \{i \in \{1, 2\} \mid ((C+FK)R^{-1})_{i\bullet} \zeta^+ + (FKR^{-1})_{i\bullet} \zeta_d^{r+}(t) + F_{i\bullet} u_d(t) = 0 \text{ for } \zeta^+ \in \phi(t)\} \quad (75)$$

and let $\Omega_i \triangleq \left\{ \zeta^+ \in \mathbb{R}^2 \mid \begin{pmatrix} 1.065 & 0.034 \\ 0.034 & 1.016 \end{pmatrix}_{i\bullet} \zeta^+ + \begin{pmatrix} 1.542u_d \\ 0.773u_d \end{pmatrix}_{i\bullet} = 0 \text{ for } \zeta^+ \in \phi(t) \right\}$, for $i \in \{1, 2\}$.

Let us examine, in the following cases, under which conditions the equality $\zeta_d^{r+} = \zeta^+$ holds, taking into account the active constraints of the closed-loop system.

6 Simple Scalar Example with State Jumps

This section and sections 7 through 13 are dedicated to analyse and mainly to present numerical simulations of the above theoretical developments. The numerical simulations are done with the INRIA software package SICONOS¹ [2], and the LMIs are solved with MOSEK 9.3.14 solver [5].

6.1 Dynamics and Closed-loop System

Consider the scalar LCS with $d = 0$:

$$\begin{cases} \dot{x}(t) = ax(t) + b\lambda(t) + u(t) \\ 0 \leq \lambda(t) \perp w(t) = cx(t) + v(t) \geq 0 \end{cases} \quad (76)$$

where a, b , and $c \in \mathbb{R}$, u and v are two inputs. The desired system is represented by an LCS as follows:

$$\begin{cases} \dot{x}_d(t) = ax_d(t) + b\lambda_d(t) + u_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = cx_d(t) + v_d(t) \geq 0 \end{cases} \quad (77)$$

If $a > 0$, then the real system in (76) and the desired system (77) are unstable (*i.e* the quadruple (a, b, c, d) has a positive real pole). But, in the context of trajectory tracking, the stability of the error dynamics is the main concern and not the stability of the real and the desired trajectories.

¹<https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

Let

$$\begin{aligned} u &= k_1(x - x_d) + g_1(\lambda - \lambda_d) + u_d \\ &\quad \text{and} \\ v &= k_2(x - x_d) + g_2(\lambda - \lambda_d) + v_d \end{aligned} \quad (78)$$

so that $K = (k_1, k_2)^\top$ and $G = (g_1, g_2)^\top$. By substituting the equations of u and v in (76), the closed-loop system is written in the form of (16) as follows:

$$\begin{cases} \dot{x}(t) = (a + k_1)x(t) + (b + g_1)\lambda(t) - k_1x_d(t) - g_1\lambda_d(t) + u_d(t) \\ 0 \leq \lambda(t) \perp w(t) = (c + k_2)x(t) + g_2\lambda(t) - k_2x_d(t) - g_2\lambda_d(t) + v_d(t) \geq 0 \end{cases} \quad (79)$$

Remark 6.1. Consider $G = 0$. Recall the conditions of well-posedness given in section 2.1, item 3 for the case when $d = 0$. Assume that there exists $p > 0$ such that the quadruple $(a, b, c, 0)$ is passive (i.e., the LMI in (7) has a solution). If the desired system in (77) is well-posed, then the condition $pb = c$ holds. In addition, if the closed-loop system in (79) is well-posed, then the condition $pb = c + k_2$ holds. Therefore, knowing that $p > 0$, the variables b , c and $c + k_2$ should have the same sign (see Example 2).

Example 2 (Ill-posed LCS). Let $b = 1 > 0$ and $c = -1 < 0$. Consider the LCS in (76) with the following linear complementarity system (LCP):

$$0 \leq \lambda(t) \perp w(t) = cx(t) + v(t) \geq 0$$

In the case when $d = 0$ and at contact (i.e., $cx(t) + v(t) = 0$), the sign of the variable w is studied. For this purpose, the following LCP is obtained:

$$0 \leq \lambda(t) \perp ca x(t) + cb\lambda(t) + cu(t) + \dot{v}(t) \geq 0 \quad (80)$$

According to Theorem A.1, the LCP in (80) is ill-posed because $cb = -1 < 0$. Therefore, the LCS in (76) is ill-posed when $cb < 0$.

Following the same steps as before for the desired system in (77) and the closed-loop system in (79), the coefficients cb and $b(c + k_2)$ have to be positive. Otherwise, the result of ill-posedness is observed in the following numerical simulation with $a = 0.5$, $b = 1$, $c = -1$, and $k_1 = -1.19$, $k_2 = 2.96$, $p = 1.96$ being the solution of the LMI of strict passivity in (88). Take $x(0) = 5$, $x_d(0) = 0$, $u_d(t) = 5 \sin 3t$ and v_d a discontinuous periodic function as shown in Figure 5b with the time step $h = 0.001$.

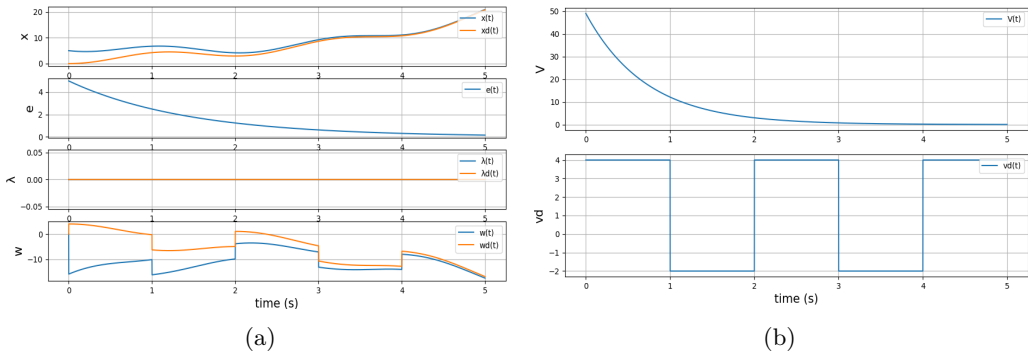


Figure 5: Numerical simulation of ill-posed LCS

As shown in Figure 5, the linear complementarity problem is not valid for both desired and closed-loop systems since the complementarity variables $w < 0$ and $w_d < 0$. Thus, the LCS in (77) and (79) is ill-posed and it is pointless to study trajectory tracking for an ill-posed system. Therefore, the conditions stated in Remark 6.1 must be respected in the following.

6.2 State-jumps Analysis

State-jumps in $x(\cdot)$ and $x_d(\cdot)$ can occur for different reasons discussed in section 3.3. The purpose of this section is to analyze state jumps in the desired and closed-loop systems in different cases.

If the controller gain $G = 0$

Notice that the dynamics (79) may not be well-posed due to the possible presence of Dirac measures that stem from both the complementarity conditions in (79) which create an impulsive multiplier λ and from the complementarity in (77) which implies an impulsive multiplier λ_d . Therefore we shall consider $G = 0$ (which means that (C1) in section 3.3 is satisfied). In order to analyze state jumps, the following sets are calculated as follows (See section 2.1). For the desired system in (77):

$$\begin{aligned} Q_{d,D} &= \{\lambda_d \in \mathbb{R} \mid 0 \leq \lambda_d \perp d\lambda_d \geq 0\} = \{\lambda_d \in \mathbb{R} \mid 0 \leq \lambda_d \perp 0 \geq 0\} = \{\lambda_d \in \mathbb{R} \mid \lambda_d \in \mathbb{R}_+\} \\ Q_{d,D}^* &= \{w_d \in \mathbb{R} \mid \langle w_d, \lambda_d \rangle \geq 0\} = \{w_d \in \mathbb{R} \mid w_d \in \mathbb{R}_+\} \\ \mathcal{K}_d &= \{x_d \in \mathbb{R} \mid cx_d + v_d(t^+) \in Q_{d,D}^*\} = \{x_d \in \mathbb{R} \mid cx_d \geq -v_d(t^+)\} \end{aligned} \quad (81)$$

A jump can occur in the desired state x_d at initial time if and only if $cx_d(0^-) < -v_d(0^+)$. The desired state x_d jumps at $t = t_c$ if and only if $cx_d(t_c^-) < -v_d(t_c^+)$. For the closed-loop system in (79):

$$\begin{aligned} Q_D &= \{\lambda \in \mathbb{R} \mid 0 \leq \lambda \perp d\lambda \geq 0\} = \{\lambda \in \mathbb{R} \mid \lambda \in \mathbb{R}_+\} \\ Q_D^* &= \{w \in \mathbb{R} \mid \langle w, \lambda \rangle \geq 0\} = \{w \in \mathbb{R} \mid w \in \mathbb{R}_+\} \\ \mathcal{K} &= \{x \in \mathbb{R} \mid (c + k_2)x - k_2x_d(t^+) + v_d(t^+) \in Q_{d,D}^*\} = \{x \in \mathbb{R} \mid (c + k_2)x \geq k_2x_d(t^+) - v_d(t^+)\} \end{aligned} \quad (82)$$

The state x of the closed-loop system in (79) jumps at any $t = t_c$, $t_c \in [0, +\infty)$, if and only if $cx(t_c^-) + k_2[x(t_c^-) - x_d(t_c^+)] < -v_d(t_c^+)$. Thus, when $G = 0$, the jumps at $t > 0$ in $x(\cdot)$ and $x_d(\cdot)$ occur only if v_d is discontinuous at t , as expected.

If the controller gain $G \neq 0$ and x_d is continuous

In this case, a state jump in x (if any) is caused by the discontinuity in λ_d , which can arise when the complementarity problem switches (*i.e.*, $cx(t) + v(t) = 0$) and $v_d(t)$ remains continuous. The desired system's state is considered continuous to avoid the Dirac measure λ_d appearing in (79). For the closed-loop system represented by $(a + k_1, b + g_1, c + k_2, g_2)$ to be strictly passive, one must have $g_2 > 0$ given that $d = 0$. But, g_2 multiplies λ as shown in (79). The LCP of the closed-loop system (79) is

$$0 \leq \lambda(t) \perp w(t) = (c + k_2)x(t) + g_2\lambda(t) - k_2x_d(t) - g_2\lambda_d(t) + v_d(t) \geq 0$$

has a unique piecewise continuous solution λ . Hence, the LCS in (79) is an ODE with an AC solution on \mathbb{R}_+ provided that $u(t)$ and $v(t)$ are continuous (see section 2.1, item 1). Thus, the closed-loop system's state is jump-free in the scalar case with $G \neq 0$ and the desired system's state x_d is continuous. Therefore, the error dynamics $e(t)$ is continuous in this case.

The explanation of this result can be further understood by analyzing the following sets of the closed-loop system.

$$\begin{aligned} Q_D &= \{\lambda \in \mathbb{R} \mid 0 \leq \lambda \perp (d + g_2)\lambda \geq 0\} = \{0\} \\ Q_D^* &= \{w \in \mathbb{R} \mid \langle w, \lambda \rangle \geq 0\} = \{\mathbb{R}\} \\ \mathcal{K} &= \{x \in \mathbb{R} \mid (c + k_2)x(t) + g_2\lambda(t) - k_2x_d(t) - g_2\lambda_d(t) + v_d(t) \in Q_{d,D}^*\} = \{\mathbb{R}\} \end{aligned} \quad (83)$$

Using the optimization problem in (9), $x(t^+) = x(t^-)$. Thus, the closed-loop system's state $x(t)$ is continuous even if λ_d is discontinuous. Therefore, the error dynamics $e(t)$ is continuous.

6.3 Stability Analysis of Error Dynamics with State-jumps ($G = 0$)

This section is dedicated to the stability analysis of the error dynamics in (84) when state jumps occur. The error dynamics $e(t) = x(t) - x_d(t)$ in the form of (18) is represented by:

$$\begin{aligned} \dot{e}(t) &= (a + k_1)e(t) + b(\lambda(t) - \lambda_d(t)) \\ 0 \leq \begin{pmatrix} \lambda(t) \\ \lambda_d(t) \end{pmatrix} \perp \begin{pmatrix} w(t) \\ w_d(t) \end{pmatrix} &= \begin{pmatrix} c + k_2 \\ -c - k_2 \end{pmatrix} e(t) + \begin{pmatrix} 0 & c \\ c + k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x(t) \\ x_d(t) \end{pmatrix} + \begin{pmatrix} v_d(t) \\ v_d(t) \end{pmatrix} \geq 0 \end{aligned} \quad (84)$$

Let us now try the passivity criterion introduced in (30) to cope with state jumps directly from the error dynamics passivity. Let $\bar{a} = a + k_1$, $\bar{b} = (b \quad -b)$ and $\bar{c} = \begin{pmatrix} c + k_2 \\ 0 \end{pmatrix}$. The passivity of the error dynamics represented by the quadruple $(\bar{a}, \bar{b}, \bar{c}, 0)$ is determined by checking analytically if the nonlinear matrix inequality in (30) has no solution for the system (84). Consider

$$M_{ext} \triangleq \begin{pmatrix} -2p(a + k_1) & -pb + c + k_2 & pb \\ -bp + c + k_2 & 0 & 0 \\ bp & 0 & 0 \end{pmatrix} \succeq 0 \quad (85)$$

has a solution for $p > 0$. The nonlinear matrix inequality in (30) has a solution for the system (84) if and only if the nonlinear matrix inequality in (85) has a solution for $p > 0$. This means that $x^\top M_{ext} x \succeq 0$ for all $x \in \mathbb{R}^3$ and $x \neq 0$. But

$$x^\top M_{ext} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\top M_{ext} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (-2p(a + k_1))x_1^2 + 2bp x_1 x_3$$

Thus, $x^\top M_{ext} x \succeq 0$ is not satisfied for all $x \in \mathbb{R}^3$ such that $x \neq 0$. Notice that it can be inferred directly from Lemma A.3 that M_{ext} can be positive semidefinite only if $-pb + c + k_2 = pb = 0$. Therefore it is inferred that this approach (imposing the passivity of the error dynamics with the LMI in (30)) is not fruitful.

The following Lemma states the stability result.

Lemma 6.2. *Consider the dynamical systems in (79) and in (77). Suppose that Assumptions 3.2 and 3.3 hold. Assume that the states $x(\cdot)$ and $x_d(\cdot)$ have jumps for $t > 0$ since $d = 0$ in the complementarity constraint. Then, the error dynamics in (84) has a globally asymptotically stable equilibrium point $e^* = 0$.*

Proof. Consider the Lyapunov candidate function $V(e(t)) = e(t)^\top P e(t)$. Recall from section 3.2, Proposition 3.8, that error dynamics in (84) has a globally exponentially stable equilibrium point $e^* = 0$ when both the states $x(\cdot)$ and $x_d(\cdot)$ are continuous. This result is proved by showing the variation of the storage function of the error dynamics $\dot{V}(e(t)) < 0$ for all $e \neq 0$.

In view of the desired dynamics (77), the closed-loop dynamics (79) and the controllers (78), let t_k where $k > 0$ be the set of time instants at which $v_d(t)$ is discontinuous. The states x and x_d undergoes a jump at $t \in \{0\} \cup \{t_k\}$. The goal is to study the sign of $\Delta V(e(t))$ at the jump time which is written as follows in scalar case:

$$\begin{aligned} \Delta V(e(t)) &= V(e(t^+)) - V(e(t^-)) \\ &= (x(t^+) - x_d(t^+))^\top p (x(t^+) - x_d(t^+)) - (x(t^-) - x_d(t^-))^\top p (x(t^-) - x_d(t^-)) \quad (86) \\ &= p (x(t^+) - x_d(t^+))^2 - p (x(t^-) - x_d(t^-))^2 \end{aligned}$$

There are three cases to consider when analyzing the sign of $\Delta V(e(t))$. To lighten notations we denote $f^+ = f(t^+)$ and $f^- = f(t^-)$.

First case: In this case, both x and x_d jump at the same time t . The values of the state jump for the desired and the closed-loop system are given by $x_d^+ = -\frac{1}{c}v_d^+$ and $x^+ = \frac{k_2 x_d^+ - v_d^+}{c + k_2}$ respectively by referring to section A.7. If we substitute the value of x_d^+ in x^+ , then $x^+ = -\frac{1}{c}v_d^+$. The variation of the storage function at the jump time is written as follows:

$$\begin{aligned} \Delta V(e(t)) &= p (x(t^+) - x_d(t^+))^2 - p (x(t^-) - x_d(t^-))^2 \\ &= p \left(-\frac{1}{c}v_d^+ + \frac{1}{c}v_d^+\right)^2 - p (x(t^-) - x_d(t^-))^2 \\ &= -p (x(t^-) - x_d(t^-))^2 \leq 0 \end{aligned}$$

Thus, when both x and x_d jump, the variation of the storage function of the error dynamics $\Delta V(e) < 0$ for all $e \neq 0$.

Second case: In this case, the closed-loop system's state x jumps and the desired system's state x_d is continuous such that $x_d^+ = x_d^- = x_d$. The value of the closed-loop system's state jump is

given by $x^+ = \frac{k_2 x_d - v_d^+}{c+k_2}$ (see section A.7). If these values are substituted in $\Delta V(e(t))$ in (86), then

$$\begin{aligned}
\Delta V(e(t)) &= p \left(\frac{k_2 x_d - v_d^+}{c+k_2} - x_d \right)^2 - p(x^- - x_d)^2 \\
&= p \left(\left(\frac{-c x_d - v_d^+}{c+k_2} \right)^2 - (x^- - x_d)^2 \right) \\
&= p \left(\left(\frac{-c x_d - v_d^+}{c+k_2} - x^- + x_d \right) \left(\frac{-c x_d - v_d^+}{c+k_2} + x^- - x_d \right) \right) \\
&= p \left(\left(\frac{-(c+k_2)x^- + k_2 x_d - v_d^+}{c+k_2} \right) \left(\frac{(c+k_2)x^- - (2c+k_2)x_d - v_d^+}{c+k_2} \right) \right) \\
&= \frac{p}{(c+k_2)^2} \left(-(c+k_2)x^- + k_2 x_d - v_d^+ \right) \left((c+k_2)x^- - (2c+k_2)x_d - v_d^+ \right)
\end{aligned}$$

Let $h(x) \triangleq -(c+k_2)x^- + k_2 x_d - v_d^+$ and $r(x) \triangleq (c+k_2)x^- - (2c+k_2)x_d - v_d^+$. Let us study the signs of $h(x)$ and $r(x)$.

Knowing that $x^- \notin \mathcal{K}$, the following inequality holds:

$$(c+k_2)x^- - k_2 x_d + v_d^+ < 0 \Leftrightarrow -(c+k_2)x^- + k_2 x_d - v_d^+ > 0 \Leftrightarrow h(x) > 0$$

Now, let us check the sign of $r(x)$. If we add and subtract v_d^+ , then

$$\begin{aligned}
r(x) \pm v_d^+ &\triangleq (c+k_2)x^- - k_2 x_d + v_d^+ - 2c x_d - 2v_d^+ \\
&\triangleq -h(x) - 2c x_d - 2v_d^+
\end{aligned}$$

Provided that $-h(x) < 0$ and $c x_d + v_d^+ \geq 0$ (i.e. $x_d \in \mathcal{K}_d$) $\Leftrightarrow -2c x_d - 2v_d^+ \leq 0$. So, $r(x) < 0$.

Thus, $h(x) > 0$ and $r(x) < 0$. Therefore, $\Delta V(e(t)) \triangleq \frac{p}{(c+k_2)^2} (h(x)r(x)) < 0$.

Third case: In this case, the desired system's state x_d jumps and the closed-loop system's state x is continuous such that $x^+ = x^- = x(t)$. The value of the desired system's state jump is $x_d^+ = -\frac{1}{c}v_d^+$ (check section A.7). If these values are substituted in $\Delta V(e(t))$ in (86), then

$$\begin{aligned}
\Delta V(e(t)) &= p \left(x + \frac{v_d^+}{c} \right)^2 - p(x - x_d^-)^2 \\
&= p \left(\left(x + \frac{v_d^+}{c} + x - x_d^- \right) \left(\frac{v_d^+}{c} + x_d^- \right) \right)
\end{aligned}$$

Let $h(x) \triangleq x + \frac{v_d^+}{c} + x - x_d^-$ and $r(x) \triangleq \frac{v_d^+}{c} + x_d^-$. Let us study the signs of $h(x)$ and $r(x)$.

Knowing that $x_d^- \notin \mathcal{K}_d$, then

$$c x_d^- + v_d^+ < 0 \Leftrightarrow x_d^- + \frac{v_d^+}{c} < 0 \Leftrightarrow r(x) < 0$$

Now, let us check the sign of $h(x)$ by adding and subtracting v_d^+ . Then,

$$\begin{aligned}
h(x) \pm v_d^+ &\triangleq 2(cx + v_d^+) - c x_d^- - v_d^+ \\
&\triangleq 2(cx + v_d^+) - r(x)
\end{aligned}$$

Let us write the following:

$$\begin{aligned}
x \in \mathcal{K} &\Leftrightarrow (c+k_2)x - k_2 x_d^+ + v_d^+ \geq 0 \\
&\Leftrightarrow (c+k_2)x + \frac{k_2}{c} v_d^+ + v_d^+ \geq 0 \\
&\Leftrightarrow \frac{(c+k_2)}{c} (cx + v_d^+) \geq 0
\end{aligned}$$

According to Remark 6.1, $\frac{c+k_2}{c} > 0$. So, $cx + v_d^+ \geq 0 \Leftrightarrow 2cx + 2v_d^+ \geq 0$ and $-r(x) > 0$. So, $h(x) > 0$.

Thus, in this case, $h(x) > 0$ and $r(x) < 0$ and $\Delta V(e(t)) \triangleq p(h(x)r(x)) < 0$.

