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ASYMPTOTIC PROPERTIES OF CONTINUOUS ASSOCIATED-KERNEL DENSITY ESTIMATORS

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We consider the general modern notion of the so-called associated-kernels for smoothing density function on a given support. We first show that the unnormalized estimator is consistent and that the normalizing random variable converges in \mathbb{L}^4 to 1. Then, we deduce the consistency of the considered normalized estimator. The comparison in favor of the normalized estimator is obtained by the mean squared error. We conclude by providing, for the first time, the general asymptotic normalities through some regularity assumptions for both unnormalized and normalized associated-kernel density estimators. The Gumbel, Weibull, lognormal, and other associated kernels are investigated for illustrating theoretically and numerically some of our results with an application to original data of automobile claim amounts from Covéa Affinity.

1. Introduction. Classical kernels have been introduced for smoothing density functions on an unbounded support. These kernels typically act as continuous and symmetric smoothers for functions defined on \mathbb{R} . They do not depend on the bandwidth or the target point where the estimated probability density function (pdf) is evaluated; see for example, Parzen [20], Epanechnikov [6], Scott [25], Wand and Jones [29], and Zougab et al. [34]. Although the method using these classical kernels is robust for most general models, the symmetry of classical kernels introduces biases, known as edge effects, in the estimation of densities with support bounded on at least one side. These biases render the estimator inconsistent near the edges. Consequently, many authors have investigated alternative kernels, known as associated kernels or non-classical kernels.

Associated kernels, which can be either discrete or continuous smoothers depending on the support of the unknown function to be estimated, represent a modern version of classical kernels. More importantly, they encompass a family of non-classical (or asymmetric) kernels. This provides a variety of kernel choices for estimating the probability density function f with support $\mathbb{T} \subseteq \mathbb{R}$. Specifically, one can select an associated kernel for each $x \in \mathbb{T}$, whether the target point is inside the support or at an

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edge, thereby achieving estimation without edge effects. For discrete kernels, we refer to the recent work of Esstafa et al. [7], which demonstrates that their normalized estimators of a probability mass function exhibit superior behavior under certain general conditions compared to unnormalized ones. Regarding continuous kernels, many authors have contributed to this field in the 2000s. Notable works include Chen [2, 3] on beta and gamma kernels, Jin and Kawczak [13] on Birnbaum-Saunders and lognormal kernels, Scaillet [24] on inverse and reciprocal inverse Gaussian kernels, Marchant et al. [18] on the generalized Birnbaum-Saunders kernel, Salha et al. [22, 23] on Weibull and Erlang kernels, Hirukawa and Sakudo [11] on a family of generalized gamma kernels, Libengué Dobélé-Kpoka and Kokonendji [17] on extended beta and lognormal kernels constructed using the mode-dispersion method, Ziane et al. [33] on the Birnbaum-Saunders power-exponential kernel, Geenens [8] on the Mellin-Meijer kernel, Khan and Akbar [14] on the Gumbel kernel, and Ouimet and Tolosana-Delgado [19] on the Dirichlet kernel. Additional contributions include the works of Hirukawa [10], Igarashi [12], and Doho et al. [5], which discuss their uses and properties among many others. Despite the extensive research, there are few general and unified (asymptotic) results related to continuous associated kernels that encompass the family of classical ones. It is in this context that we propose the current new developments.

In this paper, we primarily focus on the normalized continuous non-classical kernel estimators for any univariate probability density function f , which has support $\mathbb{T} \subseteq \mathbb{R}$, whether bounded or not. These estimators are commonly used in the literature; see for example, Wansouwé et al. [30], Kokonendji and Somé [16], Somé and Kokonendji [26], and Somé et al. [27]. However, their theoretical and comparative results are entirely missing so far. Let us first redefine the so-called (second-order) continuous associated kernel, as discussed in Kokonendji and Libengué Dobélé-Kpoka [15].

DEFINITION 1.1. *Let $\mathbb{T} \subseteq \mathbb{R}$ be the support of the p.d.f. f to be estimated, $x \in \mathbb{T}$ a target point and $h > 0$ a bandwidth. A parameterized p.d.f. $K_{x,h}(\cdot)$ on the support $\mathbb{S}_{x,h} \subseteq \mathbb{R}$ is called "continuous associated-kernel" if the following conditions are satisfied:*

$$(1.1) \quad x \in \mathbb{S}_{x,h}, \quad \mathbb{E}[Z_{x,h}] - x =: A(x,h) \xrightarrow{h \rightarrow 0} 0 \quad \text{and} \quad \text{Var}(Z_{x,h}) =: B(x,h) \xrightarrow{h \rightarrow 0} 0,$$

where $Z_{x,h}$ denotes the continuous random variable with p.d.f. $K_{x,h}(\cdot)$.

Although many kinds of associated-kernels have been proposed to solve the problem of estimation, it is not always clear how they were constructed. Thanks to Libengué Dobélé-Kpoka and Kokonendji [17], a basic idea for constructing these kernels has been revealed through the mode-dispersion method. To our knowledge, all continuous associated-kernels satisfy Definition 1.1, even if their authors have developed them through different artisanal approaches. For example, a classical or symmetric continuous kernel K is derived as an associated-kernel in the following sense for holding (1.1):

$$(1.2) \quad \mathbb{S}_{x,h} = x + h\mathbb{S}_K \quad \text{and} \quad K_{x,h}(\cdot) = \frac{1}{h} K\left(\frac{\cdot - x}{h}\right),$$

where $\mathbb{S}_K \subseteq \mathbb{R}$ designates the symmetric support of K centered at 0 (one can refer to Scott [25] and Wand and Jones [29] for a list of classical kernels). It is easy to verify that for the associated-kernel given in (1.2), the corresponding $A(x,h) = 0$ and $B(x,h) = h^2 \int_{\mathbb{S}_K} u^2 K(u) du$.

For this work, we consider an n -sample (X_1, X_2, \dots, X_n) of independent and identically distributed (i.i.d.) continuous random variables having a probability density function f on $\mathbb{T} \subseteq \mathbb{R}$. Referring to Esstafa et al. [7], we express the normalized estimator as follows:

$$(1.3) \quad \widehat{f}_n(x) = \frac{\widetilde{f}_n(x)}{C_n},$$

with

$$(1.4) \quad \widetilde{f}_n(x) = \frac{1}{n} \sum_{i=1}^n K_{x,h_n}(X_i) \quad \text{and} \quad C_n = \int_{x \in \mathbb{T}} \widetilde{f}_n(x) dx > 0.$$

In general, the basic estimator \widetilde{f}_n of f in (1.4) is usually an improper density estimate since $0 < \int_{\mathbb{T}} \widetilde{f}_n(x) dx \neq 1$ for most associated-kernels. In fact, from (1.4), the normalizing constant is defined as follows:

$$(1.5) \quad C_n = \frac{1}{n} \sum_{i=1}^n \int_{x \in \mathbb{T}} K_{x,h_n}(X_i) dx$$

and generally fails to be equal to 1 since the integral $\int_{x \in \mathbb{T}} K_{x,h_n}(X_i) dx$ from (1.5) operates on the domain of the intrinsic parameter x of the p.d.f. $K_{x,h_n}(\cdot)$. However, as shown in Kokonendji and Libengué Dobélé-Kpoka [15] and Wansouwé et al. [30], some numerical results indicate that C_n is typically close to 1. Under some general conditions, we shall prove the convergence in \mathbb{L}^4 of C_n to 1.

It should be noted that in the literature, many results on the asymptotic properties of these associated-kernel estimators are very specific to a given family of non-classical associated-kernels. Regarding a general kernel defined in Definition 1.1, we only have the following result, which we shall complete and improve in this work.

THEOREM 1.2. [15, Theorem 2.2] *Let $f \in \mathcal{C}^2(\mathbb{T})$. For any $x \in \mathbb{T}$ and under the conditions in (1.1), we have $\widetilde{f}_n(x) \xrightarrow{a.s.} f(x)$ as $n \rightarrow \infty$. Furthermore, if there exists a real number $r_2 = r_2(K_{x,h_n}) > 0$ such that*

$$h_n^{r_2} \int_{S_{x,h_n} \cap \mathbb{T}} K_{x,h_n}^2(u) du \leq c_2(x) < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} n h_n^{r_2} = \infty,$$

then $\widetilde{f}_n(x) \xrightarrow{\mathbb{L}^2} f(x)$ as $n \rightarrow \infty$, where “ $\xrightarrow{\mathbb{L}^2}$ ” (resp. “ $\xrightarrow{a.s.}$ ”) stands for “mean square” (resp. “almost sure”) convergence.

In this paper, we propose a unification of the asymptotic theory of continuous associated-kernel estimators \widetilde{f}_n in (1.4) and \widehat{f}_n in (1.3) under some regularity assumptions on the density and the associated-kernel. In addition, we show that these assumptions are satisfied by most kernels developed in the literature. Then, we study the convergence and the asymptotic normality of the (non-)normalized estimators and we provide an \mathbb{L}^2 -comparison between them.

