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Sensitivity Analysis of the Set of Sustainable Thresholds

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Abstract

In the context of constrained control-systems, the Set of Sustainable Thresholds plays in a sense the role of a dual object to the so-called Viability Kernel, because it describes all the thresholds that must be satisfied by the state of the system along a time interval, for a prescribed initial condition. This work aims at analyzing the sensitivity of the Set of Sustainable Thresholds, when it is seen as a set-valued map that depends on the initial position. In this regard, we investigate semicontinuity and Lipschitz continuity properties of this mapping, and we also study several contexts when the Set of Sustainable Thresholds is convex-valued.

Keywords: Set-valued maps, Set of Sustainable Thresholds, Discrete-time systems, Semicontinuity, Lipschitz continuity, Convexity

MSC Classification: 49J53 , 37N35 , 93C55

1 Introduction

This paper aims at contributing in the understanding of the so called *Set of Sustainable Thresholds*, which roughly speaking corresponds to the set of all possible parameters $c \in \mathbb{R}^m$, called the *thresholds*, for which the evolution of a dynamical system remains

viable for a prescribed set of constraints, which in our setting will take the form

$$g(k, x, u) \leq c, \quad \forall k \in \llbracket 0 : T \rrbracket := \{0, \dots, T\},$$

for a given mapping $g : \llbracket 0 : T \rrbracket \times \mathbb{R}^d \times \mathbf{U} \rightarrow \mathbb{R}^m$ and a given nonempty set \mathbf{U} . Here and in the sequel, the inequalities are understood in a component-wise sense.

The Set of Sustainable Thresholds plays in a sense the role of a dual object to the so-called *Viability Kernel*. To be more precise, in Viability Theory (see, e.g. [1]) one is concerned with the question of finding all the possible initial positions for which the constraints are satisfied for at least one control; in that setting the thresholds are fixed. The Set of Sustainable Thresholds is an object that focuses on a converse question: assume the initial position is known, find all the possible thresholds for which the system is viable for at least one control. In some contexts, such as in natural resources management, the sustainability question is more appropriate than the viability one, because usually one has an estimate of how many resources are in an ecosystem, so the initial position is already prescribed and can not be modified at will. However, in the same setting, decision-makers can fix different values for payoffs, scores or quality indicators, and put them in the form of the set of constraints described above. In this case, the decision-makers have more freedom for choosing these parameters, i.e. the thresholds.

The Set of Sustainable Thresholds has been studied in the literature from several points of view. For example, in [2], motivated by an epidemiology model, a method for computing this set for controlled cooperative models was investigated; see also [3–6]. In [7], a detailed study of the Set of Sustainable Thresholds was reported concerning the problem of characterizing their Pareto boundaries, and so forth, the set itself. This study was done considering the initial position as given; see also [8] for an extension to dynamics with uncertainty. We also mention [6] where another approach to compute the Pareto boundaries of the Set of Sustainable Thresholds was presented.

Our task in this paper is to study how the Set of Sustainable Thresholds changes with respect to the initial position. In mathematical terms this means studying the continuity properties of the set-valued map associated with the Set of Sustainable Thresholds. We focus on lower semicontinuity and Lipschitz continuity, and we prove that under mild condition both features can be ensured. In this paper we also discuss cases when the Set of Sustainable Thresholds is convex-valued. It is well-known that the viability kernel has convex images in the linear-convex setting. Analogously, we prove that it is also the case for the Set of Sustainable Thresholds. Furthermore, we prove that that fact also holds if some generalized notions of convexity and monotonicity are considered.

This paper is organized as follows: In section 2, we describe the dynamical systems studied in this paper, and present basic properties of the images of the Set of Sustainable Thresholds (closedness and convexity). In section 3, we focus on the sensitivity analysis, and study the continuity properties mentioned above. Finally, in section 4 we introduce a new object, called the *Set of Sustainable and Attainable Thresholds*, which can be thought as the core of the Set of Sustainable Thresholds and we study its regularity properties.

2 Mathematical background and basic properties

The focus of this paper will be on discrete-time dynamical systems. Feasible controls in our setting are all the functions $\mathbf{u} = \llbracket 0 : T \rrbracket \rightarrow \mathbf{U}$, which can be seen as *sequences of inputs* $\mathbf{u}(0), \dots, \mathbf{u}(T) \in \mathbf{U}$. The collection of all feasible controls will be denoted by \mathcal{U} . Here and in what follows, $\llbracket p : q \rrbracket := \{p, p+1, \dots, q\}$, that is, it stands for the collection of all integers between p and q (inclusive), assuming always that $p < q$. The time horizon $T > 0$ is a given nonnegative fixed integer.

For a given dynamics $F : \llbracket 0 : T \rrbracket \times \mathbb{R}^d \times \mathbf{U} \rightarrow \mathbb{R}^d$, a given initial condition $\xi \in \mathbb{R}^d$ and a given control $\mathbf{u} \in \mathcal{U}$, we are concerned with functions $x : \llbracket 0 : T+1 \rrbracket \rightarrow \mathbb{R}^d$ solution of the dynamical system

$$x(k+1) = F(k, x(k), \mathbf{u}(k)), \quad \forall k \in \llbracket 0 : T \rrbracket, \quad x(0) = \xi. \quad (D_{\xi}^{\mathbf{u}})$$

A solution of this system is uniquely determined by the initial position and the control, and therefore it is denoted by $\mathbf{x}_{\xi}^{\mathbf{u}}$.