Therefore, the storage function is strictly decreasing (i.e., $\dot{V} < 0$ when $t \neq t_k$ and $\Delta V < 0$ when $t = t_k$ for all $e \neq 0$ where t_k are the time instants of jumps) on \mathbb{R}_+ . \square

Corollary 6.3. *Following on from Lemma 6.2, assume that $v_d(\cdot)$ is discontinuous at time instants t_k , with $k = \{1, 2, \dots, n\}$ where $n \in \mathbb{N}^*$. For any t_k , if there exists $v_d(t_k^+) < \min\{-cx(t_k^-), -cx_d(t_k^-)\}$, then both x and x_d jump to the value $x_d(t_k^+) = x(t_k^+) = -\frac{1}{c}v_d(t_k^+)$. Therefore, there will be perfect tracking for all $t \geq t_k$ (i.e., finite-time tracking).*

Proof. Let the time instants at which v_d is discontinuous be denoted by t_k with $k = \{1, 2, \dots, n\}$. The goal is to show that if the input $v_d(t_k^+) < \min\{-cx(t_k^-), -cx_d(t_k^-)\}$, then both x and x_d jump to the same value at $t = t_k$. According to the set \mathcal{K}_d in (81), the state x_d performs a jump at $t = t_k$ if and only if $x_d(t_k^-) \notin \mathcal{K}_d$ which is written as follows:

$$x_d(t_k^-) \notin \mathcal{K}_d \Leftrightarrow cx_d(t_k^-) + v_d(t_k^+) < 0 \Leftrightarrow v_d(t_k^+) < -cx_d(t_k^-)$$

Also, the state x in (79) performs a jump at $t = t_k$ if and only if $x(t_k^-) \notin \mathcal{K}$ in (82). Let us write the following:

$$x(t_k^-) \notin \mathcal{K} \Leftrightarrow (c + k_2)x(t_k^-) - k_2x_d(t_k^+) + v_d(t_k^+) < 0 \quad (87)$$

Given that x_d jumps at $t = t_k$, let us substitute $x_d(t_k^+) = -\frac{1}{c}v_d(t_k^+)$ (see (150) in section A.7) in (87). Then,

$$\begin{aligned} (c + k_2)x(t_k^-) + \frac{k_2}{c}v_d(t_k^+) + v_d(t_k^+) &< 0 \\ \Leftrightarrow \frac{c+k_2}{c}(cx(t_k^-) + v_d(t_k^+)) &< 0 \end{aligned}$$

Let c and $c + k_2$ have the same sign due to the well-posedness of LCS as in Remark 6.1. So,

$$v_d(t_k^+) < -cx(t_k^-)$$

According to the solution of the optimization problem in (9) in section A.7, the closed-loop system's state x jumps at t_k such that $x(t_k^+) = \frac{k_2x_d(t_k^+) - v_d(t_k^+)}{c+k_2} = -\frac{1}{c}v_d(t_k^+)$ where $x_d(t_k^+) = -\frac{1}{c}v_d(t_k^+)$. Therefore, both x and x_d jumps at t_k to the same value $x_d(t_k^+) = x(t_k^+) = -\frac{1}{c}v_d(t_k^+)$ (see section A.7) if and only if $v_d(t_k^+) < \mu$ where μ is given by $\mu = \min\{-cx_d(t_k^-), -cx(t_k^-)\}$. \square

6.4 Numerical Applications with $G = 0$

The open-loop system with $u = v = 0$ is not strictly passive but there exist k_1 , k_2 , and p such that the closed-loop quadruple $(a + k_1, b, c + k_2, 0)$ is strictly state passive, equivalently the nonlinear matrix inequality

$$\begin{pmatrix} -2(a + k_1)p - \epsilon p & -pb + c + k_2 \\ -bp + c + k_2 & 0 \end{pmatrix} \succeq 0$$

has a solution $p > 0$ with $\epsilon > 0$. According to A.2, the following LMI is obtained

$$\begin{pmatrix} -2qa - 2N_1 - \epsilon q & -b + qc + N_2 \\ -b + cq + N_2 & 0 \end{pmatrix} \succeq 0 \quad (88)$$

where $q = p^{-1}$, $N_1 = k_1q$ and $N_2 = k_2q$. The LMI in (88) has a solution such that:

$$k_1 = -1.198, \quad k_2 = 0.96 \quad \text{and} \quad p = 1.9607$$

for $a = 0.5$, $b = 1$ and $c = 1$. For the sake of numerical simulation, take $a = 0.5$ (so that the systems in (77) and (76) diverge slowly).

Example 3 (Both x and x_d jump). *Let us take $b = 1$, and $c = 1$. The numerical simulation below for the closed-loop system and the desired system is implemented with a discontinuous controller $v_d(t)$ at t_k with $k = \{1, 2, 3, 4\}$. Take $x(0) = 3$, $x_d(0) = -3$, $u_d = 5 \sin 3t$ and $v_d = 4$ for $t \leq 1s$ with the time step $h = 0.001$. Given that $x_d(1^-) = -0.7$ and $x(1^-) = 2.28$, and following Corollary 6.3, let us choose $v_d(1^+)$ such that $v_d(1^+) = \min\{-x_d(1^-) - \delta, x(1^-) - \delta\} < \min\{-x_d(1^-), x(1^-)\}$ with $\delta = 0.5$. Based on the value of δ , $v_d(1^+) = \min\{0.2, -2.78\} = -2.78$.*

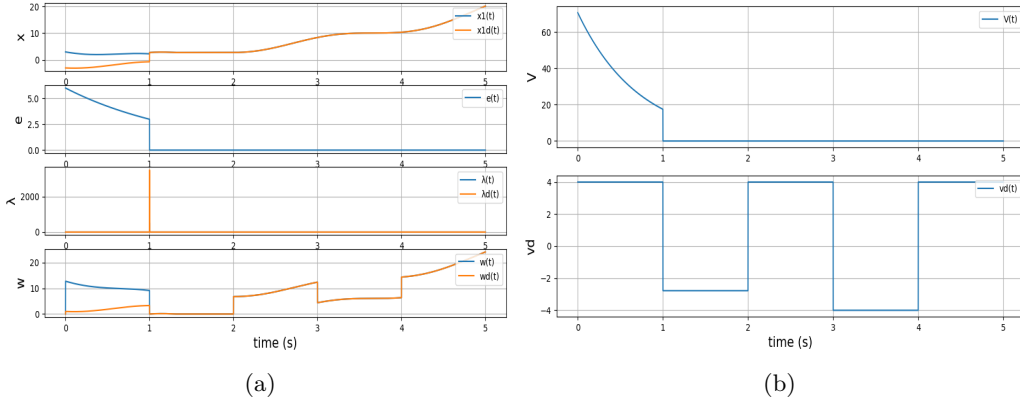


Figure 6: Numerical simulation of LCS where both x and x_d jump

Both x and x_d jumps at $t = 1$ to the same value of $-v_d(t_1^+) = 2.87$. Notice that the storage function's variation ΔV is non-positive at the time of the state jump (i.e., $V(e(1^+)) - V(e(1^-)) \leq 0$). After the first jump of both states x and x_d at the same time $t = 1$ and following Corollary 6.3, the controller v_d can take any value at the time of discontinuities such that $v_d(t_{k+1}^+) \in \mathbb{R}$ and perfect tracking is preserved (i.e., $e = 0$).

Example 4. Let us illustrate the case when x_d jumps but x does not jump. Take $x(0) = 3$, $x_d(0) = -3$. Knowing that $x_d(1^-) = -0.7$ and $x(1^-) = 2.28$ from the simulation in Figure 6, let us choose $v_d(1^+)$ such that $x_d(1^-) \notin \mathcal{K}_d$ in (81) and $x(1) \in \mathcal{K}$ in (82). Thus,

$$x_d(1^-) \notin \mathcal{K}_d \Leftrightarrow cx_d(1^-) + v_d(1^+) < 0 \Rightarrow v_d(1^+) < 0.7$$

and

$$x(1) \in \mathcal{K} \Leftrightarrow (c + k_2)x(1) - k_2x_d(1^+) + v_d(1^+) \geq 0 \Rightarrow v_d(1^+) \geq -2.28$$

with $c = 1$ and $k_2 = 0.96$ (the solution of LMI in (88)). Take $v_d(1^+) = -1$ and observe the following:

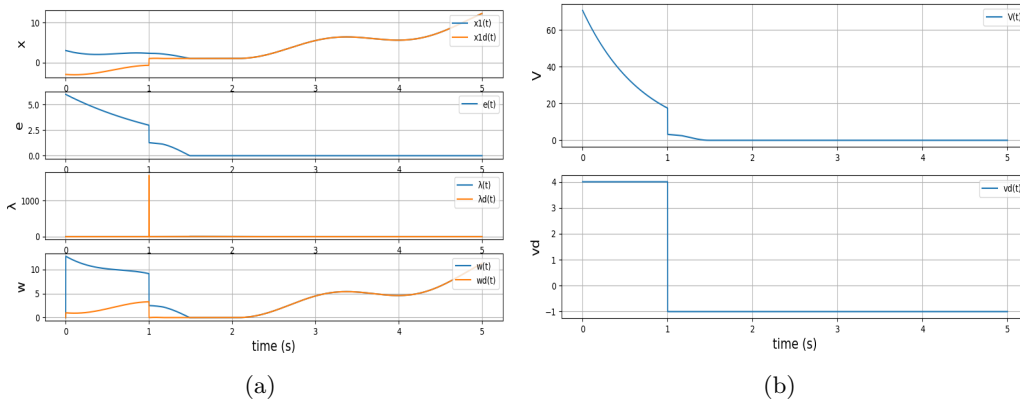


Figure 7: Numerical simulation of LCS when only x_d jumps

It is noticed from Figure 7 that the variation of the storage function V is negative when x_d jumps at $t = 1$ s even though the closed-loop system's state x does not perform a jump at $t = 1$ s. This is explained by the fact that the desired state x_d jumps toward the set of solutions of the closed-loop system x which means that the error $e = x - x_d$ is decreasing, thus the variation of storage function $V = pe^2$ is negative at the jump time.

Let us create another jump in x_d in the transient regime of the error dynamics (i.e., before $e = 0$ at $t = 1.5$ s). Knowing that $x_d(1.2^-) = 1$ and $x(1.2^-) = 2.057$, choose $-2.057 \leq v_d(1.2^+) < -1$ so that $x_d(1.2^-) \notin \mathcal{K}_d$ and $x(1.2) \in \mathcal{K}$. Take $v_d(1.2^+) = -1.5$.

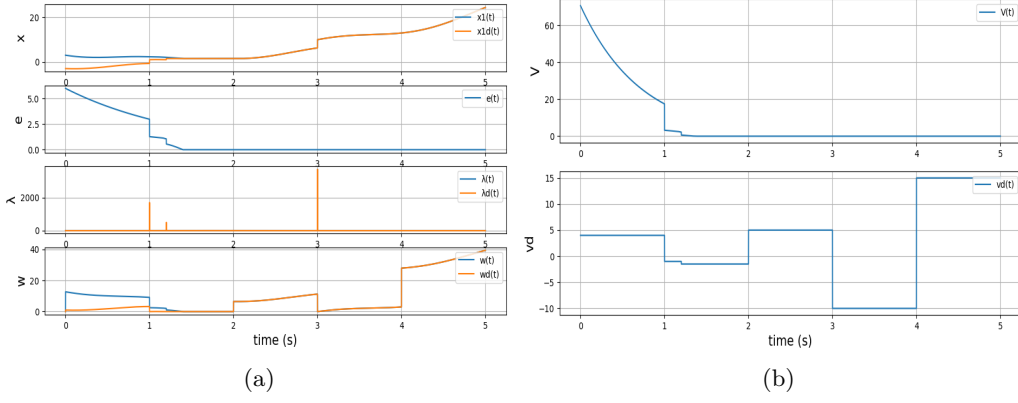


Figure 8: Numerical simulation of LCS where only x_d jumps

In Figure 8, the desired state x_d jumps at $t = 1.2s$ such that $x_d(1.2^+) = -v_d(1.2^+) = 1.5$. The value of the error $e(1.2^+) = 0.557 < e(1^-) = 1.057$, so the variation of the storage function V , defined in (86), is negative at the jump time $t = 1.2$. It is observed in Figure 8 that the error has an asymptotically stable equilibrium point $e^* = 0$ which is consistent with the result in Lemma (6.2). It is important to state that, in the case of Figures 7 and 8, the necessary and sufficient conditions to have a jump in the closed-loop state x (i.e., $v_d(1^+) < -2.28$) generate a jump as well in the desired state x_d at the same time which means that Corollary 6.3 applies.

Example 5. Let us change the initial values of the states in order to consider the case where the desired state x_d is continuous for all t but the closed-loop state x jumps. Take $x_d(0) = -1$, $x(0) = -4$ and $v_d = 4$ for $t \leq 1s$ with the time step $h = 0.001$. At $t = 1s$, the values of the states are $x_d(1^-) = 2.59$ and $x(1^-) = 1.83$. It is required to choose $v_d(1^+)$ such that $x_d(1) \in \mathcal{K}_d$ in (81) and $x(1^-) \notin \mathcal{K}$ in (82). Thus,

$$x_d(1) \in \mathcal{K}_d \Leftrightarrow cx_d(1) + v_d(1^+) \geq 0 \Rightarrow v_d(1^+) \geq -2.59$$

and

$$x(1^-) \notin \mathcal{K} \Leftrightarrow (c + k_2)x(1^-) - k_2x_d(1^+) + v_d(1^+) < 0 \Rightarrow v_d(1^+) < -1.1$$

knowing that $c = 1$, $k_2 = 0.96$ and $x_d(1^+) = x_d(1^-) = x_d(1) = 2.59$ (i.e., x_d is continuous). Let us take $v_d(1^+) = -2$ and observe the following with $v_d(t) = \begin{cases} 4, & \text{if } t = n, \\ -2, & \text{if } t = n + 1 \end{cases}$ where $n \in \{1, 2, 3, 4\}$.

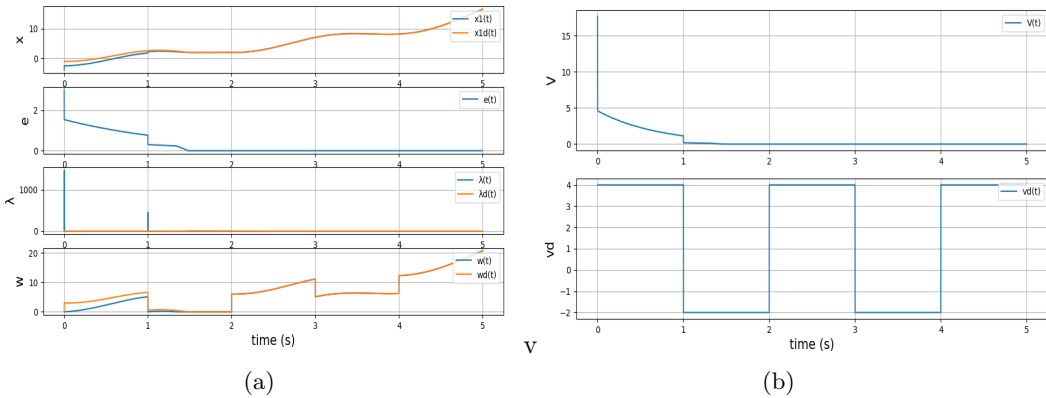


Figure 9: Numerical simulation of LCS where only x jumps

In Figure 9, the closed-loop system state x performs a jump at initial time $t = 0$ and a further jump at $t = 1s$. At the initial time $t = 0$, the state x jumps such that $x(0^+) = -2.53 \in \mathcal{K}$ which is the solution of (9) and according to Lemma 3.10, at an initial state jump we have $V(0^+) - V(0^-) \leq 0$ which is observed in the simulation shown in Figure 9.

At $t = 1.2s$, the state x performs a jump such that $x(1^+) = 2.29 \in \mathcal{K}$ which is the solution of the optimization problem in (9). Knowing that the controller is chosen such that $-2.59 \leq v_d(1^+) <$

-1.1 in order to have a jump at $t = 1$, it is noteworthy that if $v_d(1^+) < -2.59$, then both x and x_d will jump at $t = 1$ s and Corollary 6.3 applies. Whereas if $v_d(1^+) > -1.1$, then neither of the states will jump because $x(1) \in \mathcal{K}$ and $x_d(1) \in \mathcal{K}_d$ when $v_d(1^+) > -1.1$.

6.5 Conditions such that Proposition 3.11's conditions are satisfied at state jump, with $G = 0$

The above conditions that guarantee a negative Lyapunov function jump, may not match with the sufficient conditions of Proposition 3.11. The goal in this section is to find examples such that the sufficient conditions of Proposition 3.11 are satisfied, as an illustration. Let $a = 0.5$, $b = 1$, $c = 1$, $p = 1.96$ and $k_2 = 0.96$.

Example 6. Let $x(0) = 10$, $x_d(0) = -5$, $u_d = 5 \sin 3t$ and v_d is discontinuous as shown in Figure 10. Let us create a jump at $t = 1$ s such that the conditions in Proposition 3.11 are satisfied. At $t = 1^-$, the values of the states are $x_d(1^-) = -4$ and $x(1^-) = 3.45$. If we consider the case when x_d is discontinuous at $t = 1$ s, then $x_d(1^+) = -v_d(1^+)$. Assume that at $t = 1$ s:

$$0 \in \mathcal{K} \Leftrightarrow -k_2 x_d(1^+) + v_d(1^+) \geq 0 \Leftrightarrow (1 + k_2)v_d(1^+) \geq 0 \Rightarrow v_d(1^+) \geq 0$$

and

$$x_d(1^+)p x(1^-) \geq x_d(1^-)p x(1^-) \Leftrightarrow -v_d(1^+)p x(1^-) \geq x_d(1^-)p x(1^-) \Rightarrow v_d(1^+) \leq 4$$

and

$$x_d(1^+) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_P[\mathcal{K}; x(1^-)]) \Leftrightarrow x_d(1^+) \in \mathcal{T}_{\mathcal{K}}(x(1^+)) \Leftrightarrow x_d(1^+) \in (\mathcal{N}_{\mathcal{K}}(x(1^+)))^\circ$$

Knowing that $-p(x(1^+) - x(1^-)) \in \mathcal{N}_{\mathcal{K}}(x(1^+))$, then the last condition is written as

$$x_d(1^+) \in (\mathcal{N}_{\mathcal{K}}(x(1^+)))^\circ \Rightarrow -x_d(1^+)p(x(1^+) - x(1^-)) \leq 0$$

which is satisfied since $x(1^+) = x(1^-) = x(1)$ (i.e., the state x is continuous at $t = 1$ s).

Let us choose $v_d(1^+)$ such that $0 \leq v_d(1^+) \leq 4$ which is obtained from the above assumption and $-3.45 \leq v_d(1^+) < 4$ which comes from the fact that at $t = 1$ $x_d(1^-) \notin \mathcal{K}_d$ and $x(1) \in \mathcal{K}$. Take $v_d(1^+) = 2$. The numerical simulation is shown in the following with a time step $h = 0.001$.

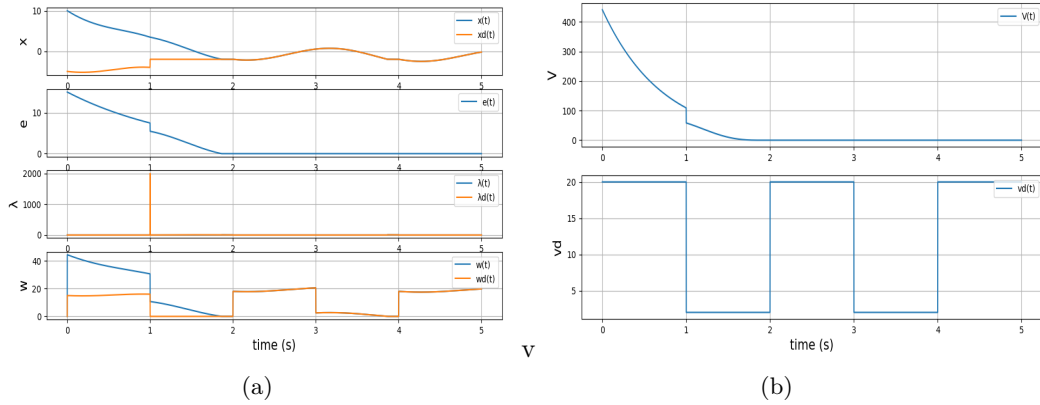


Figure 10: Numerical simulation of LCS where only x_d jumps

As shown in Figure 10, at the jump time $t = 1$ s, the variation of the storage function $\Delta V(e(1)) = V(e(1^+)) - V(e(1^-)) \leq 0$ which agrees with the result stated in Proposition 3.11.

Example 7. Let $x(0) = -15$, $x_d(0) = 1$, $u_d(t) = 5 \sin 3t$ and v_d is a discontinuous function as shown in Figure 11. Let us create a jump at $t = 1$ s such that the conditions in Proposition 3.11 are satisfied. At $t = 1^-$, the values of the states are $x(1^-) = -2.06$ and $x_d(1^-) = 5.89$. If we consider the case when x_d is continuous at $t = 1$ s (i.e., $x_d(1^+) = x_d(1^-) = x_d(1)$) and x is discontinuous at $t = 1$ (i.e., $x(1^+) = \frac{k_2 x_d^+ - v_d^+}{c + k_2}$ from Appendix A.7). Assume that at $t = 1$ s:

$$0 \in \mathcal{K} \Leftrightarrow -k_2 x_d(1^+) + v_d(1^+) \geq 0 \Leftrightarrow v_d(1^+) \geq 5.65$$

and

$$x_d(1^+)p x(1^-) \geq x_d(1^-)p x(1^-) \Leftrightarrow x_d(1)p x(1^-) = x_d(1)p x(1^-)$$

and

$$x_d(t^+) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_{\mathcal{P}}[\mathcal{K}; x(t^-)]) \Leftrightarrow x_d(t^+) \in \mathcal{T}_{\mathcal{K}}(x(t^+)) \Leftrightarrow x_d(t^+) \in (\mathcal{N}_{\mathcal{K}}(x(t^+)))^\circ$$

Knowing that $-p(x(1^+) - x(1^-)) \in \mathcal{N}_{\mathcal{K}}(x(1^+))$, then the last condition is written as

$$\begin{aligned} x_d(1^+) \in (\mathcal{N}_{\mathcal{K}}(x(1^+)))^\circ &\Rightarrow -x_d(1^+)p(x(1^+) - x(1^-)) \leq 0 \\ &\Rightarrow x_d(1^+)p\left(\frac{k_2x_d^+ - v_d^+}{c+k_2} - x(1^-)\right) \leq 0 \\ &\Rightarrow v_d(1^+) \geq 1.5 \end{aligned}$$

Let us choose $v_d(1^+) \geq 5.56$ such that the above conditions (i.e., the conditions of Proposition 3.11) are satisfied and $-5 \leq v_d(1^+) < 9.6$ (i.e., $x_d(1) \in \mathcal{K}_d$ and $x(1^-) \notin \mathcal{K}$ at $t = 1s$), so take $v_d(1^+) = 6$. The numerical simulation is shown in the following with a time step $h = 0.001$.

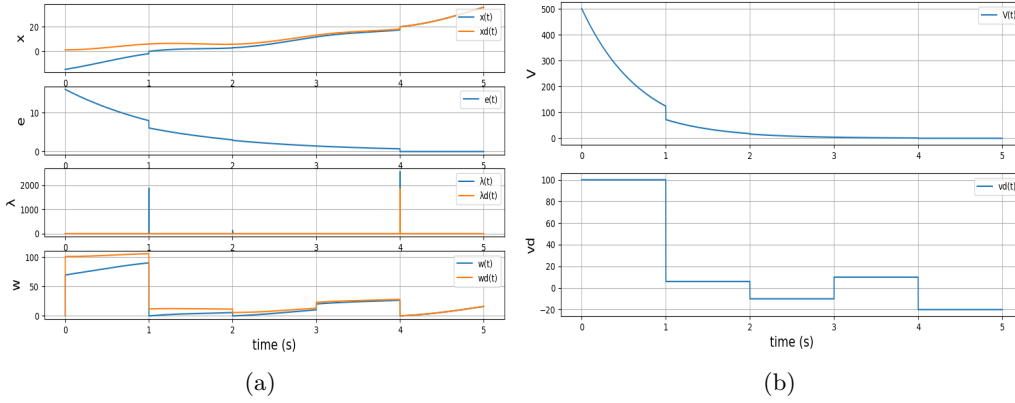


Figure 11: Numerical simulation of LCS where only x jumps

As shown in Figure 11, at the jump time $t = 1s$, the variation of the storage function $\Delta V(e(1)) = V(e(1^+)) - V(e(1^-)) \leq 0$ which agrees with the result stated in Proposition 3.11.

6.6 Recapitulation

The occurrence of "peaking phenomenon", which occurs when the jump times of two trajectories do not coincide, is not observed in our case. This is due to the fact that the desired system in (77) is derived from the dynamics of the real system in (76) and that both systems are passive.

The challenge in our work results from the different situations leading to state jumps (i.e., discontinuities in $v_d(t)$ or in $\lambda_d(t)$). It is noteworthy that, in the scalar case, the jumps only occur due to discontinuities in $v_d(t)$ as discussed in section 6.2. The closed-loop system's state $x(t)$ in (79) jumps when either $x_d(t)$ or $v_d(t)$ jumps, and since the desired system's state $x_d(t)$ only jumps when $v_d(t)$ jumps, it follows that both states jump at the same time (i.e., no jump mismatch occur).

The electrical circuits with ideal diodes, presented as examples in section 6.4, cover all the possibilities of state jumps due to the discontinuities in $v_d(t)$ (i.e., only $x_d(t)$ jumps, only $x(t)$ jumps, and both $x_d(t)$ and $x(t)$ jump) and show the stability of the error dynamics in (84), of each example, in the presence of state jumps.