The rest of the paper is organized as follows. Section 2 presents the basic assumptions on continuous associated-kernels, which we illustrate with examples of non-classical kernels. In Section 3, we show our main results on the asymptotic properties for both (non-)normalized associated-kernel estimators, in particular, pointwise

consistency and asymptotic normality are proven, along with an \mathbb{L}^2 -comparison of both estimators. Section 4 provides numerical results that corroborate our theoretical work, including an illustrative application to the original dataset from Covéa Affinity, the expert entity in affinity insurance of the Covéa group (a French mutual insurance company). Finally, Section 5 is devoted to the remaining proofs of the main results and some theoretical tools.

2. Assumptions and settings. To simplify the notation, we use \mathbb{S}_x instead of \mathbb{S}_{x,h_n} . For any fixed $x \in \mathbb{T}$, we denote

$$|\mathbb{E}[Z_{x,h_n}] - x| = A(x, h_n), \quad \text{Var}(Z_{x,h_n}) = B(x, h_n) \quad \text{and} \quad \max_{y \in \mathbb{S}_x} K_{x,h_n}(y) = C(x, h_n).$$

In what follows, we assume that the pdf f on $\mathbb{T} \subseteq \mathbb{R}$ to be estimated satisfies the following conditions:

(A₀) $f \in C^1(\mathbb{T})$ square integrable with absolutely integrable derivative on \mathbb{T} .

In addition, we suppose that the continuous associated random variable Z_{x,h_n} having the density K_{x,h_n} satisfies the following:

(A₁^α) For some $\alpha > 0$ and for any fixed $x \in \mathbb{T}$, we have

$$(2.1) \quad x \in \mathbb{S}_x, \quad A(x, h_n) = O(h_n^\alpha) \text{ and } B(x, h_n) = O(h_n^{2\alpha}) \text{ as } n \rightarrow \infty;$$

$$(2.2) \quad \max\{A(x, h_n)C(x, h_n), B(x, h_n)C^2(x, h_n)\} \leq c \text{ where } c \text{ does not depend on } x.$$

REMARK 2.1. Note that (2.1) is a refined version of Condition (1.1) in Definition 1.1. Furthermore, from (2.2), we can easily show that for any $x \in \mathbb{T}$

$$(2.3) \quad \int_{y \in \mathbb{S}_x \cap \mathbb{T}} K_{x,h_n}^2(y) dy = O(h_n^{-\alpha}) \quad \text{and} \quad \int_{y \in \mathbb{S}_x \cap \mathbb{T}} K_{x,h_n}^3(y) dy = O(h_n^{-2\alpha}).$$

Additionally, we require the following properties of K_{x,h_n} when investigating the normalized associated-kernel estimator.

(A₂^α) There exists $\alpha > 0$ such that for any $y \in \mathbb{T}$,

$$\int_{z \in \mathbb{T}} K_{z,h_n}(y) dz < c \quad \text{and} \quad \int_{z \in \mathbb{T}} K_{z,h_n}^2(y) dz \leq h_n^{-\alpha} q(y),$$

for n large enough. The constant c does not depend on y and q is some positive function on \mathbb{T} .

Conditions (A₁^α) and (A₂^α) are fulfilled by most of the associated kernels, including the ones generated by classical kernels such as Epanechnikov (or Parabolic), Cosine, Triangular, Quartic (Biweight), Triweight, Tricube, Gaussian, Logistic, Double Exponential, and Silverman kernels. In what follows, we point out some important examples.

EXAMPLE 2.2. The following continuous associated-kernels satisfy the Assumptions (A₁^α)-(A₂^α) with $\alpha = 1$.

1. *Lognormal kernel*: for $x \in \mathbb{T} \subseteq \mathbb{R}_+^*$ and $h_n > 0$, the mode-dispersion lognormal kernel on $\mathbb{S}_x = \mathbb{R}_+^*$ yields

$$K_{x,h_n}(y) = \frac{1}{yh_n \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{1}{h_n} \log(y/x) - h_n \right)^2 \right\}, \quad y \in \mathbb{S}_x = \mathbb{R}_+^*.$$

We can see that $\int_{y \in \mathbb{S}_x} K_{x,h_n}(y) dy = 1$ and that Equation (2.2) holds since $\max_{y \in \mathbb{S}_x} K_{x,h_n}(y) = K_{x,h_n}(x) = (xh_n \sqrt{2\pi})^{-1} \exp(-h_n^2/2)$. Moreover, we have

$$\mathbb{E}[Z_{x,h_n}] = x + (e^{3h_n^2/2} - 1)x \quad \text{and} \quad \text{Var}(Z_{x,h_n}) = x^2 e^{3h_n^2} (e^{h_n^2} - 1).$$

Then, the conditions in (2.1) are also satisfied. With regard to (A_2^α) , one obtains the upper bounds

$$\int_{z \in \mathbb{R}_+} K_{z,h_n}(y) dz = e^{-h_n^2/2} \quad \text{and} \quad \int_{z \in \mathbb{R}_+} K_{z,h_n}^2(y) dz = \frac{e^{-3h_n^2/4}}{2h_n y \sqrt{\pi}}, \quad \forall y \in \mathbb{R}_+^*.$$

It follows that the normalizing constant $C_n = e^{-h_n^2/2} \neq 1$. Similar results with $C_n \neq 1$ hold for many other non-classical kernel estimators, such as (generalized) gamma, (extended) beta, (reciprocal) inverse Gaussian, (generalized) Birnbaum-Saunders, Erlang, and Mellin-Meijer.

2. *Weibull kernel*: for $x \in \mathbb{T} \subseteq \mathbb{R}_+^*$ and $0 < h_n < 1$, the associated Weibull kernel is defined on $\mathbb{S}_x = \mathbb{R}_+$ by

$$K_{x,h_n}(y) = \frac{1}{xh_n} \left(\frac{y}{x} \right)^{1/h_n - 1} \exp \left\{ -\left(\frac{y}{x} \right)^{1/h_n} \right\}, \quad y \in \mathbb{S}_x = \mathbb{R}_+.$$

It is clear that $\int_{y \in \mathbb{S}_x} K_{x,h_n}(y) dy = 1$ and that (2.2) is satisfied since $\max_{y \in \mathbb{S}_x} K_{x,h_n}(y) = K_{x,h_n}((1-h_n)^{h_n} x) = (xh_n)^{-1} (1-h_n)^{1-h_n} e^{-(1-h_n)}$. Furthermore, one verifies that

$$\mathbb{E}[Z_{x,h_n}] = x\Gamma(1+h_n) \quad \text{and} \quad \text{Var}(Z_{x,h_n}) = x^2 [\Gamma(1+2h_n) - \Gamma^2(1+h_n)],$$

where $\Gamma(\cdot)$ is the gamma function.

Consequently, the conditions in (2.1) are satisfied. For (A_2^α) , one can easily check

$$\int_{z \in \mathbb{R}_+} K_{z,h_n}(y) dz = \Gamma(1-h_n) \quad \text{and} \quad \int_{z \in \mathbb{R}_+} K_{z,h_n}^2(y) dz = \frac{2^{h_n} \Gamma(2-h_n)}{4h_n}, \quad \forall y \in \mathbb{R}_+.$$

The normalizing constant is then given by $C_n = \Gamma(1-h_n) \neq 1$.

3. *Gumbel kernel*: given $x \in \mathbb{T} \subseteq \mathbb{R}$ and $h_n > 0$, the associated Gumbel kernel has been proposed on $\mathbb{S}_x = \mathbb{R}$ by

$$K_{x,h_n}(y) = \frac{1}{h_n} \exp \left[-\exp \left(-\frac{y-x}{h_n} \right) - \frac{y-x}{h_n} \right], \quad y \in \mathbb{S}_x = \mathbb{R}.$$

We easily check that $\int_{y \in \mathbb{S}_x} K_{x,h_n}(y) dy = 1$ and that Equation (2.2) holds, because $\max_{y \in \mathbb{R}} K_{x,h_n}(y) = K_{x,h_n}(x) = 1/(eh_n)$. In addition, we have

$$\mathbb{E}[Z_{x,h_n}] = x + h_n \gamma \quad \text{and} \quad \text{Var}(Z_{x,h_n}) = \frac{\pi^2}{6} h_n^2,$$

where $\gamma = 0.577215\dots$ is the Euler-Mascheroni constant. Assumption (A_2^α) holds since we have

$$\int_{z \in \mathbb{R}} K_{z, h_n}(y) dz = 1 \quad \text{and} \quad \int_{z \in \mathbb{R}} K_{z, h_n}^2(y) dz = \frac{1}{4h_n}, \quad \forall y \in \mathbb{T},$$

so $C_n = 1$. The Gumbel kernel corresponds to an asymmetric kernel and has its mode at x . It is useful for estimating extreme value distributions.

For the rest of the paper, c (respectively c_x) denotes a constant that does not depend on x or h_n (respectively depends on x), and may vary from one line to another.

3. Main results. We present our general results for the continuous associated-kernel estimators, first unnormalized and then normalized, along with a comparison.

