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Abstract In this paper, we establish an abstract framework for the approximation of the invariant probability measure for a Markov semigroup. Following Pagès and Panloup [40] we use an Euler scheme with decreasing step (unadjusted Langevin algorithm). Under some contraction property with exponential rate and some regularization properties, we give an estimate of the error in total variation distance. This abstract framework covers the main results in [40] and [14]. As a specific application we study the convergence in total variation distance to the invariant measure for jump type equations. The main technical difficulty consists in proving the regularzation properties - this is done under an ellipticity condition, using Malliavin calculus for jump processes.

Key words: Invariant measure, Unadjusted Langevin algorithm, Euler scheme with decreasing steps, Total variation distance, Malliavin calculus, Regularization lemma, Jump process

Contents

1 Introduction

The aim of this paper is to study the convergence to the invariant measure of a Markov process. We refer to [18], [35], [38] for the existence of an invariant probability measure for a general Markov process and to [41], [42] for some basic computation of the invariant probability measure for a Lévy process. Following the ideas from Pagès and Panloup [40] (see also Lamberton and Pagès [30] [31]) we use an Euler scheme with decreasing step (known in the literature as the unadjusted Langevin algorithm) in order to construct our algorithm (this has been studied in depth in [45]).

Our paper has two parts. In the first part we construct an abstract framework which is appropriate in order to state and discuss our approximation problem. We focus on the estimate of the error in total variation distance. And the main achievement is to give some sufficient regularization properties for the semigroup and for the Euler scheme, which allow to treat bounded and measurable test functions. Furthermore, in order to check such regularization properties, one has to use integration by parts techniques inspired from Malliavin calculus. We give a regularization lemma based on such arguments, which is the crucial step in our approach (it has its own interest, beyond the application in this particular framework). Let us mention that the abstract framework settled in our paper encompass the following recent results: in [40], the authors use unadjusted Langevin algorithm to approximate the invariant probability measure of a diffusion process and study the Wasserstein and total variation distance between them. In [14], the authors approximate the invariant probability measure of a Lévy process but only study the Wasserstein distance.

In the second part of the paper we illustrate our results in the case of jump type $SDE's$. In order to do it we recall the Malliavin calculus for jump processes and prove estimates of the Sobolev norms and of the Malliavin covariance matrix for the solution of such equations. These estimates are rather long and technical, but at a certain extend they come back on results already obtained in [44]. Once these estimates are proved, we apply the abstract results from the first part and obtain the estimate of the error in total variation distance.

Let us present in more detail our results. We give in Section 2 the abstract framework of the approximation for the invariant probability measure. We denote $C_b^l(\mathbb{R}^d)$ the space of $l-$times differential and bounded functions on $\mathbb{R}^d$ with bounded derivatives up to order l. We consider a semigroup $P_t, t \geq 0$ on the space $\mathcal{M}_b(\mathbb{R}^d)$ of the bounded measurable functions on $\mathbb{R}^d$ and assume that there exists at least one invariant probability measure ν for the semigroup $P_t, t \geq 0$. We assume moreover the "exponential Lipschitz property": there exists two constants $C_0 \geq 1$ and $\rho > 0$ such that for every $t > 0$ and every $\varphi \in C_b^1(\mathbb{R}^d)$

$$(L_0) \quad \|\nabla P_t \varphi\|_\infty \leq C_0 \|\nabla \varphi\|_\infty e^{-\rho t}.$$

This immediately implies that ν is unique.

In order to approximate the invariant measure ν, we introduce an Euler scheme with decreasing time steps (unadjusted Langevin algorithm). For every $\gamma > 0$ we give an operator $\overline{P}_\gamma : C_b^\infty \to C_b^\infty$ such that $\|\overline{P}_\gamma \varphi\|_\infty \leq \|\varphi\|_\infty$ and which approximates our semigroup in the following sense: for every $\gamma > 0$

$$A(k_0, \alpha) \quad \left\|(P_\gamma - \overline{P}_\gamma)\varphi\right\|_\infty \leq C_{k_0} \|\nabla \varphi\|_{k_0,\infty} \gamma^{1+\alpha}.$$

Here $\alpha > 0$ is a given number, $k_0 \in \mathbb{N}$ and $\|\psi\|_{k_0,\infty} = \sum_{|\alpha| \leq k_0} \|\partial^\alpha \psi\|_\infty$. We consider a decreasing sequence of time steps $\gamma_n \downarrow 0$ and define the time grid $\Gamma_n = \sum_{i=1}^n \gamma_i$. We assume that

$$(\Gamma) \quad \sum_{i=1}^\infty \gamma_i = \lim_{n \to \infty} \Gamma_n = \infty.$$

We also introduce

$$\overline{\omega} = \overline{\omega}((\gamma_n)_{n \in N}) = \overline{\lim_{n \to \infty}} \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2} < \infty.$$

The typical example is $\gamma_n = \frac{1}{n}$ and then $\overline{\omega} = 1$. In the following we denote $\{\Gamma\} = \{\Gamma_n, n \in \mathbb{N}\}$. And, for $\Gamma_i \leq t < \Gamma_{i+1}$ we denote $N(t) = i$ and $\tau(t) = \Gamma_i$. Then, for $s \in \{\Gamma\}$ and $t \in \{\Gamma\}$ we define the Euler scheme

$$\overline{P}_{s,t} = \prod_{i=N(s)}^{N(t)-1} \overline{P}_{\gamma_i}, \tag{1}$$

the product being understood in the sense of composition. This means that we travel from $\tau(s)$ to $\tau(t)$ by using the Euler scheme associated to the one step Euler scheme $\overline{P}_\gamma$.

So now we use the Euler scheme with decreasing time steps $\overline{P}_{0,\Gamma_n}$ (given in (1)) to approximate the invariant probability measure ν. Our aim is to estimate the total variation distance between them. To do so, we need some regularization properties. First we give the regularization hypothesis concerning the semigroup P_t:

$$\begin{array}{lll} R_P(k) & \sup\limits_{1\leq t\leq 2} \|\nabla P_t\varphi\|_{k-1,\infty} & \leq C_k \|\varphi\|_\infty, \quad and \\ R'_P(k) & \sup\limits_{1\leq t\leq 2} \|\nabla P_t\varphi\|_{k-1,\infty} & \leq C'_k \|\nabla\varphi\|_\infty. \end{array}$$

We also introduce the following variant of the Lipschitz property:

$$\overline{L}_k \quad \|\nabla P_t\varphi\|_{k,\infty} \leq C_k \|\nabla\varphi\|_{k,\infty}, \quad 1 \geq t > 0.$$

We give now the regularization properties for the Euler scheme $\overline{P}_{s,t}$. To begin, we introduce some notations. We fix a super kernel ϕ (see (18) for the precise definition), and, for $\delta \in (0,1]$ we denote $\phi_\delta(y) = \frac{1}{\delta^d}\phi(\frac{y}{\delta})$. Moreover, for a function φ we denote φ_δ the regularization by convolution with the super kernel: $\varphi_\delta = \varphi * \phi_\delta$, with $*$ denoting convolution. For $\delta > 0, \eta > 0$, and $q, \kappa, p \in \mathbb{N}$ we denote

$$A^{\delta,\eta}_{q,\kappa,p}(h) = \frac{\delta^q}{\eta^{2q}} + \eta^{-p}h^p + \eta^\kappa, \quad h > 0.$$

Let $\beta > 0$ and $p \geq 1$ be fixed and we assume the following regularization property for the Euler scheme $\overline{P}_{s,t}$: we assume that for every $q, \kappa \in \mathbb{N}$ there exists a constant $C = C_{q,\kappa,p}$ such that for every $\delta > 0, \eta > 0$, every $1 < t < r < t + 2$ and every bounded measurable function φ

$$\begin{array}{ll} R_{\overline{\mathcal{P}}}(p,\beta) & \left\|\overline{P}_{t-1,t}P_{t,r}\varphi - \overline{P}_{t-1,t}P_{t,r}\varphi_\delta\right\|_\infty + \left\|\overline{P}_{t-1,t}\overline{P}_{t,r}\varphi - \overline{P}_{t-1,t}\overline{P}_{t,r}\varphi_\delta\right\|_\infty \\ \leq & C_{q,\kappa,p} \times A^{\delta,\eta}_{q,\kappa,p}(\gamma^\beta_{N(t)}) \|\varphi\|_\infty. \end{array}$$

Now we can give our main result (see **Proposition 2.1.1**). We assume that an invariant probability measure ν exists for the semigroup $P_t, t \geq 0$. We construct an Euler scheme with decreasing time steps $\overline{P}_{s,t}$ by (1). Suppose that (L_0) holds for some ρ, $A(k_0,\alpha)$ holds for some k_0, α with $\rho > \alpha\overline{\omega}$, $R_P(k)$, $R'_P(k)$ and $\overline{L}_k$ hold for every k, and $R_{\overline{\mathcal{P}}}(p,\beta)$ holds true for some p, β. Then the invariant probability measure ν is unique and for any $\varepsilon > 0$, for every $x \in \mathbb{R}^d$ and n large enough,

$$d_{TV}(\overline{P}_{0,\Gamma_n}(x,.),\nu) \leq C_\varepsilon(\gamma_n^{((p\beta)\wedge\alpha)-\varepsilon} + \int_{\mathbb{R}^d} |x-y|\, d\nu(y) e^{-\rho\Gamma_n}).$$

We remark that we get the same speed of convergence as in [40] and [14], but in a more general framework.

We notice that we need some regularization properties (see $R_P(k)$, $R'_P(k)$ and $R_{\overline{\mathcal{P}}}(p,\beta)$). In order to obtain these properties, we introduce in Section 3 an abstract framework built on a particular case of the Dirichlet form theory (see [4] and [7]) in which such a property may be obtained by using some integration by parts techniques. Those techniques are very similar to the standard Malliavin calculus but are presented in a more general framework which goes beyond the sole case of the Wiener space. In particular, we aim at providing a minimalist setting leading to our regularization lemma. Our unified framework includes the standard Malliavin calculus and different known versions: the calculus based on the splitting method developed and used in [5], [6], [8] as well as the $\Gamma-$calculus in [4]. We also mention that our approach

applies in the case of the Malliavin calculus for jump type processes as settled by [12] and in the "lent particle" approach for Poisson point measures developed by [13].

In Section 4, we apply the results in Section 2 for jump processes. So we consider the $d-$dimensional stochastic differential equation with jumps as follows:

$$X_t = x + \int_0^t b(X_r)dr + \int_0^t \int_{\mathbb{R}^d} c(z, X_{r-})N(dz, dr), \tag{2}$$

where $N(dz, dr)$ is a Poisson point measure on the state space $\mathbb{R}^d$ with intensity measure $\widehat{N}(dz, dr) = \mu(dz)dr$, x is the initial value, μ is a positive σ-finite measure on $\mathbb{R}^d$, and $b : \mathbb{R}^d \to \mathbb{R}^d$, $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Some basic background of jump processes can be found in [15], [19], [46], [47] and [3].