7 Applications: Circuits with Ideal Diodes

The next sections are dedicated to applications on electrical circuits with ideal diodes (recall that hydraulic circuits share same components where ideal diodes are replaced by check valves [58]). Let us consider the following examples which differ mainly by the structure of the matrix D as well as the position of the controller in the dynamics (i.e the differential equations and/or complementarity problem). The following examples are provided with their numerical simulation which is done with the INRIA software package SICONOS²[2]. The algorithm of the code is explained as follows: the nominal system is defined at the beginning and it is used in order to design an optimal controller with optimal control gains by solving the LMI in (45) using MOSEK 9.3.14 solver [5]. Then, the

²<https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

real system is defined with constrained uncertainties. It is noteworthy that the uncertainties are constrained according to the values of Λ_k 's for $k \in \{A, B, C, D, E, F\}$, as shown in (41), which are unknown variables determined from the solutions ($\Lambda_1, \Lambda_2, \tilde{\Lambda}_1, \tilde{\Lambda}_2$ and Λ_F) of the LMI in (45).

8 Electrical Circuit with Parametric Uncertainties

Consider the circuit in Figure 12 with the states x_1 : the charge on the capacitor \mathbf{C} and x_2 : the current passing through the inductor \mathbf{L} . The dynamics of the electrical circuit in Figure 12 is given by the system in (89):

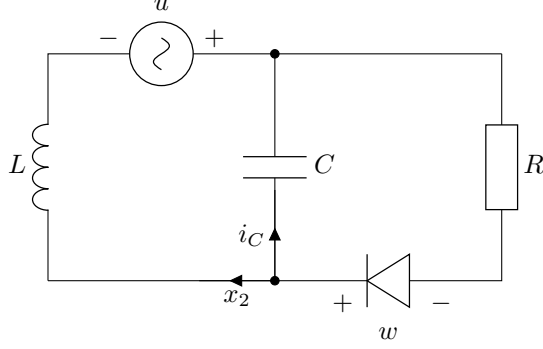


Figure 12: RLCD circuit with one ideal diode

$$\begin{cases} \dot{x}_1(t) = -x_2(t) + \lambda(t) \\ \dot{x}_2(t) = \frac{x_1(t)}{\mathbf{LC}} + \frac{u(t)}{\mathbf{L}} \\ 0 \leq \lambda(t) \perp w(t) = \frac{x_1(t)}{\mathbf{C}} + \mathbf{R}\lambda(t) \geq 0 \end{cases} \quad (89)$$

with $\mathbf{L} = 1$ H, $\mathbf{C} = 0.025$ F, $\mathbf{R} = 10$ Ω .

8.1 Study the passivity of (12) with $u = 0$

The energy stored in the circuit of Figure 12 with $u = 0$ is: $V(x) = \frac{1}{2}\mathbf{C}(\frac{x_1}{\mathbf{C}})^2 + \frac{1}{2}\mathbf{L}x_2^2$. The rate of change in stored energy is given by $\dot{V}(x) = \mathbf{C}(\frac{x_1}{\mathbf{C}})(\frac{\dot{x}_1}{\mathbf{C}}) + \mathbf{L}x_2\dot{x}_2$. If the first two lines of (89) are substituted in the equation of \dot{V} , then

$$\dot{V}(x) = \mathbf{L}x_2\frac{x_1}{\mathbf{LC}} + \frac{x_1}{\mathbf{C}}(-x_2 + \lambda) = \frac{x_1\lambda}{\mathbf{C}} \quad (90)$$

Knowing that $x_1 = \mathbf{C}V_{\mathbf{C}} = \mathbf{C}(-V_{\mathbf{R}} - V_{\mathbf{D}})$ where $V_{\mathbf{R}}$ and $V_{\mathbf{D}}$ are the voltages across the resistor and the diode respectively, the rate of change of the storage function is represented as follows:

$$\dot{V}(x) = \frac{\lambda}{\mathbf{C}}(-R\lambda - w\lambda) = -\frac{R}{\mathbf{C}}\lambda^2 \quad (91)$$

where $w\lambda = 0$ due to orthogonality. So, $\dot{V}(x) \leq 0$ (*i.e.*, the rate of change in stored energy is less than the power supplied to the system). Thus, the dynamical system in (12) with $u = 0$ is passive.

Let us check the passivity of (12) by proving that its open loop transfer function (*i.e.*, TF with $u = 0$) which is given by $H_0(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is positive real. Then,

$$H_0(s) = \frac{w(s)}{\lambda(s)} = \frac{\mathbf{R}\mathbf{L}\mathbf{C}s^2 + \mathbf{L}s + \mathbf{R}}{\mathbf{L}\mathbf{C}s^2 + 1}$$

The transfer function H_0 is Hurwitz and $\mathbf{Re}(H_0(j\omega)) = \mathbf{R} > 0$. Then, the transfer function H_0 is strictly positive real [25, Theorem 2.45]. Therefore, the open loop system in (12) is strictly passive.

8.2 Passivity of closed-loop system (12) without uncertainties

Let $u = K(x - x_d) + G(\lambda - \lambda_d) + u_d$ with $K = (k_1 \quad k_2)$ and $G = g_1$. In the case of which the system (89) has no uncertainties, the control gains are calculated by solving the BMI in (20) after

being transformed into LMI according to A.2; the solution is obtained as follows:

$$K = (493.07978501 \quad -42.07565724), \quad G = 40.65023 \quad \text{and} \quad P = \begin{pmatrix} 40.12136978 & -3.14335292 \\ -3.14335292 & 0.32336045 \end{pmatrix} \quad (92)$$

If the BMI in (20) is solved with the minimum value of the control gain G , then the solution is given by the following:

$$K = (75.0276 \quad -5.2589), \quad G = -3.9 \times 10^{-13} \quad \text{and} \quad P = \begin{pmatrix} 26.06268 & -0.99185 \\ -0.99185 & 0.25716 \end{pmatrix} \quad (93)$$

It is observed that the control gain G has a negligible value and this indicates that the additional feedback from the complementarity variable λ is useless in this example. So, in the following, the value of G is neglected.

In order to decrease the magnitude of the control gain K , the steps below are followed knowing that $K = NQ^{-1}$:

- solve the LMI derived from the LMI in (20) (see Appendix A.2) with $G = 0$ for the minimum Euclidean norm of the matrix N with fixed $Q = P^{-1}$ in (93).
- Now, fix the value of N obtained from the previous step and solve the LMI in (20) again for the maximum trace of the matrix Q .
- repeat the first two steps until there is no change in the values of N and Q obtained.

This methodology gives the following solution when followed:

$$K = (38.89363 \quad -2.145726), \quad P = \begin{pmatrix} 10.911047 & -0.26867 \\ -0.26867 & 0.12605 \end{pmatrix} \quad \text{and} \quad N = (3.31969 \quad -9.94634)$$

The plots in Figure 13 show the numerical simulation for the desired and closed loop system with two different forms of controllers. The initial state vectors are $x(0) = (1, 0)^\top$ and $x_d(0) = (-1.5, 1)^\top$, the time step $h = 10^{-3}$ and the desired input is $u_d(t) = 30 \sin 5t$.

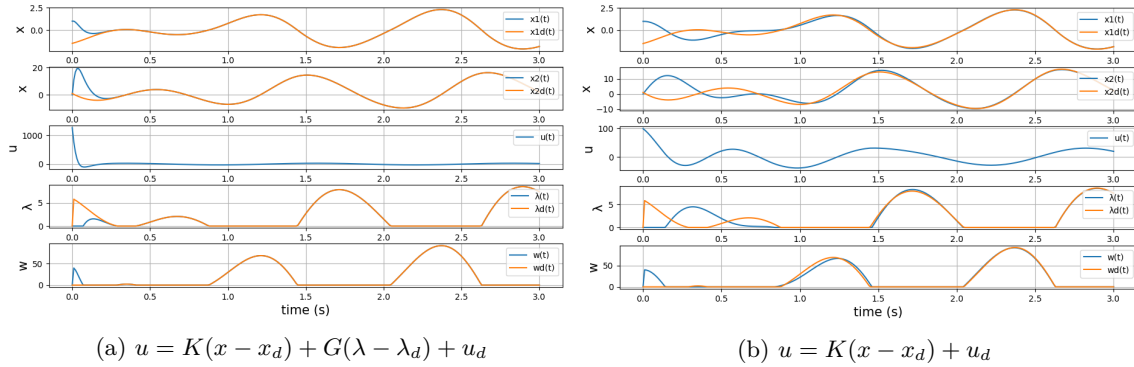


Figure 13: Numerical simulation showing the closed-loop system's trajectory x , the desired system's trajectory x_d and the error dynamics $e = x - x_d$ of the LCS in (89), without uncertainties and with two forms of controllers. The simulation also presents the controller u and the complementarity variables λ and w .

In Figures 13a and 13b, L_2 and L_∞ norms of the error between real and desired trajectories are observed at steady state. In Figure 13a, $\|e_1\|_{[2,3],2} = 0$ and $\|e_2\|_{[2,3],2} = 0$ which increases in Figure 13b such that $\|e_1\|_{[2,3],2} = 0.23$ and $\|e_2\|_{[2,3],2} = 2.01$. As well, $\|e_1\|_{[2,3],\infty} = 0$ and $\|e_2\|_{[2,3],\infty} = 0$ in Figure 13a and both values increase to $\|e_1\|_{[2,3],\infty} = 0.04$ and $\|e_2\|_{[2,3],\infty} = 0.38$ in Figure 13b. This increase shows that the transient response takes a shorter time when the controller has additional feedback from the variable λ as observed in Figure 13b even if it is shown that this additional feedback is useless in the sense of tracking.

8.3 Passivity of closed-loop system (12) in the presence of uncertainties

The uncertainties are introduced to the passive elements \mathbf{R} , \mathbf{L} and \mathbf{C} . The plots below show the numerical simulation for the desired and closed-loop systems. It shows the trajectories of

the systems, the complementarity variables λ and w , and the controller u with different values of bounded uncertainties.

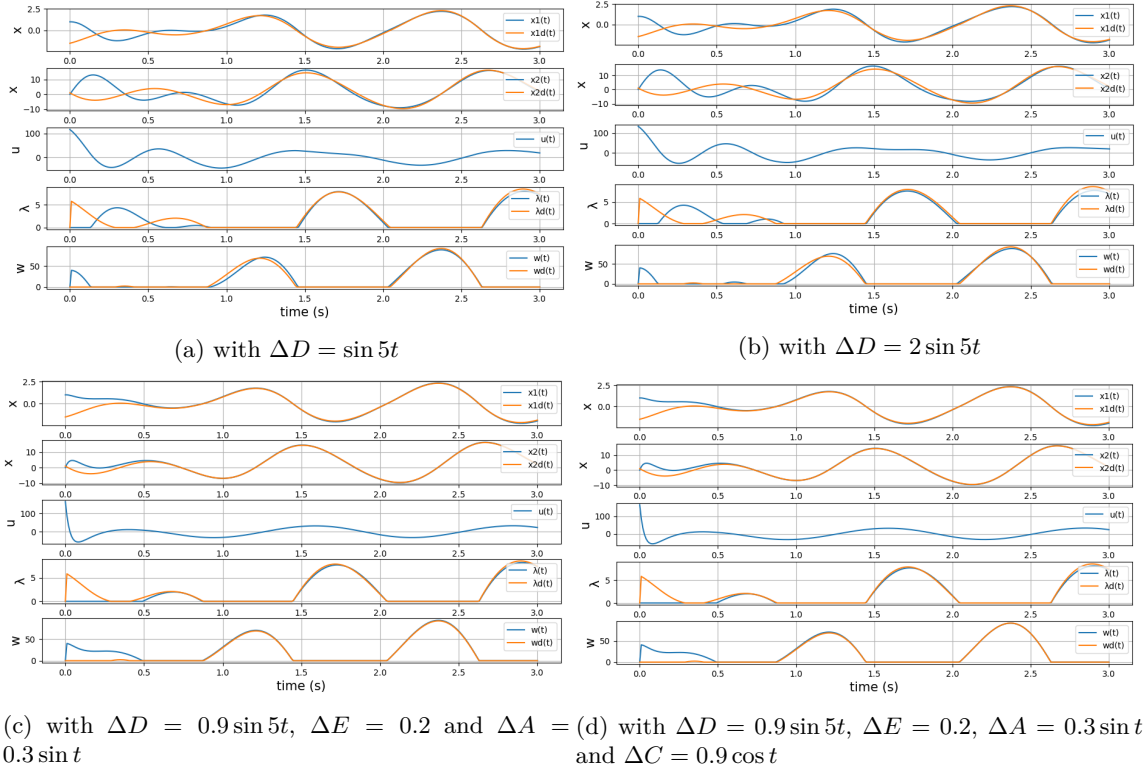


Figure 14: Numerical simulation showing the error dynamics $e = x - x_d$ of the LCS in (89), with different values of uncertainties. The simulation also presents the controller u and the complementarity variables λ and w .

In Figure 14a, the matrix D is perturbed by ΔD due to disturbance in the resistor \mathbf{R} . The perturbation is selected such that ΔD is bounded according to the value of Λ_D . For this purpose, the following LMI

$$\left(\begin{array}{cc|cc} (M_{0_{lin}})_{11} & (M_{0_{lin}})_{12} & 0 & 0 \\ (M_{0_{lin}})_{21} & (M_{0_{lin}})_{22} & I_m & I_m \\ \hline 0 & I_m & \Lambda_2^{-1} & 0 \\ 0 & I_m & 0 & \tilde{\Lambda}_2^{-1} \end{array} \right) \succ 0 \quad (94)$$

which is a special case of the LMI in (45) is solved such that $\tilde{\Lambda}_2^{-1} \leq 0.25$ and $\Lambda_2^{-1} \geq 10$; the solution is given as follows:

$$K_0 = (45.815006 \quad -2.158609), \quad P_0 = \begin{pmatrix} 13.497987 & -0.304643 \\ -0.304643 & 0.142712 \end{pmatrix}, \quad \Lambda_2^{-1} = 13.86, \quad \tilde{\Lambda}_2^{-1} = 0.15$$

Then ΔD is selected such that $|\Delta D| \leq 1.63$ as in Figure 14a. It is observed that $\|e_1\|_{[2,3],2} = 0.57$ and $\|e_2\|_{[2,3],2} = 4.89$. Also, $\|e_1\|_{[2,3],\infty} = 0.108$, $\|e_2\|_{[2,3],\infty} = 0.85$ and $\|u\|_{[2,3],\infty} = 31.98$. In order to increase the upper bound of ΔD , the constraint on $\tilde{\Lambda}_2^{-1}$ is modified such that $\tilde{\Lambda}_2^{-1} \leq 0.15$. The solutions of the LMI in (94) with a new constraint on $\tilde{\Lambda}_2^{-1}$ are given as follows:

$$K_0 = (52.60207 \quad -2.25634), \quad P_0 = \begin{pmatrix} 15.09418 & -0.32588 \\ -0.3258 & 0.14667 \end{pmatrix}, \quad \Lambda_2^{-1} = 14.03, \quad \tilde{\Lambda}_2^{-1} = 0.105.$$

Then, it is possible to choose ΔD such that $|\Delta D| \leq 2.44$. It is noticeable that a higher gain is obtained when the bound of the uncertainty is increased. As the value of ΔD increases, it is noticed in Figure 14b that L_2 and L_∞ norms of the error function increase such that $\|e_1\|_{[2,3],2} = 0.89$, $\|e_2\|_{[2,3],2} = 8.17$, $\|e_1\|_{[2,3],\infty} = 0.16$ and $\|e_2\|_{[2,3],\infty} = 1.7$ with $\|u\|_{[2,3],\infty} = 35.4$.

In Figures 14c and 14d, the uncertainties are added to more matrices of the dynamics in (89). So, the controller is designed by calculating K_0 and G_0 such that the quadruple $(A_0 + E_0 K_0, B_0 +$

E_0G_0, C_0, D_0) is strongly passive and the values of the control gains are the solutions of the LMI in (45); they are given as follows:

$$K_0 = \begin{pmatrix} 55.155 & -26.438 \end{pmatrix}, G_0 = 1.292, P_0 = \begin{pmatrix} 146.481 & -57.51 \\ -57.51 & 27.342 \end{pmatrix}, \tilde{\Lambda}_1^{-1} = \begin{pmatrix} 21.602 & -2.11 \\ -2.11 & 8.916 \end{pmatrix}$$

$$\tilde{\Lambda}_2^{-1} = 0.169, \Lambda_1^{-1} = \begin{pmatrix} 18.761 & -1.756 \\ -1.756 & 8.115 \end{pmatrix} \text{ and } \Lambda_2^{-1} = 14.015$$

The bounds of the uncertainties are calculated according to the values of $\tilde{\Lambda}_1^{-1}$ and $\tilde{\Lambda}_2^{-1}$. In Figure 14c, the matrices A , E and D are perturbed by ΔA , ΔE and ΔD respectively due to the uncertainties on the elements \mathbf{R} and \mathbf{L} . The values of L_2 norm of the error function are $\|e_1\|_{[2,3],2} = 0.43$ and $\|e_2\|_{[2,3],2} = 1.08$ and that of L_∞ norm are $\|e_1\|_{[2,3],\infty} = 0.09$ and $\|e_2\|_{[2,3],\infty} = 0.18$. These values increase when a new disturbance ΔC is introduced and the results are shown in Figure 14d. This increase is noticed such that $\|e_1\|_{[2,3],2} = 0.5$, $\|e_2\|_{[2,3],2} = 1.2$, $\|e_1\|_{[2,3],\infty} = 0.12$ and $\|e_2\|_{[2,3],\infty} = 0.25$.

9 Electrical circuit that cannot be made strictly passive

Consider the circuit depicted in Figure 15, with the states x_1 which is the charge on capacitor \mathbf{C} and x_2 is the current passing through the inductor \mathbf{L} .

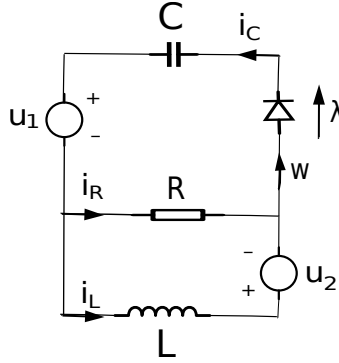


Figure 15: RLCD circuit with one ideal diode.

$$\begin{cases} \dot{x}_1(t) = -\frac{x_1(t)}{\mathbf{RC}} + x_2(t) + \frac{1}{\mathbf{R}}\lambda(t) - \frac{1}{\mathbf{R}}u_1(t) \\ \dot{x}_2(t) = -\frac{x_1(t)}{\mathbf{LC}} + \frac{1}{\mathbf{L}}\lambda(t) - \frac{1}{\mathbf{L}}u_1(t) - \frac{1}{\mathbf{L}}u_2(t) \\ 0 \leq \lambda(t) \perp w(t) = -\frac{x_1(t)}{\mathbf{RC}} + x_2(t) + \frac{1}{\mathbf{R}}\lambda(t) - \frac{1}{\mathbf{R}}u_1(t) \geq 0 \end{cases} \quad (95)$$

Take $\mathbf{L} = 1H$, $\mathbf{C} = 0.025F$ and $\mathbf{R} = 10\Omega$. Let us begin with studying the passivity of the open loop system. So, it is required to check passivity of the quadruple (A, B, C, D) with $u_1 = u_2 = 0$ by checking the positive realness of the transfer function:

$$H_0(s) = \frac{w(s)}{\lambda(s)} = \frac{s^2 + \frac{\mathbf{R}}{\mathbf{L}}s}{\mathbf{R}s^2 + \frac{1}{\mathbf{C}}s + \frac{\mathbf{R}}{\mathbf{LC}}}$$

where $H(s) = C(sI - A)^{-1}B + D$. The Nyquist plot of $H(s)$ lies in the closed right-half plane (see Figure 16), which is consistent with positive-real systems theory.

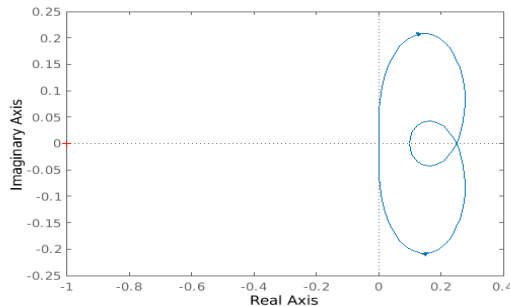


Figure 16: Nyquist Diagram

Then, from minimality and the KYP Lemma [25], the quadruple (A, B, C, D) is passive. The storage function of the circuit of Figure 15 with $u = 0$ is: $V(x) = \frac{1}{2}\mathbf{C}\left(\frac{x_1}{\mathbf{C}}\right)^2 + \frac{1}{2}\mathbf{L}x_2^2$. It follows that:

$$\dot{V}(x) = -\frac{x_1^2}{\mathbf{RC}^2} + \lambda\left(x_2 + \frac{x_1}{\mathbf{RC}}\right) \quad (96)$$

Due to complementarity conditions between λ and w , we have:

- If $\lambda = 0$, then $\dot{V}(x) = -\frac{x_1^2}{\mathbf{RC}^2} \leq 0$.
- If $\lambda > 0$, then $w = 0$ and $\lambda = \frac{x_1}{\mathbf{C}} - \mathbf{R}x_2$. Hence, $\dot{V}(x) = -\mathbf{R}x_2^2 \leq 0$.

Thus, the storage function of the circuit of Figure 15 with $u = 0$ is nonincreasing. Take $u = u_1$ and $u_2 = 0$. The transfer function of the closed-loop system is: $H_1(s) = [C + FK][sI - (A + EK)]^{-1}[B + EG] + D + FG$. Knowing that $u = K[x - x_d] + G[\lambda - \lambda_d] + u_d$ with $K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ and $G = g$, the following is obtained:

$$H_1(s) = \frac{w(s)}{\lambda(s)} = (1 - g) \frac{\mathbf{LC}s^2 + \mathbf{RC}s}{\mathbf{RLC}s^2 + (\mathbf{L} + k_1\mathbf{LC} + k_2\mathbf{RC})s + k_1\mathbf{RC} + \mathbf{R}} \quad (97)$$

Let us consider $u_1 = 0$ and $u = u_2$. The transfer function of the systems is represented as follows:

$$H_2(s) = \frac{w(s)}{\lambda(s)} = \frac{\mathbf{LC}s^2 + (k_2\mathbf{C} + \mathbf{RC} - g\mathbf{RC})s}{\mathbf{RLC}s^2 + (k_2\mathbf{RC} + \mathbf{L})s + k_2 + k_1\mathbf{RC} + \mathbf{R}} \quad (98)$$

If $u = (u_1, u_2)^\top$ with $K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ and $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, then the transfer function of the system is represented as follows:

$$H_3(s) = \frac{w(s)}{\lambda(s)} = \frac{(1 - g_1)(\mathbf{LC}s^2 + [(k_2 + k_4)\mathbf{C} - g_2(\mathbf{RC} + k_2\mathbf{C})]s)}{\mathbf{RLC}s^2 + [\mathbf{RC}(k_2 + k_4) + \mathbf{L} + k_1\mathbf{LC}]s + \mathbf{RC}(k_1 + k_3) + \mathbf{C}(k_1k_4 + k_2k_3) + \mathbf{R} + k_4} \quad (99)$$

We can see from (97), (98) and (99) that the transfer functions of the closed loop system cannot be made strictly positive real, thus the circuit shown in Figure 15 cannot be made strictly passive with u_1 and u_2 of the general form $u = K[x - x_d] + G[\lambda - \lambda_d] + u_d$. In terms of zero dynamics of the closed-loop system, when $w = 0$, there is always one zero at zero (i.e $\dot{x}_1 = 0$).