3.1. *Non-normalized associated-kernel estimator.* This section is divided into two main parts. In the first, we prove the mean square convergence of the non-normalized estimator \tilde{f}_n to the unknown probability density function f . Building on this result, the second part is dedicated to proving the corresponding central limit theorem.

Now, proceeding is the same ideas as in [15], we provide the next technical proposition.

PROPOSITION 3.1. *Assume that (A_0) , (A_1^α) and (A_2^α) hold for some $\alpha > 0$, then, for any $x \in \mathbb{T}$ and n large enough, we have*

$$\left| \mathbb{E} [\tilde{f}_n(x) - f(x)] \right| \leq g_n(x) h_n^{\alpha/2}$$

where $(g_n)_{n \geq 1}$ are positive integrable functions on \mathbb{T} .

PROOF. First, for any $x \in \mathbb{T}$, let $\mathcal{E}_x^n = [\ell_x^n, r_x^n]$ be the compact subset of \mathbb{T} with $\ell_x^n = \begin{cases} x - h_n^{\alpha/2}, & x \in \text{int}(\mathbb{T}) \cup \{\partial(\mathbb{T})^+\} \\ x, & x = \partial(\mathbb{T})^- \end{cases}$ and $r_x^n = \begin{cases} x + h_n^{\alpha/2}, & x \in \text{int}(\mathbb{T}) \cup \{\partial(\mathbb{T})^-\} \\ x, & x = \partial(\mathbb{T})^+ \end{cases}$ where $\partial(\mathbb{T})^-$ (respectively $\partial(\mathbb{T})^+$) denotes the lower (respectively the upper) bound of \mathbb{T} when there exists and $\text{int}(\mathbb{T})$ denotes the interior of \mathbb{T} . In addition, let the complement of a set A be denoted as A^c . Then, using the i.i.d. property of $(X_i)_{1 \leq i \leq n}$ combined with $\mathbb{T}^c \subset (\mathcal{E}_x^n)^c$, we obtain

$$\begin{aligned} \mathbb{E} [\tilde{f}_n(x) - f(x)] &= \mathbb{E} [K_{x, h_n}(X_1)] - f(x) = \int_{y \in \mathbb{S}_x \cap \mathbb{T}} f(y) K_{x, h_n}(y) dy - f(x) \\ &:= \mathbf{C}_{1, n} + \mathbf{C}_{2, n} + \mathbf{C}_{3, n}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{C}_{1, n} &:= \int_{y \in \mathbb{S}_x \cap \mathcal{E}_x^n} K_{x, h_n}(y) (f(y) - f(x)) dy, \\ \mathbf{C}_{2, n} &:= \int_{y \in \mathbb{S}_x \cap \mathbb{T} \cap (\mathcal{E}_x^n)^c} K_{x, h_n}(y) (f(y) - f(x)) dy, \\ \mathbf{C}_{3, n} &:= -f(x) \int_{y \in \mathbb{S}_x \cap \mathbb{T}^c} K_{x, h_n}(y) dy. \end{aligned} \tag{3.1}$$

Considering $\mathbf{C}_{1,n}$: using Taylor's theorem, there exists a constant $\theta \in (0, 1)$ such that

$$\begin{aligned}
& \left| \int_{y \in \mathbb{S}_x \cap \mathcal{E}_x^n} (f(y) - f(x)) K_{x,h_n}(y) dy \right| \\
& \leq \int_{y \in \mathbb{S}_x \cap \mathcal{E}_x^n} |f'(\theta x + (1 - \theta)y)| |y - x| K_{x,h_n}(y) dy \\
(3.2) \quad & \leq \max_{y \in \mathbb{S}_x \cap \mathcal{E}_x^n} |f'(\theta x + (1 - \theta)y)| h_n^{\alpha/2}.
\end{aligned}$$

The last inequality in (3.2) is obtained from the continuity of f' , the compactness of the set \mathcal{E}_x^n and the fact that K_{x,h_n} is a probability density.

Considering $\mathbf{C}_{2,n}$ and $\mathbf{C}_{3,n}$: first of all, using the fact that f is null on \mathbb{T}^c , we have

$$|\mathbf{C}_{2,n}| + |\mathbf{C}_{3,n}| \leq \int_{y \in \mathbb{S}_x \cap (\mathcal{E}_x^n)^c} |f(y) - f(x)| K_{x,h_n}(y) dy.$$

Second, we can rewrite the formula on the r.h.s. of this inequality as follows

$$\int_{y \in \mathbb{S}_x \cap (\mathcal{E}_x^n)^c} |f(y) - f(x)| K_{x,h_n}(y) dy := \mathbf{S}_{1,n} + \mathbf{S}_{2,n} + \mathbf{S}_{3,n},$$

where

$$\begin{aligned}
\mathbf{S}_{1,n} &:= \int_{y \in \mathbb{S}_x} |f(y) - f(x)| K_{x,h_n}(y) \mathbf{1}_{|y-x| > h_n^{\alpha/2}} dy, \\
\mathbf{S}_{2,n} &:= \int_{y \in \mathbb{S}_x} |f(y) - f(x)| K_{x,h_n}(y) \mathbf{1}_{0 \leq y-x \leq h_n^{\alpha/2}} dy \mathbf{1}_{x=\partial(\mathbb{T})^+}, \\
\mathbf{S}_{3,n} &:= \int_{y \in \mathbb{S}_x} |f(y) - f(x)| K_{x,h_n}(y) \mathbf{1}_{-h_n^{\alpha/2} \leq y-x \leq 0} dy \mathbf{1}_{x=\partial(\mathbb{T})^-}.
\end{aligned}$$

For $\mathbf{S}_{1,n}$: to begin with, from the Markov inequality and (A₁^α), we have

$$\mathbb{P}(|Z_{x,h_n} - x| > h_n^{\alpha/2}) \leq \frac{B(x, h_n) + A^2(x, h_n)}{h_n^\alpha} \leq c_x h_n^\alpha.$$

This implies that $\lim_{n \rightarrow \infty} \mathbb{P}(|Z_{x,h_n} - x| > h_n^{\alpha/2}) = 0$. Then, for $\delta = h_n^\alpha > 0$ and n large enough, using Cauchy-Schwarz's inequality and the fact that K_{x,h_n} is a density function, we have

$$\begin{aligned}
|\mathbf{S}_{1,n}| &\leq \sqrt{\int_{y \in \mathbb{S}_x \cap (\mathcal{E}_x^n)^c} (f(y) - f(x))^2 K_{x,h_n}(y) dy} \times \mathbb{P}(|Z_{x,h_n} - x| > h_n^{\alpha/2}) \\
(3.3) \quad &\leq \sqrt{\delta} \sqrt{\int_{y \in \mathbb{T}} f^2(y) K_{x,h_n}(y) dy + \int_{y \in \mathbb{S}_x} f^2(x) K_{x,h_n}(y) dy} \\
&\leq \sqrt{\delta} \sqrt{\int_{y \in \mathbb{T}} f^2(y) K_{x,h_n}(y) dy + f^2(x)}.
\end{aligned}$$

For $\mathbf{S}_{2,n}$ and $\mathbf{S}_{3,n}$: similarly as for $\mathbf{C}_{1,n}$, using simply Taylor's theorem, there exist $t, s \in (0, 1)$ such that

$$|\mathbf{S}_{2,n}| \leq \max_{y \in [x, x+h_n^{\alpha/2}]} |f'(tx + (1-t)y)| h_n^{\alpha/2},$$

$$|\mathbf{S}_{3,n}| \leq \max_{y \in [x-h_n^{\alpha/2}, x]} |f'(sx + (1-s)y)| h_n^{\alpha/2}.$$

Therefore, the bias of the non-normalized estimator

$$\left| \mathbb{E} \left[\widetilde{f}_n(x) - f(x) \right] \right| \leq |\mathbf{C}_{1,n}| + |\mathbf{C}_{2,n}| + |\mathbf{C}_{3,n}| \leq g_n(x) h_n^{\alpha/2},$$

where the functions $g_n(x)$, $n \geq 1$, can be defined as

$$\max \left\{ \max_{y \in \mathcal{S}_x \cap \mathcal{E}_x^n} |f'(\theta x + (1-\theta)y)|, \sqrt{\int_{y \in \mathbb{T}} f^2(y) K_{x,h_n}(y) dy + f^2(x)} \right\}.$$

Now, in the above definition of $g_n(x)$, on the one hand, since the function $\max_{y \in \mathcal{S}_x \cap \mathcal{E}_x^n} |f'(\theta x + (1-\theta)y)|$ is decreasing and converges to $|f'(x)|$ as $n \rightarrow +\infty$, it is integrable with respect to x by monotone convergence theorem. On the other hand, from square integrability of f in (A_0) and the first inequality in (A_2^α) , the function $\int_{y \in \mathbb{T}} f^2(y) K_{x,h_n}(y) dy + f^2(x)$ is also integrable on x . This completes the proof. \square

In what follows, we state that $\widetilde{f}_n(x)$ is consistent and its proof is postponed in Section 5.