We need to give sufficient conditions to ensure the existence of an invariant probability measure for the jump equation (2). We recall by [18] the classical results of the existence of an invariant probability measure for a general Markov process. Recently, [33] gives some specific criterias for the existence of an invariant probability measure of a jump process and also discuss some ergodicity properties. Here we suppose that (**Hypothesis 2.5**)

$$\begin{aligned} i) \quad \langle x - y, b(x) - b(y)\rangle &\leq -\bar{b}\,|x - y|^2 \\ ii) \quad |c(z, x) - c(z, y)| &\leq \bar{c}(z)\,|x - y| \end{aligned}$$

and

$$iii) \quad 2\bar{b} - \int_{\mathbb{R}^d} (2\bar{c}(z) + \bar{c}^2(z))\mu(dz) := \theta > 0.$$

Our conditions are based on [18] and are essentially the same as the conditions in [33]. Indeed, the conditions above implies that for some $\bar{\beta}, \bar{\alpha} > 0$ and a Lyapunov function $V(x) = |x|^2$, we have $LV \leq \bar{\beta} - \bar{\alpha}V$,with L denoting the infinitesimal operator of (2). This guarantees the existence of an invariant probability measure ν.

Moreover, in order to apply the Malliavin framework in Section 3 and obtain regularization properties, we assume (see **Hypothesis 2.4** b)) that the measure μ is absolutely continuous with respect to the Lebesgue measure: $\mu(dz) = h(z)dz$, where h is infinitely differentiable and $\ln h$ has bounded derivatives of any order. We also need some regularity and ellipticity conditions on the coefficients (see **Hypothesis 2.1**$\sim$**2.3** for details). We mention that for every multi-indices β_1, β_2, we assume that there exists a non-negative function $\bar{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that

$$|c(z, x)| + |\partial_z^{\beta_2} \partial_x^{\beta_1} c(z, x)| \leq \bar{c}(z),$$

with $\int_{\mathbb{R}^d} |\bar{c}(z)|^p \mu(dz) < \infty$, $\forall p \geq 1$. We also assume that there exists a non-negative function $\underline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that for every $\zeta \in \mathbb{R}^d$,

$$\sum_{j=1}^{d} \langle \partial_{z_j} c(z, x), \zeta\rangle^2 \geq \underline{c}(z)|\zeta|^2.$$

Now we construct the Euler scheme. We take a partition with decreasing time steps $\mathcal{P} = \{0 = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{n-1} < \Gamma_n < \cdots\}$ with the time steps $\gamma_n = \Gamma_n - \Gamma_{n-1}$, $n \in \mathbb{N}$ verifying some suitable conditions (see Section 4.3 for details). For $\Gamma_n \leq t < \Gamma_{n+1}$ we denote $\tau(t) = \Gamma_n$. We consider the Euler scheme:

$$X_t^{\mathcal{P}} = x + \int_0^t b(X_{\tau(r)}^{\mathcal{P}})dr + \int_0^t \int_{\mathbb{R}^d} c(z, X_{\tau(r)-}^{\mathcal{P}})N(dz, dr).$$

Some results concerning the convergence of the Euler scheme of a jump equation can be found for example in [43], [22], [24], [23], [21], [25] and [2].

Since $\mu(\mathbb{R}^d) = \infty$ (which is a consequence of **Hypothesis 2.4** a)), we have infinitely many jumps. So we construct the truncated Euler scheme in order to have finite numbers of jumps for the sake of simulation and Malliavin calculus. For $m \in \mathbb{N}$, we denote $B_m = \{z \in \mathbb{R}^d : |z| \leq m\}$ and denote

$$\varepsilon_m := \int_{\{|z|>m\}} |\bar{c}(z)|^2 \mu(dz) + \Big| \int_{\{|z|>m\}} \bar{c}(z)\mu(dz)\Big|^2.$$

For every $\gamma > 0$, we define the truncation function $M(\gamma) \in \mathbb{N}$ to be the smallest integer such that

$$\varepsilon_{M(\gamma)} \leq \gamma^2.$$

For $\Gamma_n < t \leq \Gamma_{n+1}$, we denote $M_{\mathcal{P}}(t) = M(\gamma_{n+1})$. Now we cancel the "big jumps" (the jumps of size $|z| > M_{\mathcal{P}}(t)$):

$$X_t^{\mathcal{P},M_{\mathcal{P}}} = x + \int_0^t b(X_{\tau(r)}^{\mathcal{P},M_{\mathcal{P}}})dr + \int_0^t \int_{B_{M_{\mathcal{P}}(r)}} c(z, X_{\tau(r)-}^{\mathcal{P},M_{\mathcal{P}}})N(dz, dr). \tag{3}$$

We remark that the solution of the equation (3) can be constructed in an explicit way.

Then we apply the abstract framework in Section 2 for $X_{\Gamma_n}^{\mathcal{P},M_{\mathcal{P}}}$ and obtain the following main result (see **Theorem 4.1**): An invariant probability measure ν of the jump equation (2) exists and is unique, and for any $\varepsilon > 0$, there exists a constant C_ε such that for every $x \in \mathbb{R}^d$ and n large enough, we have

$$d_{TV}(\mathcal{L}(X_{\Gamma_n}^{\mathcal{P},M_{\mathcal{P}}}), \nu) \leq C_\varepsilon(\gamma_n^{1-\varepsilon} + \int_{\mathbb{R}^d} |x-y|\, d\nu(y) e^{-\frac{\theta}{2}\Gamma_n}),$$

with $\mathcal{L}(X)$ denoting the law of a random variable X. We notice that we obtain the same speed of convergence as in [40] but [40] concern the diffusion process driven by a Brownian motion while here we consider the jump process. Comparing with the results in [14], we also obtain the same speed of convergence but [14] only deals with the Wasserstein distance while in our paper, we deal with the total variation distance.

2 Approximation of the invariant measure: Abstract framework

2.1 The semigroup and the invariant measure

We consider a semigroup $P_t, t \geq 0$ on the space $\mathcal{M}_b(\mathbb{R}^d)$ of the bounded measurable functions on $\mathbb{R}^d$. We denote $C_b^l(\mathbb{R}^d)$ the space of $l-$times differential and bounded functions on $\mathbb{R}^d$ with bounded derivatives up to order l. We will use the following two hypotheses:

(I) We assume that there exists at least one invariant distribution for the semigroup $P_t, t \geq 0$.

Moreover we assume the following "exponential Lipschitz property": we assume that there exists two constants $C_0 \geq 1$ and $\rho > 0$ such that for every $t > 0$ and every $\varphi \in C_b^1(\mathbb{R}^d)$

$$(L_0) \quad \|\nabla P_t \varphi\|_\infty \leq C_0 \|\nabla \varphi\|_\infty e^{-\rho t}. \tag{4}$$

We also denote by $\mathcal{P}_1$ the space of the probability measures on $\mathbb{R}^d$ which have finite moment of order one $\int_{\mathbb{R}^d} |x|\, \nu(dx) < \infty$. This is a Banach space under the Wasserstein distance W_1:

$$W_1(\nu, \mu) = \sup\{\left|\int_{\mathbb{R}^d} \varphi d(\nu - \mu)\right| : \|\nabla \varphi\|_\infty \leq 1\}.$$

Proposition 2.0.1. *Suppose that the semigroup $P_t, t \geq 0$ has at least an invariant probability measure ν and that (4) holds true. Then the invariant probability measure is unique and moreover, for every $x \in \mathbb{R}^d$*

$$W_1(\nu, P_t(x, \cdot)) \leq C \int_{\mathbb{R}^d} |x-y|\, \nu(dy) \times e^{-\rho t}. \tag{5}$$

Proof. Step 1 We will prove that for sufficiently large t, the application $\nu \mapsto \nu P_t$ is a strict contraction on the Wassertein space: using (4),

$$\begin{aligned} \left|\int_{\mathbb{R}^d} \varphi(y) d(\nu P_t - \mu P_t)(dy)\right| &= \left|\int_{\mathbb{R}^d} P_t\varphi(x) d(\nu(x) - \mu(x))\right| \\ &\leq \|\nabla P_t \varphi\|_\infty W_1(\nu, \mu) \\ &\leq C_0 \|\nabla \varphi\|_\infty e^{-\rho t} W_1(\nu, \mu). \end{aligned}$$

This means that, for large t

$$W_1(\nu P_t, \mu P_t) \leq C_0 e^{-\rho t} W_1(\nu, \mu) \leq \frac{1}{2} W_1(\nu, \mu)$$

and this guarantees the uniqueness of the invariant measure.

Step 2 Since ν is an invariant measure

$$\int_{\mathbb{R}^d} \varphi(z)\nu(dz) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_t(z, dy)\varphi(y)\nu(dz)$$

which gives, for every fixed $x \in \mathbb{R}^d$ (ν is a probability)

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(z)\nu(dz) - \int_{\mathbb{R}^d} P_t(x, dy)\varphi(y) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (P_t(z, dy) - P_t(x, dy))\varphi(y)\nu(dz) \\ &= \int_{\mathbb{R}^d} (P_t\varphi(z) - P_t\varphi(x))\nu(dz) \end{aligned} \tag{6}$$

so that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \varphi(z)\nu(dz) - \int_{\mathbb{R}^d} P_t(x, dy)\varphi(y) \right| &\leq \|\nabla P_t\varphi\|_\infty \int_{\mathbb{R}^d} |x - z| \, \nu(dz) \\ &\leq C_0 e^{-\rho t} \|\nabla\varphi\|_\infty \int_{\mathbb{R}^d} |x - z| \, \nu(dz) \end{aligned}$$

which yields (5). □

2.2 The Euler scheme

We introduce now an Euler scheme with decreasing steps. First, for every $\gamma > 0$ we give an operator $\overline{P}_\gamma : C_b^\infty \to C_b^\infty$ such that $\|\overline{P}_\gamma\varphi\|_\infty \leq \|\varphi\|_\infty$ and which approximates our semigroup in the following sense: for every $\gamma > 0$

$$A(k_0, \alpha) \quad \left\|(P_\gamma - \overline{P}_\gamma)\varphi\right\|_\infty \leq C_{k_0} \|\nabla\varphi\|_{k_0,\infty} \gamma^{1+\alpha}. \tag{7}$$

Here $\alpha > 0$ is a given number, $k_0 \in \mathbb{N}$ and

$$\|\psi\|_{k_0,\infty} = \sum_{|\alpha| \leq k_0} \|\partial^\alpha \psi\|_\infty .$$