10 Study of passivity of the circuit in Figure 15 with current source

The problem that appeared in section 9, which states that the system in (95) cannot be made strictly state passive even with $u = (u_1, u_2)^\top$ motivates us to change the type and the connection of the controller for the circuit of Figure 15. If the controller is designed as a current source connected in parallel with the capacitor, then the following is obtained:

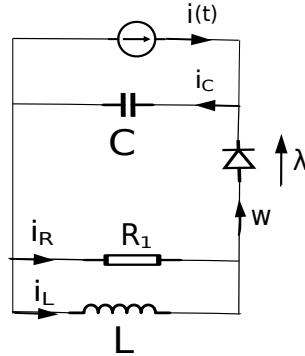


Figure 17: RLCD circuit with current source

Let $i(t) \triangleq u(t)$ and $i_d(t) \triangleq u_d(t)$ be the controller and the desired controller respectively. The dynamics are represented as follows:

$$\begin{cases} \dot{x}_1(t) = -\frac{x_1(t)}{\mathbf{RC}} + x_2(t) + \frac{1}{\mathbf{R}}\lambda(t) + u(t) \\ \dot{x}_2(t) = -\frac{x_2(t)}{\mathbf{LC}} + \frac{1}{\mathbf{L}}\lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = -\frac{x_1(t)}{\mathbf{RC}} + x_2(t) + \frac{1}{\mathbf{R}}\lambda(t) \geq 0 \end{cases} \quad (100)$$

If $u(t) = K[x - x_d] + u_d(t)$, then the transfer function of the closed loop system is: $H_4(s) = C[sI - (A + EK)]^{-1}B + D$ and represented as follows:

$$H_4(s) = \frac{\mathbf{L}\mathbf{C}s^2 + (\mathbf{R}\mathbf{C} - k_1\mathbf{L}\mathbf{C})s - k_1\mathbf{R}\mathbf{C}}{\mathbf{R}\mathbf{L}\mathbf{C}s^2 + (\mathbf{L} - k_1\mathbf{R}\mathbf{L}\mathbf{C})s + \mathbf{R} + k_2\mathbf{R}} \quad (101)$$

It is noticeable that the zeros of the transfer function are replaced by the feedback controller designed and implemented in the RLCD circuit as shown in Figure 17. This is observed through the zero dynamics of the system presented in (100). If $w = 0$, then $\dot{x}_1 = u$ which means that the zero dynamics is set in the desired place using the feedback controller.

Proposition 10.1. *The transfer function in (101) is strongly strictly positive real (SSPR) if and only if there exists $K = (k_1 \ k_2)$ such that*

- $k_1 < 0$
- $k_2 \in \left] -1, k_1(k_1\mathbf{L}\mathbf{C} - \frac{\mathbf{L}}{\mathbf{R}} - 2\mathbf{R}\mathbf{C}) + 2\sqrt{-k_1^3\mathbf{R}\mathbf{L}\mathbf{C}^2 + k_1^2\mathbf{L}\mathbf{C} - k_1\mathbf{R}^2\mathbf{C} - k_1\mathbf{R}} \right[$

Proof. In order to check the strong strict positive realness of the transfer function $H_4(s)$, it is required to present the transfer function in its frequency domain as $H_4(j\omega) = \mathbf{Re}[H_4(j\omega)] + \mathbf{Im}[H_4(j\omega)]$ such that:

$$\mathbf{Re}[H_4(j\omega)] = \frac{\mathbf{R}\mathbf{L}^2\mathbf{C}^2\omega^4 + (-k_2\mathbf{R}\mathbf{L}\mathbf{C} + k_1^2\mathbf{R}\mathbf{L}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C})\omega^2 - k_1\mathbf{R}^2\mathbf{C}(1 + k_2)}{(-\mathbf{R}\mathbf{L}\mathbf{C}\omega^2 + \mathbf{R} + k_2\mathbf{R})^2 + (\mathbf{L} - k_1\mathbf{R}\mathbf{L}\mathbf{C})^2\omega^2} \quad (102)$$

$$\mathbf{Im}[H_4(j\omega)] = j \frac{(\mathbf{L}^2\mathbf{C} - \mathbf{R}^2\mathbf{L}\mathbf{C}^2)\omega^3 + (\mathbf{R}^2\mathbf{C} + k_2\mathbf{R}^2\mathbf{C} - k_1^2\mathbf{R}^2\mathbf{L}\mathbf{C}^2 - k_1k_2\mathbf{R}\mathbf{L}\mathbf{C})\omega}{(-\mathbf{R}\mathbf{L}\mathbf{C}\omega^2 + \mathbf{R} + k_2\mathbf{R})^2 + (\mathbf{L} - k_1\mathbf{R}\mathbf{L}\mathbf{C})^2\omega^2} \quad (103)$$

The transfer function $H_4(s)$ is strongly strictly positive real if and only if the following conditions are satisfied [25, Definition 2.78]:

- $H_4(s)$ is Hurwitz,
- $\mathbf{Re}[H_4(j\omega)] \geq \delta > 0$ for all $\omega \in [-\infty, \infty]$ and some $\delta \in \mathbb{R}$.

The transfer function $H_4(s)$ is Hurwitz if the poles of H_4 have real part in the open left half-plane (LHP). The quadratic function: $\mathbf{R}\mathbf{L}\mathbf{C}s^2 + (\mathbf{L} - k_1\mathbf{R}\mathbf{L}\mathbf{C})s + \mathbf{R} + k_2\mathbf{R}$ has roots with negative real part if and only if k_1 and k_2 satisfy:

- $1 + k_2 > 0$
- $\mathbf{L} - k_1\mathbf{R}\mathbf{L}\mathbf{C} > 0$

The function $\mathbf{Re}[H_4(j\omega)] \geq \delta > 0$ if and only if the function $g(\omega) \triangleq \mathbf{R}\mathbf{L}^2\mathbf{C}^2\omega^4 + (-k_2\mathbf{R}\mathbf{L}\mathbf{C} + k_1^2\mathbf{R}\mathbf{L}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C})\omega^2 - k_1\mathbf{R}^2\mathbf{C}(1 + k_2)$ has a global minimum at $\omega = \omega_0$ such that $g(\omega_0) > 0$. Knowing that $1 + k_2 > 0$, it is necessary to have:

$$-k_1\mathbf{R}^2\mathbf{C}(1 + k_2) > 0 \Rightarrow k_1 < 0 \quad (104)$$

so that the function $g(0) > 0$. The minimum of $g(\omega)$ is studied as follows:

$$\frac{\partial g}{\partial \omega}(\omega) = 4\mathbf{R}\mathbf{L}^2\mathbf{C}^2\omega^3 + 2(-k_2\mathbf{R}\mathbf{L}\mathbf{C} + k_1^2\mathbf{R}\mathbf{L}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C})\omega$$

At this step, there are two cases:

First case: If

$$-k_2\mathbf{R}\mathbf{L}\mathbf{C} + k_1^2\mathbf{R}\mathbf{L}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C} > 0 \Leftrightarrow k_2 < k_1 \frac{\mathbf{L}}{\mathbf{R}} (k_1\mathbf{R}\mathbf{C} - 1) \quad (105)$$

then the function $g(\omega)$ is convex and has a global minimum at $\omega = 0$ where $\frac{\partial g}{\partial \omega}(0) = 0$. The function $g(\omega) > 0 \forall \omega \in [-\infty, +\infty]$ if and only if the following holds:

$$g(0) = -k_1\mathbf{R}^2\mathbf{C}(1 + k_2) > 0 \Leftrightarrow k_1 < 0 \quad (106)$$

Second case: If

$$-k_2\mathbf{RLC} + k_1^2\mathbf{RL}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C} < 0 \Leftrightarrow k_2 > k_1\frac{\mathbf{L}}{\mathbf{R}}(k_1\mathbf{RC} - 1) \quad (107)$$

then the roots of $\frac{\partial g}{\partial \omega}(\omega)$ are: $\omega_1 = 0$ and $\omega_{2,3} = \pm\sqrt{-\frac{-k_2\mathbf{RLC} + k_1^2\mathbf{RL}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C}}{2\mathbf{RL}^2\mathbf{C}^2}}$. The second derivative of $g(\omega)$ is:

$$\frac{\partial^2 g}{\partial \omega^2}(\omega) = 12\mathbf{RL}^2\mathbf{C}^2\omega^2 + 2(-k_2\mathbf{RLC} + k_1^2\mathbf{RL}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C})$$

The value of $\frac{\partial^2 g}{\partial \omega^2}(\omega_{2,3}) = -4(-k_2\mathbf{RLC} + k_1^2\mathbf{RL}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C}) > 0$. So, the function $g(\omega)$ has global minima at $\omega = \pm\sqrt{-\frac{-k_2\mathbf{RLC} + k_1^2\mathbf{RL}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C}}{2\mathbf{RL}^2\mathbf{C}^2}}$. The value of $g(\omega_{2,3})$ must be positive. It follows that:

$$g(\omega_{2,3}) = \frac{-(-k_2\mathbf{RLC} + k_1^2\mathbf{RL}^2\mathbf{C}^2 - k_1\mathbf{L}^2\mathbf{C})^2 + 4\mathbf{RL}^2\mathbf{C}^2(-k_1\mathbf{R}^2\mathbf{C}(1 + k_2))}{4\mathbf{RL}^2\mathbf{C}^2} > 0 \quad (108)$$

Then,

$$-\frac{\mathbf{R}}{4}k_2^2 + k_2\left(k_1^2\frac{\mathbf{RLC}}{2} - k_1\frac{\mathbf{L}}{2} - k_1\mathbf{R}^2\mathbf{C}\right) - k_1^4\frac{\mathbf{RL}^2\mathbf{C}^2}{4} + k_1^3\frac{\mathbf{L}^2\mathbf{C}}{2} - k_1^2\frac{\mathbf{L}^2}{4\mathbf{R}} - k_1\mathbf{R}^2\mathbf{C} > 0 \quad (109)$$

The function $g(\omega_{2,3})$ in (109) is a concave function in k_2 with the discriminant:

$$\Delta = 16\mathbf{R}^4\mathbf{C}(-k_1^3\mathbf{RLC} + k_1^2\mathbf{L} + k_1^2\mathbf{R}^2\mathbf{C} - k_1\mathbf{R}) > 0 \text{ since } k_1 < 0$$

The roots of the quadratic function $h(k_2)$ are:

$$k_{2_1} = k_1^2\mathbf{LC} - k_1\left(\frac{\mathbf{L}}{\mathbf{R}} + 2\mathbf{RC}\right) + 2\sqrt{-k_1^3\mathbf{RLC}^2 + k_1^2\mathbf{LC} - k_1\mathbf{R}^2\mathbf{C} - k_1\mathbf{R}}$$

or

$$k_{2_2} = k_1^2\mathbf{LC} - k_1\left(\frac{\mathbf{L}}{\mathbf{R}} + 2\mathbf{RC}\right) - 2\sqrt{-k_1^3\mathbf{RLC}^2 + k_1^2\mathbf{LC} - k_1\mathbf{R}^2\mathbf{C} - k_1\mathbf{R}}$$

So, the function $h(k_2)$ is positive iff $k_{2_2} < k_2 < k_{2_1}$. Hence, the function $g(\omega)$, in this case, is positive, if and only if the conditions (104),(107) and (108) are satisfied.

It is noteworthy that if $k_2 = k_1\frac{\mathbf{L}}{\mathbf{R}}(k_1\mathbf{RC} - 1)$, then $\mathbf{RL}^2\mathbf{C}^2\omega^4 - k_1\mathbf{R}^2\mathbf{C}(1 + k_1^2\mathbf{LC} - k_1\frac{\mathbf{L}}{\mathbf{R}}) > 0 \forall \omega \in [-\infty, +\infty]$. \square

If the transfer function in (101) is SSPR for k_1 and k_2 satisfying Proposition 10.1, then the nonlinear matrix inequality in (20) always has a solution. Thus, the system is strongly passive.

Let us check strong passivity by solving the NLMI in (20). Take $\mathbf{L} = 1$ H, $\mathbf{C} = 0.025$ F, and $\mathbf{R} = 10$ Ω . The solution is shown below:

$$K = (-11.78022858 \quad 0.54878816) \text{ and } P = \begin{pmatrix} 27.65215362 & -0.26929383 \\ -0.26929383 & 0.99594605 \end{pmatrix}$$

Using the given values of \mathbf{L} , \mathbf{R} , \mathbf{C} and K parameters, the plot of $\mathbf{Re}[H_4(j\omega)]$ in (102) is depicted in the following figure:

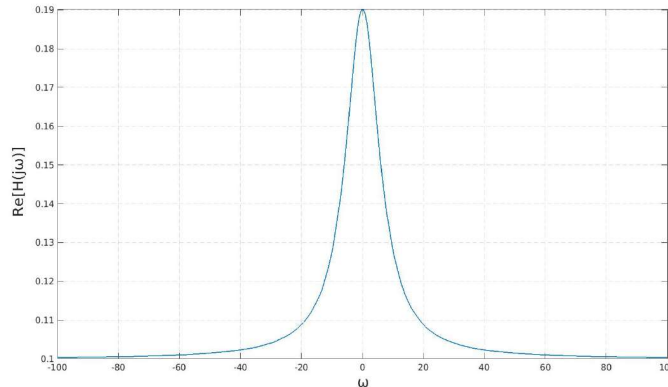


Figure 18: Graphical representation of (102) with given parameters

which shows that $\text{Re}[H_4(j\omega)] \geq \delta > 0$ with $\delta = \text{Re}[H_4(j\infty)] = \frac{1}{\mathbf{R}} = 0.1$. After designing the controller, the simulation of the system is performed with the desired current source $u_d = 30 \sin 5t$ and the initial state vectors are $x(0) = (-5 \ 1)^\top$ and $x_d(0) = (-3 \ -1)^\top$ with the time step $h = 10^{-3}$.

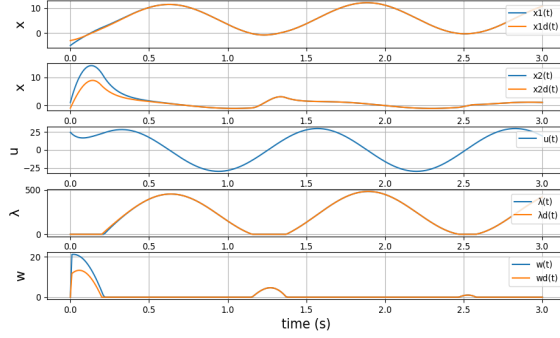


Figure 19: Solution of (100) with a controller represented by the current source

11 Electrical circuit with state jumps ($D = D^\top \succeq 0$)

The circuits of the foregoing section have the matrix $D = D^\top \succeq 0$ which gives the possibility of having state jumps. Let x_1, x_2, x_3 and x_4 be the voltages across the capacitors $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ and \mathbf{C}_4 respectively. Take $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{C}_3 = \mathbf{C}_4 = \mathbf{C}$ and $\mathbf{C} = 0.025$ F, $\mathbf{R} = 10 \ \Omega$. Consider the circuit of Figure 20 with its dynamics in (110)

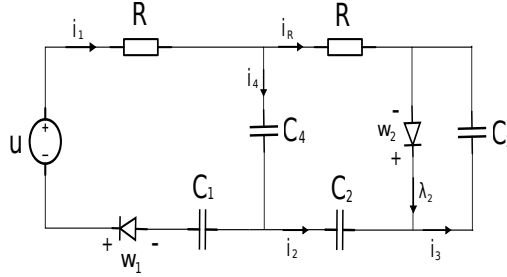


Figure 20: RC circuit with two ideal diodes

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} \\ 0 & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} \\ 0 & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{\mathbf{C}} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \\ \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \mathbf{R} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u \geq 0 \end{array} \right. \quad (110)$$

Note that there exists $P \succeq 0$ such that the quadruple (A, B, C, D) is passive (i.e the LMI in (7) has a solution). In order to check the strict state passivity of (110). Let us observe the strict positive realness (SPR) of the transfer function matrix given by:

$$G(s) = \begin{pmatrix} \frac{\mathbf{R}^2 \mathbf{C} s^4 + s^3 (5\mathbf{R} - k_1 \mathbf{R} - k_4 \mathbf{R}) + \frac{s^2}{\mathbf{C}} (-3k_1 + k_2 + k_3 - 2k_4 + 5)}{\mathbf{RC} s^4 + 3s^3} & \frac{k_3 \mathbf{RC} s^3 + s^2 (k_2 + 2k_3 + k_4 - 1)}{\mathbf{RC}^2 s^4 + 3\mathbf{C} s^3} \\ -s^2 & \frac{\mathbf{R} s^3 + \frac{2s^2}{\mathbf{C}}}{\mathbf{RC} s^4 + 3s^3} \end{pmatrix} \quad (111)$$

where $G(s) \triangleq \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \triangleq \begin{pmatrix} \frac{w_1}{\lambda_1} & \frac{w_1}{\lambda_2} \\ \frac{w_2}{\lambda_1} & \frac{w_2}{\lambda_2} \end{pmatrix}$.

In view of the matrix of the transfer functions in (111), it is noticeable that the transfer function is not Hurwitz, hence not strictly positive real due to the pole at zero of multiplicity 3.

Remark 11.1. According to [25, Definition 2.70], all the principal sub-matrices of the transfer function matrix G_{11} , G_{22} and $|G_{11} * G_{22} - G_{21} * G_{12}|$ are of index = 1 (i.e all the principal sub-matrices are proper). So, the transfer matrix $G(s)$ is totally index 1. As well, according to [25, Proposition 2.71], the transfer matrix $G(s)$ (111) is totally of index 1 knowing that (A, B, C, D) is passive, (A, B) is controllable and the matrix $\begin{pmatrix} B \\ D + D^\top \end{pmatrix}$ has full column rank.

Thus, the circuit of Figure 20 is not strictly state passive (SSP) and we cannot analyze the stability of the LCS in (110) in the presence of state jumps.

11.1 Build a strict state passive circuit without adding controllers

Knowing that strict state passivity is a sufficient condition for stability analysis in the case of state jumps, the aim is to find an SSP circuit with a matrix $D \succeq 0$. One approach to make the circuit of Figure 20 strictly passive would be by adding proper resistors, i.e adding dissipativity in the circuit as shown in Figure 21.

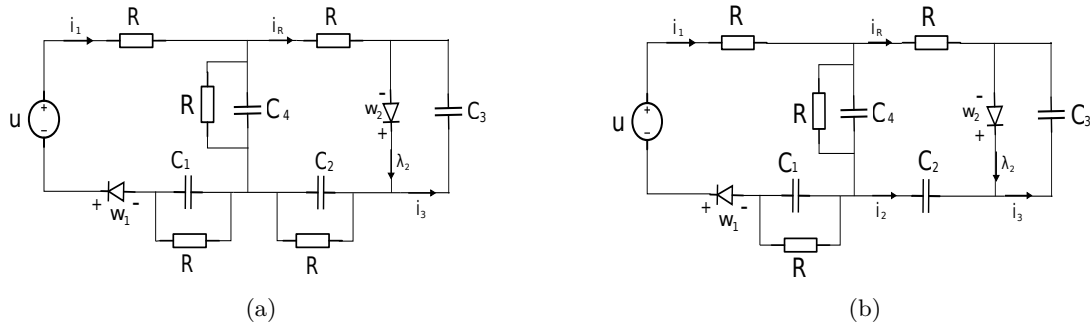


Figure 21: RLCD circuit with two ideal diodes.

Their dynamical equations are given as follows:

$$(b) \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{RC} & 0 & 0 & 0 \\ 0 & -\frac{2}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{2}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u \geq 0 \end{array} \right. \quad (112)$$

$$(c) \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{RC} & 0 & 0 & 0 \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{2}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u \geq 0 \end{array} \right. \quad (113)$$

It is noteworthy that connecting a resistor across the capacitor C_3 would cancel the property of positive semi-definiteness (PSD) for the matrix D which is a necessary condition for the existence of state jumps.

The transfer function matrix of the closed-loop system in (112) is represented as follows:

$$H(s) \triangleq \begin{pmatrix} \frac{w_1}{\lambda_1} & \frac{w_1}{\lambda_2} \\ \frac{w_2}{\lambda_1} & \frac{w_2}{\lambda_2} \end{pmatrix} \triangleq \begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix}$$

where

$$\begin{aligned} H_{11}(s) &= \frac{\mathbf{R}^4 \mathbf{C}^3 (k_1 + k_2 + 2)s^3 + \mathbf{R}^3 \mathbf{C}^2 (5k_1 - k_2 - k_3 + 4k_4 + 9)s^2 + \mathbf{R}^2 \mathbf{C} (5k_1 - k_2 - 2k_3 + 4k_4 + 9)s + \mathbf{R} (k_1 - k_3 + k_4 + 2)}{\mathbf{R}^4 \mathbf{C}^4 s^4 + 6\mathbf{R}^3 \mathbf{C}^3 s^3 + 10\mathbf{R}^2 \mathbf{C}^2 s^2 + 6\mathbf{R} \mathbf{C} s + 1} + \mathbf{R} \\ H_{12}(s) &= \frac{\mathbf{R}^4 \mathbf{C}^3 k_3 s^3 + \mathbf{R}^3 \mathbf{C}^2 (-k_2 + 5k_3 - k_4 - 1)s^2 + \mathbf{R}^2 \mathbf{C} (-2k_2 + 4k_3 - 2k_4 - 2)s + \mathbf{R} (-k_2 + 3k_3 - k_4 - 1)}{\mathbf{R}^4 \mathbf{C}^4 s^4 + 6\mathbf{R}^3 \mathbf{C}^3 s^3 + 10\mathbf{R}^2 \mathbf{C}^2 s^2 + 6\mathbf{R} \mathbf{C} s + 1} \\ H_{21}(s) &= \frac{-\mathbf{R}^3 \mathbf{C}^2 s^2 - 2\mathbf{R}^2 \mathbf{C} s - \mathbf{R}}{\mathbf{R}^4 \mathbf{C}^4 s^4 + 6\mathbf{R}^3 \mathbf{C}^3 s^3 + 10\mathbf{R}^2 \mathbf{C}^2 s^2 + 6\mathbf{R} \mathbf{C} s + 1} \quad \text{and} \quad H_{22}(s) = \frac{\mathbf{R}^4 \mathbf{C}^3 s^3 + 5\mathbf{R}^3 \mathbf{C}^2 s^2 + 4\mathbf{R}^2 \mathbf{C} s + 3\mathbf{R}}{\mathbf{R}^4 \mathbf{C}^4 s^4 + 6\mathbf{R}^3 \mathbf{C}^3 s^3 + 10\mathbf{R}^2 \mathbf{C}^2 s^2 + 6\mathbf{R} \mathbf{C} s + 1} \end{aligned}$$

The poles are the roots of the characteristic equation:

$$\mathbf{R}^4 \mathbf{C}^4 s^4 + 6\mathbf{R}^3 \mathbf{C}^3 s^3 + 10\mathbf{R}^2 \mathbf{C}^2 s^2 + 6\mathbf{R} \mathbf{C} s + 1$$

and they are represented as follows:

- $s = -\frac{1}{\mathbf{R} \mathbf{C}}$ (double root),
- $s = -\frac{2}{\mathbf{R} \mathbf{C}} + \sqrt{\frac{3}{\mathbf{R}^2 \mathbf{C}^2}}$, and
- $s = -\frac{2}{\mathbf{R} \mathbf{C}} - \sqrt{\frac{3}{\mathbf{R}^2 \mathbf{C}^2}}$.

Hence, the problem of the pole at zero with multiplicity 3 which appears in the system represented in Figure 20 is solved in system represented in Figure 21a since all the poles have negative real part.