Mathematically speaking, the Set of Sustainable Thresholds (SST for short), denoted by $\mathbb{S}(\xi)$ in the sequel, is defined in the following way:

$$\mathbb{S}(\xi) := \{c \in \mathbb{R}^m \mid \exists \mathbf{u} \in \mathcal{U} \text{ such that } g(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)) \leq c, \quad \forall k \in \llbracket 0 : T \rrbracket\}$$

where, as pointed out in the introduction, $g : \llbracket 0 : T \rrbracket \times \mathbb{R}^d \times \mathbf{U} \rightarrow \mathbb{R}^m$ is given.

Our task in this paper is to provide a sensitivity analysis for the SST. In other words, we aim at understanding how the set $\mathbb{S}(\xi)$ changes with respect to the variable ξ . Therefore, the SST will be deemed as a set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ and we will study continuity and regularity properties of this set-valued map. Properties such as convexity, closedness, semi-continuity and Lipschitz continuity will be investigated.

In this work, and in compliance with [7], we assume that the data of the dynamical system with mixed constraints satisfy the following basic conditions, which we term *standing assumptions*:

- (H1) $F(k, \cdot, \cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in \llbracket 0 : T \rrbracket$
- (H2) $g(k, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in \llbracket 0 : T \rrbracket$.
- (H3) \mathbf{U} is a nonempty compact metric space.

In the next section, these assumptions may be accordingly strengthened to get stronger regularity conditions.

2.1 Basic properties

In this part we present some basic regularity properties that are satisfied by the graph and the images of the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$. We also show that, under suitable additional assumptions, this set-valued map also satisfies a monotonicity property.

2.1.1 Closedness

The first property we study is closedness.

Proposition 1. *The set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has closed graph with closed and nonempty images.*

Proof. Consider two sequences, $\{\xi_n\}_n \subset \mathbb{R}^d$ and $\{c_n\}_n \subset \mathbb{R}^m$, so that $c_n \in \mathbb{S}(\xi_n)$ for any $n \in \mathbb{N}$, with $\xi_n \rightarrow \xi$ and $c_n \rightarrow c$.

By definition, there is a sequence $\{\mathbf{u}_n\}_n \subset \mathcal{U}$ such that

$$g\left(k, \mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k), \mathbf{u}_n(k)\right) \leq c_n, \quad \forall k \in \llbracket 0 : T \rrbracket, \forall n \in \mathbb{N}.$$

Since $\mathcal{U} \cong \mathbf{U}^{T+1}$, by Tychonoff's theorem we have that \mathcal{U} is compact, and so, passing into a subsequence (which we do not relabel), we can assume that there is $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{u}_n \rightarrow \mathbf{u}$; that is $\mathbf{u}_n(k) \rightarrow \mathbf{u}(k)$ for any $k \in \llbracket 0 : T \rrbracket$. Since we also have that $\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(0) = \xi_n$, it follows that $\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(0) \rightarrow \xi$. Thus, by the Induction Principle and by **(H1)** (the continuity of the dynamics), it is not difficult to see that for any $k \in \llbracket 0 : T \rrbracket$ fixed we have

$$\mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k+1) = F\left(k, \mathbf{x}_{\xi_n}^{\mathbf{u}_n}(k), \mathbf{u}_n(k)\right) \rightarrow F\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) = \mathbf{x}_{\xi}^{\mathbf{u}}(k+1).$$

Consequently, in the light of **(H2)** and the fact that $c_n \rightarrow c$, we get

$$g\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) \leq c, \quad \forall k \in \llbracket 0 : T \rrbracket,$$

or in other words, $c \in \mathbb{S}(\xi)$, and so the graph of \mathbb{S} is closed. It is then a straightforward matter to check that $\mathbb{S}(\xi)$ is closed for each $\xi \in \mathbb{R}^d$ fixed.

Finally, notice that for any $\xi \in \mathbb{R}^d$ and any $\mathbf{u} \in \mathcal{U}$, the threshold $c = (c_1 \dots, c_m)$ given by

$$c_i = \max_{k \in \llbracket 0 : T \rrbracket} \left(g_i\left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k)\right) \right),$$

belongs to $\mathbb{S}(\xi)$. This means that for any $\xi \in \mathbb{R}^d$, we have that $\mathbb{S}(\xi) \neq \emptyset$. \square

2.1.2 Convexity

Similarly as for the viability kernel, when the system is linear-convex, it can be proven that the set-valued mapping \mathbb{S} verifies some convexity properties, as described below.

Proposition 2. *Assume that the standing assumptions are strengthened as follows:*
(H1) for any $k \in \llbracket 0 : T \rrbracket$ fixed, $F(k, x, u) = A(k)x + B(k)u$, where $A(k)$ and $B(k)$ are matrices of dimension $d \times d$ and $d \times n$, respectively.

(H2) for any $k \in \llbracket 0 : T \rrbracket$ and $i \in \llbracket 1 : m \rrbracket$ fixed, $g_i(k, \cdot)$ is convex on $\mathbb{R}^d \times \mathbf{U}$.

(H3) $\mathbf{U} \subset \mathbb{R}^n$ is a nonempty convex subset.

Then, the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has convex graph with convex images.