Moreover, we consider a decreasing sequence of time steps $\gamma_n \downarrow 0$ and define the time grid $\Gamma_n = \sum_{i=1}^n \gamma_i$. We assume that

$$(\Gamma) \quad \sum_{i=1}^\infty \gamma_i = \lim_{n\to\infty} \Gamma_n = \infty. \tag{8}$$

We also introduce

$$\overline{\omega} = \overline{\omega}((\gamma_n)_{n\in N}) = \overline{\lim_{n\to\infty}} \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2} < \infty.$$

The typical example is $\gamma_n = \frac{1}{n}$ and then $\overline{\omega} = 1$. In the following we denote $\{\Gamma\} = \{\Gamma_n, n \in \mathbb{N}\}$. And, for $\Gamma_i \leq t < \Gamma_{i+1}$ we denote

$$N(t) = i \quad and \quad \tau(t) = \Gamma_i.$$

In particular, for $t = \Gamma_i \in \{\Gamma\}$ we have $N(t) = i$ such that $t = \Gamma_{N(t)}$. Then, for $s \in \{\Gamma\}$ and $t \in \{\Gamma\}$ we define the Euler scheme

$$\overline{P}_{s,t} = \prod_{i=N(s)}^{N(t)-1} \overline{P}_{\gamma_i} \tag{9}$$

the product being understood in sense of composition. This means that we travel from $\tau(s)$ to $\tau(t)$ by using the Euler scheme associated to the one step Euler scheme $\overline{P}_\gamma$. In the appendix 7.1 we will prove the following lemma (which is a slight generalisation of the lemma given by Pages and Panloup [40]): for every $\rho > \alpha\overline{\omega}$, there exists n_ρ and C_ρ such that for $n \geq n_\rho$

$$\sum_{i=1}^{n} \gamma_i^{1+\alpha} e^{-\rho(\Gamma_n - \Gamma_i)} \leq C_\rho \gamma_n^\alpha. \tag{10}$$

Moreover, there exists n_* such that, for $n_* \leq i \leq n$

$$\gamma_i \leq e^{2\overline{\omega}(\Gamma_n - \Gamma_i)} \gamma_n. \tag{11}$$

Notice that $P_t, t \geq 0$ is a homogeneous semigroup, and we may define $P_{s,t} = P_{t-s} = P_{0,t-s}$. In contrast, $\overline{P}_{s,t}, s < t$, is not homogeneous: we do not have $\overline{P}_{s,t} = \overline{P}_{0,t-s}$. This is due to the fact that the greed $\Gamma_i, i \in \mathbb{N}$ is not uniform.

Finally we assume the following stronger variant of the Lipschitz property L_0:

$$(L_{k_0}) \quad \|\nabla P_t \varphi\|_{k_0,\infty} \leq C_{k_0} \|\nabla \varphi\|_{k_0,\infty} e^{-\rho t} \tag{12}$$

where k_0 is the one from $A(k_0, \alpha)$.

Proposition 2.0.2. *Suppose that (7) and (12) hold true with $\rho > \alpha\overline{\omega}$. Then for $N(t) > n_\rho + 1$, we have*

$$\left\|(P_{s,t} - \overline{P}_{s,t})\varphi\right\|_\infty \leq C_{k_0} \|\nabla \varphi\|_{k_0,\infty} \gamma_{N(t)}^\alpha. \tag{13}$$

Proof We use (7) first and (12) then

$$\begin{aligned}
\left\|(P_{s,t} - \overline{P}_{s,t})\varphi\right\|_\infty &\leq \sum_{i=N(s)}^{N(t)-1} \left\|\overline{P}_{s,\Gamma_{i-1}}(\overline{P}_{\gamma_i} - P_{\gamma_i})P_{\Gamma_i,t}\varphi\right\|_\infty \\
&\leq \sum_{i=N(s)}^{N(t)-1} \left\|(\overline{P}_{\gamma_i} - P_{\gamma_i})P_{\Gamma_i,t}\varphi\right\|_\infty \\
&\leq C_{k_0} \sum_{i=N(s)}^{N(t)-1} \|\nabla P_{\Gamma_i,t}\varphi\|_{k_0,\infty} \gamma_i^{1+\alpha} \\
&\leq C'_{k_0} \sum_{i=N(s)}^{N(t)-1} \|\nabla \varphi\|_{k_0,\infty} \gamma_i^{1+\alpha} e^{-\rho(\Gamma_{N(t)} - \Gamma_i)} \\
&\leq C''_{k_0} \|\nabla \varphi\|_{k_0,\infty} \gamma_{N(t)}^\alpha.
\end{aligned}$$

For the last inequality we have used (10). □

Remark. Suppose that (7) and (12) hold with $k_0 = 0$. We also suppose that an invariant probability measure ν of the semigroup $P_t, t \geq 0$ exists and that (4) holds true. Then **Proposition 2.0.1** and **Proposition 2.0.2** give that for every $x \in \mathbb{R}^d$, we have

$$W_1(\nu, \overline{P}_{0,t}(x, \cdot)) \leq C(\gamma_{N(t)}^\alpha + \int_{\mathbb{R}^d} |x - y| \, \nu(dy) \times e^{-\rho t}).$$

For this result, we do not need any regularization properties. In order to obtain the result for the total variation distance, we give some regularization properties in the next subsection.

2.3 Regularization properties

In this section we will assume that the semigroup and the Euler scheme have some regularization properties which allow to obtain convergence in total variation distance.

First we give the regularization hypothesis concerning the semigroup:

$$R_P(k) \quad \sup_{1\le t\le 2} \|\nabla P_t\varphi\|_{k-1,\infty} \le C_k \|\varphi\|_\infty, \quad and \tag{14}$$

$$R'_P(k) \quad \sup_{1\le t\le 2} \|\nabla P_t\varphi\|_{k-1,\infty} \le C'_k \|\nabla\varphi\|_\infty, \tag{15}$$

Such a regularization property is proved using the integration by parts formula in Malliavin calculus. Moreover, we suppose that we have the following variant of the Lipschitz property:

$$\begin{array}{lll} \overline{L}_k \quad i) & \|\nabla P_t\varphi\|_\infty \le C_k \|\nabla\varphi\|_{k,\infty} e^{-\rho t}, & t\ge 1, \\ \quad ii) & \|\nabla P_t\varphi\|_{k,\infty} \le C_k \|\nabla\varphi\|_{k,\infty}, & 1\ge t>0. \end{array} \tag{16}$$

Notice that $\overline{L}_k, i)$ is weaker then L_0 (see (4)) because we have $\|\nabla\varphi\|_{k,\infty}$ instead of $\|\nabla\varphi\|_\infty$. However, if the regularization property $R'_P(k)$ holds then $\overline{L}_k, i)$ implies L_0 (for $t\ge 1$). Indeed, $\overline{L}_k$ gives

$$\begin{aligned} \|\nabla P_t\varphi\|_\infty &= \|\nabla(P_{t-1}P_1\varphi)\|_\infty \le C\|\nabla P_1\varphi\|_{k,\infty} e^{-\rho(t-1)} \\ &\le C\|\nabla\varphi\|_\infty e^{-\rho(t-1)}, \end{aligned}$$

the last inequality being the consequence of $R'_P(k)$. In particular, if an invariant probability measure ν exists, then it is unique and we have (5).

Remark. We also notice that $R'_P(k+1)$ and $\overline{L}_k$ imply L_k. Indeed, for $t\le 1, \overline{L}_k\ ii)$ gives

$$\|\nabla P_t\varphi\|_{k,\infty} \le C_k\|\nabla\varphi\|_{k,\infty} \le e^\rho C_k \|\nabla\varphi\|_{k,\infty} e^{-\rho t}$$

and for $t\ge 1$

$$\begin{aligned} \|\nabla P_t\varphi\|_{k,\infty} &= \|\nabla(P_1P_{t-1}\varphi)\|_{k,\infty} \le C\|\nabla P_{t-1}\varphi\|_\infty \\ &\le C\|\nabla\varphi\|_{k,\infty} e^{-\rho(t-1)}. \end{aligned}$$

Moreover, for $t\ge 1$, $\overline{L}_k$ and $R_P(k+1)$ give

$$d_{TV}(P_t(x,.),\nu) \le C(\int_{\mathbb{R}^d} |x-y|\, d\nu(y)) e^{-\rho t}, \tag{17}$$

where d_{TV} denotes the total variation distance:

$$d_{TV}(\mu,\nu) = \sup_{\|f\|_\infty\le 1} \big| \int_{\mathbb{R}^d} f(x)\mu(dx) - \int_{\mathbb{R}^d} f(x)\nu(dx)\big|.$$

Indeed,

$$\begin{aligned} |P_t\varphi(x) - P_t\varphi(y)| &= |P_{t-1}P_1\varphi(x) - P_{t-1}P_1\varphi(y)| \\ &\le C_k\|\nabla P_1\varphi\|_{k,\infty} e^{-\rho(t-1)}|x-y| \\ &\le C_kC_{k+1}e^\rho\|\varphi\|_\infty e^{-\rho t}|x-y|. \end{aligned}$$

Then we come back to (6) and we obtain

$$\left|\int_{\mathbb{R}^d}\varphi(z)\nu(dz) - \int_{\mathbb{R}^d} P_t(x,dy)\varphi(y)\right| \le C\|\varphi\|_\infty \int_{\mathbb{R}^d} e^{-\rho t}|x-y|\,\nu(dy)$$

so (17) is proved. □

We give now the regularization properties for the Euler scheme; this is a more delicate subject, because we have some difficulties in order to use directly the Malliavin calculus for the Euler scheme (the reason is that the decomposition using the inverse of the tangent flow does not work, and so the proof of the non degeneracy property is more difficult) .

We introduce some notations. We recall that a super kernel $\phi : \mathbb{R}^d \to \mathbb{R}$ is a function which belongs to the Schwartz space and such that for every multi-indexes β_1 and β_2, one has

$$\int_{\mathbb{R}^d} \phi(x)dx = 1, \quad \int_{\mathbb{R}^d} y^{\beta_1}\phi(y)dy = 0 \quad \text{for} \quad |\beta_1| \geq 1, \quad \int_{\mathbb{R}^d} |y|^m |\partial_{\beta_2}\phi(y)|dy < \infty \quad \text{for} \quad m \in \mathbb{N}. \tag{18}$$

We fix a super kernel ϕ. For $\delta \in (0,1]$, we denote $\phi_\delta(y) = \frac{1}{\delta^d}\phi(\frac{y}{\delta})$ and φ_δ the regularization by convolution with a super kernel:

$$\varphi_\delta = \varphi * \phi_\delta, \tag{19}$$

with $*$ denoting convolution.

As usual, for a multi-index $\beta_1 = (\beta_1^1, \cdots, \beta_1^m) \in \{1, \cdots, d\}^m$, one denotes $|\beta_1| = m$ and $y^{\beta_1} = \prod_{i=1}^m y_{\beta_1^i}$.