THEOREM 3.2. *Assume that (A_0) and (A_1^α) with some $\alpha > 0$ hold. Then, if the sequence $(h_n)_{n \geq 1}$ is chosen such that $nh_n^\alpha \xrightarrow[n \rightarrow \infty]{} \infty$, we have, for any $x \in \mathbb{T}$,*

$$\widetilde{f}_n(x) \xrightarrow[n \rightarrow \infty]{\mathbb{L}^2} f(x).$$

The limiting distribution of the non-normalized estimator is given in the following theorem, and its proof is provided in Section 5. This is a novel result and the first general theorem in the direction of continuous associated-kernel estimators.

THEOREM 3.3. *Assume that (A_0) and (A_1^α) with some $\alpha > 0$ hold. Then, if the sequence $(h_n)_{n \geq 1}$ is chosen such that $\sqrt{nh_n^\alpha} \xrightarrow[n \rightarrow \infty]{} 0$ and $nh_n^\alpha \xrightarrow[n \rightarrow \infty]{} \infty$, we have, for any $x \in \mathbb{T}$,*

$$\sqrt{nh_n^\alpha} (\widetilde{f}_n(x) - f(x)) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, f(x) \lambda_{x,\alpha}),$$

where $\lambda_{x,\alpha} = \lim_{n \rightarrow \infty} h_n^\alpha \int_{y \in \mathbb{T} \cap \mathcal{S}_x} K_{x,h_n}^2(y) dy$ and “ $\xrightarrow{\mathcal{L}}$ ” stands for “convergence in law”.

3.2. Normalized associated-kernel estimator. As the discrete associated-kernel case treated by Esstafa et al. [7], we prove in Section 5 the following proposition, which relates to the normalizing random variable for the continuous estimator (1.3) from a non-classical associated-kernel.

PROPOSITION 3.4. *Assume that (A_0) , (A_1^α) and (A_2^α) hold for some $\alpha > 0$, $\int_{\mathbb{T}} q(x) f(x) dx < \infty$ and $nh_n^\alpha \rightarrow \infty$. Then, the normalizing random variable C_n converges in \mathbb{L}^4 to 1.*

In what follows, we show that $\widehat{f}_n(x)$ is consistent.

THEOREM 3.5. Assume that (A_0) , (A_1^α) and (A_2^α) hold for some $\alpha > 0$, $\int_{\mathbb{T}} q(x)f(x)dx < \infty$ and $nh_n^\alpha \rightarrow \infty$. Then, for any $x \in \mathbb{T}$, we have

$$\widehat{f}_n(x) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} f(x),$$

where “ $\xrightarrow{\mathbb{P}}$ ” indicate “convergence in probability”.

PROOF. To begin with, we note that for any $x \in \mathbb{T}$, the estimation error is given by

$$\widehat{f}_n(x) - f(x) = \frac{1}{C_n} \left\{ (\widetilde{f}_n(x) - f(x)) + (1 - C_n)f(x) \right\}.$$

Second, from Theorem 3.2, we recall that $\widetilde{f}_n(x)$ converges in mean square to $f(x)$. Therefore, we complete the proof using Proposition 3.4 and the Slutsky theorem. \square

In addition, within our context, we can use the asymptotic properties of $\widetilde{f}_n(x)$ to derive a central limit theorem for the normalized estimator $\widehat{f}_n(x)$.

THEOREM 3.6. Assume that (A_0) hold, $\int_{\mathbb{T}} q(x)f(x)dx < \infty$, (A_1^α) and (A_2^α) hold for some $\alpha > 0$. Then, if the sequence $(h_n)_{n \geq 1}$ is chosen such that $\sqrt{nh_n^\alpha} \xrightarrow[n \rightarrow \infty]{} 0$ and $nh_n^\alpha \xrightarrow[n \rightarrow \infty]{} \infty$, we have, for any $x \in \mathbb{T}$,

$$\sqrt{nh_n^\alpha} (\widehat{f}_n(x) - f(x)) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, f(x)\lambda_{x,\alpha}),$$

where $\lambda_{x,\alpha} = \lim_{n \rightarrow \infty} h_n^\alpha \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy$.

PROOF. Let us rewrite our normalized error as follows

$$(3.4) \quad \sqrt{nh_n^\alpha} (\widehat{f}_n(x) - f(x)) = \frac{1}{C_n} \left(\frac{\widetilde{f}_n(x) - \mathbb{E}[\widetilde{f}_n(x)]}{\sqrt{\text{Var}(\widetilde{f}_n(x))}} \right) \sqrt{nh_n^\alpha \text{Var}(\widetilde{f}_n(x))} \\ + \frac{1}{C_n} \sqrt{nh_n^\alpha} \left\{ \mathbb{E}[\widetilde{f}_n(x)] - f(x) \right\} + \frac{f(x)}{C_n} \sqrt{nh_n^\alpha} (1 - C_n).$$

Then, the convergence of the first term in the r.h.s. (right hand side) of (3.4) is straightforward by Proposition 3.4, the inequality (5.9) from the proof of the central limit theorem of the non-normalized estimator, and Slutsky's theorem. More precisely, we get

$$\frac{1}{C_n} \left(\frac{\widetilde{f}_n(x) - \mathbb{E}[\widetilde{f}_n(x)]}{\sqrt{\text{Var}(\widetilde{f}_n(x))}} \right) \sqrt{nh_n^\alpha \text{Var}(\widetilde{f}_n(x))} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left(0, f(x) \lim_{n \rightarrow \infty} h_n^\alpha \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy \right).$$

The convergence to zero of the second term in the r.h.s. of (3.4) is also straightforward through Proposition 3.1 (or (5.10)) and Proposition 3.4. Therefore, it is left to show the convergence in probability to zero of the last term in the r.h.s. of (3.4). First, let us denote by

$$S_n = \sqrt{nh_n^\alpha} (1 - C_n) = \frac{\sqrt{h_n^\alpha}}{\sqrt{n}} \sum_{i=1}^n Y_{i,n}, \quad Y_{i,n} = \int_{x \in \mathbb{T}} \{f(x) - K_{x,h_n}(X_i)\} dx.$$

Then, by Assumption (A_2^α) , there exists a positive constant c such that $|Y_{i,n}| < c$ a.s. and by Hoeffding's inequality, we get for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^n (Y_{i,n} - \mathbb{E}[Y_{i,n}])\right| \geq \varepsilon\right) \leq c \exp\left(-\frac{\varepsilon^2}{n}\right).$$

Thus, on the one hand, for any $\delta > 0$, taking $\varepsilon = \frac{\sqrt{n}}{\sqrt{h_n^\alpha}}\delta > 0$, we get

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geq \delta) \leq c \exp\left(-\frac{\delta^2}{h_n^\alpha}\right) \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand, by Proposition 3.1, for n large enough, we obtain

(3.5)

$$|\mathbb{E}[S_n]| \leq \sqrt{nh_n^\alpha} |\mathbb{E}[1 - C_n]| \leq \sqrt{nh_n^\alpha} \int_{x \in \mathbb{T}} |\mathbb{E}[f(x) - \widetilde{f}_n(x)]| dx \leq h_n^\alpha \sqrt{n} \int_{x \in \mathbb{T}} g_n(x) dx.$$

Then, as $\sqrt{nh_n^\alpha} \xrightarrow{n \rightarrow \infty} 0$, we deduce the convergence in probability to zero of $\sqrt{nh_n^\alpha} (1 - C_n)$ and also of the last term in the r.h.s. of (3.4). This completes the proof. \square

3.3. \mathbb{L}^2 -comparison. In order to make a global comparison between \widehat{f}_n and \widetilde{f}_n , we use the \mathbb{L}^2 error rather than the \mathbb{L}^1 in Esstafa et al. [7] for discrete case. The following proposition emphasizes that, in terms of the \mathbb{L}^2 error, the normalized estimator outperforms the unnormalized one. Similar conclusions have been drawn in the continuous setting of estimating a probability density function with classical kernels; see for instance, Devroye and Lugosi [4, Section 5.6] and Glad et al. [9] for results based on the \mathbb{L}^2 criterion.

Note that we require the convergence of C_n to 1 in \mathbb{L}^4 to obtain the following analogous comparison between \widehat{f}_n and \widetilde{f}_n . From Proposition 3.4, we can easily derive the comparison between the mean integrated squared error of the normalized continuous estimator \widehat{f}_n and that of the unnormalized associated version \widetilde{f}_n in the following proposition.