Proof. Let $\xi_1, \xi_2 \in \mathbb{R}^d$, and $c_1, c_2 \in \mathbb{R}^m$, such that $c_1 \in \mathbb{S}(\xi_1)$ and $c_2 \in \mathbb{S}(\xi_2)$. Let $\lambda \in [0, 1]$. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}$, be such that

$$g\left(k, \mathbf{x}_{\xi_i}^{\mathbf{u}_i}(k), \mathbf{u}_i(k)\right) \leq c_i, \quad \forall k \in \llbracket 0 : T \rrbracket, \forall i = 1, 2.$$

By convexity of the constraints mapping we have

$$g\left(k, \lambda \mathbf{x}_{\xi_1}^{\mathbf{u}_1}(k) + (1 - \lambda) \mathbf{x}_{\xi_2}^{\mathbf{u}_2}(k), \lambda \mathbf{u}_1(k) + (1 - \lambda) \mathbf{u}_2(k)\right) \leq \lambda c_1 + (1 - \lambda) c_2, \quad \forall k \in \llbracket 0 : T \rrbracket.$$

By convexity of \mathbf{U} , we have that $\lambda \mathbf{u}_1(k) + (1 - \lambda) \mathbf{u}_2(k) \in \mathbf{U}$ for any $k \in \llbracket 0 : T \rrbracket$, and since the dynamics is linear, we also have that

$$\mathbf{x}_{\lambda \xi_1 + (1-\lambda)\xi_2}^{\lambda \mathbf{u}_1 + (1-\lambda)\mathbf{u}_2}(k) = \lambda \mathbf{x}_{\xi_1}^{\mathbf{u}_1}(k) + (1 - \lambda) \mathbf{x}_{\xi_2}^{\mathbf{u}_2}(k), \quad \forall k \in \llbracket 0 : T + 1 \rrbracket.$$

From here we conclude that

$$\lambda c_1 + (1 - \lambda) c_2 \in \mathbb{S}(\lambda \xi_1 + (1 - \lambda) \xi_2).$$

Therefore, the graph of \mathbb{S} is convex. Notice that by taking $\xi_1 = \xi_2$ we get that the images of \mathbb{S} are convex as well. \square

In Proposition 2 it is fundamental that the control space \mathbf{U} is convex as the following example demonstrates.

Example 1. Take $F(k, x, u) = u$, $g_1(k, x, u) = -u$ and $g_2(k, x, u) = x$. Notice that $(\tilde{\mathbf{H}}1)$ and $(\tilde{\mathbf{H}}2)$ hold. Take $\mathbf{U} = \{-1, 1\}$, which is not convex, the final horizon $T > 2$ and the initial position $\xi > -1$. There are three types of admissible controls that we need to analyze:

- $\mathbf{u}(k) = -1$ for any $k \in \llbracket 0 : T \rrbracket$: Here we have that the minimal value that c_1 can take is $c_1^* = 1$, and since $\mathbf{x}_{\xi}^{\mathbf{u}}(k+1) = -1$ for any $k \in \llbracket 0 : T \rrbracket$ the minimal value that c_2 can take is $c_2^* = \xi$, because $\mathbf{x}_{\xi}^{\mathbf{u}}(0) = \xi > -1$. Therefore, $(1, \xi) \in \mathbb{S}(\xi)$.
- $\mathbf{u}(k) = 1$ for any $k \in \llbracket 0 : T \rrbracket$: Here we have that the minimal value that c_1 can take is $c_1^* = -1$, and since $\mathbf{x}_{\xi}^{\mathbf{u}}(k+1) = 1$ for any $k \in \llbracket 0 : T \rrbracket$ the minimal value that c_2 can take is $c_2^* = 1$. Thus, $(-1, 1) \in \mathbb{S}(\xi)$.
- \mathbf{u} is not constant: : Here we have that the minimal value that c_1 can take is $c_1^* = 1$, because $\mathbf{u}(k) = -1$ for some $k \in \llbracket 0 : T \rrbracket$. Notice that if $\mathbf{u}(j) = 1$ for some $j \in \llbracket 0 : T - 1 \rrbracket$, the minimal value that c_2 can take is $c_2^* = 1$, otherwise it is $c_2^* = \xi$. Notice that $(-1, 1), (1, 0) \in \mathbb{S}(0)$, but $(0, \frac{1}{2}) \notin \mathbb{S}(0)$. If it was the case, then the associated control must satisfy $\mathbf{u}(k) \geq 0$ for any $k \in \llbracket 0 : T \rrbracket$, and so \mathbf{u} will be the control constantly equal to 1. But, we have seen that in this case the minimal value that c_2 can take is $c_2^* = 1$. It follows then that $\mathbb{S}(0)$ is not a convex set.

As a matter of fact, one can check that for $\xi > -1$ we have

$$\mathbb{S}(\xi) = \{(c_1, c_2) \in \mathbb{R}^2 \mid c_1 \geq -1, c_2 \geq 1\} \cup \{(c_1, c_2) \in \mathbb{R}^2 \mid c_1 \geq 1, c_2 \geq \xi\}.$$

Therefore, $\mathbb{S}(\xi)$ is not a convex set for any $\xi > -1$.

Example 2. The assumption on convexity over the constraints mapping $(\tilde{\mathbf{H}}2)$ may seem rather strong at first sight. However, it appears somewhat naturally in renewable resource management models. Consider for instance an example taken from [9], where each x_k represents the stock of a renewable resource at a time period k and the control $\mathbf{u}(k)$ is the catch. In this model a regulatory agency has the social objective of ensuring a minimal stock and a minimal catch. In mathematical terms this means that for some thresholds x^{lim} and h^{lim} we must have:

$$-x_k \leq -x^{\text{lim}} \quad \text{and} \quad -\mathbf{u}(k) \leq -h^{\text{lim}}, \quad \forall k \in \llbracket 0 : T \rrbracket.$$