For $\delta > 0, \eta > 0$, and $q, \kappa, p \in \mathbb{N}$ we denote

$$A^{\delta,\eta}_{q,\kappa,p}(h) = \frac{\delta^q}{\eta^{2q}} + \eta^{-p}h^p + \eta^\kappa, \quad h > 0.$$

Then we assume the following:

Let $\beta > 0$ and $p \geq 1$ be fixed. We assume that for every $q, \kappa \in \mathbb{N}$ there exists a constant $C = C_{q,\kappa,p}$ such that for every $\delta > 0, \eta > 0$, every $1 < t < r < t+2$ and every bounded measurable function φ

$$\begin{aligned} R_{\overline{\mathcal{P}}}(p,\beta) \qquad & \left\|\overline{P}_{t-1,t}P_{t,r}\varphi - \overline{P}_{t-1,t}P_{t,r}\varphi_\delta\right\|_\infty + \left\|\overline{P}_{t-1,t}\overline{P}_{t,r}\varphi - \overline{P}_{t-1,t}\overline{P}_{t,r}\varphi_\delta\right\|_\infty \\ & \leq C_{q,\kappa,p} \times A^{\delta,\eta}_{q,\kappa,p}(\gamma^\beta_{N(t-1)}) \left\|\varphi\right\|_\infty . \end{aligned} \tag{20}$$

This represents the "regularization property for $\overline{P}_{t-1,t}$". In order to prove it, one employs **Lemma 3.5** (see (38)) in Section 3.1.

As a consequence of these properties, we obtain the following lemma. We recall n_ρ and n_* in (10) and (11).

Lemma 2.1. *We fix $\beta > 0$ and $p \geq 1$.Suppose that (7) (12) hold with $\rho > \alpha\overline{\omega}$, and $R_{\overline{\mathcal{P}}}(p,\beta)$ (see (20)) holds. Then, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 1$ such that for every $s < t-1 < t < r < t+2$ with $N(r) > n_\rho + 1$ and $N(t-1) > n_*$, and for every bounded measurable function φ*

$$\left\|\overline{P}_{s,t}(\overline{P}_{t,r} - P_{t,r})\varphi\right\|_\infty \leq C_\varepsilon \left\|\varphi\right\|_\infty \gamma^{((p\beta)\wedge\alpha)-\varepsilon}_{N(t)}. \tag{21}$$

Proof We use (20) and (11) in order to get

$$\begin{aligned} \left\|\overline{P}_{s,t}(\overline{P}_{t,r} - P_{t,r})\varphi\right\|_\infty &\leq \left\|\overline{P}_{t-1,t}(\overline{P}_{t,r} - P_{t,r})\varphi\right\|_\infty \\ &\leq C_{q,\kappa,p} \left\|\varphi\right\|_\infty \times A^{\delta,\eta}_{q,\kappa,p}(\gamma^\beta_{N(t-1)}) + b_\delta \\ &\leq C_{q,\kappa,p} \left\|\varphi\right\|_\infty \times A^{\delta,\eta}_{q,\kappa,p}(\gamma^\beta_{N(t)}) + b_\delta \end{aligned}$$

with

$$\begin{aligned} b_\delta &= \left\|\overline{P}_{t-1,t}(\overline{P}_{t,r} - P_{t,r})\varphi_\delta\right\|_\infty \leq \left\|(\overline{P}_{t,r} - P_{t,r})\varphi_\delta\right\|_\infty \leq \\ &\leq C \left\|\nabla\varphi_\delta\right\|_{k_0,\infty} \gamma^\alpha_{N(r)} \leq \frac{C}{\delta^{1+k_0}} \left\|\varphi\right\|_\infty \gamma^\alpha_{N(t)}. \end{aligned}$$

Here we used (13) and $\gamma_{N(r)} \leq \gamma_{N(t)}$. We conclude that

$$\left\|\overline{P}_{s,t}(\overline{P}_{t,r} - P_{t,r})\varphi\right\|_\infty \leq C_{q,\kappa,p} \left\|\varphi\right\|_\infty \times (A^{\delta,\eta}_{q,\kappa,p}(\gamma^\beta_{N(t)}) + \frac{1}{\delta^{1+k_0}}\gamma^\alpha_{N(t)}).$$

Optimization For some fixed $\alpha, \beta, p, k_0, \varepsilon$, we optimize over δ, η, κ, q. Let $\Delta = \gamma^{\beta}_{N(t)}$. First we choose $\eta = \Delta^{\frac{p}{p+\kappa}}$ so that $\eta^{-p}\Delta^p = \eta^{\kappa}$. Then

$$A^{\delta,\eta}_{q,\kappa,p}(\gamma^{\beta}_{N(t)}) = \frac{\delta^q}{\Delta^{\frac{2pq}{p+\kappa}}} + 2\Delta^{\frac{p\kappa}{p+\kappa}}.$$

Take now $\delta = \Delta^{\frac{3p}{p+\kappa}}$ so that

$$A^{\delta,\eta}_{q,\kappa,p}(\gamma^{\beta}_{N(t)}) = \Delta^{\frac{pq}{p+\kappa}} + 2\Delta^{\frac{p\kappa}{p+\kappa}}.$$

With this choice

$$\begin{aligned} A^{\delta,\eta}_{q,\kappa,p}(\gamma^{\beta}_{N(t)}) + \frac{\gamma^{\alpha}_{N(t)}}{\delta^{1+k_0}} &= \Delta^{\frac{pq}{p+\kappa}} + 2\Delta^{\frac{p\kappa}{p+\kappa}} + \Delta^{-\frac{3p(1+k_0)}{p+\kappa}}\gamma^{\alpha}_{N(t)} \\ &= \gamma^{\frac{pq\beta}{p+\kappa}}_{N(t)} + 2\gamma^{\frac{p\kappa\beta}{p+\kappa}}_{N(t)} + \gamma^{-\frac{3p(1+k_0)\beta}{p+\kappa}}_{N(t)} \times \gamma^{\alpha}_{N(t)} \end{aligned}$$

We need

$$\begin{aligned} i) \quad \frac{3p(1+k_0)\beta}{p+\kappa} &< \varepsilon, \\ ii) \quad \frac{\kappa}{p+\kappa} &\geq 1-\varepsilon \\ iii) \quad \frac{q}{p+\kappa} &\geq 1-\varepsilon. \end{aligned}$$

We first choose $\kappa(\varepsilon)$ such that $i)$ and $ii)$ hold true. Then we choose $q(\varepsilon)$ such that $\frac{q(\varepsilon)}{p+\kappa(\varepsilon)} \geq 1-\varepsilon$. With this choice we have

$$\begin{aligned} \left\|\overline{P}_{s,t}(\overline{P}_{t,r} - P_{t,r})\varphi\right\|_{\infty} &\leq C_{q,\kappa,p}\left\|\varphi\right\|_{\infty} \times (A^{\delta,\eta}_{q,\kappa,p}(\gamma^{\beta}_{N(t)}) + \gamma^{-\frac{3p(1+k_0)\beta}{p+\kappa}}_{N(t)}\gamma^{\alpha}_{N(t)}) \\ &\leq C'_{q(\varepsilon),\kappa(\varepsilon),p}\left\|\varphi\right\|_{\infty} \times (\gamma^{p\beta(1-\varepsilon)}_{N(t)} + \gamma^{\alpha-\varepsilon}_{N(t)}) \\ &\leq C'_{q(\varepsilon),\kappa(\varepsilon),p}\left\|\varphi\right\|_{\infty} \times \gamma^{((p\beta)\wedge\alpha)-\bar{\varepsilon}}_{N(t)}, \end{aligned}$$

with $\bar{\varepsilon} = p\beta\varepsilon \vee \varepsilon$. □

We give now **the main result**. We recall n_ρ and n_* in (10) and (11).

Proposition 2.1.1. *Let $\beta > 0$ and $p \geq 1$ be fixed. Suppose that (7) holds for some α, k_0, (14),(15),(16) hold for every k and some ρ with $\rho > \alpha\overline{\omega}$, and $R_{\overline{\mathcal{P}}}(p,\beta)$ (see (20)) holds. For every $\varepsilon > 0$ and every measurable and bounded function φ, for n large enough such that $N(\Gamma_n - 3) > n_*$ and $N(\Gamma_n - 2) > n_\rho + 1$, we have*

$$\left\|(\overline{P}_{0,\Gamma_n} - P_{0,\Gamma_n})\varphi\right\|_{\infty} \leq C_\varepsilon \left\|\varphi\right\|_{\infty} \gamma_n^{((p\beta)\wedge\alpha)-\varepsilon}. \tag{22}$$

Moreover, if an invariant probability measure ν exists, then the invariant probability measure ν is unique and for every $x \in \mathbb{R}^d$, we have

$$d_{TV}(\overline{P}_{0,\Gamma_n}(x,.),\nu) \leq C_\varepsilon(\gamma_n^{((p\beta)\wedge\alpha)-\varepsilon} + \int_{\mathbb{R}^d} |x-y|\, d\nu(y) e^{-\rho\Gamma_n}). \tag{23}$$

Proof We fix $i < n$ such that $1 < \Gamma_i$ and $\Gamma_i + 1 \leq \Gamma_n \leq \Gamma_i + 2$ and we write

$$\begin{aligned} & \left\|(\overline{P}_{0,\Gamma_n} - P_{0,\Gamma_n})\varphi\right\|_{\infty} \\ \leq\ & \left\|(\overline{P}_{0,\Gamma_i}\overline{P}_{\Gamma_i,\Gamma_n} - \overline{P}_{0,\Gamma_i}P_{\Gamma_i,\Gamma_n})\varphi\right\|_{\infty} + \left\|(\overline{P}_{0,\Gamma_i}P_{\Gamma_i,\Gamma_n} - P_{0,\Gamma_i}P_{\Gamma_i,\Gamma_n})\varphi\right\|_{\infty} \\ =\ & : A + B. \end{aligned}$$

First, since $\Gamma_i > 1$, using (21) with $s = 0, t = \Gamma_i$ and $r = \Gamma_n$ we obtain

$$A \leq C_\varepsilon \left\|\varphi\right\|_{\infty} \times \gamma_i^{((p\beta)\wedge\alpha)-\varepsilon} \leq C_\varepsilon \left\|\varphi\right\|_{\infty} \times \gamma_n^{((p\beta)\wedge\alpha)-\varepsilon},$$

where in the last inequality, we have used (11).

Moreover, we recall that (15) and (16) imply (12). So using (13) and the regularization property (14) (notice that $\Gamma_n - \Gamma_i \geq 1$) we obtain

$$B \leq C \left\| \nabla P_{\Gamma_i, \Gamma_n} \varphi \right\|_{k_0,\infty} \gamma_i^\alpha \leq C \left\| \varphi \right\|_\infty \gamma_i^\alpha \leq C \left\| \varphi \right\|_\infty \gamma_n^\alpha,$$

the last inequality being obtained by (11) (because $\Gamma_n - \Gamma_i \leq 2$).