PROPOSITION 3.7. *Assume that (A_1^α) and (A_2^α) hold for some $\alpha > 0$, $nh_n^\alpha \rightarrow \infty$, $f \in C^1(\mathbb{T})$, $\int_{\mathbb{T}} q(x)f(x)dx < \infty$ and \mathbb{T} is compact. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$\mathbb{E}\left[\int_{\mathbb{T}} |\widehat{f}_n(x) - f(x)|^2 dx\right] < \mathbb{E}\left[\int_{\mathbb{T}} |\widetilde{f}_n(x) - f(x)|^2 dx\right] + \varepsilon.$$

PROOF. First, we successively have

$$\begin{aligned} \mathbb{E}\left[\int_{\mathbb{T}} |\widehat{f}_n(x) - f(x)|^2 dx\right] &= \mathbb{E}\left[\int_{\mathbb{T}} \left|\frac{\widetilde{f}_n(x)}{C_n} - f(x)\right|^2 dx\right] \\ &= \mathbb{E}\left[\int_{\mathbb{T}} \left|\frac{\widetilde{f}_n(x)}{C_n} - \frac{f(x)}{C_n} + \frac{f(x)}{C_n} - f(x)\right|^2 dx\right] \\ &= \mathbb{E}\left[\frac{1}{C_n^2} \int_{\mathbb{T}} |\widetilde{f}_n(x) - f(x)|^2 dx\right] + \int_{\mathbb{T}} f^2(x) dx \mathbb{E}\left[\left|\frac{C_n - 1}{C_n}\right|^2\right] \end{aligned}$$

$$\begin{aligned}
& + 2\mathbb{E} \left[C_n^{-2} \int_{\mathbb{T}} (\tilde{f}_n(x) - f(x))(1 - C_n)f(x)dx \right] \\
& = \mathbb{E} \left[\int_{\mathbb{T}} |\tilde{f}_n(x) - f(x)|^2 dx \right] + \mathbb{E} \left[\frac{1 - C_n^2}{C_n^2} \int_{\mathbb{T}} |\tilde{f}_n(x) - f(x)|^2 dx \right] \\
& + \mathbb{E} \left[\left| \frac{C_n - 1}{C_n} \right|^2 \right] \int_{\mathbb{T}} f^2(x)dx + 2\mathbb{E} \left[C_n^{-2}(1 - C_n) \int_{\mathbb{T}} (\tilde{f}_n(x) - f(x))f(x)dx \right].
\end{aligned}$$

We note that, on \mathbb{T} compact, $f \in C^1(\mathbb{T})$ satisfies Assumption (A₀). Now, since f and $\{K_{\cdot, h_n}(X_i)\}_{i \in \{1, \dots, n\}}$ are everywhere bounded on the compact set \mathbb{T} , $\int_{\mathbb{T}} f^2(x)dx \leq c$ and $\int_{\mathbb{T}} \tilde{f}_n^2(x)dx \leq c$ a.s., one can use the Jensen and Cauchy-Schwarz inequalities in order to obtain

$$\begin{aligned}
& \mathbb{E} \left[\int_{\mathbb{T}} |\widehat{f}_n(x) - f(x)|^2 dx \right] \\
& \leq \mathbb{E} \left[\int_{\mathbb{T}} |\tilde{f}_n(x) - f(x)|^2 dx \right] + c \mathbb{E} \left[\frac{1 - C_n^2}{C_n^2} \right] + c \mathbb{E} \left[\left| \frac{C_n - 1}{C_n} \right|^2 \right] \\
& + c \mathbb{E} [C_n^{-2}|1 - C_n|(C_n + 1)] \\
& \leq \mathbb{E} \left[\int_{\mathbb{T}} |\tilde{f}_n(x) - f(x)|^2 dx \right] + c \mathbb{E} \left[\left| \frac{C_n - 1}{C_n} \right|^2 \right] + c \mathbb{E} \left[\left| \frac{(1 - C_n)(C_n + 1)}{C_n^2} \right| \right] \\
(3.6) \quad & \leq \mathbb{E} \left[\int_{\mathbb{T}} |\tilde{f}_n(x) - f(x)|^2 dx \right] + c \sqrt{\mathbb{E}[(C_n - 1)^4] \mathbb{E}[C_n^{-4}]} \\
& + c \sqrt{\mathbb{E}[(1 - C_n)^2] \mathbb{E} \left[\frac{(C_n + 1)^2}{C_n^4} \right]} \\
& \leq \mathbb{E} \left[\int_{\mathbb{T}} |\tilde{f}_n(x) - f(x)|^2 dx \right] + c \sqrt{\mathbb{E}[(C_n - 1)^4] \mathbb{E}[C_n^{-4}]} \\
& + c \sqrt{\mathbb{E}[(1 - C_n)^2] \left\{ \mathbb{E}[C_n^{-2}] + \mathbb{E}[C_n^{-3}] + \mathbb{E}[C_n^{-4}] \right\}}.
\end{aligned}$$

Then, on the one hand, from the almost sure absolute boundedness of C_n and C_n^{-1} combined with Proposition 3.4, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mathbb{E}(C_n^j)$ with $j \in \{-4, -3, -2\}$ are bounded by a finite constant $M > 0$. On the other hand, for some arbitrarily small $\varepsilon > 0$, from the proof of Proposition 3.4, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we respectively have $\mathbb{E}[(C_n - 1)^2] < \varepsilon^2/(12Mc^2)$ and $\mathbb{E}[(C_n - 1)^4] < \varepsilon^2/(4c^2M)$. Therefore, taking $N = \max\{n_0, n_1\}$, for all $n \geq N$, the upper bound (3.6) provides the desired result of the proposition. \square

4. Simulation studies and an illustrative application. In this section, all numerical studies are performed in the context (non-)normalized associated-kernel estimators. Thus, we provide simulation results which are conducted for evaluating the performance of the proposed approaches. Computations have been done by using the R software of R Core Team [21]. The following numerical studies have two objectives concerning the simulation schemes and an application to original real data. We shall use the support of lognormal and Weibull kernel estimators.

4.1. *Numerical simulations.* Three following scenarios denoted by A, B and C are considered for the simulation studies.

- Scenario A is generated by using the gamma distribution

$$f_A(x) = xe^{-x}, \quad x \in \mathbb{R}_+;$$

- Scenario B comes from the Weibull distribution

$$f_B(x) = \frac{3}{2} \left(\frac{x}{2}\right)^2 \exp\left(-\left(\frac{x}{2}\right)^3\right), \quad x \in \mathbb{R}_+;$$

- Scenario C is from the lognormal distribution

$$f_C(x) = \frac{1}{0.25x\sqrt{2\pi}} \exp(-8(\log x)^2), \quad x \in \mathbb{R}_+^*.$$

We evaluate the performances of these continuous associated-kernel estimators with the cross-validation selections of the optimal bandwidth parameter. In fact, for all the considering associated-kernel estimators, $\alpha = 1$ and the optimal bandwidth of h by the cross-validation method is determined by

$$\tilde{h}_{cv} = \arg \min_{h>0} \left[\int_{x \in \mathbb{T}} \{\tilde{f}_n(x)\}^2 dx - \frac{2}{n} \sum_{i=1}^n \tilde{f}_{n,h,-i}(X_i) \right],$$

where $\tilde{f}_{n,h,-i}(X_i) = (n-1)^{-1} \sum_{\ell=1, \ell \neq i}^n K_{X_i,h}(X_\ell)$ is being computed as $\tilde{f}_n(X_i)$ without the observation X_i .

The efficiency of these estimators is examined via the empirical estimates of \widehat{C}_n , the integrated squared errors \widehat{ISE}_n (in normalized case) with

$$\widehat{C}_n := \frac{1}{N_{sim}} \sum_{t=1}^{N_{sim}} \int_{x \in \mathbb{T}} \tilde{f}_n(x) dx$$

and

$$\widehat{ISE}_n := \frac{1}{N_{sim}} \sum_{t=1}^{N_{sim}} \int_{x \in \mathbb{T}} \{\widehat{f}_n(x) - f(x)\}^2 dx,$$

where N_{sim} represents the number of replications and n corresponds to the sample size which shall be chosen to be small, medium and large.

Table 1 reports some empirical mean values of \widehat{C}_n and \widehat{ISE}_n with their standard deviations using $N_{sim} = 750$ replications from Scenarios A, B and C to the corresponding sample sizes $n = 100, 500, 1000$ having the support of lognormal and Weibull kernel estimators. For each given subsample and the continuous associated-kernel, we need to compute the related bandwidth \tilde{h}_{cv} through the cross-validation method in advance and use it to produce \tilde{f}_n , \widehat{C}_n , \widehat{f}_n and, therefore, \widehat{ISE}_n . With regard to the three scenarios, the estimate of \widehat{C}_n converges to 1 and \widehat{ISE}_n decreases when the sample size increases. However, it is to be noticed that the lognormal kernel estimator is more accurate than the one based on the Weibull one; especially, when the sample size is relatively large.