In order to check if there exist K and P such that the quadruple $(A, B, C + FK, D)$ of the closed loop system (112) is strictly passive, the NLMI in (19) is solved and the solution is:

$$K = \begin{pmatrix} -0.304 & -0.658 & 0 & -0.962 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0.0032 & 0 & 0 & 0 \\ 0 & 0.005 & 0 & 0.002 \\ 0 & 0 & 0.0025 & 0 \\ 0 & 0.002 & 0 & 0.005 \end{pmatrix} \quad (114)$$

with $\epsilon = 0.1$.

The variable ϵ is chosen arbitrarily by the user such that the system is strictly passive with a desired degree of strict passivity (as ϵ increases, the degree of strict passivity increases). The value of ϵ can be increased until it reaches a critical value, at which the LMI becomes infeasible again. If $F = 0$, then the open loop system of (112) is strictly passive as well. So, in our case, the open loop system is strictly passive and the role of ϵ is to increase the degree of strict passivity by pushing the eigenvalues of the matrix $A + EK$ to be more negative. But, $E = 0$ in this system, thus varying the value of ϵ does not have an interesting effect on the control gain. It is noteworthy that the critical value of ϵ (the last value at which the system is strictly passive) is $\epsilon = 2$ in the closed-loop system represented by (112).

In order to check if it is necessary to connect three resistors across C_1 , C_2 and C_4 , let us remove one of these resistors as shown in Figure 21b. The poles of the transfer function can be viewed as the poles of $\det[sI - A]^{-1}$ which is given by:

$$\frac{\mathbf{R}^3 \mathbf{C}^3}{s(\mathbf{R} \mathbf{C} s + 1)(\mathbf{R}^2 \mathbf{C}^2 s^2 + 4\mathbf{R} \mathbf{C} s + 2)}$$

with one pole at zero. Hence, the transfer function is not strictly positive real and the system represented by (113) is not strictly state passive.

Thus, the closed-loop as well as the open-loop of the system in (112) of the circuit in Figure 21a are strictly state passive.

11.2 Build a strictly state passive circuit from the circuit in Figure 20 by adding controllers

Let us now add controllers (current sources) to the circuit in Figure 20 in different numbers and places in order to study strict state passivity of the closed-loop system as shown in circuits of Figure 22.

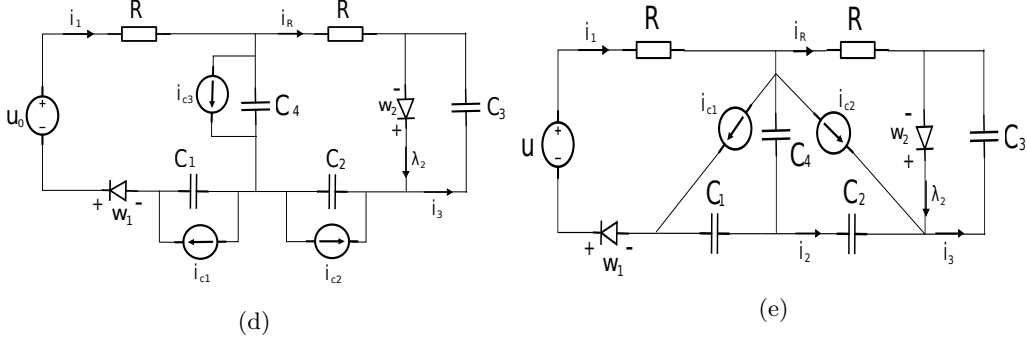


Figure 22: RLCD circuit with two ideal diodes and both voltage and current sources.

Their dynamics are given as follows:

$$(d) \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{C} & 0 & 0 \\ 0 & 0 & -\frac{1}{C} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{C} \end{pmatrix} \begin{pmatrix} u_0 \\ i_{c1} \\ i_{c2} \\ i_{c3} \end{pmatrix} \\ \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ i_{c1} \\ i_{c2} \\ i_{c3} \end{pmatrix} \geq 0 \end{array} \right. \quad (115)$$

$$(e) \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{C} & 0 \\ 0 & 0 & -\frac{1}{C} \\ 0 & 0 & 0 \\ 0 & -\frac{1}{C} & -\frac{1}{C} \end{pmatrix} \begin{pmatrix} u \\ i_{c1} \\ i_{c2} \end{pmatrix} \\ \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ i_{c1} \\ i_{c2} \end{pmatrix} \geq 0 \end{array} \right. \quad (116)$$

If $E = 0$ and $F = 0$, then the NLMI in (19) has no solution and the open loop system (A, B, C, D) for circuits in Figure 22 is not strictly passive. In order to avoid Dirac measure in the presence of state jumps, let us take $G = 0$ in the following study (so that (C1) is satisfied). The NLMI in (19) for strict state passivity has a solution when solved for the closed loop system $(A + EK, B, C + FK, D)$ represented by (115). The solution is:

$$K = \begin{pmatrix} 0.52 & -0.01 & 0 & 0.504 \\ 0.014 & -0.00001 & 0 & -0.0006 \\ 0 & -0.088 & -0.302 & -0.025 \\ -0.0006 & 0.18 & 0.302 & 0.09 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0.012 & 0 & 0 & 0 \\ 0 & 0.012 & 0 & 0.00025 \\ 0 & 0 & 0.025 & 0 \\ 0 & 0.00025 & 0 & 0.012 \end{pmatrix} \quad (117)$$

with $\epsilon = 0.1$. Let $K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}$. The closed loop system with $u = (u_0 \ i_{c1} \ i_{c2} \ i_{c3})^\top =$

$K[x - x_d] + u_d$ is represented as follows:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} &= \begin{pmatrix} \frac{-k_{21}}{\mathbf{C}} & \frac{-k_{22}}{\mathbf{C}} & \frac{-k_{23}}{\mathbf{C}} & \frac{-k_{24}}{\mathbf{C}} \\ \frac{-k_{31}}{\mathbf{C}} & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{32}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{33}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{34}) \\ 0 & \frac{1}{\mathbf{RC}} & \frac{1}{\mathbf{RC}} & \frac{1}{\mathbf{RC}} \\ -\frac{k_{41}}{\mathbf{C}} & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{42}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{43}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{44}) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{\mathbf{C}} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \\
&- \begin{pmatrix} \frac{-k_{21}}{\mathbf{C}} & \frac{-k_{22}}{\mathbf{C}} & \frac{-k_{23}}{\mathbf{C}} & \frac{-k_{24}}{\mathbf{C}} \\ \frac{-k_{31}}{\mathbf{C}} & \frac{-k_{32}}{\mathbf{C}} & \frac{-k_{33}}{\mathbf{C}} & \frac{-k_{34}}{\mathbf{C}} \\ 0 & 0 & 0 & 0 \\ -\frac{k_{41}}{\mathbf{C}} & \frac{-k_{42}}{\mathbf{C}} & \frac{-k_{43}}{\mathbf{C}} & \frac{-k_{44}}{\mathbf{C}} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\mathbf{C}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\mathbf{C}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\mathbf{C}} \end{pmatrix} \begin{pmatrix} u_{0d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \\
0 \leq \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} &= \begin{pmatrix} 1 - k_{11} & -k_{12} & -k_{13} & 1 - k_{14} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \mathbf{R} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \\
&- \begin{pmatrix} -k_{11} & -k_{12} & -k_{13} & -k_{14} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{0d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \geq 0
\end{aligned} \tag{118}$$

The desired dynamics are represented as follows:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_{1d}(t) \\ \dot{x}_{2d}(t) \\ \dot{x}_{3d}(t) \\ \dot{x}_{4d}(t) \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} \\ 0 & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} \\ 0 & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{\mathbf{C}} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\mathbf{C}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\mathbf{C}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\mathbf{C}} \end{pmatrix} \begin{pmatrix} u_{0d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \\
0 \leq \begin{pmatrix} \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \perp \begin{pmatrix} w_{1d}(t) \\ w_{2d}(t) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{R} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{0d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \geq 0
\end{aligned} \tag{119}$$

The error dynamics of (115) is written in the form of (18) as follows:

$$\begin{aligned}
\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{pmatrix} &= \begin{pmatrix} \frac{-k_{21}}{\mathbf{C}} & \frac{-k_{22}}{\mathbf{C}} & \frac{-k_{23}}{\mathbf{C}} & \frac{-k_{24}}{\mathbf{C}} \\ \frac{-k_{31}}{\mathbf{C}} & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{32}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{33}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{34}) \\ 0 & \frac{1}{\mathbf{RC}} & \frac{1}{\mathbf{RC}} & \frac{1}{\mathbf{RC}} \\ -\frac{k_{41}}{\mathbf{C}} & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{42}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{43}) & -\frac{1}{\mathbf{C}}(\frac{1}{\mathbf{R}} + k_{44}) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{\mathbf{C}} & 0 \\ 0 & 0 \\ 0 & \frac{1}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_{1d} \\ \lambda_2 - \lambda_{2d} \end{pmatrix} \\
0 \leq \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_{1d}(t) \\ w_{2d}(t) \end{pmatrix} &= \begin{pmatrix} 1 - k_{11} & -k_{12} & -k_{13} & 1 - k_{14} \\ 0 & 0 & 1 & 0 \\ k_{11} - 1 & k_{12} & k_{13} & k_{14} - 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} + \begin{pmatrix} \mathbf{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 - k_{11} & -k_{12} & -k_{13} & 1 - k_{14} & k_{11} & k_{12} & k_{13} & k_{14} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_{1d} \\ x_{2d} \\ x_{3d} \\ x_{4d} \end{pmatrix} + \begin{pmatrix} -u_{0d} \\ 0 \\ -u_{0d} \\ 0 \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \geq 0
\end{aligned} \tag{120}$$

Let us study state jumps in the system represented by the circuit in Figure 22d whose dynamics is in (119). For the corresponding desired system, the set:

$$Q_{d,D} = \{\lambda_d \in \mathbb{R}^2 \mid 0 \leq \lambda_d \perp D\lambda_d \geq 0\} = \{\lambda_d \in \mathbb{R}^2 \mid \lambda_{1d} = 0 \text{ and } \lambda_{2d} \in \mathbb{R}_+\}$$

The dual cone of $Q_{d,D}$ is

$$Q_{d,D}^* = \{w_d \in \mathbb{R}^2 \mid \langle w_d, \lambda_d \rangle \geq 0\} = \{w_d \in \mathbb{R}^2 \mid w_{1d} \in \mathbb{R} \text{ and } w_{2d} \in \mathbb{R}_+\}$$

and

$$\mathcal{K}_d = \{x_d \in \mathbb{R}^4 \mid Cx_d + Fu_d(t^+) \in Q_{d,D}^*\} = \{x_d \in \mathbb{R}^4 \mid x_{1d}, x_{2d}, x_{4d} \in \mathbb{R} \text{ and } x_{3d} \geq 0\}$$

According to the conditions stated in the set \mathcal{K} , a state jump exists only at initial time in the desired system trajectory x_{3d} , and it occurs if and only if $x_{3d}(0^-) < 0$. It is notable that, by solving the minimization problem in (9), if $x_{3d}(0^-) < 0$, then $x_{3d}(0^+) = 0$. Otherwise, if $x_{3d}(0^-) \geq 0$, then $x_{3d}(0^+) = x_{3d}(0^-)$.

For the closed-loop system whose dynamics is in (118), the sets are calculated as :

$$Q_D = \{\lambda \in \mathbb{R}^2 \mid 0 \leq \lambda \perp D\lambda \geq 0\} = \{\lambda \in \mathbb{R}^2 \mid \lambda_1 = 0 \text{ and } \lambda_2 \in \mathbb{R}_+\}$$

with a dual cone:

$$Q_D^* = \{w \in \mathbb{R}^2 \mid \langle w, \lambda \rangle \geq 0\} = \{w \in \mathbb{R}^2 \mid w_1 \in \mathbb{R} \text{ and } w_2 \in \mathbb{R}_+\}$$

and the set:

$$\begin{aligned} \mathcal{K} &= \{x \in \mathbb{R}^4 \mid (C + FK)x - FKx_d + Fu_d(t) \in Q_D^*\} \\ &= \{x \in \mathbb{R}^4 \mid x_3 \geq 0 \text{ and } x_1, x_2 \text{ and } x_4 \in \mathbb{R}\} \end{aligned}$$

In the closed-loop system, it is observed from the set \mathcal{K} and the state jump rule in (9) that a state jump occurs in the state x_3 only at $t = 0$, and it occurs if and only if $x_3(0^-) < 0$ so that $x_3(0^+) = 0$.

For the error dynamics in (120):

$$Q_{e,D} = \left\{ \lambda \in \mathbb{R}^4 \mid 0 \leq \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_{1d} \\ \lambda_{2d} \end{pmatrix} \perp \begin{pmatrix} \mathbf{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_{1d} \\ \lambda_{2d} \end{pmatrix} \geq 0 \right\} = \{\lambda \in \mathbb{R}^4 \mid \lambda_1 = 0, \lambda_{1d} = 0, \lambda_2 \text{ and } \lambda_{2d} \in \mathbb{R}_+\}$$

with the dual cone given by:

$$Q_{e,D}^* = \{w \in \mathbb{R}^4 \mid \langle w, \lambda \rangle \geq 0\} = \{w \in \mathbb{R}^4 \mid w_1 \in \mathbb{R}, w_2 \in \mathbb{R}_+, w_{1d} \in \mathbb{R} \text{ and } w_{2d} \in \mathbb{R}_+\}$$

and the set

$$\begin{aligned} \mathcal{K}_e &= \left\{ e \in \mathbb{R}^4 \mid \begin{pmatrix} C + FK \\ -C - FK \end{pmatrix} e + \begin{pmatrix} 0 & C \\ C + FK & -FK \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} Fu_d \\ Fu_d \end{pmatrix} \in Q_{e,D}^* \right\} \\ &= \{e \in \mathbb{R}^4 \mid e_1, e_2, e_4 \in \mathbb{R}, e_3 + x_{3d}(0^+) \geq 0 \text{ and } -e_3 + x_3(0^+) \geq 0\} \\ &= \{e \in \mathbb{R}^4 \mid e_1, e_2, e_4 \in \mathbb{R} \text{ and } e_3 \in [-x_{3d}(0^+), x_3(0^+)]\} \end{aligned}$$

Knowing that the minimization problem in (9) is the same at initial time for the states x_3 and x_{3d} , the jump in e_3 vanishes when $x_3(0^-) = x_{3d}(0^-)$. A jump occurs at $t = 0$, in the error dynamics in e_3 if and only if $e_3(0^-) < -x_{3d}(0^-)$ or $e_3(0^-) > x_3(0^-)$. If x_{3d} and x_3 have state jumps at $t = 0$ and if $x_3(0^-) \neq x_{3d}(0^-)$, then e_3 jumps.

Consider the following numerical simulation with the initial state $x(0^-) = (-1, 1, -2, 2)$ and $x_d(0^-) = (1, 0, 1, -2)$ and with a time step $h = 0.001$. Take $u_{0d} = 5 \sin 10t$, $i_{c1d} = \sin 5t$, $i_{c2d} = 3 \sin 5t$ and $i_{c3d} = 2 \sin 3t$.

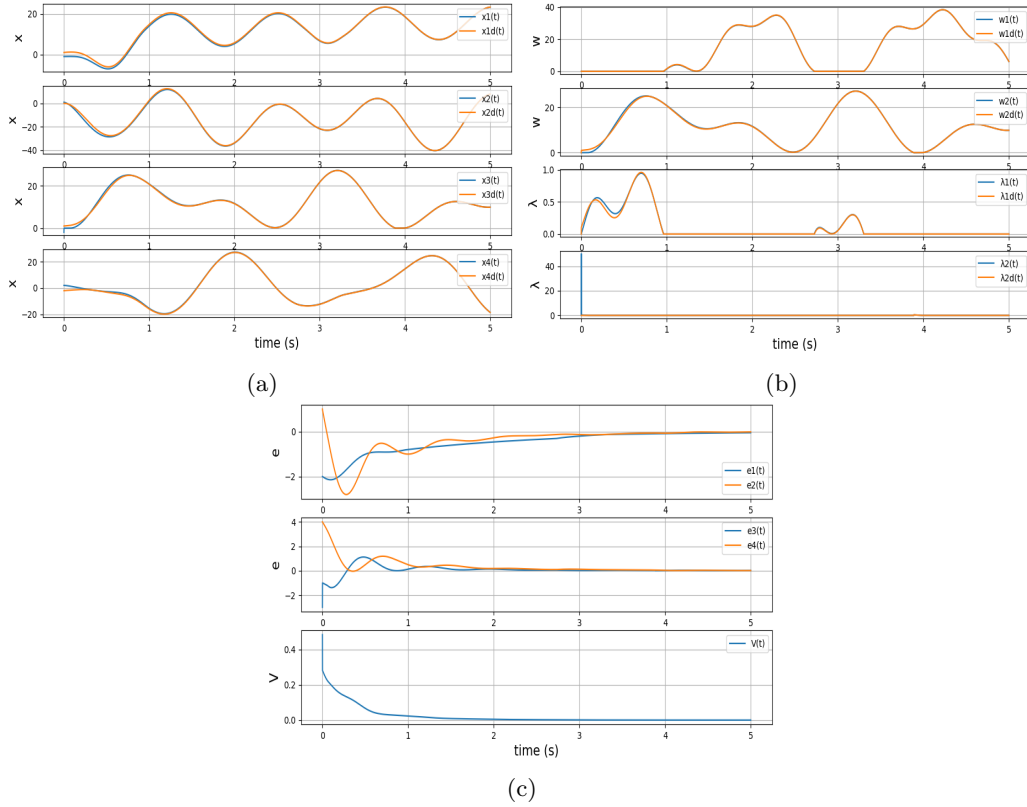


Figure 23: Numerical Simulation of desired, closed-loop and error dynamics of LCS in (115)

It is observed from the numerical simulation in Figure 23 that the state x_3 of the closed-loop system performs a state jump from $x_3(0^-) = -2$ to $x_3(0^+) = 0$. This numerical result confirms the jump rule given by the minimization problem in (9).

In Figure 23b, a synchronization is noticed between the complementarity variables λ and w in the desired and the real systems. The reason of this result is the fact that the term $\lambda - \lambda_d$ in (120) converges to $\text{Ker}(B + EG)$ which is zero in this example.

It is observable in Figure 23c that the storage function jumps at initial time such that $V(0^+) - V(0^-) < 0$. By applying Lemma 3.10, it is given in our example that $u_{0d}(0)$ is time continuous, x_d is continuous at $t = 0$ and $FKx_d(0) - Fu_d(0) \geq 0 \Leftrightarrow 0 \in \{x \in \mathbb{R}^4 \mid (C + FK)x + FKx_d(0) - Fu_d(0) \geq 0\}$, then at an initial state jump we have $V(0^+) - V(0^-) \leq 0$ where $P = P^T \succ 0$ given in (117).

Let us decrease the number of controllers (current sources) as shown in Figure 22e. The NLMI in (19) has no solution for the closed-loop system in (116) which means that the stability of the LCS cannot be analyzed in the presence of state jumps.

11.3 Strictly state passive circuit with state jumps at $t = t_c > 0$

Let us add a voltage source in series with the capacitor C_3 to the strictly passive circuit in Figure 22d so that a jump can occur at $t = t_c > 0$ when u_d is discontinuous at $t = t_c$.

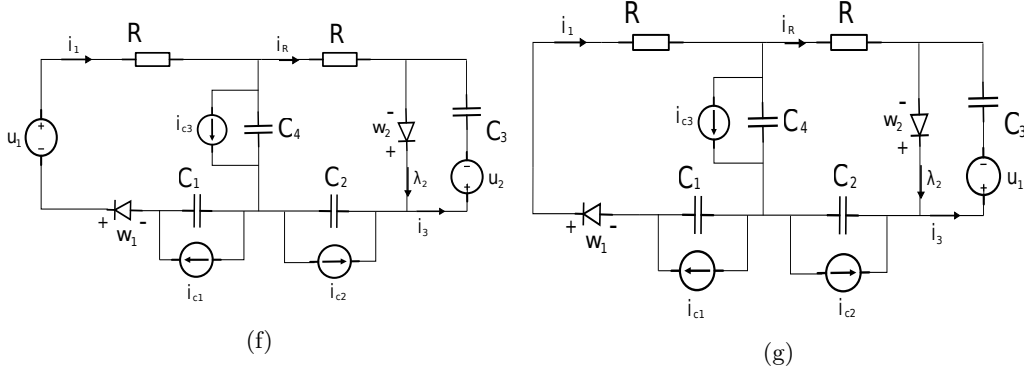


Figure 24: RLCD circuit with two ideal diodes and both voltage and current sources.