Finally, in order to obtain (23) we use (17). The uniqueness of the invariant probability measure ν comes directly from **Proposition 2.0.1**. □

3 Abstract integration by parts framework

Here we recall the abstract integration by parts framework in [7].

We denote $C_p^\infty(\mathbb{R}^d)$ to be the space of smooth functions which, together with all the derivatives, have polynomial growth. We also denote $C_p^q(\mathbb{R}^d)$ to be the space of $q-$times differentiable functions which, together with all the derivatives, have polynomial growth.

We consider a probability space (Ω,$\mathcal{F}$,$\mathbb{P}$), and a linear subset $\mathcal{S} \subset \bigcap_{p=1}^{\infty} L^p(\Omega;\mathbb{R})$ such that for every $\phi \in C_p^\infty(\mathbb{R}^d)$ and every $F \in \mathcal{S}^d$, we have $\phi(F) \in \mathcal{S}$. A typical example of $\mathcal{S}$ is the space of simple functionals, as in the standard Malliavin calculus. Another example is the space of "Malliavin smooth functionals", usually denoted by $\mathcal{D}_\infty$ (see [37]).

Given a separable Hilbert space $\mathcal{H}$, we assume that we have a derivative operator $D : \mathcal{S} \to \bigcap_{p=1}^{\infty} L^p(\Omega;\mathcal{H})$ which is a linear application which satisfies

$a)$

$$D_h F := \langle DF, h \rangle_{\mathcal{H}} \in \mathcal{S}, \ for\ any\ h \in \mathcal{H}, \tag{24}$$

$b)$ $\underline{Chain\ Rule}$: For every $\phi \in C_p^1(\mathbb{R}^d)$ and $F = (F_1, \cdots, F_d) \in \mathcal{S}^d$, we have

$$D\phi(F) = \sum_{i=1}^{d} \partial_i \phi(F) DF_i, \tag{25}$$

Since $D_h F \in \mathcal{S}$, we may define by iteration the derivative operator of higher order $D^q : \mathcal{S} \to \bigcap_{p=1}^{\infty} L^p(\Omega;\mathcal{H}^{\otimes q})$ which verifies $\langle D^q F, \otimes_{i=1}^q h_i \rangle_{\mathcal{H}^{\otimes q}} = D_{h_q} D_{h_{q-1}} \cdots D_{h_1} F$. We also denote $D^q_{h_1,\cdots,h_q} F := \langle D^q F, \otimes_{i=1}^q h_i \rangle_{\mathcal{H}^{\otimes q}}$, for any $h_1, \cdots, h_q \in \mathcal{H}$. Then, $D^q_{h_1,\cdots,h_q} F = D_{h_q} D^{q-1}_{h_1,\cdots,h_{q-1}} F$ ($q \geq 2$).

We notice that since $\mathcal{H}$ is separable, there exists a countable orthonormal base $(e_i)_{i\in\mathbb{N}}$. We denote

$$D_i F = D_{e_i} F = \langle DF, e_i \rangle_{\mathcal{H}}.$$

Then

$$DF = \sum_{i=1}^{\infty} D_i F \times e_i \quad \text{and} \quad D^q F = \sum_{i_1,\cdots,i_q} D_{i_1,\cdots,i_q} F \times \otimes_{j=1}^q e_j.$$

For $F = (F_1, \cdots, F_d) \in \mathcal{S}^d$, we associate the Malliavin covariance matrix

$$\sigma_F = (\sigma_F^{i,j})_{i,j=1,\cdots,d}, \quad \text{with} \quad \sigma_F^{i,j} = \langle DF_i, DF_j \rangle_{\mathcal{H}}. \tag{26}$$

And we denote

$$\Sigma_p(F) = \mathbb{E}(1/\det \sigma_F)^p. \tag{27}$$

We say that the covariance matrix of F is non-degenerated if $\Sigma_p(F) < \infty$, $\forall p \geq 1$.

We also assume that we have an Ornstein-Uhlenbeck operator $L : \mathcal{S} \to \mathcal{S}$ which is a linear operator satisfying

$a)$ $\underline{Duality}$: For every $F, G \in \mathcal{S}$,

$$\mathbb{E}\langle DF, DG\rangle_{\mathcal{H}} = \mathbb{E}(FLG) = \mathbb{E}(GLF), \tag{28}$$

$b)$ $\underline{Chain\ Rule}$: For every $\phi \in C_p^2(\mathbb{R}^d)$ and $F = (F_1, \cdots, F_d) \in \mathcal{S}^d$, we have

$$L\phi(F) = \sum_{i=1}^{d} \partial_i \phi(F) LF_i - \sum_{i=1}^{d}\sum_{j=1}^{d} \partial_i\partial_j \phi(F)\langle DF_i, DF_j\rangle_{\mathcal{H}}.$$

As an immediate consequence of the duality formula (28), we know that $L : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega)$ is closable. But it is not clear that D is also closable. We have to assume this and to check it for each particular example.

Definition 3.1. If $D^q : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega; \mathcal{H}^{\otimes q})$, $\forall q \geq 1$, are closable, then the triplet $(\mathcal{S}, D, L)$ is called an IbP (Integration by Parts) framework.

Remark. The bilinear forms $\Gamma(F, G) = \langle DF, DG\rangle_{\mathcal{H}}$ is called "carré du champ" operator in the theory of Dirichlet form. And $\mathcal{E}(F, G) = \mathbb{E}(\Gamma(F, G))$ is the Dirichlet form associated to Γ. So our Integration by Parts framework appears as a particular case of the $\Gamma-$calculus, presented in [4] and [7].

Now, we introduce the Sobolev norms. For any $l \geq 1$, $F \in \mathcal{S}$,

$$|F|_{1,l} = \sum_{q=1}^{l} |D^q F|_{\mathcal{H}^{\otimes q}}, \quad |F|_l = |F| + |F|_{1,l}, \tag{29}$$

We put $|F|_0 = |F|$, $|F|_l = 0$ for $l < 0$, and $|F|_{1,l} = 0$ for $l \leq 0$. For $F = (F_1, \cdots, F_d) \in \mathcal{S}^d$, we set

$$|F|_{1,l} = \sum_{i=1}^{d} |F_i|_{1,l}, \quad |F|_l = \sum_{i=1}^{d} |F_i|_l,$$

Moreover, we associate the following norms. For any $l \geq 0, p \geq 1$,

$$\begin{aligned} \|F\|_{l,p} &= (\mathbb{E}\,|F|_l^p)^{1/p}, \quad \|F\|_p = (\mathbb{E}\,|F|^p)^{1/p}, \\ \|F\|_{L,l,p} &= \|F\|_{l,p} + \|LF\|_{l-2,p}. \end{aligned} \tag{30}$$

With these notations, we have the following lemma from [9] (lemma 8 and lemma 10), which is a consequence of the chain rule.

Lemma 3.1. *Let $F \in \mathcal{S}^d$. For every $l \in \mathbb{N}$, if $\phi : \mathbb{R}^d \to \mathbb{R}$ is a $C^l(\mathbb{R}^d)$ function ($l-$times differentiable function), then there is a constant C_l dependent on l such that*

$$a) \quad |\phi(F)|_{1,l} \leq |\nabla\phi(F)||F|_{1,l} + C_l \sup_{2\leq|\beta|\leq l} |\partial^\beta \phi(F)||F|_{1,l-1}^l.$$

If $\phi \in C^{l+2}(\mathbb{R}^d)$, then

$$b) \quad |L\phi(F)|_l \leq |\nabla\phi(F)||LF|_l + C_l \sup_{2\leq|\beta|\leq l+2} |\partial^\beta \phi(F)|(1 + |F|_{l+1}^{l+2})(1 + |LF|_{l-1}).$$

For $l = 0$, we have

$$c) \quad |L\phi(F)| \leq |\nabla\phi(F)||LF| + \sup_{|\beta|=2} |\partial^\beta \phi(F)||F|_{1,1}^2.$$

We denote by $\mathcal{D}_{l,p}$ the closure of $\mathcal{S}$ with respect to the norm $\|\circ\|_{L,l,p}$:

$$\mathcal{D}_{l,p} = \overline{\mathcal{S}}^{\|\circ\|_{L,l,p}}, \tag{31}$$

and

$$\mathcal{D}_{\infty} = \bigcap_{l=1}^{\infty} \bigcap_{p=1}^{\infty} \mathcal{D}_{l,p}, \quad \mathcal{H}_l = \mathcal{D}_{l,2}. \tag{32}$$

For an IbP framework $(\mathcal{S}, D, L)$, we now extend the operators from $\mathcal{S}$ to $\mathcal{D}_{\infty}$. For $F \in \mathcal{D}_{\infty}$, $p \geq 2$, there exists a sequence $F_n \in \mathcal{S}$ such that $\|F - F_n\|_p \to 0$, $\|F_m - F_n\|_{q,p} \to 0$ and $\|LF_m - LF_n\|_{q-2,p} \to 0$. Since D^q and L are closable, we can define

$$D^q F = \lim_{n\to\infty} D^q F_n \quad in \quad L^p(\Omega; \mathcal{H}^{\otimes q}), \quad LF = \lim_{n\to\infty} LF_n \quad in \quad L^p(\Omega). \tag{33}$$

We still associate the same norms and covariance matrix introduced above for $F \in \mathcal{D}_{\infty}$.

Lemma 3.2. *The triplet $(\mathcal{D}_{\infty}, D, L)$ is an IbP framework.*

Proof. The proof is standard and we refer to the lemma 3.1 in [10] for details. □

The following lemma is useful in order to control the Sobolev norms and covariance matrices when passing to the limit.