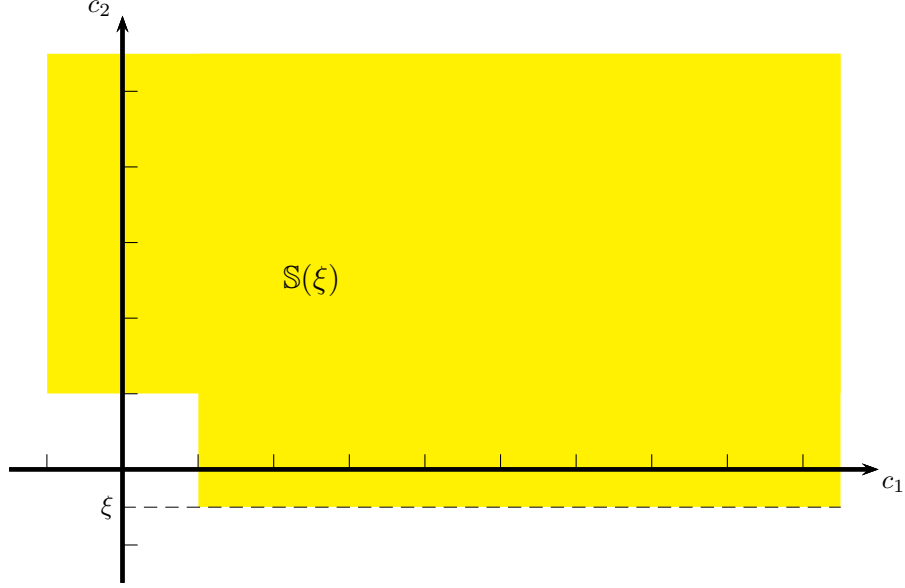


Fig. 1 Sketch of the SST of Example 1.

Notice that this example lies within the setting of $(\tilde{H}2)$.

On the other hand, the assumption over the dynamics $(\tilde{H}1)$ may indeed rule out several interesting cases of study; it is not very often that dynamics in natural resource management are linear. Fortunately, the convexity of the STT (its images) can still be preserved if linearity is replaced with some notions of generalized convexity and monotonicity for vector fields.

We now present a criterion, which covers several cases of interest, that allows to ensure that the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has convex images. The underlying idea is the following. From [7], we have that $\mathbb{S}(\xi) = \{x \in \mathbb{R}^m \mid \omega_\xi(c) \leq 0\}$, where

$$\omega_\xi(c) = \min_{\mathbf{u} \in \mathcal{U}} \max_{k \in \llbracket 0:T \rrbracket} \max_{i \in \llbracket 1:m \rrbracket} (g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)) - c_i), \quad \forall c \in \mathbb{R}^m.$$

Therefore, the convexity of the SST can be obtained from studying the convexity of the function $c \mapsto \omega_\xi(c)$. To do so in a nonlinear dynamics framework, we need to introduce some definitions.

Definition 1. For a nonempty set $K \subset \mathbb{R}^d$, let us consider the relation \preceq_K on \mathbb{R}^d defined as follows

$$x \preceq_K y \iff y - x \in K.$$

- Given a nonempty convex set $S \subset \mathbb{R}^p$, we say that a vector field $\Psi : S \rightarrow \mathbb{R}^d$ is K -convex if

$$\Psi(\lambda x + (1 - \lambda)y) \preceq_K \lambda \Psi(x) + (1 - \lambda)\Psi(y), \quad \forall x, y \in S, \lambda \in [0, 1].$$

Similarly, we say that it is K -concave if

$$\lambda\Psi(x) + (1 - \lambda)\Psi(y) \preceq_K \Psi(\lambda x + (1 - \lambda)y), \quad \forall x, y \in S, \lambda \in [0, 1].$$

- A mapping $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to be a K -monotone vector field if

$$\Phi(x) \preceq_K \Phi(y), \quad \forall x, y \in \mathbb{R}^d, \text{ such that } x \preceq_K y.$$

- A function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be a K -monotone function if

$$\varphi(x) \leq \varphi(y), \quad \forall x, y \in \mathbb{R}^d, \text{ such that } x \preceq_K y.$$

Example 3. The monotonicity character of the dynamics is a rather common property encountered in applications. Consider for instance a particular case of the renewable resource management model mentioned in Example 2, where the dynamical system is then given by

$$x_{k+1} = f(x_k) - \mathbf{u}(k), \quad \forall k \in k \in \llbracket 0 : T \rrbracket.$$

where f is the so-called Beverton-Holt population dynamics, which is given by

$$f(x) = (1 + r)x \left(1 + \frac{r}{\kappa}x\right)^{-1}. \quad (1)$$

The parameters r and κ are both positive, the first one being the intrinsic growth and the second one the carrying capacity κ . It is not difficult to see that f is strictly increasing and concave on $(-\frac{\kappa}{r}, +\infty)$. In particular, it verifies the conditions of Theorem 4 stated below for $K = \mathbb{R}_+$.

Remark 1. If $K = \mathbb{R}_+^d$, then K -convexity, K -concavity and K -monotonicity mean component-wise convexity, concavity and monotonicity in the usual sense on \mathbb{R} , respectively. Note as well that in the case $K = \{0\}$, the K -monotonicity is trivial.

It is well-known that if K is a closed convex cone, \preceq_K is a reflexive and transitive relation, which is also antisymmetric if for instance K is pointed (K contains no lines). However, for our purposes, none of these features are required.

Let us now present the main results of this section.

Theorem 3. Let $K \subset \mathbb{R}^d$ be a given nonempty set and assume that

- (A1) for any $k \in \llbracket 0 : T \rrbracket$ fixed, $F(k, \cdot)$ is K -convex and $F(k, \cdot, u)$ is a K -monotone vector field for any $u \in \mathbf{U}$ fixed;
- (A2) for any $k \in \llbracket 0 : T \rrbracket$ and $i \in \llbracket 1 : m \rrbracket$ fixed, $g_i(k, \cdot)$ is convex on $\mathbb{R}^d \times \mathbf{U}$ and $g_i(k, \cdot, u)$ is a K -monotone function for any $u \in \mathbf{U}$ fixed;
- (A3) $\mathbf{U} \subset \mathbb{R}^n$ is a nonempty convex subset.