Their dynamics are given by:

$$(f) \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{1}{C} & 0 & 0 \\ 0 & \frac{-1}{RC} & 0 & -\frac{1}{C} & 0 \\ 0 & \frac{-1}{RC} & 0 & 0 & 0 \\ 0 & \frac{-1}{RC} & 0 & 0 & -\frac{1}{C} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ i_{c1} \\ i_{c2} \\ i_{c3} \end{pmatrix} \\ \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ i_{c1} \\ i_{c2} \\ i_{c3} \end{pmatrix} \geq 0 \end{array} \right. \quad (121)$$

$$(g) \left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{C} & 0 & 0 \\ -\frac{1}{RC} & 0 & -\frac{1}{C} & 0 \\ -\frac{1}{RC} & 0 & 0 & 0 \\ -\frac{1}{RC} & 0 & 0 & -\frac{1}{C} \end{pmatrix} \begin{pmatrix} u_1 \\ i_{c1} \\ i_{c2} \\ i_{c3} \end{pmatrix} \\ \\ 0 \leq \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \perp \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ i_{c1} \\ i_{c2} \\ i_{c3} \end{pmatrix} \geq 0 \end{array} \right. \quad (122)$$

The open loop quadruple (A, B, C, D) (i.e $E = 0$ and $F = 0$) of circuits in Figure 24 is not strictly passive. But, while checking if there exist K and P such that the closed loop quadruple $(A + EK, B, C + FK, D)$ is strictly passive, the NLMI in (19) has a solution for circuit in Figure 24f such that:

$$K = \begin{pmatrix} -1.955 & 0.017 & 0.213 & -1.942 \\ 0.001 & 0.212 & -1.089 & 0.212 \\ 0.013 & 0 & 0 & 0 \\ 0 & -0.1 & -0.086 & -0.015 \\ 0 & 0.21 & 0.072 & 0.994 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0.074 & 0 & 0 & 0 \\ 0 & 0.074 & -0.005 & 0 \\ 0 & -0.005 & 0.052 & -0.005 \\ 0 & 0 & -0.005 & 0.074 \end{pmatrix}$$

As well as the NLMI (19) has a solution for the circuit in Figure 24g such that

$$K = \begin{pmatrix} -0.238 & 0.625 & 0.298 & 0.389 \\ 0.009 & 0.006 & 0.009 & 0.003 \\ 0.042 & -0.22 & -0.25 & -0.109 \\ 0.045 & -0.31 & -0.303 & -0.17 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0.029 & 0.001 & -0.006 & -0.004 \\ 0.001 & 0.033 & 0.016 & 0.001 \\ -0.006 & 0.016 & 0.032 & 0.01 \\ -0.004 & 0.001 & 0.01 & 0.0304 \end{pmatrix} \quad (123)$$

with $\epsilon = 0.01$. Consider the circuit of Figure 24g which is strictly passive with less number of controllers. The desired dynamics of (122) is given as follows:

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_{1d}(t) \\ \dot{x}_{2d}(t) \\ \dot{x}_{3d}(t) \\ \dot{x}_{4d}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \\ 0 & -\frac{1}{RC} & -\frac{1}{RC} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{C} & 0 & 0 \\ -\frac{1}{RC} & 0 & -\frac{1}{C} & 0 \\ -\frac{1}{RC} & 0 & 0 & 0 \\ -\frac{1}{RC} & 0 & 0 & -\frac{1}{C} \end{pmatrix} \begin{pmatrix} u_{1d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \\ \\ 0 \leq \begin{pmatrix} \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \perp \begin{pmatrix} w_{1d}(t) \\ w_{2d}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \geq 0 \end{array} \right. \quad (124)$$

For the desired system, the set:

$$Q_{d,D} = \{\lambda_d \in \mathbb{R}^2 \mid 0 \leq \lambda_d \perp D\lambda_d \geq 0\} = \{\lambda_d \in \mathbb{R}^2 \mid \lambda_{1d} = 0 \text{ and } \lambda_{2d} \in \mathbb{R}_+\}$$

The dual cone of $Q_{d,D}$ is

$$Q_{d,D}^* = \{w_d \in \mathbb{R}^2 \mid \langle w_d, \lambda_d \rangle \geq 0\} = \{w_d \in \mathbb{R}^2 \mid w_{1d} \in \mathbb{R} \text{ and } w_{2d} \in \mathbb{R}_+\}$$

and

$$\mathcal{K}_d = \{x_d \in \mathbb{R}^4 \mid Cx_d + Fu_d(t^+) \in Q_{d,D}^*\} = \{x_d \in \mathbb{R}^4 \mid x_{1d}, x_{2d}, x_{4d} \in \mathbb{R} \text{ and } x_{3d} + u_{1d}(t^+) \geq 0\}$$

Let $u = (u_1 \quad i_{c1} \quad i_{c2} \quad i_{c3})^\top = K(x - x_d) + u_d$ with $K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}$. The closed-loop

system is represented as follows:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} &= \begin{pmatrix} -\frac{k_{21}}{C} & -\frac{k_{22}}{C} & -\frac{k_{23}}{C} & -\frac{k_{24}}{C} \\ -\frac{k_{11}}{RC} - \frac{k_{31}}{C} & -\frac{1}{RC}(1+k_{12}) - \frac{k_{32}}{C} & -\frac{1}{RC}(1+k_{13}) - \frac{k_{33}}{C} & -\frac{1}{RC}(1+k_{14}) - \frac{k_{34}}{C} \\ -\frac{k_{11}}{RC} & -\frac{1}{RC}(1+k_{12}) & -\frac{1}{RC}(1+k_{13}) & -\frac{1}{RC}(1+k_{14}) \\ -\frac{k_{11}}{RC} - \frac{k_{41}}{C} & -\frac{1}{RC} - \frac{k_{12}}{RC} - \frac{k_{42}}{C} & -\frac{1}{RC} - \frac{k_{13}}{RC} - \frac{k_{43}}{C} & -\frac{1}{RC} - \frac{k_{14}}{RC} - \frac{k_{44}}{C} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{C} & 0 \\ 0 & 0 \\ 0 & \frac{1}{C} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} - \begin{pmatrix} -\frac{k_{21}}{C} & -\frac{k_{22}}{C} & -\frac{k_{23}}{C} & -\frac{k_{24}}{C} \\ -\frac{k_{11}}{RC} - \frac{k_{31}}{C} & -\frac{k_{12}}{RC} - \frac{k_{32}}{C} & -\frac{k_{13}}{RC} - \frac{k_{33}}{C} & -\frac{k_{14}}{RC} - \frac{k_{34}}{C} \\ -\frac{k_{11}}{RC} & -\frac{k_{12}}{RC} & -\frac{k_{13}}{RC} & -\frac{k_{14}}{RC} \\ -\frac{k_{11}}{RC} - \frac{k_{41}}{C} & -\frac{k_{12}}{RC} - \frac{k_{42}}{C} & -\frac{k_{13}}{RC} - \frac{k_{43}}{C} & -\frac{k_{14}}{RC} - \frac{k_{44}}{C} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -\frac{1}{C} & 0 & 0 \\ -\frac{1}{RC} & 0 & -\frac{1}{C} & 0 \\ -\frac{1}{RC} & 0 & 0 & 0 \\ -\frac{1}{RC} & 0 & 0 & -\frac{1}{C} \end{pmatrix} \begin{pmatrix} u_{1d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \\ 0 \leq \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ k_{11} & k_{12} & 1+k_{13} & k_{14} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ k_{11} & k_{12} & k_{13} & k_{14} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \\ x_{4d}(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1d} \\ i_{c1d} \\ i_{c2d} \\ i_{c3d} \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} \geq 0 \end{aligned} \quad (125)$$

For the closed-loop system whose dynamics is in (125), the sets are calculated as :

$$Q_D = \{\lambda \in \mathbb{R}^2 \mid 0 \leq \lambda \perp D\lambda \geq 0\} = \{\lambda \in \mathbb{R}^2 \mid \lambda_1 = 0 \text{ and } \lambda_2 \in \mathbb{R}_+\}$$

with a dual cone:

$$Q_D^* = \{w \in \mathbb{R}^2 \mid \langle w, \lambda \rangle \geq 0\} = \{w \in \mathbb{R}^2 \mid w_1 \in \mathbb{R} \text{ and } w_2 \in \mathbb{R}_+\}$$

and the set:

$$\begin{aligned} \mathcal{K} &= \{x \in \mathbb{R}^4 \mid (C + FK)x - FKx_d + Fu_d(t) \in Q_D^*\} \\ &= \{x \in \mathbb{R}^4 \mid x_1 + x_4 \in \mathbb{R} \text{ and } k_{11}(x_1 - x_{1d}) + k_{12}(x_2 - x_{2d}) + k_{13}(x_3 - x_{3d}) + k_{14}(x_4 - x_{4d}) + x_3 + u_{1d}(t^+) \geq 0\} \end{aligned}$$

The error dynamics is written in the form of the dynamics in (25) as follows:

$$\begin{aligned} \begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{pmatrix} &= \begin{pmatrix} \frac{-k_{21}}{\mathbf{C}} & \frac{-k_{22}}{\mathbf{C}} & \frac{-k_{23}}{\mathbf{C}} & \frac{-k_{24}}{\mathbf{C}} \\ -\frac{k_{11}}{\mathbf{RC}} - \frac{k_{31}}{\mathbf{C}} & -\frac{1}{\mathbf{RC}}(1 + k_{12}) - \frac{k_{32}}{\mathbf{C}} & -\frac{1}{\mathbf{RC}}(1 + k_{13}) - \frac{k_{33}}{\mathbf{C}} & -\frac{1}{\mathbf{RC}}(1 + k_{14}) - \frac{k_{34}}{\mathbf{C}} \\ -\frac{k_{11}}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}}(1 + k_{12}) & -\frac{1}{\mathbf{RC}}(1 + k_{13}) & -\frac{1}{\mathbf{RC}}(1 + k_{14}) \\ -\frac{k_{11}}{\mathbf{RC}} - \frac{k_{41}}{\mathbf{C}} & -\frac{1}{\mathbf{RC}} - \frac{k_{12}}{\mathbf{RC}} - \frac{k_{42}}{\mathbf{C}} & -\frac{1}{\mathbf{RC}} - \frac{k_{13}}{\mathbf{RC}} - \frac{k_{43}}{\mathbf{C}} & -\frac{1}{\mathbf{RC}} - \frac{k_{14}}{\mathbf{RC}} - \frac{k_{44}}{\mathbf{C}} \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{\mathbf{C}} & 0 & -\frac{1}{\mathbf{C}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mathbf{C}} & 0 & -\frac{1}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & 0 & -\frac{1}{\mathbf{C}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \\ 0 \leq \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_{1d}(t) \\ w_{2d}(t) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ k_{11} & k_{12} & 1 + k_{13} & k_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} + \begin{pmatrix} \mathbf{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \\ &+ \begin{pmatrix} x_{1d}(t) + x_{4d}(t) \\ x_{3d}(t) + u_{1d}(t) \\ x_{1d}(t) + x_{4d}(t) \\ x_{3d}(t) + u_{1d}(t) \end{pmatrix} \perp \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_{1d}(t) \\ \lambda_{2d}(t) \end{pmatrix} \geq 0 \end{aligned} \tag{126}$$

Numerically, it is checked that the nonlinear matrix inequality (30) has no solution. Hence, the quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of the error dynamics in (126) is not passive with $\bar{A} = A + EK$, $\bar{B} = (B + EG \quad -B - EG)$, $\bar{C} = \begin{pmatrix} C + FK \\ 0 \end{pmatrix}$ and $\bar{D} = \begin{pmatrix} D + FG & -FG \\ 0 & D \end{pmatrix}$. Take $i_{c1d}(t) = \sin 5t$, $i_{c2d} = 0.5 \sin 5t$, $i_{c3d}(t) = 0.2 \sin 3t$ and $u_{1d}(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ -10 & \text{if } t \geq 1 \end{cases}$

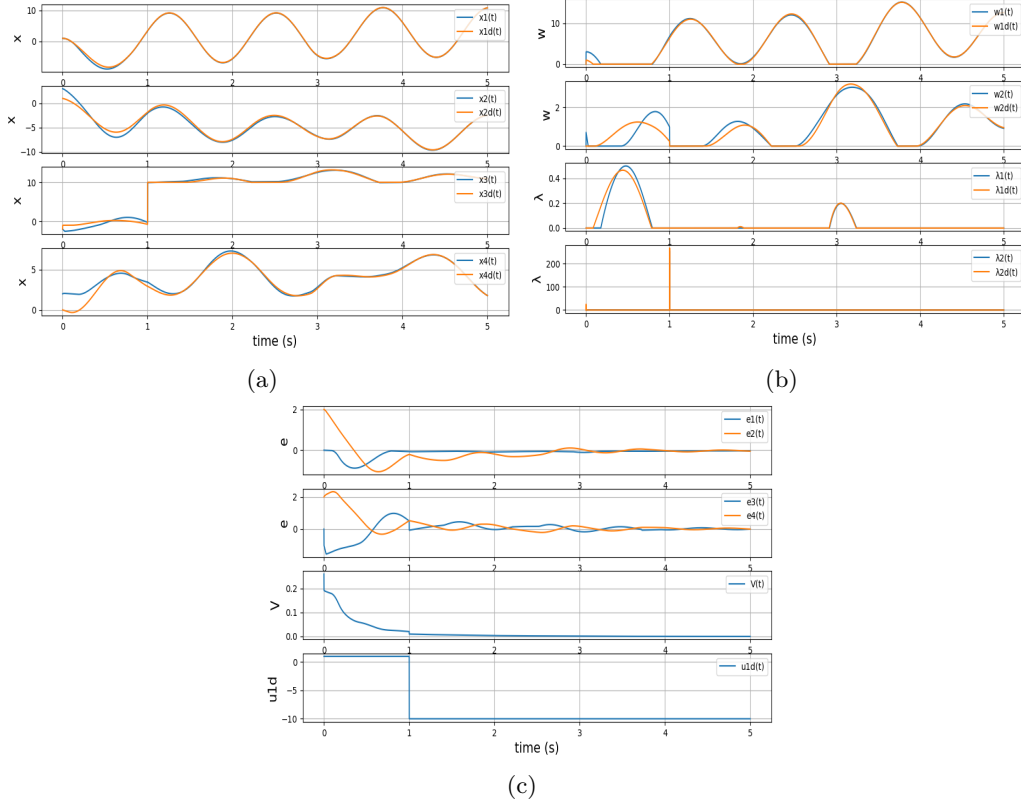


Figure 25: Numerical Simulation

12 Electrical circuit with state jumps ($D = 0$)

For the circuit in Figure 26, the states x_1 and x_2 are the charge on the capacitor \mathbf{C} and the current passing through the inductor \mathbf{L} . The diode is connected in parallel with a voltage w and current λ .

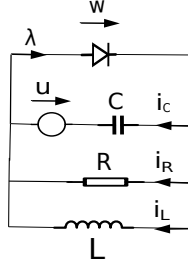


Figure 26: RLCD circuit

$$\begin{cases} \dot{x}_1(t) = -\frac{x_1(t)}{\mathbf{RC}} - x_2(t) + \lambda(t) - \frac{1}{\mathbf{R}}u(t) \\ \dot{x}_2(t) = \frac{x_1(t)}{\mathbf{LC}} + \frac{1}{\mathbf{L}}u(t) \\ 0 \leq \lambda(t) \perp w(t) = \frac{x_1(t)}{\mathbf{C}} + u(t) \geq 0 \end{cases} \quad (127)$$

Take $\mathbf{L} = 1H$, $\mathbf{R} = 10\Omega$ and $\mathbf{C} = 0.25F$. The storage function of the circuit of Figure 15 with $u = 0$ is: $V(x) = \frac{1}{2}\mathbf{C}(\frac{x_1}{\mathbf{C}})^2 + \frac{1}{2}\mathbf{L}x_2^2$. It follows that:

$$\dot{V}(x) = -\frac{x_1}{\mathbf{RC}^2} + \lambda w$$

Due to orthogonality between λ and w , the rate of change of the storage function $\dot{V}(x) = -\frac{x_1}{\mathbf{RC}^2} \leq 0$. If $u(t) = K[x - x_d] + u_d(t)$, then the transfer function of the closed loop system is: $H_4(s) = [C + FK][sI - (A + EK)]^{-1}B + D$ and represented as follows:

$$H(s) = \frac{s(\mathbf{RL} + k_1\mathbf{RLC})}{s^2 - s(\mathbf{L} + k_1\mathbf{LC} - k_2\mathbf{RC}) + \mathbf{R} + k_1\mathbf{RC}}$$

The transfer function $H(s)$ cannot be made strictly state passive by the feedback u . In terms of zero dynamics of the closed-loop system, when $w = 0$, there is always one zero at zero (i.e $\dot{x}_2 = 0$). In this example, $D = 0$. So, the desired system has the following sets: $Q_D = \{z \in \mathbb{R} \mid 0 \leq z \perp Dz \geq 0\} = \mathbb{R}_+ = Q_D^*$ and $\mathcal{K}_d = \{x_d \in \mathbb{R}^2 \mid Cx_d + Fu_d(t) \geq 0\} = \{x_d \in \mathbb{R}^2 \mid x_{1d} \geq -Cu_d(t)\}$. For the closed loop system, when $u = K[x - x_d] + u_d$, the set $\mathcal{K} = \{x \in \mathbb{R}^2 \mid (C + FK)x - FKx_d + Fu_d(t) \geq 0\} = \{x \in \mathbb{R}^2 \mid (\frac{1}{\mathbf{C}} + k_1)x_1 + k_2x_2 \geq k_1x_{1d} + k_2x_{2d} - u_d\}$.

13 Diode Bridge Example

Consider the diode bridge rectifier circuit in Figure 27. Let x_1 be the current passing through the inductor \mathbf{L} , x_2 be the voltage across the capacitor \mathbf{C}_1 , and x_3 be the voltage across the capacitor \mathbf{C}_2 . Take $\lambda(t) \triangleq (i_{DF1}, i_{DR1}, -v_{DF2}, i_{DR2})^\top$ and $w(t) \triangleq (-v_{DF1}, -v_{DR1}, i_{DF2}, -v_{DR2})^\top$. The dynamics are written as a linear complementarity system LCS and are given in (128).

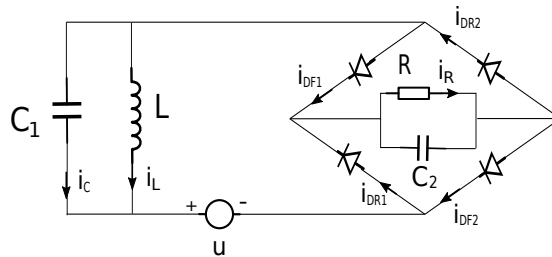


Figure 27: Diode Bridge

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} & 0 \\ -\frac{1}{C} & 0 & 0 \\ 0 & 0 & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{C} & 0 & 0 & \frac{1}{C} \\ \frac{1}{C} & \frac{1}{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} \\ 0 \leq \lambda(t) \perp w(t) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t) \geq 0 \end{array} \right. \quad (128)$$

Clearly $PB = C^\top$ with $P = CI_3$, and $D + D^\top = 0$ (*i.e.*, D is a skew-symmetric matrix). The closed-loop system is not strictly passive.

Let us add more inputs (*i.e.*, voltage sources) in the circuit of the diode bridge in Figure 27 as shown in the following Figure:

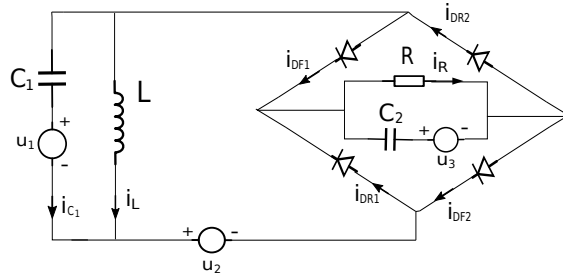


Figure 28: Diode Bridge

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} & 0 \\ -\frac{1}{C} & 0 & 0 \\ 0 & 0 & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{C} & 0 & 0 & \frac{1}{C} \\ \frac{1}{C} & \frac{1}{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \\ 0 \leq \lambda(t) \perp w(t) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} + \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \geq 0 \end{array} \right. \quad (129)$$

The closed-loop system of the LCS with the external input $u = (u_1 \quad u_2 \quad u_3)^\top = K(x - x_d) + u_d$

where $K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}$ is given by:

$$\left\{ \begin{array}{l}
\dot{x} = \begin{pmatrix} \frac{k_{11}}{\mathbf{L}} & \frac{1}{\mathbf{L}} + \frac{k_{12}}{\mathbf{L}} & \frac{k_{13}}{\mathbf{L}} \\ -\frac{1}{\mathbf{C}} & 0 & 0 \\ -\frac{k_{31}}{\mathbf{RC}} & -\frac{k_{32}}{\mathbf{RC}} & -\frac{1}{\mathbf{RC}} - \frac{k_{33}}{\mathbf{RC}} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\mathbf{C}} & 0 & 0 & \frac{1}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & \frac{1}{\mathbf{C}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} \\
- \begin{pmatrix} \frac{k_{11}}{\mathbf{L}} & \frac{k_{12}}{\mathbf{L}} & \frac{k_{13}}{\mathbf{L}} \\ 0 & 0 & 0 \\ -\frac{k_{31}}{\mathbf{RC}} & -\frac{k_{32}}{\mathbf{RC}} & -\frac{k_{33}}{\mathbf{RC}} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{\mathbf{L}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\mathbf{RC}} \end{pmatrix} \begin{pmatrix} u_{1d}(t) \\ u_{2d}(t) \\ u_{3d}(t) \end{pmatrix} \\
0 \leq \lambda(t) \perp w(t) = \begin{pmatrix} -k_{11} - k_{21} + k_{31} & -1 - k_{12} - k_{22} + k_{32} & 1 - k_{13} - k_{23} + k_{33} \\ k_{31} & k_{32} & 1 + k_{33} \\ 0 & 0 & 0 \\ k_{11} + k_{21} & 1 + k_{12} + k_{22} & k_{13} + k_{23} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \\
+ \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \lambda_4(t) \end{pmatrix} - \begin{pmatrix} -k_{11} - k_{21} + k_{31} & -k_{12} - k_{22} + k_{32} & -k_{13} - k_{23} + k_{33} \\ k_{31} & k_{32} & k_{33} \\ 0 & 0 & 0 \\ k_{11} + k_{21} & k_{12} + k_{22} & k_{13} + k_{23} \end{pmatrix} \begin{pmatrix} x_{1d}(t) \\ x_{2d}(t) \\ x_{3d}(t) \end{pmatrix} \\
+ \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_{1d}(t) \\ u_{2d}(t) \\ u_{3d}(t) \end{pmatrix} \geq 0
\end{array} \right. \quad (130)$$

is strictly passive since there exist K and P such that the NLMI in (19) has a solution given by:

$$P = \begin{pmatrix} 0.093 & -0.001 & 0 \\ -0.001 & 0.093 & 0 \\ 0 & 0 & 0.048 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} -0.99 & 39 & -0.015 \\ 0.95 & -36 & 0.014 \\ 0.0002 & -0.001 & 0.93 \end{pmatrix} \quad (131)$$

where $PB = (C + FK)^\top$ holds.

14 Nonsmooth Mechanical Systems with Unilateral Springs

It is known that unilateral spring/dashpot contact/impact models, can be written in a complementarity framework [15, 23, 16, 7]. This class of contact/impact models significantly differs from unilateral constraints which yield complementarity constraints and impact models as in Remark 14.1.

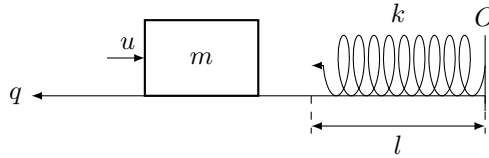


Figure 29: Mechanical system with unilateral spring.

Consider the mechanical system in Figure 29 which possesses the following dynamics:

$$\begin{cases} m\ddot{q}(t) = u(t) + \lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = \lambda(t) + k(q(t) - l) \geq 0 \end{cases} \quad (132)$$

where $q-l$ is the deformation of the spring, $k > 0$ is the stiffness of the spring. The complementarity constraint is written between the contact force and the signed distance between the spring and the

mass. Clearly the dynamics (132) fits with our general framework. Let $x = (x_1, x_2)^\top = (q - l, \dot{q})^\top$, (132) is rewritten equivalently as the LCS:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \lambda(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u(t) \\ 0 \leq \lambda(t) \perp w(t) = \lambda(t) + kx_1(t) \geq 0 \end{cases} \quad (133)$$

Here $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = E = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$, $C = (k, 0)$, $D = 1$, $F = 0$, and the system has AC solutions with uniqueness, see section 2.1 item 1. The quadruple (A, B, C, D) can be made strongly passive with state feedback $u = Kx = k_1x_1 + k_2x_2$. This means that there exists a control gain $K = (k_1, k_2)$ such that the quadruple $(A + EK, B, C, D)$ of the closed-loop system is strongly passive. Equivalently, the BMI in (20) has a solution after being transformed into an LMI, as detailed in A.2 and the solution is given by:

$$P = \begin{pmatrix} 7.863 & 0.538 \\ 0.538 & 0.337 \end{pmatrix} \quad \text{and} \quad K = (-21.38 \quad -2.148)$$

with $m = 1\text{kg}$ and $k = 5\text{N/m}$. As the stiffness of the spring k increases, the control gain gives very large numerical values like $k_1 = -1.2 \times 10^7$ for $k = 100\text{N/m}$. This is explained analytically by explicitly writing the matrix inequality in (20) and verifying the positive definiteness conditions. Following the matrix M in (19), the matrix inequality is written as:

$$\begin{pmatrix} -\frac{2p_{12}k_1}{m} & -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} & -\frac{p_{12}}{m} + k \\ -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} & -2p_{12} - \frac{2p_{22}k_2}{m} & -\frac{p_{22}}{m} \\ -\frac{p_{12}}{m} + k & -\frac{p_{22}}{m} & \frac{2}{m} \end{pmatrix} \succ 0$$

with $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \succ 0$. One of the necessary conditions to be satisfied is:

$$2 \times \begin{pmatrix} -\frac{2p_{12}k_1}{m} & -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} \\ -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} & -2p_{12} - \frac{2p_{22}k_2}{m} \end{pmatrix} - \begin{pmatrix} (-\frac{2p_{12}}{m} + k)^2 & 0 \\ 0 & 0 \end{pmatrix} \succ 0$$

Given that $-\frac{p_{12}k_1}{m} > 0$ which is a necessary condition for positive definiteness, thus for strong passivity of the quadruple $(A + EK, B, C, D)$, it is required to prove that: $-\frac{4p_{12}k_1}{m} - (-\frac{2p_{12}}{m} + k)^2 > 0$. This condition implies that as the stiffness of the spring k increases, the control gain k_1 increases to satisfy the condition. It is seen that in a certain sense this class of mechanical systems (with $D > 0$ and $F = 0$) lies in-between the unilaterally constrained mechanical systems (with always $D = 0$ and $F = 0$), and LCS (with possibly both D and F nonzero).