Lemma 3.3. ***(A)*** *We fix $p \geq 2, l \geq 2$. Let $F \in L^1(\Omega; \mathbb{R}^d)$ and let $F_n \in \mathcal{S}^d, n \in \mathbb{N}$ such that*

$$\begin{aligned} i) \quad \mathbb{E}\,|F_n - F| &\to 0, \\ ii) \quad \sup_n \|F_n\|_{L,l,p} &\leq K_{l,p} < \infty. \end{aligned}$$

Then for every $1 \leq \bar{p} < p$, we have $F \in \mathcal{D}^d_{l,\bar{p}}$ and $\|F\|_{L,l,\bar{p}} \leq K_{l,\bar{p}}$. Moreover, there exists a convex combination

$$G_n = \sum_{i=n}^{m_n} \gamma_i^n \times F_i \in \mathcal{S}^d,$$

with $\gamma_i^n \geq 0, i = n, ..., m_n$ and $\sum_{i=n}^{m_n} \gamma_i^n = 1$, such that

$$\|G_n - F\|_{L,l,2} \to 0.$$

(B) *For $F \in \mathcal{D}^d_{\infty}$, we denote*

$$\lambda(F) = \inf_{|\zeta|=1} \langle \sigma_F \zeta, \zeta \rangle$$

*the lowest eigenvalue of the covariance matrix σ_F. We consider some F and F_n which verify $i), ii)$ in **(A)**. We also suppose that*

$$iii) \quad (DF_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } L^2(\Omega; \mathcal{H}),$$

and for every $p \geq 1$,

$$iv) \quad \sup_n \mathbb{E}(\lambda^{-p}(F_n)) \leq Q_p < \infty. \tag{34}$$

Then we have

$$\mathbb{E}(\lambda^{-p}(F)) \leq Q_p < \infty, \quad \forall p \geq 1.$$

(C) *We suppose that we have $(F, \bar{F})$ and $(F_n, \bar{F}_n)$ which verify the hypotheses of **(A)**. If we also have*

$$v) \quad \sup_n \|DF_n - D\bar{F}_n\|_{L^2(\Omega;\mathcal{H})} \leq \bar{\varepsilon}, \tag{35}$$

then

$$\|DF - D\bar{F}\|_{L^2(\Omega;\mathcal{H})} \leq \bar{\varepsilon}.$$

Proof. **Proof of (A)** For the sake of the simplicity of notations, we only prove for the one dimensional case. We recall the notations in Section 3. The Hilbert space $\mathcal{H}_l = \mathcal{D}_{l,2}$ equipped with the scalar product

$$\begin{aligned}\langle U, V\rangle_{L,l,2} &:= \sum_{q=1}^{l} \mathbb{E}\langle D^q U, D^q V\rangle_{\mathcal{H}^{\otimes q}} + \mathbb{E}(UV) \\ &+ \sum_{q=1}^{l-2} \mathbb{E}\langle D^q LU, D^q LV\rangle_{\mathcal{H}^{\otimes q}} + \mathbb{E}(LU \times LV)\end{aligned}$$

is the space of the functionals which are $l-$times differentiable in L^2 sense. By $ii)$, for $p \geq 2$, $\|F_n\|_{L,l,2} \leq \|F_n\|_{L,l,p} \leq K_{l,p}$. Then, applying Banach Alaoglu theorem, there exists $G \in \mathcal{H}_l$ and a subsequence (we still denote it by n), such that $F_n \to G$ weakly in the Hilbert space $\mathcal{H}_l$. This means that for every $Q \in \mathcal{H}_l$, $\langle F_n, Q\rangle_{L,l,2} \to \langle G, Q\rangle_{L,l,2}$. Therefore, by Mazur theorem, we can construct some convex combination

$$G_n = \sum_{i=n}^{m_n} \gamma_i^n \times F_i \in \mathcal{S}$$

with $\gamma_i^n \geq 0, i = n, ..., m_n$ and $\sum\limits_{i=n}^{m_n} \gamma_i^n = 1$, such that

$$\|G_n - G\|_{L,l,2} \to 0.$$

In particular we have

$$\mathbb{E}\,|G_n - G| \leq \|G_n - G\|_{L,l,2} \to 0.$$

Also, we notice that by i),

$$\mathbb{E}\,|G_n - F| \leq \sum_{i=n}^{m_n} \gamma_i^n \times \mathbb{E}\,|F_i - F| \to 0.$$

So we conclude that $F = G \in \mathcal{H}_l$. We also have

$$\|G_n\|_{L,l,p} \leq \sum_{i=n}^{m_n} \gamma_i^n \|F_i\|_{L,l,p} \leq K_{l,p}.$$

Then a standard argument gives, for every $\bar{p} \in [1, p)$,

$$\|F\|_{L,l,\bar{p}} \leq K_{l,\bar{p}}.$$

Proof of (B) We consider for a moment some general $F, G \in \mathcal{D}_\infty^d$. Notice that $\langle \sigma(F)\zeta, \zeta\rangle = |\langle DF, \zeta\rangle|^2_{\mathcal{H}}$, so $\lambda(F) = \inf_{|\zeta|=1} |\langle DF, \zeta\rangle|^2_{\mathcal{H}}$. It is easy to check that

$$|\sqrt{\lambda(F)} - \sqrt{\lambda(G)}| \leq |D(F - G)|_{\mathcal{H}}. \tag{36}$$

We now come back to our framework. Recalling that $G_n = \sum\limits_{i=n}^{m_n} \gamma_i^n \times F_i$, we observe that

$$\|DG_n - DF_n\|_{L^2(\Omega;\mathcal{H})} \leq \sum_{i=n}^{m_n} \gamma_i^n \|DF_i - DF_n\|_{L^2(\Omega;\mathcal{H})} \to 0.$$

Here we use the fact that $(DF_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega;\mathcal{H})$. Meanwhile, we know from **(A)** that $\|DG_n - DF\|_{L^2(\Omega;\mathcal{H})} \to 0$. So we conclude that $\|DF - DF_n\|_{L^2(\Omega;\mathcal{H})} \to 0$. Thus, by (36), $\mathbb{E}|\sqrt{\lambda(F)} - \sqrt{\lambda(F_n)}| \to 0$. This gives that there exists a subsequence (also denote by n) such that $\sqrt{\lambda(F_n)}$ converges

to $\sqrt{\lambda(F)}$ almost surely, and consequently $|\lambda(F_n)|^{-p}$ converges to $|\lambda(F)|^{-p}$ almost surely. Since we have (34), $(|\lambda(F_n)|^{-p})_{n\in\mathbb{N}}$ is uniformly integrable. It follows that

$$\mathbb{E}(|\lambda(F)|^{-p}) = \lim_{n\to\infty} \mathbb{E}(|\lambda(F_n)|^{-p}) \leq Q_p.$$

Proof of (C) Since the couples $(F, \bar{F})$ and $(F_n, \bar{F}_n)$ verify the hypotheses of **(A)**, we know by **(A)** that we may find a convex combination such that

$$\overline{\lim}_{n\to\infty} \| \sum_{i=n}^{m_n} \gamma_i^n (DF_i, D\bar{F}_i) - (DF, D\bar{F}) \|_{L^2(\Omega;\mathcal{H})} = 0.$$

Then it follows by (35) that

$$\begin{aligned} \|DF - D\bar{F}\|_{L^2(\Omega;\mathcal{H})} &\leq \overline{\lim}_{n\to\infty} \| \sum_{i=n}^{m_n} \gamma_i^n (DF_i - D\bar{F}_i) \|_{L^2(\Omega;\mathcal{H})} \\ &\leq \overline{\lim}_{n\to\infty} \sum_{i=n}^{m_n} \gamma_i^n \|DF_i - D\bar{F}_i\|_{L^2(\Omega;\mathcal{H})} \\ &\leq \bar{\varepsilon}. \end{aligned}$$

□

3.1 Main consequences

We will use the abstract framework presented above for the IbP framework $(\mathcal{D}_\infty, D, L)$, with D and L defined in (33). We recall the notations $\|F\|_{L,l,p}$ in (30), $\Sigma_p(F)$ in (27) and σ_F in (26). For any $\eta > 0$, we take $\Upsilon_\eta(x) : (0,\infty) \to \mathbb{R}$ to be a function of class C_b^∞ such that

$$\mathbb{1}_{[\frac{\eta}{2},\infty)} \leq \Upsilon_\eta \leq \mathbb{1}_{[\eta,\infty)}.$$

We remark that σ_F is invertible on the set $\{\Upsilon_\eta(\det \sigma_F) > 0\}$. We give the following lemma, which is stated in lemma 2.4 of [7] and is proved in the Appendix of [6], based on some integration by parts formula.

Lemma 3.4. *Let $F = (F_1, \cdots, F_d) \in \mathcal{D}_\infty^d$ and $G \in \mathcal{D}_\infty$. We fix $q \in \mathbb{N}$.*

***(A)** Suppose that there exists a constant C_q (dependent on q, d) such that $\|F\|_{L,q+2,8dq} + \Sigma_{4q}(F) + \|G\|_{q,4} \leq C_q$. Then for any multi-index β with $|\beta| = q$ and any function $f \in C_b^q(\mathbb{R}^d)$,*

$$(\mathbf{B}_q) \quad |\mathbb{E}(\partial^\beta f(F) G)| \leq C_q \|f\|_\infty, \quad \forall |\beta| = q. \tag{37}$$

***(B)** Suppose that there exists a constant C'_q (dependent on q, d) such that $\|F\|_{L,q+2,(4d+1)q} + \|G\|_{q,4} \leq C'_q$. Then for any $\eta > 0$, any multi-index β with $|\beta| = q$ and any function $f \in C_b^q(\mathbb{R}^d)$,*

$$(\mathbf{B}'_q) \quad |\mathbb{E}(\partial^\beta f(F) \Upsilon_\eta(\det \sigma_F) G)| \leq C'_q \|f\|_\infty \times \frac{1}{\eta^{2q}}, \quad \forall |\beta| = q.$$

Remark. In **(A)**, we assume the non-degeneracy condition for F, so we can give the estimate based on the standard integration by parts formula. In **(B)**, we no longer suppose non-degeneracy condition for F, so we can only obtain an estimate based on a localized form of integration by parts formula.

Remark. If the property $(\mathbf{B}_q)$ (respectively $(\mathbf{B}'_q)$) holds for a random variable F, then it also holds for $F + x$ for every x in $\mathbb{R}^d$, with the same constant C_q (respectively C'_q). In order to see this, given a test function f, one defines $f_x(y) = f(x + y)$ so that $f(F + x) = f_x(F)$. And one notice that the infinite norm of f_x is the same as the infinite norm of f.

We give now a regularization lemma which plays a crucial role in our paper. We consider the $d-$dimensional super kernel ϕ_δ in (18) and (19) and we denote

$$f_\delta(x) = f * \phi_\delta(x) = \int_{\mathbb{R}^d} f(y)\phi_\delta(x-y)dy.$$

Then we have the following regularization lemma.

Lemma 3.5. *We fix some $q, d \in \mathbb{N}$ and $\kappa, p \geq 1$. We suppose that $F \in \mathcal{D}_\infty^d$ such that $\|F\|_{L,q+2,(4d+1)q} < \infty$. We also consider an auxiliary random variable $Q \in \mathcal{D}_\infty^d$ such that $\Sigma_\kappa(Q) < \infty$. Then there exists a constant C depending on p, q, κ and d (but not on Q) such that for any $\eta > 0$ and $\delta > 0$, for any function $f \in C_b^q(\mathbb{R}^d)$, we have*

$$|\mathbb{E}(f(F)) - \mathbb{E}(f_\delta(F))| \leq C \|f\|_\infty \times (\frac{\delta^q}{\eta^{2q}} + \eta^{-p}\mathbb{E}(|\det\sigma_F - \det\sigma_Q|^p) + \eta^\kappa \mathbb{E}(|\det\sigma_Q|^{-\kappa})). \tag{38}$$

Remark. We remark that we do not assume the non-degeneracy condition for F, but we need to assume that we have another random variable Q which is non-degenerated such that $\det\sigma_Q$ is close to $\det\sigma_F$. Then we obtain the regularization lemma (38). The regularization lemma here is originally from the paper [7].