Then, for each $\xi \in \mathbb{R}^d$ the set $\mathbb{S}(\xi)$ is convex and $\mathbb{S}(\xi) \subset \mathbb{S}(\xi')$ whenever $\xi' \preceq_K \xi$.

Proof. The conclusion follows from proving that $c \mapsto \omega_\xi(c)$ is convex and $\omega_{\xi'}(c) \leq \omega_\xi(c)$ for any $c \in \mathbb{R}^m$ fixed and $\xi', \xi \in \mathbb{R}^d$ such that $\xi' \preceq_K \xi$.

Let us begin by recalling that the value function ω_ξ can be computed by means of the Dynamic Programming Principle. Indeed, define for any $n \in \llbracket 0 : T \rrbracket$

$$V_n^c(\xi) = \min_{\mathbf{u} \in \mathcal{U}} \left\{ \max_{k=n, \dots, T} \Phi^c(k, x_k, \mathbf{u}(k)) \mid x_{k+1} = F(k, x_k, \mathbf{u}(k)), k \in \llbracket n : T \rrbracket, x_n = \xi \right\},$$

where

$$\Phi^c(k, x, u) = \max_{i=1, \dots, m} g_i(k, x, u) - c_i.$$

Notice that $V_0^c(\xi) = \omega_\xi(c)$ and $V_T^c(\xi) = \min_{u \in \mathbf{U}} \Phi^c(T, \xi, u)$. From [7, Proposition 3] we have

$$V_n^c(\xi) = \min_{u \in \mathbf{U}} \max \{ V_{n+1}^c(F(n, \xi, u)), \Phi^c(n, \xi, u) \}. \quad (2)$$

From (A2), we have that for any $k \in \llbracket 0 : T \rrbracket$ fixed, the mapping $(c, \xi, u) \mapsto \Phi^c(k, \xi, u)$ is convex, because each function $(c, x, u) \mapsto g_i(k, x, u) - c_i$ is so. Moreover, for any given $u \in \mathbf{U}$ and $c \in \mathbb{R}^m$, the function $\xi \mapsto \Phi^c(k, \xi, u)$ is K -monotone because each function $g_i(k, \cdot, u)$ has that property. Indeed, it is enough to notice that, given $x, y \in \mathbb{R}^d$ such that $x \preceq_K y$ we have

$$g_i(k, x, u) - c_i \leq g_i(k, y, u) - c_i \leq \Phi^c(k, y, u), \quad \forall i \in \llbracket 1 : m \rrbracket.$$

It follows then that the mapping $(c, \xi) \mapsto V_n^c(\xi)$ is convex and the function $\xi \mapsto V_n^c(\xi)$ is K -monotone for any given $c \in \mathbb{R}^m$; the last statement comes from observing that if $\xi', \xi \in \mathbb{R}^d$ are such that $\xi' \preceq_K \xi$, we have

$$V_n^c(\xi') \leq \Phi^c(k, \xi', u) \leq \Phi^c(k, \xi, u), \quad \forall u \in \mathbf{U}.$$

Assume by induction that for some $n \in \llbracket 0 : T - 1 \rrbracket$ the mapping $(c, \xi) \mapsto V_{n+1}^c(\xi)$ is convex and $\xi \mapsto V_{n+1}^c(\xi)$ is a K -monotone function. We claim that the mapping $(c, \xi, u) \mapsto V_{n+1}^c(F(n, \xi, u))$ is convex. Indeed, this is a direct consequence of the fact that $(\xi, u) \mapsto F(n, \xi, u)$ is K -convex and the induction hypothesis: Observe that for any $c \in \mathbb{R}^d$, $x, y \in \mathbb{R}^d$, $u, v \in \mathbf{U}$ and $\lambda \in [0, 1]$ we have that

$$V_{n+1}^c(F(n, \lambda x + (1 - \lambda)y, \lambda u + (1 - \lambda)v)) \leq V_{n+1}^c(\lambda F(n, x, u) + (1 - \lambda)F(n, y, v)).$$

Evaluating at $c = \lambda a + (1 - \lambda)b$ for $a, b \in \mathbb{R}^m$ and using the convexity of the mapping $(c, \xi) \mapsto V_{n+1}^c(\xi)$, we get the claim. Moreover, since $\xi \mapsto F(n, \xi, u)$ is a K -monotone vector field, by the induction hypothesis it follows that $\xi \mapsto V_{n+1}^c(F(n, \xi, u))$ is a K -monotone function:

$$\xi' \preceq_K \xi \implies F(n, \xi', u) \preceq_K F(n, \xi, u) \implies V_{n+1}^c(F(n, \xi', u)) \leq V_{n+1}^c(F(n, \xi, u)).$$

Therefore, by (2), it follows that $(c, \xi) \mapsto V_n^c(\xi)$ is convex and $\xi \mapsto V_n^c(\xi)$ is a K -monotone function. The last affirmation comes from the fact that the mapping $\xi \mapsto \max \{ V_{n+1}^c(F(n, \xi, u)), \Phi^c(n, \xi, u) \}$ is a K -monotone function for $u \in \mathbf{U}$ fixed. Therefore, the conclusion follows from the Induction Principle. \square

Remark 2. Notice that the data in Example 1 satisfy the assumptions (A1) and (A2) with $K = \{0\}$. Thus the convexity assumption (A3) is also essential for the validity of Proposition 3.

Similar arguments as the ones presented above can be used to prove the following result. Notice that in this theorem, the monotonicity behavior of the mapping $\xi \mapsto \mathbb{S}(\xi)$ changes with respect to Theorem 3.