TABLE 1
Empirical mean values of \widehat{C}_n and \widehat{ISE}_n with their standard deviations in parentheses for lognormal and the Weibull kernels under Scenarios A, B and C.

	n	$\widehat{C}_{n, \text{Lognormal}}$	$\widehat{ISE}_{n, \text{Lognormal}}$	$\widehat{C}_{n, \text{Weibull}}$	$\widehat{ISE}_{n, \text{Weibull}}$
A	100	0.9504 (0.0197)	0.0106 (0.0083)	1.1774 (0.0585)	0.0161 (0.0098)
	500	0.9754 (0.0091)	0.0031 (0.0035)	1.0893 (0.0331)	0.0060 (0.0033)
	1000	0.9818 (0.0081)	0.0020 (0.0034)	1.0669 (0.0285)	0.0038 (0.0019)
B	100	0.9866 (0.0065)	0.0159 (0.0145)	1.0721 (0.0202)	0.0233 (0.0145)
	500	0.9938 (0.0030)	0.0043 (0.0032)	1.0406 (0.0129)	0.0080 (0.0037)
	1000	0.9956 (0.0026)	0.0026 (0.0020)	1.0321 (0.0111)	0.0050 (0.0022)
C	100	0.9879 (0.0065)	0.0337 (0.0299)	1.0364 (0.0113)	0.0554 (0.0396)
	500	0.9930 (0.0028)	0.0093 (0.0073)	1.0207 (0.0069)	0.0186 (0.0106)
	1000	0.9940 (0.0025)	0.0055 (0.0042)	1.0160 (0.0054)	0.0116 (0.0059)

Now, we illustrate both empirical distributions of $\sqrt{nh_n}\{\widetilde{f}_n(3) - f_A(3)\}$ (Figure 1) and $\sqrt{nh_n}\{\widehat{f}_n(3) - f_A(3)\}$ (Figure 2) for the lognormal and Weibull kernels over 5,000 replications of Scenario A with the sample size $n = 1,000$ and the choice of bandwidth $h_n = n^{-2/3}$. It is to observe that we have a similarity of behavior of the pointwise empirical distribution to both normalized and non-normalized estimators. This confirms the results of Theorems 3.3 and 3.6.

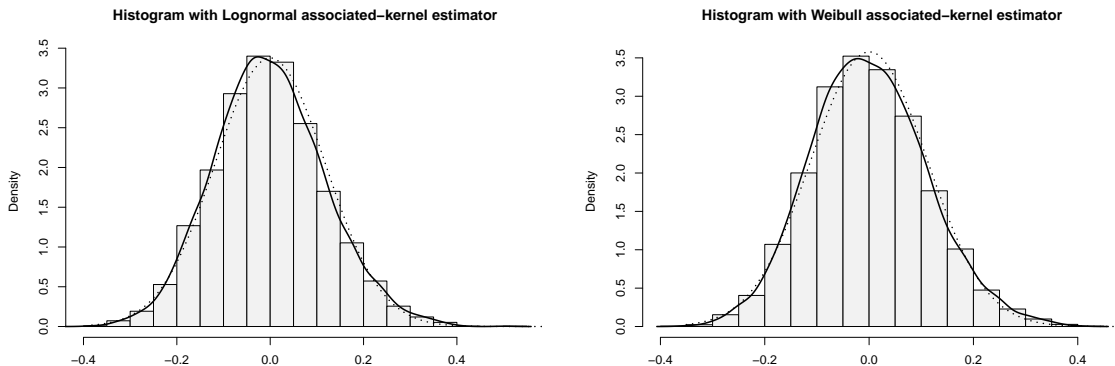


Fig 1: Empirical distributions of $\sqrt{nh_n}\{\widetilde{f}_n(3) - f_A(3)\}$ using the non-normalized lognormal (left) and Weibull (right) kernel estimators. The smoothed kernel density is displayed in full line, and the centered Gaussian density with the limit variance of Theorem 3.3 is plotted in the dotted line.

4.2. *An example of application on original data.* In this subsection, we compare the performance of our estimators on data of automobile claim amounts occurring between 2016 and 2020 provided by Covéa Affinity. This dataset is composed of $n = 76,414$ observations distributed between 3.6 and 6,985.51 EUR.

In Figure 3, we fit the empirical distribution of claim amounts using the normalized estimators, which are computed from the lognormal (solid line) and Weibull (dotted

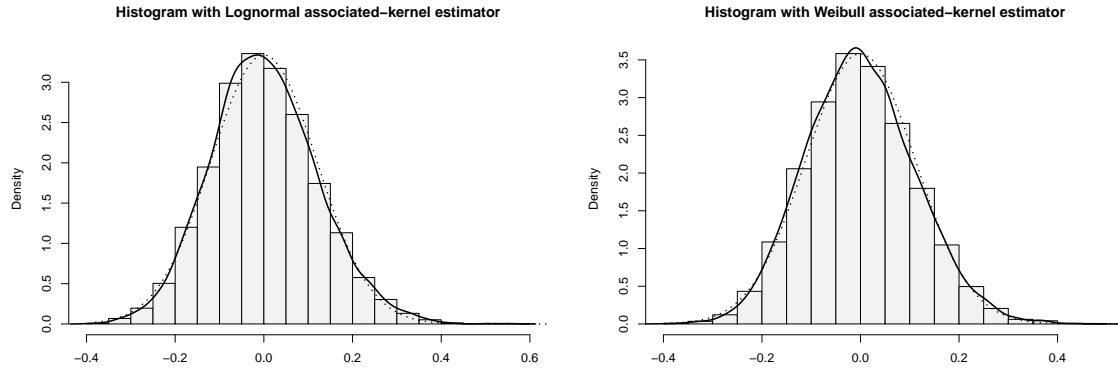


Fig 2: Empirical distributions of $\sqrt{nh_n}\{\widehat{f}_n(3) - f_A(3)\}$ using the normalized lognormal (left) and Weibull (right) kernel estimators. The smoothed kernel density is displayed in full line, and the centered Gaussian density with the limit variance of Theorem 3.6 is plotted in the dotted line.

line) kernels. The optimal bandwidth h is selected by the cross-validation method. It is pointed out that the lognormal kernel is more appropriate and provides a very satisfactory adjustment of the distribution of claim amounts (see also the below panel of Figure 3 for a zoom on the distribution of claim amounts between 3.6 and 800 EUR).

Table 2 confirms this finding for which other kernels (given in Section 2) have been implemented. Here, practical performances are examined via the cross-validation method and the empirical criterion of ISE : $\widehat{ISE}_0 := \int_0^\infty \{\widehat{f}_n(x) - f_0(x)\}^2 dx$, where $f_0(\cdot)$ is the empirical or naive estimator. Recall that the estimators corresponding to the last three kernels of Table 2 are normalized by construction (their C_n is equal to 1). We notice by referring to their \widehat{ISE}_0 that they remain less efficient than the estimator with the lognormal kernel but do slightly better than the one computed from the Weibull kernel.

Relying upon practical reasons, we evaluate the right tail of the distribution (from 1,000 EUR) of claim amounts using the different kernels of Table 2. These tail probabilities are estimated to be 0.1331, 0.1321, 0.1615, 0.1342, 0.1337 and 0.1344 for the empirical estimator f_0 , lognormal, Weibull, Gaussian, Gumbel and Laplace kernel estimations, respectively. Again, the lognormal kernel appears to be competitive and provides a very good fit to the tail distribution of this data set.

TABLE 2

Empirical values of \widehat{C}_n and \widehat{ISE}_0 of the normalized estimator of the distribution of claim amounts using different associated-kernels.

Associated-kernel	\widehat{C}_n	$\widehat{ISE}_0 (\times 10^4)$
Lognormal	0.9967	8.5482
Weibull	1.0700	9.1646
Gaussian	1.0000	9.0724
Gumbel	1.0000	9.0822
Laplace	1.0000	9.0383

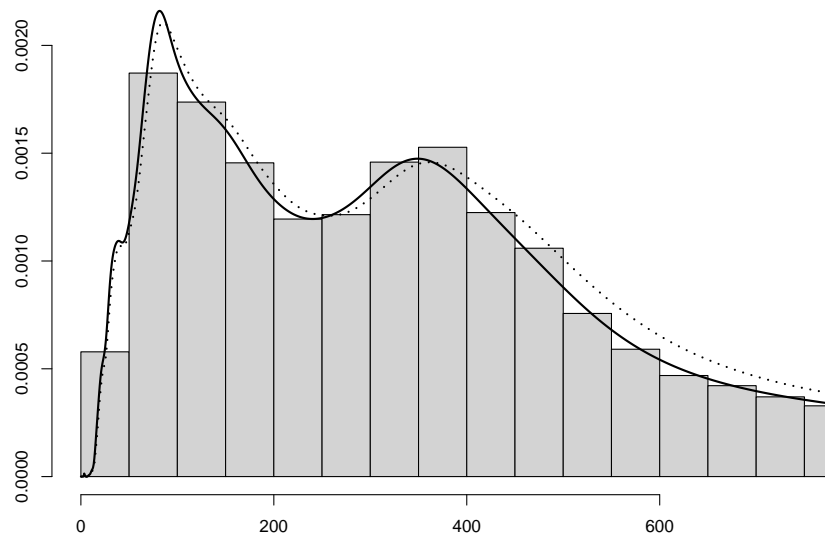
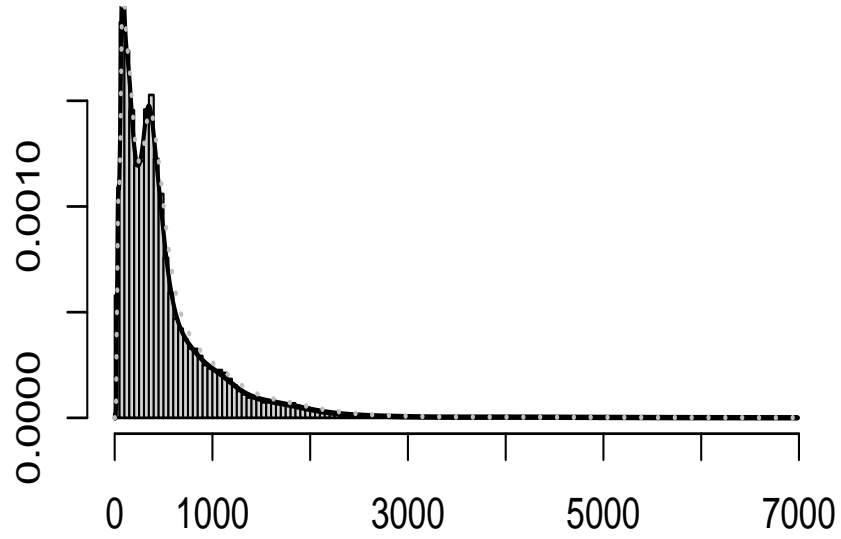


Fig 3: Empirical distribution of Covéa data and its estimates using both normalized lognormal (solid line) and Weibull (dotted line) kernel estimators. The graph below depicts a zoom of the region of claim amounts from 3.6 to 800 EUR of the graph on the top.