Remark. If the property (38) holds for a random variable F, then it also holds for $F + x$ for every x in $\mathbb{R}^d$, with the same constant C.

Proof. We denote

$$R_q(\delta, x) = \frac{1}{q!}\sum_{|\alpha|=q}\int_0^1 d\lambda(1-\lambda)^q \int_{\mathbb{R}^d} dy\phi_\delta(y)y^\alpha \partial^\alpha f(x+\lambda y)$$

with $y^\alpha = \prod_{i=1}^q y_{\alpha_i}$ for $\alpha = (\alpha_1, ..., \alpha_q)$. Notice that if F satisfies $(\mathbf{B}'_q)$ with $G = 1$, then

$$|\mathbb{E}(R_q(\delta, F)\Upsilon_\eta(\det\sigma_F))| \leq C'_q \frac{\|f\|_\infty}{\eta^{2q}} \int_{\mathbb{R}^d} dy\phi_\delta(y)\,|y|^q = C'_q \int_{\mathbb{R}^d} \phi(y)\,|y|^q\,dy\,\|f\|_\infty \frac{\delta^q}{\eta^{2q}}. \tag{39}$$

We use a development in Taylor series of order q in order to get

$$\begin{aligned}\mathbb{E}(f(F)\Upsilon_\eta(\det\sigma_F)) - \mathbb{E}(f_\delta(F)\Upsilon_\eta(\det\sigma_F)) &= \mathbb{E}(\int_{\mathbb{R}^d} dy\phi_\delta(y)(f(F+y) - f(y))\Upsilon_\eta(\det\sigma_F)) \\ &= \mathbb{E}(R_q(\delta, F)\Upsilon_\eta(\det\sigma_F)).\end{aligned}$$

Here we have used the property of a super kernel: $\int_{\mathbb{R}^d} y^\beta\phi(y)dy = 0,\ \forall|\beta| \leq q$. Using (39), we have

$$|\mathbb{E}(f(F)\Upsilon_\eta(\det\sigma_F)) - \mathbb{E}(f_\delta(F)\Upsilon_\eta(\det\sigma_F))| \leq C\|f\|_\infty \frac{\delta^q}{\eta^{2q}}. \tag{40}$$

Following the idea from [11] p14, we denote

$$R = \frac{\det\sigma_F - \det\sigma_Q}{\det\sigma_Q}.$$

For an arbitrary η, we write

$$\mathbb{P}(\det\sigma_F < \eta) \leq \mathbb{P}(\det\sigma_F < \eta, |R| < \frac{1}{4}) + \mathbb{P}(|R| \geq \frac{1}{4}). \tag{41}$$

When $|R| < \frac{1}{4}$, $|\det\sigma_F - \det\sigma_Q| < \frac{1}{4}\det\sigma_Q$. This implies that $\det\sigma_F > \frac{1}{2}\det\sigma_Q$. Recalling that Q is non-degenerated and using Markov inequality, for every $\kappa \in \mathbb{N}$, it follows that

$$\mathbb{P}(\det\sigma_F < \eta, |R| < \frac{1}{4}) \leq \mathbb{P}(\det\sigma_Q < 2\eta) \leq 2^\kappa\eta^\kappa\mathbb{E}(|\det\sigma_Q|^{-\kappa}). \tag{42}$$

For any $\eta > 0$, $\kappa \in \mathbb{N}$, we write

$$\begin{aligned}\mathbb{P}(|R| \geq \frac{1}{4}) &= \mathbb{P}(|\det \sigma_F - \det \sigma_Q| \geq \frac{1}{4} \det \sigma_Q) \\ &\leq \mathbb{P}(\det \sigma_Q \leq \eta) + \mathbb{P}(|\det \sigma_F - \det \sigma_Q| > \frac{1}{4}\eta) \\ &\leq C(\eta^\kappa \mathbb{E}(|\det \sigma_Q|^{-\kappa}) + \eta^{-p}\mathbb{E}(|\det \sigma_F - \det \sigma_Q|^p)). \end{aligned} \tag{43}$$

So we conclude that

$$\mathbb{P}(\det \sigma_F < \eta) \leq C(\eta^\kappa \mathbb{E}(|\det \sigma_Q|^{-\kappa}) + \eta^{-p}\mathbb{E}(|\det \sigma_F - \det \sigma_Q|^p)). \tag{44}$$

Then we have

$$|\mathbb{E}((1 - \Upsilon_\eta(\det \sigma_F))f(F))| \leq \|f\|_\infty \mathbb{P}(\det \sigma_F < \eta) \leq C\|f\|_\infty(\eta^\kappa \mathbb{E}(|\det \sigma_Q|^{-\kappa}) + \eta^{-p}\mathbb{E}(|\det \sigma_F - \det \sigma_Q|^p)). \tag{45}$$

Similarly, we also have

$$|\mathbb{E}((1 - \Upsilon_\eta(\det \sigma_F))f_\delta(F))| \leq C\|f\|_\infty(\eta^\kappa \mathbb{E}(|\det \sigma_Q|^{-\kappa}) + \eta^{-p}\mathbb{E}(|\det \sigma_F - \det \sigma_Q|^p)). \tag{46}$$

We conclude by combining (40), (45) and (46). □

4 Application for jump equations

4.1 Basic notations and the main equation

To begin, we introduce some notations which will be used in the following sections. For a multi-index β, we denote $|\beta|$ to be the length of β. We denote $C_b^l(\mathbb{R}^d)$ the space of $l-$times differential and bounded functions on $\mathbb{R}^d$ with bounded derivatives up to order l, and $\|f\|_{l,\infty} := \sum_{|\beta| \leq l} \left\|\partial^\beta f\right\|_\infty$ for a function $f \in C_b^l(\mathbb{R}^d)$. We also denote $\mathcal{P}_l(\mathbb{R}^d)$ the space of all probability measures on $\mathbb{R}^d$ with finite $l-$moment. For $\rho_1, \rho_2 \in \mathcal{P}_1(\mathbb{R}^d)$, we define the Wasserstein distance W_1 by

$$W_1(\rho_1, \rho_2) = \sup_{Lip(f) \leq 1} \Big| \int_{\mathbb{R}^d} f(x)\rho_1(dx) - \int_{\mathbb{R}^d} f(x)\rho_2(dx)\Big|, \tag{47}$$

with $Lip(f) := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$ the Lipschitz constant of f, and we define the total variation distance d_{TV} by

$$d_{TV}(\rho_1, \rho_2) = \sup_{\|f\|_\infty \leq 1} \Big| \int_{\mathbb{R}^d} f(x)\rho_1(dx) - \int_{\mathbb{R}^d} f(x)\rho_2(dx)\Big|. \tag{48}$$

For $F, G \in L^1(\Omega)$, we also denote $W_1(F, G) = W_1(\mathcal{L}(F), \mathcal{L}(G))$ and $d_{TV}(F, G) = d_{TV}(\mathcal{L}(F), \mathcal{L}(G))$, with $\mathcal{L}(F)$(respectively $\mathcal{L}(G)$) the law of the random variable F(respectively G). We refer to [49] and [34] the basic properties of these distances. In addition, along the paper, C will be a constant which may change from a line to another. It may depend on some parameters and sometimes the dependence is precised in the notation (ex. C_l is a constant depending on l).

In this paper, we consider the $d-$dimensional stochastic differential equation with jumps

$$X_t = x + \int_0^t b(X_r)dr + \int_0^t \int_{\mathbb{R}^d} c(z, X_{r-})N(dz, dr), \tag{49}$$

where $N(dz, dr)$ is a Poisson point measure on the state space $\mathbb{R}^d$ with intensity measure $\widehat{N}(dz, dr) = \mu(dz)dr$, x is the initial value, μ is a positive σ-finite measure on $\mathbb{R}^d$, and $b : \mathbb{R}^d \to \mathbb{R}^d$, $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$.

4.2 Hypotheses

Here we give our hypotheses.

Hypothesis 2.1 (**Regularity**) We assume that the function $x \mapsto b(x)$ is infinitely differentiable with bounded derivatives of any orders. We also assume that the function $(z,x) \mapsto c(z,x)$ is infinitely differentiable and for every multi-indices β_1, β_2, there exists a function $\bar{c} : \mathbb{R}^d \to \mathbb{R}_+$ depending on β_1, β_2 such that we have

$$\sup_{x\in\mathbb{R}^d} (|c(z,x)| + |\partial_z^{\beta_2}\partial_x^{\beta_1} c(z,x)|) \leq \bar{c}(z), \quad \forall z \in \mathbb{R}^d, \tag{50}$$

$$with \quad \int_{\mathbb{R}^d} |\bar{c}(z)|^p \mu(dz) := \bar{c}_p < \infty, \quad \forall p \geq 1. \tag{51}$$

Remark. We will use several times the following consequence of (51) and of Burkholder inequality (see for example the Theorem 2.11 in [28], see also [29]): Let $\Phi(s,z,\omega) : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}_+$ and $\varphi(s,\omega) : [0,T] \times \Omega \to \mathbb{R}_+$ be two non-negative functions. The Burkholder inequality states that for any $p \geq 2$, we have

$$\begin{aligned}
&\mathbb{E}|\int_0^t \int_{\mathbb{R}^d} \Phi(s,z,\omega) N(dz,ds)|^p \\
&\leq C[\mathbb{E}(\int_0^t \int_{\mathbb{R}^d} |\Phi(s,z,\omega)|^2 \mu(dz)ds)^{\frac{p}{2}} + \mathbb{E}\int_0^t \int_{\mathbb{R}^d} |\Phi(s,z,\omega)|^p \mu(dz)ds \\
&+\mathbb{E}|\int_0^t \int_{\mathbb{R}^d} |\Phi(s,z,\omega)|\mu(dz)ds|^p].
\end{aligned} \tag{52}$$

If we have

$$|\Phi(s,z,\omega)| \leq |\bar{c}(z)||\varphi(s,\omega)|,$$

then for any $p \geq 2$,

$$\mathbb{E}\Big|\int_0^t \int_{\mathbb{R}^d} \Phi(s,z,\omega) N(dz,ds)\Big|^p \leq C\mathbb{E}\int_0^t |\varphi(s,\omega)|^p ds, \tag{53}$$

where C is a constant depending on p, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_p$ and T.