Theorem 4. Let $K \subset \mathbb{R}^d$ be a given nonempty set and assume that

- (A1) for any $k \in \llbracket 0 : T \rrbracket$ fixed, $F(k, \cdot)$ is K -concave and $F(k, \cdot, u)$ is a K -monotone vector field for any $u \in \mathbf{U}$ fixed;
- (A2) for any $k \in \llbracket 0 : T \rrbracket$ and $i \in \llbracket 1 : m \rrbracket$ fixed, $g_i(k, \cdot)$ is convex on $\mathbb{R}^d \times \mathbf{U}$ and $-g_i(k, \cdot, u)$ is a K -monotone function for any $u \in \mathbf{U}$ fixed;
- (A3) $\mathbf{U} \subset \mathbb{R}^n$ is a nonempty convex subset.

Then, for each $\xi \in \mathbb{R}^d$ the set $\mathbb{S}(\xi)$ is convex and $\mathbb{S}(\xi') \subset \mathbb{S}(\xi)$ whenever $\xi' \preceq_K \xi$.

Proof. The idea in this case is to prove as well that $c \mapsto \omega_\xi(c)$ is convex, but now that $\omega_\xi(c) \leq \omega_{\xi'}(c)$ for any $c \in \mathbb{R}^m$ fixed and $\xi', \xi \in \mathbb{R}^d$ such that $\xi' \preceq_K \xi$.

The proof is essentially the same as for Theorem 3, however noticing that in this case for any given $u \in \mathbf{U}$ and $c \in \mathbb{R}^m$, the function $\xi \mapsto -\Phi^c(k, \xi, u)$ is K -monotone, which implies that the function $\xi \mapsto -V_\xi^c(\xi)$ is K -monotone for any given $c \in \mathbb{R}^m$.

Therefore, the Induction Hypothesis in this case says that for $n \in \llbracket 0 : T - 1 \rrbracket$ the mapping $(c, \xi) \mapsto V_{n+1}^c(\xi)$ is convex and $\xi \mapsto -V_{n+1}^c(\xi)$ is a K -monotone function. The conclusion then is obtained arguing as in the proof of Theorem 3. \square

3 Sensitivity analysis of the SST

In this part we focus on the sensitivity analysis of the SST, by studying several notions of continuity for the set-valued map $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$.

3.1 Lower semicontinuity

Recall that a set-valued map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is said to be *lower semicontinuous* at $x \in \mathbb{R}^d$ if and only if for every $y \in F(x)$ and for every sequence $\{x_n\}_n$ converging to x , there exists a sequence $\{y_n\}_n$ converging to y such that $y_n \in F(x_n)$, for any $n \in \mathbb{N}$.

Theorem 5. Suppose that $g(k, \cdot, u)$ is also continuous on \mathbb{R}^d for any $u \in \mathbf{U}$ and $k \in \llbracket 0 : T \rrbracket$ fixed. Then, $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is lower semicontinuous.

Proof. Let $c \in \mathbb{S}(\xi)$ and $\mathbf{u} \in \mathbf{U}$ the corresponding control given by the definition of the SST. Take a sequence $\{\xi_n\}_n$ such that $\xi_n \rightarrow \xi$. Notice first that for every $k \in \llbracket 0 : T \rrbracket$ fixed, we have $\mathbf{x}_{\xi_n}^{\mathbf{u}}(k) \rightarrow \mathbf{x}_\xi^{\mathbf{u}}(k)$; this is a straightforward consequence of (H1). Also, by assumption, since each mapping $g(k, \cdot, \mathbf{u}(k))$ is also continuous at $x = \mathbf{x}_\xi^{\mathbf{u}}(k)$, we have that

$$\max_{k=0, \dots, T} g_i(k, \mathbf{x}_{\xi_n}^{\mathbf{u}}(k), \mathbf{u}(k)) \rightarrow \max_{k=0, \dots, T} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)), \quad \forall i \in \llbracket 1 : m \rrbracket.$$

Define now

$$c_{n,i} = c_i - \max_{k=0,\dots,T} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)) + \max_{k=0,\dots,T} g_i(k, \mathbf{x}_{\xi_n}^{\mathbf{u}}(k), \mathbf{u}(k))$$

Clearly, $c_{n,i} \rightarrow c_i$ for any $i \in \llbracket 1 : m \rrbracket$ and also since $c_i \geq \max_{k=0,\dots,T} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k))$, we have that $(c_{n,1}, \dots, c_{n,m}) \in \mathbb{S}(\xi_n)$, which completes the proof. \square

Another concept that may be worth studying is the *continuity* of the mapping $\xi \mapsto \mathbb{S}(\xi)$. Recall that a set valued-map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is said to be continuous at x if it is lower semicontinuous at x and upper semicontinuous at x in the sense that for every neighborhood V of $\Psi(x)$, there exists $\eta > 0$ such that:

$$\Psi(x') \subset V, \quad \forall x' \in \mathbb{R}^d \text{ such that } \|x - x'\| < \eta.$$

Upper semicontinuity for set-valued maps is a notion that does not fit well with non-compact valued maps, as can be inferred from [10, Theorem 1.1.2]. Notice that the SST is by definition unbounded because

$$\mathbb{S}(\xi) + \mathbb{R}_+^d \subset \mathbb{S}(\xi), \quad \forall \xi \in \mathbb{R}^d.$$

This suggests that upper semicontinuity is not a property commonly satisfied by the SST, even for very simple cases. Indeed, it is not difficult to see that in Example 1, the mapping $\xi \mapsto \mathbb{S}(\xi)$ is not upper semicontinuous at $\xi = 0$.

Therefore, in general the mapping $\xi \mapsto \mathbb{S}(\xi)$ will not be continuous. However, contrarily to the single-valued case, continuity of a set-valued map is not mandatory for that mapping to be Lipschitz continuous.