5. Proofs of Theorems 3.2 and 3.3 and of Proposition 3.4.

PROOF OF THEOREM 3.2. In order to obtain the theorem, for any $x \in \mathbb{T}$, we demonstrate the point-wise convergence to zero of the following mean-squared error

$$(5.1) \quad \mathbb{E} \left[\left(\widetilde{f}_n(x) - f(x) \right)^2 \right] = \left(\mathbb{E} \left[\widetilde{f}_n(x) - f(x) \right] \right)^2 + \text{Var} \left(\widetilde{f}_n(x) \right).$$

Considering the bias term in the r.h.s. of (5.1): from Proposition 3.1, we have

$$(5.2) \quad \text{Bias} := \mathbb{E} \left[\widetilde{f}_n(x) \right] - f(x) = \mathcal{O}(h_n^{\alpha/2})$$

and the convergence to zero of the bias.

Considering the variance term in the r.h.s. of (5.1): we have

$$\text{Var}(\widetilde{f}_n(x)) = \frac{1}{n} \text{Var}(K_{x,h_n}(X_1)) \leq \frac{1}{n} \mathbb{E} \left[K_{x,h_n}^2(X_1) \right].$$

In order to prove the convergence of this variance term, we demonstrate that

$$(5.3) \quad \mathbb{E} \left[K_{x,h_n}^2(X_1) \right] = f(x) \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy + \mathcal{O}(h_n^{-\alpha/2}).$$

Indeed, using the second Part (2.2) of Assumption (A₁^α), combined with the same arguments as for $\mathbb{C}_{1,n}$ and $\mathbb{C}_{2,n}$ in the proof of Proposition 3.1, we obtain

$$\begin{aligned} & \left| \mathbb{E} \left[K_{x,h_n}^2(X_1) \right] - f(x) \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy \right| = \left| \int_{y \in \mathbb{S}_x \cap \mathbb{T}} K_{x,h_n}^2(y) (f(y) - f(x)) dy \right| \\ & \leq \max_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}(y) \int_{y \in \mathbb{S}_x \cap \mathbb{T}} K_{x,h_n}(y) |f(y) - f(x)| dy \leq c_x h_n^{-\alpha/2}. \end{aligned}$$

Therefore, Equation (5.3) is verified. From this, combined with (A₂^α), one gets

$$\frac{1}{n} \mathbb{E} \left[K_{x,h_n}^2(X_1) \right] \leq c_x \frac{1}{nh_n^\alpha}$$

which leads to the convergence to zero of the variance since we assume that nh_n^α goes to infinity.

The proof is completed. \square

PROOF OF THEOREM 3.3. To begin with, let us rewrite our error as follows

$$(5.4) \quad \begin{aligned} & \sqrt{nh_n^\alpha} (\widetilde{f}_n(x) - f(x)) \\ & = \left(\frac{\widetilde{f}_n(x) - \mathbb{E}[\widetilde{f}_n(x)]}{\sqrt{\text{Var}(\widetilde{f}_n(x))}} \right) \sqrt{nh_n^\alpha \text{Var}(\widetilde{f}_n(x))} + \sqrt{nh_n^\alpha} \{ \mathbb{E}[\widetilde{f}_n(x)] - f(x) \}. \end{aligned}$$

Concerning the first term in the r.h.s. of (5.4): first, let us identify the limit of the normalized variance using the results in (5.3) and (5.2). In particular, we have

$$\begin{aligned} nh_n^\alpha \text{Var}(\widetilde{f}_n(x)) & = h_n^\alpha \text{Var}(K_{x,h_n}(X_1)) \\ & = h_n^\alpha \left\{ \mathbb{E} \left[K_{x,h_n}^2(X_1) \right] - \left(\mathbb{E} \left[K_{x,h_n}(X_1) \right] \right)^2 \right\} \end{aligned}$$

$$\begin{aligned}
&= f(x)h_n^\alpha \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy + \mathcal{O}(h_n^\alpha) - h_n^\alpha \left(\mathbb{E} \left[\widetilde{f}_n(x) \right] \right)^2 \\
(5.5) \quad &= f(x)h_n^\alpha \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy + \mathcal{O}(h_n^\alpha).
\end{aligned}$$

Therefore, Part (2.3) of Assumption (A_1^α) implies

$$(5.6) \quad nh_n^\alpha \text{Var}(\widetilde{f}_n(x)) \xrightarrow{n \rightarrow \infty} f(x) \lim_{n \rightarrow \infty} h_n^\alpha \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^2(y) dy < \infty.$$

Now, let $(Y_{n,i})$ be the row-wise i.i.d. triangular array defined by

$$Y_{n,i} = \frac{K_{x,h_n}(X_i) - \mathbb{E}[K_{x,h_n}(X_i)]}{\sqrt{n \text{Var}(K_{x,h_n}(X_i))}}, \quad i \in \{1, \dots, n\}.$$

It is clear that for all $n \geq 1$ and any $i \in \{1, \dots, n\}$,

$$\mathbb{E}[Y_{n,i}] = 0 \text{ and } \sum_{i=1}^n \mathbb{E}[Y_{n,i}^2] = 1.$$

Moreover, the Jensen inequality allows to write

$$\begin{aligned}
(5.7) \quad \sum_{i=1}^n \mathbb{E}[|Y_{n,i}|^3] &= \frac{\mathbb{E} \left[|K_{x,h_n}(X_1) - \mathbb{E}(K_{x,h_n}(X_1))|^3 \right]}{n^{1/2} \{ \text{Var}(K_{x,h_n}(X_1)) \}^{3/2}} \\
&\leq \frac{c \mathbb{E} \left[|K_{x,h_n}(X_1)|^3 \right]}{n^{1/2} \{ \text{Var}(K_{x,h_n}(X_1)) \}^{3/2}} = \frac{ch_n^{3\alpha/2} \mathbb{E} \left[K_{x,h_n}^3(X_1) \right]}{n^{1/2} \{ nh_n^\alpha \text{Var}(\widetilde{f}_n(x)) \}^{3/2}}.
\end{aligned}$$

Similar arguments as in (5.3) lead to

$$(5.8) \quad \mathbb{E} \left[K_{x,h_n}^3(X_1) \right] = f(x) \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^3(y) dy + \mathcal{O}(h_n^{-3\alpha/2}).$$

More precisely, using again the second Part (2.2) of Assumption (A_1^α) , combined with the same arguments as for $\mathbb{C}_{1,n}$ and $\mathbb{C}_{2,n}$ in the proof of Proposition 3.1, we can write

$$\begin{aligned}
&\left| \mathbb{E} \left[K_{x,h_n}^3(X_1) \right] - f(x) \int_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}^3(y) dy \right| = \left| \int_{y \in \mathbb{S}_x \cap \mathbb{T}} K_{x,h_n}^3(y) (f(y) - f(x)) dy \right| \\
&\leq \left(\max_{y \in \mathbb{T} \cap \mathbb{S}_x} K_{x,h_n}(y) \right)^2 \int_{y \in \mathbb{S}_x \cap \mathbb{T}} K_{x,h_n}(y) |f(y) - f(x)| dy \leq c_x h_n^{-3\alpha/2}.
\end{aligned}$$

Then, from Equation (5.8) and Part (2.3) from Assumption (A_1^α) , we get $\mathbb{E}[K_{x,h_n}^3(X_1)] = \mathcal{O}(h_n^{-2\alpha})$ which is combined with (5.6) and $nh_n^\alpha \rightarrow \infty$ give

$$\frac{h_n^{3\alpha/2} \mathbb{E} \left[K_{x,h_n}^3(X_1) \right]}{n^{1/2} \{ nh_n^\alpha \text{Var}(\widetilde{f}_n(x)) \}^{3/2}} \xrightarrow{n \rightarrow \infty} 0$$

and, the Lyapunov condition in (5.7) holds. According to the Lindeberg theorem, we obtain

$$\frac{\widetilde{f}_n(x) - \mathbb{E}[\widetilde{f}_n(x)]}{\sqrt{\text{Var}(\widetilde{f}_n(x))}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, 1).$$