Remark 14.1 (Linear Complementarity Lagrangian Systems). *Linear Lagrangian systems with unilateral constraints have the nonlinear nonsmooth dynamics:*

$$\begin{cases} M\ddot{q}(t) + R\dot{q}(t) + Sq(t) = u(t) + C_q^\top \lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = C_q q(t) + F_q \geq 0 \\ C_q \dot{q}(t^+) = -e_n C_q \dot{q}(t) \text{ if } C_q q(t) = 0 \text{ and } C_q \dot{q}(t^-) \leq 0, \end{cases} \quad (134)$$

where $q(t) \in \mathbb{R}^n$, $M = M^\top \succ 0$, the restitution coefficient $e_n \in [0, 1]$, $R \succcurlyeq 0$ is a Rayleigh dissipation matrix [25, Definition 6.12], $S \succcurlyeq 0$ is a stiffness matrix, and it is assumed that \dot{q} has left and right limits. For simplicity we also assume that there is a unique unilateral constraint, i.e., $C_q \in \mathbb{R}^{1 \times n}$, $\lambda \in \mathbb{R}$ and $F_q \in \mathbb{R}$ is a constant (the system's admissible domain in the configuration space, is a convex polyhedral set, assumed to be nonempty). Denoting $x = (q^\top, \dot{q}^\top)^\top$, $x_1 = q$, $x_2 = \dot{q}$, we obtain:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & I_n \\ -M^{-1}S & -M^{-1}R \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ M^{-1}C_q^\top \end{pmatrix} \lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = Cx(t) + F_q = (C_q, 0)x(t) + F_q \geq 0 \\ C\dot{x}(t^+) = -e_n C\dot{x}(t^-) \text{ if } Cx(t) = 0 \text{ and } C\dot{x}(t^-) \leq 0. \end{cases} \quad (135)$$

Therefore $A = \begin{pmatrix} 0 & I_n \\ -M^{-1}S & -M^{-1}R \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ M^{-1}C_q^\top \end{pmatrix}$, $E = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}$, $F = 0$, $C = (C_q, 0)$, $D = 0$. Since $D = 0$ and no control acts in the complementarity constraint, passivity implies that $PB = C^\top$ whatever the controller $u(x) = Kx$. It can be checked that this is not possible with $P \succ 0$. Adding a multiplier feedback $u(x, \lambda) = Kx + G\lambda$ does not change the conclusion. Fundamentally, systems as (135) have a relative degree 2 when λ is seen as the input and w is seen as the output [15] (hence hampering passivity [25]), while systems as in (133) have a relative degree 0 between the same input/output variables.

15 Networks with Unilateral Interactions

Let us present an example of a network with unilateral interactions, which is introduced in [63]. These systems can be treated as LCS [23], and by adding exogenous signals, they can be formulated as LCS with external inputs as in (2). Network systems with unilateral interactions have applications in various fields such as sensor networks, robotics, and game theory. This example focuses on the study of trajectory tracking for a system of networks with unilateral interactions and inputs with the following dynamics:

$$\begin{cases} \dot{x}_1 = \max(0, x_3 - x_1 - u_1) + u_2 \\ \dot{x}_2 = x_1 - x_2 + \min(0, x_3 - x_2 + u_3) + u_4 \\ \dot{x}_3 = \max(0, x_1 - x_3 - u_5) + u_6 \end{cases} \quad (136)$$

which is written equivalently in the form of LCS as:

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda + \begin{pmatrix} u_2 \\ u_4 \\ u_6 \end{pmatrix} \\ 0 \leq \lambda \perp \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda + \begin{pmatrix} u_1 \\ u_3 \\ u_5 \end{pmatrix} \geq 0 \end{cases} \quad (137)$$

The desired system is given by the following LCS:

$$\begin{cases} \dot{x}_d = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_d + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda_d + \begin{pmatrix} u_{2d} \\ u_{4d} \\ u_{6d} \end{pmatrix} \\ 0 \leq \lambda_d \perp \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} x_d + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda_d + \begin{pmatrix} u_{1d} \\ u_{3d} \\ u_{5d} \end{pmatrix} \geq 0 \end{cases} \quad (138)$$

Remark 15.1. *In this analysis, the most general form of inputs is chosen (they appear in both the linear and the nonsmooth parts of the dynamics). It is out of the scope of this analysis to justify the feasibility of such inputs.*

15.1 First case: Complete controller

Let $u = (u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6)^\top$ and the matrices E and F in (16) are defined as

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The closed-loop system with $u = K(x - x_d) + u_d$ where $K = \begin{pmatrix} k_{11} & k_{21} & k_{31} & k_{41} & k_{51} & k_{61} \\ k_{12} & k_{22} & k_{32} & k_{42} & k_{52} & k_{62} \\ k_{13} & k_{23} & k_{33} & k_{43} & k_{53} & k_{63} \end{pmatrix}^\top$

is written in the form of the LCS in (16) as follows:

$$\begin{cases} \dot{x} = \begin{pmatrix} k_{21} & k_{22} & k_{23} \\ 1 + k_{41} & -1 + k_{42} & k_{43} \\ k_{61} & k_{62} & k_{63} \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda - \begin{pmatrix} k_{21} & k_{22} & k_{23} \\ k_{41} & k_{42} & k_{43} \\ k_{61} & k_{62} & k_{63} \end{pmatrix} x_d + \begin{pmatrix} u_{2d} \\ u_{4d} \\ u_{6d} \end{pmatrix} \\ 0 \leq \lambda \perp \begin{pmatrix} 1 + k_{11} & k_{12} & -1 + k_{13} \\ k_{31} & -1 + k_{32} & 1 + k_{33} \\ -1 + k_{51} & k_{52} & 1 + k_{53} \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda - \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{31} & k_{32} & k_{33} \\ k_{51} & k_{52} & k_{53} \end{pmatrix} x_d + \begin{pmatrix} u_{1d} \\ u_{3d} \\ u_{5d} \end{pmatrix} \geq 0 \end{cases} \quad (139)$$

The solution of the NLMI in (19) for the closed-loop system in (139)

$$K = \begin{pmatrix} -0.43 & -0.497 & 0.003 & -0.97 & 0.996 & 0.002 \\ -0.003 & -0.017 & 0.431 & 0.5 & -0.003 & 0.009 \\ 0.996 & 0.0029 & -0.996 & -0.003 & -0.43 & -0.497 \end{pmatrix}^T \quad \text{and} \quad P = \begin{pmatrix} 0.57 & -0.003 & -0.003 \\ -0.0032 & 0.57 & -0.003 \\ -0.003 & -0.003 & 0.57 \end{pmatrix} \quad (140)$$

Hence, the quadruple of the closed-loop system $(A + EK, B, C + FK, D)$ is strictly passive. Take $x(0) = (-1, 1, 0)^T$, $x_d(0) = (1, 2, -1)^T$ and $u_d = (\sin t \ \cos t \ \sin 2t \ \cos 2t \ \sin 3t \ \cos 3t)^T$. The numerical simulation is shown below with time step $h = 0.001$. It is noteworthy that in this

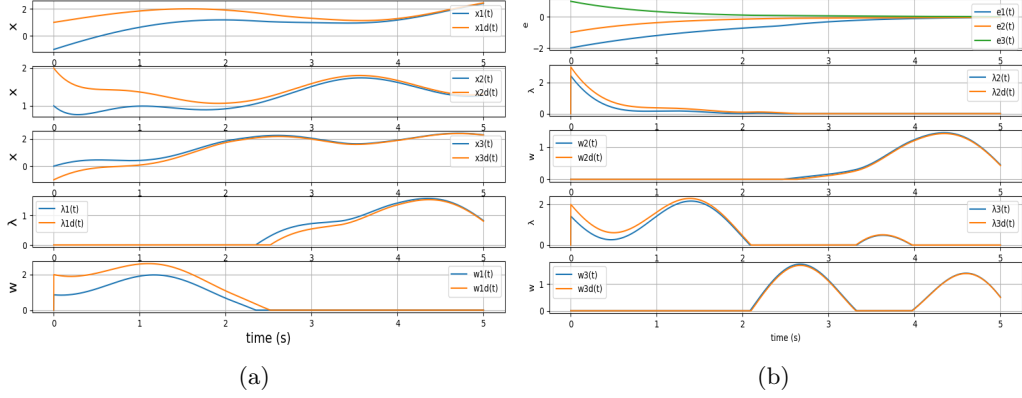


Figure 30: Network with complete controller

case, the closed-loop quadruple $(A + EK, B, C + FK, D)$ is strongly passive. Thus, the system can be studied within the framework of section 4 which allows the consideration of uncertainties.

15.2 Second case: $u = (u_2 \ u_4 \ u_6)^T$

The solution is:

$$K = \begin{pmatrix} 0 & -0.87 & 0 & -0.16 & 0 & 0.26 \\ 0 & -0.42 & 0 & 0.327 & 0 & 0.233 \\ 0 & 0.18 & 0 & -0.0001 & 0 & -1.27 \end{pmatrix}^T \quad \text{and} \quad P = \begin{pmatrix} 0.99 & -0.085 & -0.26 \\ -0.085 & 0.72 & -0.2 \\ -0.26 & -0.2 & 1.18 \end{pmatrix} \quad (141)$$

Hence, the closed-loop system $(A + EK, B, C, D)$ is strictly passive.

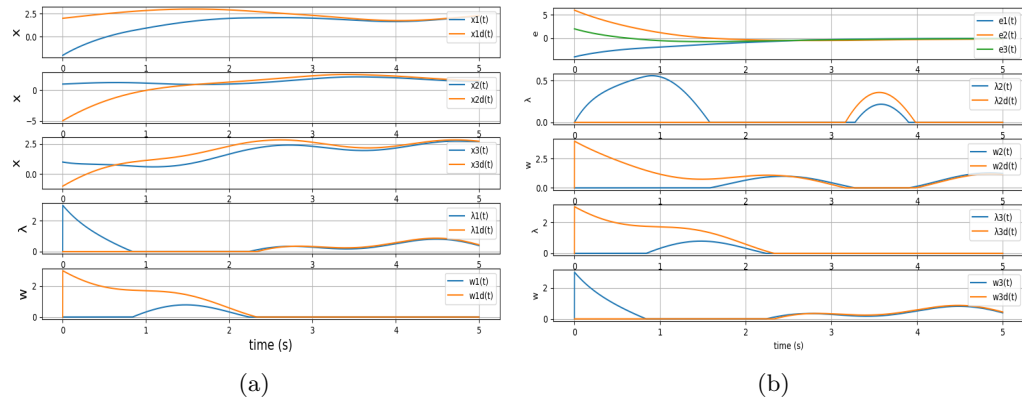


Figure 31: Network with input $(u_2, u_4, u_6)^T$

It is important to note that, in this case, the closed-loop quadruple $(A + EK, B, C, D)$ is strongly passive. This property allows us to analyse the problem of trajectory tracking in the presence of uncertainties (see section 4). However, whether or not considering such uncertainties in networks with unilateral interactions makes sense, is out of the scope of this work.

15.3 Third case: $u = (u_1 \ u_3 \ u_5)^\top$

The NLMI in (19) has no solution. Hence, the quadruple of the closed-loop system $(A, B, C + FK, D)$ cannot be made strictly passive with such a set of inputs and the chosen feedback, and the framework developed in the foregoing sections does not apply. It is noteworthy that the conclusions are still valid even if we consider $G \neq 0$.

16 Conclusions

Open issues: robustness with respect to Shockley's model of diodes, output tracking [71] (tracking of an output $y_d = Hx_d + J\lambda_d + Lu_d$ with state feedback), output feedback (same problem as in this paper but with only $y = Hx + J\lambda$ available for feedback), state observation, adaptive control when parameters are uncertain (does the fact that uncertain parameters are present only in the ODE part and not in the complementarity part –as for friction oscillators– play a role?).

A Appendix

A.1 LCP and P-matrices

Definition 8. A matrix M is said to be *P-matrix* if and only if all its principal minors are positive.

The complementarity between two vectors $x, y \in \mathbb{R}^n$ is denoted as $0 \leq x \perp y \geq 0$ where the notation $x \perp y$ means that $x^\top y = 0$, and nonnegativeness is understood componentwise. When $w = M\lambda + q$, this gives rise to a Linear Complementarity Problem (LCP), denoted as $\text{LCP}(M, q)$.

Theorem A.1 ([1], Theorem 12.14). A matrix $M \in \mathbb{R}^{n \times n}$ is a *P-matrix* if and only if $\text{LCP}(M, q)$ has a unique solution for all vectors $q \in \mathbb{R}^n$.

A.2 Transformation of NLMI to an LMI

The inequalities in (20) and (35) are nonlinear matrix inequalities (NLMI). In order to solve these inequalities, the NLMI must be reformulated to an LMI (following a classical technique, *e.g.*, [78, Example 5.5 p.136]). First, the left-hand and the right-hand sides of (20) are multiplied by $\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$, where $Q \triangleq P^{-1}$, and let $N \triangleq KQ$. It follows that

$$\begin{pmatrix} QA^\top + AQ + N^\top E^\top + EN + \epsilon Q & B + EG - QC^\top - N^\top F^\top \\ B^\top + G^\top E^\top - CQ - FN & -D - FG - (D + FG)^\top \end{pmatrix} \preceq 0 \quad (142)$$

This LMI is a feasible problem and it is solved in the new variables $Q = Q^\top \succ 0$, G , and N where $K = NQ^{-1}$.

A.3 Motivation for feedback from λ in the controller in u (15)

Let us consider the LCS in (2) with the following matrices:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E = 0 \quad \text{and} \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (143)$$

Let $u = K(x - x_d) + u_d$, then the closed-loop system's quadruple $(A, B, C + FK, D)$ is strictly state passive with $K = (1.12, -1)$ and $P = \begin{pmatrix} 1.12 & -0.006 \\ -0.006 & 2.012 \end{pmatrix}$ given by solving the LMI in (19). However, the LMI for strong passivity in (20) does not have a solution for the closed-loop quadruple $(A, B, C + FK, D)$.

Let us introduce feedback from the complementarity variable λ , resulting in the extended controller $u = K(x - x_d) + G(\lambda - \lambda_d) + u_d$. In order to check if the closed-loop quadruple $(A, B, C + FK, D + FG)$ with the extended controller is strongly passive, let us check if the LMI in (20) has a solution. It appears that the LMI in (20) for strong passivity has a solution given by:

$$K = (1.13 \quad -0.99), \quad G = 0.59 \quad \text{and} \quad P = \begin{pmatrix} 1.17 & -0.014 \\ -0.014 & 2.16 \end{pmatrix} \quad (144)$$

Strong passivity of the closed-loop system is a fundamental assumption in robustness analysis in section 4 in order to guarantee the boundedness of the tracking error e . Therefore, there is a need to introduce a feedback from λ in the controller u to enhance the passivity of the closed-loop system in certain applications.

A.4 Uniform and Ultimate Boundedness

The following Lyapunov-like theorem is for showing uniform boundedness and ultimate boundedness.

Theorem A.2. [57, Theorem 4.18] *Let $D \subset \mathbb{R}^n$ be a domain that contains the origin and $V : [0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (145)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{\partial x}{\partial t} \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0 \quad (146)$$

for all $t \geq 0$, for all $x \in D$, where α_1 and α_2 are class \mathcal{K} functions and $W_3(\cdot)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that,

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \quad (147)$$

Then, there exists a class \mathcal{KL} function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there exists $T > 0$ (dependent on $x(t_0)$ and μ) such that the solution satisfies

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(r)), \quad \forall t \geq t_0 + T \quad (148)$$

Moreover, if $D = \mathbb{R}^n$ and α_1 belongs to class \mathcal{K}_∞ , then (148) holds for any initial state $x(t_0)$, with no restrictions on how large μ is.

A.5 Conditions for Positive (Semi) Definiteness

Lemma A.3. [8, Proposition 8.2.4] *Consider the matrix $M = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$. Assume that $Q = Q^\top$ and $R = R^\top$. Then $M \succcurlyeq 0$ if and only if:*

1. $R \succcurlyeq 0$,
2. $SR^\dagger R = S \Leftrightarrow \text{Im}(S^\top) \subseteq \text{Im}(R)$,
3. $Q \succcurlyeq SR^\dagger S^\top$.

The equivalence in item 2 follows from [8, Remark after Fact 6.4.5]. The next lemma provides some necessary conditions for item 3 to hold.

Lemma A.4. *Assume that $Q = Q^\top \succcurlyeq 0$ and $R = R^\top \succcurlyeq 0$. Then $Q \succcurlyeq SR^\dagger S^\top$:*

1. $\Rightarrow \text{Im}(Q) \supseteq \text{Im}(SR^\dagger S^\top) \Leftarrow \text{Im}(Q) \supseteq \text{Im}(S)$.
2. $\Rightarrow \lambda_i(Q) \geq \lambda_i(SR^\dagger S^\top)$ for all $i \in \{1, \dots, n\}$, and $\text{tr}(Q) \geq \text{tr}(SR^\dagger S^\top)$ and $\det(Q) \geq \det(SR^\dagger S^\top) \geq 0$.

Proof. 1. From [9, Fact 7.17.24], there exists matrices L such that $Q = LL^\top$ and N such that $SR^\dagger S^\top = NN^\top$, since R^\dagger is symmetric [8, Proposition 6.1.6]. From [9, Theorem 10.6.1] it follows that $LL^\top \succcurlyeq NN^\top \Rightarrow \text{Im}(N) \subseteq \text{Im}(L)$. Using [8, Theorem 2.4.3] we have that $\text{Im}(N) = \text{Im}(NN^\top)$ and $\text{Im}(L) = \text{Im}(LL^\top)$. The first implication is proved. Using [8, Lemma 2.4.1] we have $\text{Im}(SR^\dagger S^\top) \subseteq \text{Im}(S)$. The second implication is proved.

2. From [9, Theorem 10.4.9, Corollary 10.4.10].

□

Notice that $\lambda_i(SR^\dagger S^\top) = \lambda_i(NN^\top) = \sigma_i^2(N)$ (because $R^\dagger \succ 0$), and similarly $\lambda_i(Q) = \sigma_i^2(L)$. The matrix inequality in Lemma A.3 item 3 can therefore be transformed into a singular values inequality, provided the matrices N and L are computed (Cholesky decomposition [9, Fact 10.10.42], or the Gram matrix decomposition [9, Fact 10.10.41] can be chosen).

An interesting result follows from [9, Theorem 10.6.2] and the same reasoning as in the proof of Lemma A.5 item 1: there exists $\alpha > 0$ such that $\alpha Q \succ SR^\dagger S^\top \iff \text{Im}(Q) \supseteq \text{Im}(SR^\dagger S^\top)$. A first step may be to check the ranges inclusion, then to calculate an $\alpha > 0$, following the reasoning in the proof of [9, Theorem 10.6.2]: compute the matrices N and L , compute the matrix T such that $N = LT$, and compute $\alpha = \lambda_{\max}(TT^\top)$. If $\alpha \leq 1$ then Lemma A.3 item 3 holds true. Following the proof of [9, Theorem 10.6.2], T can be calculated.

Let us provide an excerpt of [35, Theorem 2.11], and a corollary of it. Let us recall that for a given $M \in \mathbb{R}^{n \times n}$, $\|M\|_{2,2}$ is the induced matricial norm such that $\|M\|_{2,2} = \sigma_{\max}(M)$ (the largest singular value) [8, Proposition 9.4.9], it is a submultiplicative norm [8, Corollary 9.4.12].

Theorem A.5. [35] *Let $M \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Then every matrix*

$$A \in \left\{ A \in \mathbb{R}^{n \times n} \mid \left\| \left(\frac{M+M^\top}{2} \right)^{-1} \right\|_{2,2} \|M - A\|_{2,2} < 1 \right\}$$

is positive definite.

Corollary A.6. [14] *Let $D = P + N$, where D , P and N are $n \times n$ real matrices, and $P \succ 0$, not necessarily symmetric. If*

$$\|N\|_{2,2} < \frac{1}{\left\| \left(\frac{P+P^\top}{2} \right)^{-1} \right\|_{2,2}} \quad (149)$$

then $D \succ 0$.

A.6 Numerical computation of state-jump times in (122)

Let us check numerically that the state jumps and the state-jump times are independent of the matrix solution of the passivity LMI, as predicted by theoretical arguments [27, 23, 50]. It is also shown in [50] that the event-capturing time-stepping Moreau-Jean scheme that is implemented in the SICONOS software package, does approximate this state-jump rule. Hence the numerical results obtained from SICONOS can be taken as the correct state-jump times when the chosen time-step is small enough.

For the desired system: Knowing that at the time of discontinuity $t = 1$, the values of the desired state from simulation in Figure 25 at $t = 1^-$ is given by the numerical solver SICONOS as follows:

$$x_d(1^-) = \begin{pmatrix} 3.4615397 \\ -1.832126 \\ 0.238885 \\ 2.91881 \end{pmatrix}$$

In order to calculate the values of the states at $t = 1^+$, the optimization problem in (9) is solved using MOSEK solver with $P = P_d$ which is the solution of the LMI for the passivity of the desired system in (7). We solved the optimization problem for different values of the solution P_d (*i.e.*, the solution P_d is not unique). The results is obtained below:

Values of P_d	Values of $x_d(1^+)$
$P_{d1} = \begin{pmatrix} 1.2165869 & 1.1915869 & 0 & -1.1915867 \\ 1.1915869 & 1.2165866 & 0 & -1.191587 \\ 0 & 0 & 0.02499 & 0 \\ -1.1915867 & -1.191587 & 0 & 1.2165868 \end{pmatrix}$	$\begin{pmatrix} 3.4615397 \\ -1.832121 \\ 10 \\ 2.91881 \end{pmatrix}$
$P_{d2} = \begin{pmatrix} 0.9388826 & 0.9138827 & 0 & -0.9138835 \\ 0.9138827 & 0.9388826 & 0 & -0.9138835 \\ 0 & 0 & 0.024999 & 0 \\ 0.9138835 & -0.9138835 & 0 & 0.9388818 \end{pmatrix}$	$\begin{pmatrix} 3.4615397 \\ -1.832126 \\ 10 \\ 2.91881 \end{pmatrix}$
$P_{d3} = \begin{pmatrix} 0.016668 & -0.008336 & 0 & 0.008332 \\ -0.008336 & 0.016673 & 0 & 0.008336 \\ 0 & 0 & 0.025 & 0 \\ 0.008332 & 0.008336 & 0 & 0.016668 \end{pmatrix}$	$\begin{pmatrix} 3.4615397 \\ -1.832126 \\ 10 \\ 2.91881 \end{pmatrix}$

It is noteworthy that the different values P_{d1} , P_{d2} and P_{d3} comes from solving the LMI in (7) using MOSEK, CVXOPT and SCS solvers respectively. The numerical solver SICONOS computes the

jump automatically and gives $x_d(1^+) = \begin{pmatrix} 3.499868 \\ -1.798264 \\ 10 \\ 2.932387 \end{pmatrix}$ when the time step $h = 0.001$ as well as

$x_d(1^+) = \begin{pmatrix} 3.466 \\ -1.834 \\ 10 \\ 2.92 \end{pmatrix}$ when the time step $h = 0.00001$.

For the closed-loop system: At the time of discontinuity $t = 1$, the value of the closed-loop state is given by:

$$x(1^-) = \begin{pmatrix} 3.391813 \\ -2.0267 \\ 0.723447 \\ 3.42975 \end{pmatrix}$$

The value of the state jump is the solution of the optimization problem in (9). The table below shows the values of $x(1^+)$ for different value of the storage function matrix P being the solution of NLMI in (19) for strict passivity.