Remark 3. *A set-valued map can be Lipschitz continuous without being upper semicontinuous. For example, one might consider the set-valued map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, such that $\Psi(x) = \{y \in \mathbb{R}^d \mid y = f(x) + v, v \in \mathbb{R}_+^d\}$, where the application $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz continuous on \mathbb{R}^d .*

3.2 Lipschitz continuity

Recall that a set valued-map $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is said to be *Lipschitz continuous* at \bar{x} if there is a constant $\kappa_\Psi \geq 0$ such that

$$\sup_{y \in \Psi(x)} \text{dist}(y, \Psi(\bar{x})) \leq \kappa_\Psi \|x - \bar{x}\|, \quad \forall x, \bar{x} \in \mathbb{R}^d.$$

Theorem 6. *Suppose that there exist $\kappa_F, \kappa_g \geq 0$ such that $F(k, \cdot, u)$ and $g(k, \cdot, u)$ are Lipschitz continuous on \mathbb{R}^d of modulus κ_F and κ_g , respectively, for any $u \in \mathbf{U}$ and $k \in \llbracket 0 : T \rrbracket$ fixed. Then, $\xi \mapsto \mathbb{S}(\xi)$ is Lipschitz continuous on \mathbb{R}^d .*

Proof. Let $\bar{\xi}, \xi \in \mathbb{R}^d$ and take $c \in \mathbb{S}(\xi)$. Let $\mathbf{u} \in \mathbf{U}$ be a control such that

$$g(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)) \leq c, \quad \forall k \in \llbracket 0 : T \rrbracket.$$

Let $\bar{c} \in \mathbb{R}^m$ be given by

$$\bar{c}_i = c_i + \max_{k \in \llbracket 0 : T \rrbracket} \left(g_i \left(k, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(k), \mathbf{u}(k) \right) - g_i \left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k) \right) \right)^+,$$

where a^+ stands for the positive part of $a \in \mathbb{R}$.

It follows that $\bar{c} \in \mathbb{S}(\bar{\xi})$. To conclude, we need to prove that there is $\kappa_{\mathbb{S}} > 0$ (independent of c and ξ) such that

$$\|c - \bar{c}\| \leq \kappa_{\mathbb{S}} \|\xi - \bar{\xi}\|.$$

Notice that if

$$g_i \left(k, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(k), \mathbf{u}(k) \right) \leq g_i \left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k) \right), \quad \forall k \in \llbracket 0 : T \rrbracket, i \in \llbracket 1 : m \rrbracket,$$

then $c = \bar{c}$ and the conclusion follows. So, let us assume that there is $\hat{k} \in \llbracket 0 : T \rrbracket$ and $i \in \llbracket 1 : m \rrbracket$ such that

$$g_i \left(\hat{k}, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(\hat{k}), \mathbf{u}(\hat{k}) \right) > g_i \left(\hat{k}, \mathbf{x}_{\xi}^{\mathbf{u}}(\hat{k}), \mathbf{u}(\hat{k}) \right).$$

In particular, for some $k \in \llbracket 0 : T \rrbracket$

$$|\bar{c}_i - c_i| = \bar{c}_i - c_i = g_i \left(k, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(k), \mathbf{u}(k) \right) - g_i \left(k, \mathbf{x}_{\xi}^{\mathbf{u}}(k), \mathbf{u}(k) \right).$$

Notice that

$$\|g(0, \xi, \mathbf{u}(0)) - g(0, \bar{\xi}, \mathbf{u}(0))\| \leq \kappa_g \|\xi - \bar{\xi}\|.$$

Fix now $j \in \llbracket 0 : T - 1 \rrbracket$. Then, we have that

$$\begin{aligned} & \|g(j+1, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j+1), \mathbf{u}(j+1)) - g(j+1, \mathbf{x}_{\xi}^{\mathbf{u}}(j+1), \mathbf{u}(j+1))\| \\ & \leq \kappa_g \|\mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j+1) - \mathbf{x}_{\xi}^{\mathbf{u}}(j+1)\| \\ & \leq \kappa_g \|F(j, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j), \mathbf{u}(j)) - F(j, \mathbf{x}_{\xi}^{\mathbf{u}}(j), \mathbf{u}(j))\| \\ & \leq \kappa_g \kappa_F \|\mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j) - \mathbf{x}_{\xi}^{\mathbf{u}}(j)\|. \end{aligned}$$

It is not difficult then to get by the Induction Principle that for any $j \in \llbracket 0 : T \rrbracket$

$$\|g(j, \mathbf{x}_{\bar{\xi}}^{\mathbf{u}}(j), \mathbf{u}(j)) - g(j, \mathbf{x}_{\xi}^{\mathbf{u}}(j), \mathbf{u}(j))\| \leq \kappa_g \kappa_F^{j-1} \|\xi - \bar{\xi}\|.$$

Consequently,

$$|\bar{c}_i - c_i| \leq \kappa_g \kappa_F^{k-1} \|\xi - \bar{\xi}\|.$$

From here the conclusion follows easily. \square

Remark 4. *If in the preceding result we change Lipschitz continuity with local Lipschitz continuity, the result holds as well, with $\xi \mapsto \mathbb{S}(\xi)$ being now locally Lipschitz*

continuous. *This is due to the fact that the set of admissible trajectories is locally bounded with respect to the initial position; recall that the dynamics is continuous and the control space is compact.*

4 The Attainable Thresholds

As we have discussed in the preceding section, upper semicontinuity of the SST is unlikely to hold, even in very simple cases as in Example 1. This is mainly due to the fact that the SST is by definition unbounded. Now, this unboundedness is not necessarily due to the data of the problem, but to the structure of the inequalities, because we have that if $c \in \mathbb{S}(\xi)$, then $c + v \in \mathbb{S}(\xi)$ for any $v \in \mathbb{R}_+^m$. This hints that some useful information can also be obtained from the thresholds that can be realized by some control.