Consequently, from (5.6) and using the Slutsky theorem, we get

$$(5.9) \quad \left(\frac{\tilde{f}_n(x) - \mathbb{E}[\tilde{f}_n(x)]}{\sqrt{\text{Var}(\tilde{f}_n(x))}} \right) \sqrt{nh_n^\alpha \text{Var}(\tilde{f}_n(x))} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}\left(0, f(x) \lim_{n \rightarrow \infty} h_n^\alpha \int_{y \in \mathbb{T} \cap \mathcal{S}_x} K_{x,h_n}^2(y) dy\right).$$

As for the second term in the r.h.s. of (5.4): directly followed from (5.2) and the assumption that $\sqrt{nh_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$, we have

$$(5.10) \quad \sqrt{nh_n^\alpha} \left| \mathbb{E}[\tilde{f}_n(x) - f(x)] \right| \leq c \sqrt{nh_n^\alpha} \xrightarrow[n \rightarrow \infty]{} 0,$$

which concludes the convergence to zero for the second term in the r.h.s. of (5.4) when n goes to infinity. This completes the proof. \square

PROOF OF PROPOSITION 3.4. For the proof of this proposition, first we prove the convergence in \mathbb{L}^2 , then we deduce its extension to the convergence in \mathbb{L}^4 . For this purpose, we rewrite the considered mean square error as the sum of two terms

$$(5.11) \quad \mathbb{E}[(C_n - 1)^2] = \text{Var}(C_n) + (\mathbb{E}[C_n] - 1)^2.$$

Considering the first term $\text{Var}(C_n)$ in (5.11), we have the following development under the i.i.d. property of $(X_i)_{1 \leq i \leq n}$:

$$(5.12) \quad \begin{aligned} \text{Var}(C_n) &= \mathbb{E} \left[\int_{x,y \in \mathbb{T}} \tilde{f}_n(x) \tilde{f}_n(y) dx dy \right] - \mathbb{E} \left[\int_{x \in \mathbb{T}} \tilde{f}_n(x) dx \right] \mathbb{E} \left[\int_{y \in \mathbb{T}} \tilde{f}_n(y) dy \right] \\ &= \int_{x,y \in \mathbb{T}} \left\{ \mathbb{E}[\tilde{f}_n(x) \tilde{f}_n(y)] - \mathbb{E}[\tilde{f}_n(x)] \mathbb{E}[\tilde{f}_n(y)] \right\} dx dy \\ &= \frac{1}{n^2} \int_{x,y \in \mathbb{T}} \text{Cov} \left(\sum_{i=1}^n K_{x,h_n}(X_i), \sum_{i=1}^n K_{y,h_n}(X_i) \right) dx dy \\ &= \frac{1}{n^2} \int_{x,y \in \mathbb{T}} \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(K_{x,h_n}(X_i), K_{y,h_n}(X_j)) dx dy \\ &= \frac{1}{n^2} \int_{x,y \in \mathbb{T}} \sum_{i=1}^n \text{Cov}(K_{x,h_n}(X_i), K_{y,h_n}(X_i)) dx dy \\ &= \frac{1}{n} \int_{x,y \in \mathbb{T}} \text{Cov}(K_{x,h_n}(X_1), K_{y,h_n}(X_1)) dx dy \\ &= \frac{1}{n} \int_{x \in \mathbb{T}} \text{Var}(K_{x,h_n}(X_1)) dx + \frac{1}{n} \int_{\substack{x,y \in \mathbb{T} \\ x \neq y}} \text{Cov}(K_{x,h_n}(X_1), K_{y,h_n}(X_1)) dx dy. \end{aligned}$$

By (A₂^α) and $\int_{\mathbb{T}} q(x)f(x)dx < \infty$, the first term in (5.12) satisfies

$$(5.13) \quad \begin{aligned} \frac{1}{n} \int_{x \in \mathbb{T}} \text{Var}(K_{x,h_n}(X_1))dx &\leq \frac{1}{n} \int_{y \in \mathbb{T}} \int_{x \in \mathbb{T}} K_{x,h_n}^2(y)dx f(y)dy \\ &\leq \frac{1}{nh_n^\alpha} \int_{y \in \mathbb{T}} q(y)f(y)dy \leq \frac{c}{nh_n^\alpha}, \end{aligned}$$

where c denotes our generic constant that can change from line to line. Similarly, using Assumption (A₂^α), one easily see that the second term in (5.12) is bounded as follows

$$(5.14) \quad \begin{aligned} \frac{1}{n} \left| \int_{\substack{x,y \in \mathbb{T} \\ x \neq y}} \text{Cov}(K_{x,h_n}(X_1), K_{y,h_n}(X_1)) dx dy \right| &\leq \frac{1}{n} \int_{z \in \mathbb{T}} \int_{\substack{x,y \in \mathbb{T} \\ x \neq y}} K_{x,h_n}(z) K_{y,h_n}(z) dx dy f(z) dz \\ &\quad + \frac{1}{n} \int_{z \in \mathbb{T}} \int_{z' \in \mathbb{T}} \int_{\substack{x,y \in \mathbb{T} \\ x \neq y}} K_{x,h_n}(z) K_{y,h_n}(z') dx dy f(z') dz' f(z) dz \\ &\leq \frac{c}{n} \int_{z \in \mathbb{T}} f(z) dz + \frac{c}{n} \int_{z \in \mathbb{T}} \int_{z' \in \mathbb{T}} f(z') f(z) dz' dz \leq \frac{c}{n}. \end{aligned}$$

Therefore, we conclude from (5.13) and (5.14) that $\text{Var}(C_n) \xrightarrow{n \rightarrow \infty} 0$.

As for the second term $(\mathbb{E}[C_n] - 1)^2$ in the right hand side of (5.11): from Proposition 3.1, for n large enough, $|\mathbb{E}[\tilde{f}_n(x) - f(x)]|$ is bounded by $g_n(x) h_n^{\alpha/2}$ for any $x \in \mathbb{T}$ and g_n is an integrable function on \mathbb{T} . Therefore, we get

$$|\mathbb{E}[C_n - 1]| \leq \int_{x \in \mathbb{T}} |\mathbb{E}[\tilde{f}_n(x) - f(x)]| dx \leq c h_n^{\alpha/2} \xrightarrow{n \rightarrow \infty} 0.$$

This completes the prove for the convergence in \mathbb{L}^2 of C_n to 1.

Now, by the convergence in \mathbb{L}^2 , there exists a constant $M > 0$ such that $\mathbb{P}(|C_n| \leq M) = 1$ for n large enough. In view of the almost sure absolute boundedness of C_n , one gets $\mathbb{E}[|C_n|^k] \leq M^k$ for any $k \in \mathbb{N}^*$. Combined with the Cauchy-Schwarz inequality, we successively have

$$\begin{aligned} \text{Var}(C_n^2) &= \mathbb{E}[C_n^4 - 1] - \left\{ (\mathbb{E}[C_n^2])^2 - 1 \right\} \\ &\leq \left| \mathbb{E}[(C_n - 1)(C_n + 1)(C_n^2 + 1)] \right| + (\mathbb{E}[C_n^2] + 1) |\mathbb{E}[(C_n - 1)(C_n + 1)]| \\ &\leq \sqrt{\mathbb{E}[(C_n - 1)^2]} \left\{ \sqrt{\mathbb{E}[(C_n + 1)^2(C_n^2 + 1)^2]} + (\mathbb{E}[C_n^2] + 1) \sqrt{\mathbb{E}[(C_n + 1)^2]} \right\} \\ &\leq c \sqrt{\mathbb{E}[(C_n - 1)^2]} \end{aligned}$$

and

$$\text{Cov}(C_n^2, C_n) = \mathbb{E}[C_n^2(C_n - \mathbb{E}[C_n])] \leq \sqrt{\mathbb{E}[C_n^4] \text{Var}(C_n)} \leq c \sqrt{\text{Var}(C_n)},$$

where c denotes our generic constant that can change from line to line. Write again the considering fourth-moment as follows

$$\begin{aligned}\mathbb{E}\left[(C_n - 1)^4\right] &= \text{Var}\left\{(C_n - 1)^2\right\} + \left(\mathbb{E}\left[(C_n - 1)^2\right]\right)^2 \\ &= \text{Var}(C_n^2) + 4\text{Var}(C_n) - 4\text{Cov}(C_n^2, C_n) + \left(\mathbb{E}\left[(C_n - 1)^2\right]\right)^2.\end{aligned}$$

From the above arguments, when n is large enough, we obtain

$$\mathbb{E}\left[(C_n - 1)^4\right] \leq c \sqrt{\mathbb{E}\left[(C_n - 1)^2\right]} + 4\text{Var}(C_n) + 4c \sqrt{\text{Var}(C_n)} + \left(\mathbb{E}\left[(C_n - 1)^2\right]\right)^2.$$

Hence, using the convergence in \mathbb{L}^2 proven above, $\text{Var}(C_n)$ and $\mathbb{E}[(C_n - 1)^2]$ converge to zeros as n approaches infinity, we complete the proof of the Proposition 3.4. \square

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