Values of P	Values of $x(1^+)$
$P_1 = \begin{pmatrix} 0.029 & 0.001 & -0.006 & -0.004 \\ 0.001 & 0.033 & 0.016 & 0.001 \\ -0.006 & 0.016 & 0.032 & 0.01 \\ -0.004 & 0.001 & 0.01 & 0.0304 \end{pmatrix}$	$\begin{pmatrix} 3.397125 \\ -2.0267 \\ 9.9417 \\ 3.431 \end{pmatrix}$
$P_2 = \begin{pmatrix} 0.04589 & 0.006245 & -0.005963 & -0.006264 \\ 0.006245 & 0.03969 & 0.015618 & -0.001698 \\ -0.005963 & 0.015618 & 0.032439 & 0.009719 \\ -0.006264 & -0.001698 & 0.009719 & 0.034227 \end{pmatrix}$	$\begin{pmatrix} 3.3952 \\ -2.0269 \\ 9.9415 \\ 3.431 \end{pmatrix}$
$P_3 = \begin{pmatrix} 0.03844 & 0.00339 & -0.00596 & -0.00489 \\ 0.00339 & 0.03742 & 0.01562 & 0 \\ -0.005963 & 0.015618 & 0.032439 & 0.009719 \\ -0.00489 & 0 & 0.009719 & 0.03262 \end{pmatrix}$	$\begin{pmatrix} 3.3958 \\ -2.0268 \\ 9.9415 \\ 3.431 \end{pmatrix}$

The numerical solver SICONOS computes the jump automatically and gives $x(1^+) = \begin{pmatrix} 3.4299 \\ -1.9915 \\ 9.9262 \\ 3.4462 \end{pmatrix}$

when the time step $h = 0.001$ as well as $x(1^+) = \begin{pmatrix} 3.3986 \\ -2.029 \\ 9.93 \\ 3.4233 \end{pmatrix}$ when the time step $h = 0.00001$.

According to the results shown above, the value of the jump in the desired state and closed-loop state x_d and x respectively given by SICONOS converges to the value of the jump which is the solution of the optimization problem in (9) as the time step $h \rightarrow 0$ [50, Definition 8].

A.7 Solve the optimization problem in (9) for the simple example in section 6

In this section, the optimization problem in (9) is solved using KKT conditions since we are dealing with quadratic optimization problems with inequality constraints. In this case, KKT conditions are necessary and sufficient conditions for the optimal value because the following optimization problem

$$x(t^+) = \operatorname{argmin}_{x \in \mathcal{K}} \frac{1}{2} (x - x(t^-))^T P (x - x(t^-)).$$

is convex knowing that $P = P^T \succ 0$, $x(t)$ and $x(t^-) \in \mathbb{R}$. The optimization problem is solved for the two systems as shown in the following.

For the desired system in (77)

It is required to find $x_d(t^+)$ which is the optimal value to minimize the following

$$\begin{aligned} f_d(x_d) &= \min \frac{1}{2} (x_d - x_d^-)^2 \\ \text{s.t. } x_d &\in \mathcal{K}_d \end{aligned}$$

where $P = 1$ and $\mathcal{K}_d = \{x_d \in \mathbb{R} \mid cx_d + v_d(t^+) \geq 0\} = \{x_d \in \mathbb{R} \mid -cx_d - v_d^+ \leq 0\}$. Let us write the Lagrangian function of the minimization problem

$$\mathcal{L}(x_d, \mu) = \frac{1}{2} (x_d - x_d^-)^2 - \mu (-cx_d - v_d^+)$$

where μ is the lagrangian multiplier. Let us state KKT conditions to be satisfied:

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{L}}{\partial x_d} = 0 \Leftrightarrow x_d - x_d^- + c\mu = 0 \Leftrightarrow x_d = x_d^- - c\mu \\ \mu(-cx_d - v_d^+) = 0 \Leftrightarrow \mu(c^2\mu - cx_d^- - v_d^+) = 0 \\ \mu \leq 0 \\ -cx_d - v_d^+ \leq 0 \Leftrightarrow c^2\mu - cx_d^- - v_d^+ \leq 0 \end{array} \right.$$

KKT conditions are written in the form of a linear complementarity problem LCP as shown below:

$$0 \leq -\mu \perp -c^2\mu + cx_d^- + v_d^+ \geq 0$$

The linear complementarity problem (LCP) is denoted by $\text{LCP}(c^2, cx_d^- + v_d^+)$ and it has a unique solution defined by:

$$-\mu = \begin{cases} 0 & \text{if } cx_d^- + v_d^+ \geq 0 \\ -\frac{1}{c}x_d^- - \frac{1}{c^2}v_d^+ & \text{if } cx_d^- + v_d^+ < 0 \end{cases}$$

Let us substitute the values of the solution in the KKT condition $\frac{\partial \mathcal{L}}{\partial x_d} = 0$ to calculate x_d as follows:

$$x_d = \begin{cases} x_d^- & \text{if } cx_d^- + v_d^+ \geq 0 \\ -\frac{1}{c}v_d^+ & \text{if } cx_d^- + v_d^+ < 0 \end{cases} \quad (150)$$

For the closed-loop system in (79)

It is required to find the optimal value $x(t^+)$ in order to minimize the following

$$f(x) = \min \frac{1}{2}(x - x^-)^2$$

s.t. $x \in \mathcal{K}$

where $\mathcal{K} = \{x \in \mathbb{R} \mid (c+k_3)x - k_3x_d(t^+) + v_d(t^+) \geq 0\} = \{x \in \mathbb{R} \mid -(c+k_3)x + k_3x_d(t^+) - v_d(t^+) \leq 0\}$. Let us write the Lagrangian function

$$\mathcal{L}(x, \mu) = \frac{1}{2}(x - x^-)^2 - \mu(-(c+k_3)x + k_3x_d^+ - v_d^+)$$

The necessary and sufficient conditions to be satisfied are the following:

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{L}}{\partial x} = 0 \Leftrightarrow x - x^- + \mu(c+k_3) = 0 \Leftrightarrow x = x^- - \mu(c+k_3) \\ \mu(-(c+k_3)x + k_3x_d^+ - v_d^+) = 0 \Leftrightarrow \mu(-(c+k_3)x^- + \mu(c+k_3)^2 + k_3x_d^+ - v_d^+) = 0 \\ \mu \leq 0 \\ -(c+k_3)x + k_3x_d^+ - v_d^+ \leq 0 \Leftrightarrow -(c+k_3)x^- + \mu(c+k_3)^2 + k_3x_d^+ - v_d^+ = 0 \end{array} \right.$$

KKT conditions are written in the form of a linear complementarity problem LCP as shown below:

$$0 \leq -\mu \perp -(c+k_3)^2\mu + (c+k_3)x^- - k_3x_d^+ + v_d^+ \geq 0$$

The linear complementarity problem (LCP) is denoted by $\text{LCP}((c+k_3)^2, (c+k_3)x^- - k_3x_d^+ + v_d^+)$ and it has a unique solution defined by:

$$-\mu = \begin{cases} 0 & \text{if } (c+k_3)x^- - k_3x_d^+ + v_d^+ \geq 0 \\ -\frac{1}{(c+k_3)}x^- + \frac{k_3}{(c+k_3)^2}x_d^+ - \frac{1}{(c+k_3)^2}v_d^+ & \text{if } (c+k_3)x^- - k_3x_d^+ + v_d^+ < 0 \end{cases}$$

Let us substitute the values of the solution in the KKT condition $\frac{\partial \mathcal{L}}{\partial x} = 0$ to calculate x as follows:

$$x = \begin{cases} x^- & \text{if } (c+k_3)x^- - k_3x_d^+ + v_d^+ \geq 0 \\ -\frac{k_3x_d^+ - v_d^+}{(c+k_3)} & \text{if } (c+k_3)x^- - k_3x_d^+ + v_d^+ < 0 \end{cases}$$

A.8 Proof of the DI in (3.3)

Let us remind the transformation of an LCP into a differential inclusion of the first-order sweeping process (FOSwP) with perturbation type, as proposed in [23, 12, 24]. It is assumed that the constraint qualification stated after (3.3) holds true. Recall the closed-loop system in (16) with $D = 0$ and $G = 0$ as follows:

$$\begin{cases} \dot{x}(t) = (A + EK)x(t) + B\lambda(t) - EKx_d(t) + Eu_d(t) \\ 0 \leq \lambda(t) \perp w(t) = (C + FK)x(t) - FKx_d(t) + Fu_d(t) \geq 0 \end{cases} \quad (151)$$

As a result of convex analysis, the linear complementarity problem (LCP) is written in the form of a differential inclusion DI as follows:

$$0 \leq w \perp \lambda \geq 0 \Leftrightarrow w \in -\mathcal{N}_{\mathbb{R}_+^m}(\lambda) \Leftrightarrow \lambda \in -\mathcal{N}_{\mathbb{R}_+^m}(w) \quad (152)$$

By applying this property to the LCP in (151), then

$$\lambda(t) \in -\mathcal{N}_{\mathbb{R}_+^m}((C + FK)x(t) - FKx_d(t) + Fu_d(t)) \Leftrightarrow \lambda(t) \in -\mathcal{N}_{S(t)}((C + FK)x(t)) \quad (153)$$

where $S(t) = \{v \in \mathbb{R}_+^m \mid v - FKx_d(t) + Fu_d(t) \in \mathbb{R}_+^m\}$. Given that the quadruple of the closed-loop system $(A+EK, B, C+FK, 0)$ is strictly passive, then $PB = (C+FK)^\top$ for some $P = P^\top \succ 0$. The closed-loop system in (151) is written as DI:

$$\dot{x} \in (A + EK)x(t) - B\mathcal{N}_{S(t)}((C + FK)x(t)) - EKx_d(t) + Eu_d(t) \quad (154)$$

Let $R^2 = P$ such that $R = R^\top \succ 0$ and $\zeta = Rx$, then the DI in (154) is written as:

$$\dot{\zeta}(t) = R\dot{x}(t) \in R(A+EK)R^{-1}\zeta(t) - RB\mathcal{N}_{S(t)}((C+FK)R^{-1}\zeta(t)) - REKx_d(t) + REu_d(t) \quad (155)$$

But

$$RB = R^{-1}PB = R^{-1}(C+FK)^\top$$

By using the following property from the convex analysis (the chain rule):

$$M^\top \mathcal{N}_{S(t)}(Mx) = \mathcal{N}_{\phi(t)}(x)$$

where $\phi(t) = \{z \mid Mz \in S(t)\}$. Then,

$$R^{-1}(C+FK)^\top \mathcal{N}_{S(t)}((C+FK)R^{-1}\zeta(t)) = \mathcal{N}_{\phi(t)}(\zeta(t)) \quad (156)$$

where $\phi(t) = \{Rx \mid (C+FK)x(t) \in S(t)\}$. Thus, the differential inclusion DI in (155) is written as follows:

$$\dot{\zeta} \in R(A+EK)R^{-1}\zeta(t) + R(-EKx_d(t) + Eu_d(t)) - \mathcal{N}_{\phi(t)}(\zeta(t)) \quad (157)$$

B Python Code

There are two files used in order to simulate the electrical circuits in section 7. The first Python file is implemented to solve the matrix inequalities in (19), (20) or (42) after being transformed into linear matrix inequalities using the solver MOSEK 9.3.14. The solution of the LMI gives the values of the control gain as well as that of the upper bounds of the uncertainties (in the case of robustness analysis). Then, the real system is defined.

```

import siconos.kernel as sk
import numpy as np
import cvxpy as cvx
import math

Lvalue = 1          # inductance
Cvalue = 0.025     # capacitance
Rvalue = 10        # resistance

def compute_K0G0():
    #Nominal System
    A_0 = np.array(n,n)
    B_0 = np.array(n,m)
    C_0 = np.array(m,n)
    D_0 = np.array(m,m)
    E_0 = np.array(n,p)
    F_0 = np.array(m,p)
    eps = 1e-2

    #Structures of Q, N and G: parameters of LMI
    Q_0 = cvx.Variable((n, n), symmetric=True)
    N_0 = cvx.Variable((p, n))
    G_0 = np.zeros((p,m))

    #Structure of Lambdas in case of uncertainties
    Lam1_inv = cvx.Variable((n,n))
    Lam1_tilde_inv = cvx.Variable((n,n))
    Lam2_inv = cvx.Variable((m,m))
    Lam2_tilde_inv = cvx.Variable((m,m))
    LamF_inv = cvx.Variable((m,p))

    #Structure of LMI in the case of uncertainties
    LMI_1 = cvx.bmat(
        [
            [
                -(Q_0 @ (A_0.transpose())) - (A_0 @ Q_0) - ((N_0.T) @ (E_0.T)) - (E_0 @ N_0),
                -B_0 - (E_0 @ G_0) + (Q_0 @ (C_0.T)) + ((N_0.T) @ (F_0.T)),
                np.eye(n),
                np.eye(n),
                (N_0.T),
                Q_0,
                np.zeros((n, p)),
                np.zeros((n, m)),
                np.zeros((n, m)),
            ],
            [
                -(B_0.T) - ((G_0.T) @ (E_0.T)) + (C_0 @ Q_0) + (F_0 @ N_0),
                D_0 + (F_0 @ G_0) + np.transpose(D_0) + ((G_0.T) @ (F_0.T)),
                np.zeros((m, n)),
                np.zeros((m, n)),
                np.zeros((m, p)),
                np.zeros((m, n)),
                G_0.T,
                np.eye(m),
                np.eye(m),
            ],
            [
                np.eye(n),
                np.zeros((n, m)),
                Lam1_inv,
                np.zeros((n, n)),
                np.zeros((n, p)),
                np.zeros((n, n)),
                np.zeros((n, p)),
                np.zeros((n, m)),
                np.zeros((n, m)),
            ],
        ],
    )

```

```

    [
        np.eye(n),
        np.zeros((n, m)),
        np.zeros((n, n)),
        Lam1_tilde_inv,
        np.zeros((n, p)),
        np.zeros((n, n)),
        np.zeros((n, p)),
        np.zeros((n, m)),
        np.zeros((n, m)),
    ],
    [
        N_0,
        np.zeros((p, m)),
        np.zeros((p, n)),
        np.zeros((p, n)),
        LamF_inv,
        np.zeros((p, n)),
        np.zeros((p, p)),
        np.zeros((p, m)),
        np.zeros((p, m)),
    ],
    [
        Q_0,
        np.zeros((n, m)),
        np.zeros((n, n)),
        np.zeros((n, n)),
        np.zeros((n, p)),
        0.5 * np.eye((n)),
        np.zeros((n, p)),
        np.zeros((n, m)),
        np.zeros((n, m)),
    ],
    [
        np.zeros((p, n)),
        G_0,
        np.zeros((p, n)),
        np.zeros((p, n)),
        np.zeros((p, p)),
        np.zeros((p, n)),
        np.zeros((p, p)),
        np.zeros((p, m)),
        np.zeros((p, m)),
    ],
    [
        np.zeros((m, n)),
        np.eye((m)),
        np.zeros((m, n)),
        np.zeros((m, n)),
        np.zeros((m, p)),
        np.zeros((m, n)),
        np.zeros((m, p)),
        Lam2_inv,
        np.zeros((m, m)),
    ],
    [
        np.zeros((m, n)),
        np.eye((m)),
        np.zeros((m, n)),
        np.zeros((m, n)),
        np.zeros((m, p)),
        np.zeros((m, n)),
        np.zeros((m, p)),
        np.zeros((m, m)),
        Lam2_tilde_inv,
    ],
)

# Constraints
constraints = [
    LMI_1 >> 1.0e-3,    #LMI is positive definite
    Q_0 >> 1.0e-2,    #the inverse of P is PD
]

```

```

#
obj = cvx.Minimize(0)
prob = cvx.Problem(obj, constraints)

# solve
prob.solve(solver=cvx.MOSEK, verbose = 1)
K_0 = N_0 @ np.linalg.inv(Q_0.value)          #K = NQ-1

#print solution if exists
if prob.status == cvx.OPTIMAL:
    print("P_0=", np.linalg.inv(Q_0.value))
    print("N_0=", N_0.value)
    print("K_0=", K_0.value)
    print("G_0=", G_0.value)
else:
    print("No solution")

return K_0, G_0

#Define real system dynamics in closed loop
class my_dynamics(sk.FirstOrderLinearDS):
    def __init__(self, K_0):
        ndof = n # number of states
        x0 = np.zeros(ndof)
        #Define initial values
        x0[0] =
        x0[1] =
        :
        x0[n-1] =
        self.K_0 = K_0

        A = np.zeros((ndof, ndof))
        b0 = np.zeros(ndof)

        # This line means that you build the underlying
        # FirstOrderLinearDS with A, b0 and x0.
        # A is empty and will be updated at each time step with a call to computeA.
        # Same for b with computeb...
        super(my_dynamics, self).__init__(x0, A, b0)
    #to recall the values of K and G, use self.K ...
    def computeA(self, time):
        self.A()[0, 0] =
        self.A()[0, 1] =
        :
        self.A()[n-1, n-1] =

    #The function b in siconos is equivalent to Eu in our LCS
    def computeb(self, time):
        self.b()[0] =
        :
        self.b()[n-1] =

class my_relation(sk.FirstOrderLinearR):
    # def __init__(self, arguments)
    def __init__(self):
        #self.K_0 = K_0
        #Define relation (constraints)
        # y = Cx + Dlambda + e(t)
        # r = B lambda
        # First create the operators (C, D, ...) of the extended system which contains the desired and

        C = np.zeros((2*m, 2*n))

        D = np.zeros((2*m, 2*m))

        B = np.zeros((2*n,2*m))

        e = np.zeros(2*m)

        # And then build the underlying relation with these operators
        super(my_relation, self).__init__(C, B)

```

```

        self.setDPtr(D)
        self.setePtr(e)      #if F ≠ 0
#define the matrices B and C of the extended system
def computeC(self, time, z, C):
    self.C()[0, 0] =
        :
    self.C()[2*m-1, 2*n-1] =

def computeB(self, time, z, B):
    self.B()[0, 0] =
        :
    self.B()[2*n-1, 2*m-1] =

def computeD(self, time, z, D):
    self.D()[0, 0] = (1.0/ Rvalue)
    self.D()[0, 1] = 0.0
    self.D()[1, 0] = 0.0
    self.D()[1, 1] = (1.0/ Rvalue)

#Fu
def computee(self, time, z, e):
    self.e()[0] =
        :
    self.e()[2*m-1] =

def computeh(self, time, x, ll, z, y):
    self.computeC(time, z, self.C());
    np.matmul(self.C(), x, y)
    self.computeD(time, z, self.D());
    self.computee(time, z, self.e());
    y[...] += np.matmul(self.D(), ll); + self.e();

def computeg(self, time, ll, z, r):
    self.computeB(time, z, self.B());
    r[...] += np.matmul(self.B(), ll)

```

Now, the second file is implemented and the first is imported.

```

import numpy as np
import scipy as sp
from scipy.integrate import quad
import siconos.kernel as sk
#import the file where the dynamics systems are defined
import System as ex
import matplotlib.pyplot as plt
import math

# A function to build the nsds and launch a simulation
# - between start_time and end_time with a time step size equal to time_step
# - for some given values of K_0 and G_0
def build_and_run(time_step, start_time, end_time, K_0, G_0):
    # create dynamics
    ds = ex.my_dynamics(K_0)

    # create interactions
    relation = ex.my_relation(K_0)

    # Declare a nonsmooth law. It's size must be the size of your algebraic
    # system defining the constraints
    # i.e the length of y and lambda vectors of your relation.
    nslaw = sk.ComplementarityConditionNSL(m)      #m is the number of complementarity variables
    inter = sk.Interaction(nslaw, relation)

    # nonsmooth dynamical system
    nsds = sk.NonSmoothDynamicalSystem(start_time, end_time)
    nsds.insertDynamicalSystem(ds)
    nsds.link(inter, ds)

    # simulation setup
    theta = 0.5
    integrator = sk.EulerMoreauOSI(theta)
    osnspb = sk.LCP()
    time_discr = sk.TimeDiscretisation(start_time, time_step)

```



```

simu = sk.TimeStepping(nsds, time_discr, integrator, osnspb)

# estimate the number of time steps
N = math.ceil((end_time - start_time) / simu.timeStep()) + 1

# dataPlot: an array where each line k is used to save
# values at time-step k
dataPlot = np.zeros((N, 12))          # numpy array to save data for plotting.

# store initial values
k = 0
x = ds.x() # state
w1 = inter.y(0)[0]
w2 = inter.y(0)[1]
lamb1 = inter.lambda_(0)[0]
lamb2 = inter.lambda_(0)[1]          # pointer on the variable lambda in the interaction

# store initial values of x, lambda and w
dataPlot[k, 0] = start_time
dataPlot[k, 1] = x[0]
\dots

while simu.hasNextEvent():
    k = k + 1
    simu.computeOneStep() # compute one step
    time = simu.nextTime()
    #print("time =", time)
    dataPlot[k, 0] = simu.nextTime()
    dataPlot[k, 1] = x[0]
    dataPlot[k, 2] = x[1]
    \dots
    simu.nextStep()

# Save current vars and prepare next step
return dataPlot

def filter_datas(start_time, end_time, dataPlot):
    t = dataPlot[:, 0]
    imax = t.argmax()
    dataPlot = dataPlot[:, imax + 1, :]
    print(dataPlot[0, 0], dataPlot[-1, 0])
    assert dataPlot[0, 0] == start_time
    assert dataPlot[-1, 0] == end_time
    return dataPlot

if __name__ == "__main__":

    # This is where things are really done

    start_time = 0.0
    end_time = 50.0 #* 10e-4

    # Build the nsds and run the simu for a given time_step
    K_0, G_0 = ex.compute_K0G0()
    current_time_step = 1e-2
    dataPlot = build_and_run(current_time_step, start_time, end_time, K_0, G_0)

#plot figures

```

C Circuits' Dynamics

Let us recall some useful mathematical relations between the voltage V and the current i of the electrical components \mathbf{R} , \mathbf{C} and \mathbf{L} as follows:

- For the resistor, let us recall Ohm's Law: $V_{\mathbf{R}} = \mathbf{R}i_{\mathbf{R}}$ where \mathbf{R} is the resistance of the resistor.
- For the capacitor, we have the relations: $Q_{\mathbf{C}} = \mathbf{C}V_{\mathbf{C}}$ where \mathbf{C} is the capacitance and $Q_{\mathbf{C}}$ is the charge on the capacitor and $\dot{Q}_{\mathbf{C}} = i_{\mathbf{C}}$
- For the inductor, the relation is $V_{\mathbf{L}} = \mathbf{L}\frac{di_{\mathbf{L}}}{dt}$ where \mathbf{L} is the inductance of the inductor.

C.1 Circuit in Figure 12

Recall that the state x_1 is the charge on the capacitor and the state x_2 is the current passing through the inductor. By applying KVL (Kirchhoff's Voltage Law) we can write the following equations:

$$\begin{aligned} -u + V_C - V_L = 0 &\Leftrightarrow -u + \frac{x_1}{C} - L\dot{x}_2 = 0 \Leftrightarrow \dot{x}_2 = \frac{1}{LC}x_1 - \frac{u}{L} \\ -V_C - V_R + w = 0 &\Leftrightarrow -\frac{x_1}{C} - R\lambda + w = 0 \Leftrightarrow w = \frac{x_1}{C} + R\lambda \end{aligned}$$

Let us apply KCL (Kirchhoff's Current Law) at the point of connection between the capacitor C and the voltage source u , then:

$$i_C + i_L = i_R \Leftrightarrow \dot{x}_1 + x_2 = \lambda \Leftrightarrow \dot{x}_1 = -x_2 + \lambda$$

The equations derived, along with the complementarity relation (*i.e.*, $0 \leq \lambda \perp w \geq 0$), are consistent with the LCS in (89).

C.2 Circuit in Figure 15

C.3 Circuit in Figure 20

C.4 Circuit in Figure 27

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