Accordingly, let us define the *Set of Sustainable and Attainable Thresholds* (SSAT for short) as all the possible thresholds in the SST that can be realized by some control:

$$\mathbb{S}^A(\xi) := \left\{ c \in \mathbb{S}(\xi) \mid \exists \mathbf{u} \in \mathcal{U}, \forall i \in \llbracket 0 : T \rrbracket \text{ such that } \max_{k \in \llbracket 0 : T \rrbracket} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)) = c_i \right\}.$$

It is not difficult to see that

$$\mathbb{S}(\xi) = \mathbb{S}^A(\xi) + \mathbb{R}_+^d, \quad \forall \xi \in \mathbb{R}^d.$$

Moreover, similar arguments used to prove Theorem 5 and Theorem 6 can be developed for proving continuity properties of the SSAT. Here we provide the statement and the sketch of the proofs.

Theorem 7. *Suppose that*

1. $g(k, \cdot, u)$ is also continuous on \mathbb{R}^d for any $u \in \mathbf{U}$ and $k \in \llbracket 0 : T \rrbracket$ fixed. Then, $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is lower semicontinuous.
2. there exist $\kappa_F, \kappa_g \geq 0$ such that $F(k, \cdot, u)$ and $g(k, \cdot, u)$ are Lipschitz continuous on \mathbb{R}^d of modulus κ_F and κ_g , respectively, for any $u \in \mathbf{U}$ and $k \in \llbracket 0 : T \rrbracket$ fixed. Then, $\xi \mapsto \mathbb{S}^A(\xi)$ is Lipschitz continuous on \mathbb{R}^d .

Proof. It is enough to use the same arguments as in the proofs Theorem 5 and Theorem 6, with the fact that now $c_i = \max_{k=0, \dots, T} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k))$. \square

Proposition 1 also holds for the SSAT, however now the continuity of the constraints mapping is required.

Proposition 8. *Suppose that $g(k, \cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in \llbracket 0 : T \rrbracket$ fixed. Then, $\mathbb{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has closed graph with closed and nonempty images.*

Proof. It is enough to use the same arguments as in the proof Proposition 1, with the sequence of controls $\{\mathbf{u}_n\}_n$ such that $g(k, \mathbf{x}_\xi^{\mathbf{u}_n}(k), \mathbf{u}_n) = c_n$ for some $k \in \llbracket 0 : T \rrbracket$. \square

4.1 Upper semicontinuity

The question that rises now is what can we say about upper semicontinuity. To answer this, notice that under the standing assumptions, we have that

$$\mathbf{m}_i(\xi) := \min_{\mathbf{u} \in \mathcal{U}} \min_{k \in \llbracket 0 : T \rrbracket} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)) > -\infty, \quad \forall i \in \llbracket 1 : m \rrbracket.$$

Consequently, if $\mathbf{m}_i(\xi) = (\mathbf{m}_1(\xi), \dots, \mathbf{m}_m(\xi))$ then

$$\mathbb{S}^A(\xi) \subset \mathbf{m}(\xi) + \mathbb{R}_+^m.$$

In a similar way, if any g_i is bounded from above or lower semicontinuity in **(H2)** is strengthened to continuity, then we would also have that

$$\mathbb{S}^A(\xi) \subset \mathfrak{M}(\xi) + \mathbb{R}_-^m,$$

where $\mathfrak{M}(\xi) = (\mathfrak{M}_1(\xi), \dots, \mathfrak{M}_m(\xi))$ and

$$\mathfrak{M}_i(\xi) := \sup_{\mathbf{u} \in \mathcal{U}} \max_{k \in \llbracket 0 : T \rrbracket} g_i(k, \mathbf{x}_\xi^{\mathbf{u}}(k), \mathbf{u}(k)) < +\infty, \quad \forall i \in \llbracket 1 : m \rrbracket.$$

This would mean that the SSAT has compact images, which can be useful to gain properties such as upper semicontinuity. This is actually the case as we show next.

Theorem 9. *Suppose that $g(k, \cdot)$ is continuous on $\mathbb{R}^d \times \mathbf{U}$ for any $k \in \llbracket 0 : T \rrbracket$ fixed. Then, $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is upper semicontinuous.*

Proof. As pointed out above we have that

$$\mathbb{S}^A(\xi) \subset \mathbb{K}(\xi) := \{c \in \mathbb{R}^m \mid \mathbf{m}_i(\xi) \leq c_i \leq \mathfrak{M}_i(\xi), \forall i \in \llbracket 0 : T \rrbracket\}, \quad \forall \xi \in \mathbb{R}^d.$$

On the other hand, it also follows that $\xi \mapsto \mathbf{m}_i(\xi)$ and $\xi \mapsto \mathfrak{M}_i(\xi)$ are lower and upper semicontinuous, respectively. This means that for any $\bar{\xi} \in \mathbb{R}^d$ and $\delta > 0$ fixed, there is a compact set $\mathbf{K} \subset \mathbb{R}^m$ such that

$$\mathbb{K}(\xi) \subseteq \mathbf{K}, \quad \forall \xi \in \mathbb{R}^d, \text{ such that } \|\xi - \bar{\xi}\| \leq \delta.$$

Therefore, since the map $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ has closed graph (Proposition 8), in the light of [10, Corollary 1.1.1], we conclude that $\mathbb{S}^A : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ is upper semicontinuous. \square

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