Proof. By compensating N and using Burkholder inequality and (51), we have

$$\begin{aligned}
&\mathbb{E}|\int_0^t \int_{\mathbb{R}^d} \Phi(s,z,\omega) N(dz,ds)|^p \\
&\leq C[\mathbb{E}(\int_0^t \int_{\mathbb{R}^d} |\Phi(s,z,\omega)|^2 \mu(dz)ds)^{\frac{p}{2}} + \mathbb{E}\int_0^t \int_{\mathbb{R}^d} |\Phi(s,z,\omega)|^p \mu(dz)ds \\
&+\mathbb{E}|\int_0^t \int_{\mathbb{R}^d} |\Phi(s,z,\omega)|\mu(dz)ds|^p] \\
&\leq C\mathbb{E}\int_0^t |\varphi(s,\omega)|^p ds.
\end{aligned}$$

□

For the sake of simplicity of notations, in the following, for a constant C, we do not precise the dependence on the regularity constants of the function b and c (such as $\|\nabla_x b\|_\infty$, L_b and $\bar{c}_p$).

Hypothesis 2.2 We assume that there exists a non-negative function $\breve{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} |\breve{c}(z)|^p \mu(dz) := \breve{c}_p < \infty,\ \forall p \geq 1$, and

$$\big\|\nabla_x c(z,x)(I_d + \nabla_x c(z,x))^{-1}\big\| \leq \breve{c}(z), \quad \forall x \in \mathbb{R}^d, z \in \mathbb{R}^d,$$

with I_d the $d-$dimensional identity matrix. To avoid overburdening notation, since both hypotheses 2.1 and 2.2 apply, we take $\breve{c}(z) = \bar{c}(z)$ and $\breve{c}_p = \bar{c}_p$.

Remark. We need this hypothesis to prove the regularity of the inverse tangent flow (see Section 5.2).

Hypothesis 2.3 (**Ellipticity**) There exists a non-negative function $\underline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that for every $x \in \mathbb{R}^d, z \in \mathbb{R}^d, \zeta \in \mathbb{R}^d$, we have

$$\sum_{j=1}^{d} \langle \partial_{z_j} c(z,x), \zeta \rangle^2 \geq \underline{c}(z)|\zeta|^2.$$

Remark. We notice that together with **Hypothesis 2.1**, we have $\underline{c}(z) \leq |\bar{c}(z)|^2, \ \forall z \in \mathbb{R}^d$.

Hypothesis 2.4

We give some supplementary hypotheses concerning the function $\underline{c}$ and the measure μ.

$a)$ We assume that

$$\underline{\lim}_{u \to +\infty} \frac{1}{\ln u} \overline{\mu}\{\underline{c} \geq \frac{1}{u}\} = \infty, \tag{54}$$

with

$$\overline{\mu}(dz) = \sum_{k=1}^{\infty} \mathbb{1}_{[k-\frac{3}{4}, k-\frac{1}{4}]}(|z|)\mu(dz).$$

This means that $\underline{c}$ could not be too small so that we could have enough noises to deduce the non-degeneracy of the Malliavin covariance matrix (see Section 5.2).

Remark. If $\mu(\mathbb{R}^d) < \infty$, then $\underline{\lim}_{u \to +\infty} \frac{1}{\ln u} \overline{\mu}\{\underline{c} \geq \frac{1}{u}\} = 0$. So (54) implies that $\mu(\mathbb{R}^d) = \infty$.

$b)$ We assume that μ is absolutely continuous with respect to the Lebesgue measure: $\mu(dz) = h(z)dz$, where h is infinitely differentiable and $\ln h$ has bounded derivatives of any order.

Remark. We need this hypothesis to construct the integration by parts framework for the jump equations.

Hypothesis 2.5

We give some conditions which ensure the existence and uniqueness of the invariant measure and the "exponential Lipschitz property" (4).

Suppose that

$$\begin{aligned} i) \quad \langle x-y, b(x)-b(y) \rangle &\leq -\overline{b}\,|x-y|^2 \\ ii) \quad |c(z,x)-c(z,y)| &\leq \bar{c}(z)\,|x-y| \end{aligned} \tag{55}$$

and

$$iii) \quad 2\overline{b} - \int_{\mathbb{R}^d} (2\bar{c}(z) + \bar{c}^2(z))\mu(dz) := \theta > 0. \tag{56}$$

Hypothesis 2.6

We assume that $\mathcal{P}$ is a partition with decreasing time steps: $\mathcal{P} = \{0 = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{n-1} < \Gamma_n < \cdots\}$. We denote $\gamma_n = \Gamma_n - \Gamma_{n-1}, \ n \in \mathbb{N}$ and assume that $\gamma_n \downarrow 0$. We also introduce

$$\overline{\omega} = \overline{\lim_{n \to \infty}} \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2},$$

and assume that $\overline{\omega} < \frac{\theta}{2}$, with θ given in (56).

Remark. A typical example is $\gamma_n = \frac{1}{n}$ and so $\overline{\omega} = 1$.

4.3 The truncated Euler scheme

Now we construct the Euler scheme. For some technical reasons, we take a general partition $\mathcal{P} = \{0 = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{n-1} < \Gamma_n < \cdots\}$ (without assuming **Hypothesis 2.6** at this moment). We denote $\gamma_n = \Gamma_n - \Gamma_{n-1},\ n \in \mathbb{N}$ and denote $|\mathcal{P}| := \max_{n\in\mathbb{N}}(\Gamma_{n+1} - \Gamma_n)$. We assume that $|\mathcal{P}| \leq 1$, and

$$\sum_{i=1}^{\infty} \gamma_i = \lim_{n\to\infty} \Gamma_n = \infty.$$

For $\Gamma_n \leq t < \Gamma_{n+1}$ we denote $N(t) = n$ and $\tau(t) = \Gamma_n$. We consider the Euler scheme:

$$X_t^{\mathcal{P}} \quad = \quad x + \int_0^t b(X_{\tau(r)}^{\mathcal{P}})dr + \int_0^t \int_{\mathbb{R}^d} c(z, X_{\tau(r)-}^{\mathcal{P}})N(dz, dr). \tag{57}$$

Since we have $\mu(\mathbb{R}^d) = \infty$ (which is a consequence of (54)), we have infinitely many jumps. We use a truncation argument in order to have finite numbers of jumps and obtain a representation by means of a compound Poisson process. This is necessary in order to obtain a scheme which may be simulated. We construct the truncated Euler scheme as below. To begin, we give some notations.

We denote

$$\varepsilon_m := \int_{\{|z|>m\}} |\bar{c}(z)|^2 \mu(dz) + |\int_{\{|z|>m\}} \bar{c}(z)\mu(dz)|^2, \quad \forall m \in \mathbb{N}. \tag{58}$$

For every $\gamma > 0$, we define the truncation function $M(\gamma) \in \mathbb{N}$ to be the smallest integer such that

$$\varepsilon_{M(\gamma)} \leq \gamma^2. \tag{59}$$

For $m \in \mathbb{N}$, we denote $B_m = \{z \in \mathbb{R}^d : |z| \leq m\}$. For $\Gamma_{n-1} < t \leq \Gamma_n$, we denote $M_{\mathcal{P}}(t) = M(\gamma_n)$. We remark that we have $\lim_{\gamma\to 0} M(\gamma) = \infty$ and for $\Gamma_{n-1} < t \leq \Gamma_n$, we have $M_{\mathcal{P}}(t) = M(\gamma_n) \geq M(|\mathcal{P}|) \to \infty,$ as $|\mathcal{P}| \to 0$. Now we discard the "big jumps" (the jumps of size $|z| > M_{\mathcal{P}}(t)$):

$$X_t^{\mathcal{P},M_{\mathcal{P}}} \quad = \quad x + \int_0^t b(X_{\tau(r)}^{\mathcal{P},M_{\mathcal{P}}})dr + \int_0^t \int_{B_{M_{\mathcal{P}}(r)}} c(z, X_{\tau(r)-}^{\mathcal{P},M_{\mathcal{P}}})N(dz, dr). \tag{60}$$

The advantage of considering $X_t^{\mathcal{P},M_{\mathcal{P}}}$ is that we may represent it by means of compound Poisson processes. For $k \in \mathbb{N}$, we denote $I_1 = B_1$, $I_k = B_k \backslash B_{k-1}$ for $k \geq 2$ and take $(J_t^k)_{t\geq 0}$ a Poisson process of intensity $\mu(I_k)$. We denote by $(T_i^k)_{i\in\mathbb{N}}$ the jump times of $(J_t^k)_{t\geq 0}$ and we consider a sequences of independent random variables $Z_i^k \sim \mathbb{1}_{I_k}(z)\frac{\mu(dz)}{\mu(I_k)}, k, i \in \mathbb{N}$. Moreover, $(J_t^k)_{\substack{t\geq 0\\ k\in\mathbb{N}}}$ and $(Z_i^k)_{k,i\in\mathbb{N}}$ are taken to be independent. Then we represent the jump's part of the equation (60) by compound Poisson processes. We write

$$\begin{aligned} X_t^{\mathcal{P},M_{\mathcal{P}}} &= x + \int_0^t b(X_{\tau(r)}^{\mathcal{P},M_{\mathcal{P}}})dr + \sum_{k=1}^{\infty}\sum_{i=1}^{J_t^k} \mathbb{1}_{B_{M_{\mathcal{P}}(T_i^k)}}(Z_i^k)c(Z_i^k, X_{\tau(T_i^k)-}^{\mathcal{P},M_{\mathcal{P}}}) \\ &= x + \int_0^t b(X_{\tau(r)}^{\mathcal{P},M_{\mathcal{P}}})dr + \sum_{k=1}^{\infty}\sum_{i=1}^{J_t^k}\sum_{n=0}^{N(t)} \mathbb{1}_{B_{M(\gamma_{n+1})}}(Z_i^k)c(Z_i^k, X_{\tau(T_i^k)-}^{\mathcal{P},M_{\mathcal{P}}}). \end{aligned}$$

Since $Z_i^k \in B_k \backslash B_{k-1}$, it follows that $Z_i^k \in B_{M(\gamma_{n+1})}$ is equivalent to $k \leq M(\gamma_{n+1})$. Then

$$X_t^{\mathcal{P},M_{\mathcal{P}}} \quad = \quad x + \int_0^t b(X_{\tau(r)}^{\mathcal{P},M_{\mathcal{P}}})dr + \sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k} \mathbb{1}_{\{\Gamma_n < T_i^k \leq \Gamma_{n+1}\wedge t\}}c(Z_i^k, X_{\tau(T_i^k)-}^{\mathcal{P},M_{\mathcal{P}}}). \tag{61}$$

We remark that the solution of the equation (61) can be constructed in an explicit way.

We recall the notation θ in **Hypothesis 2.5**. We also recall $n_\rho = n_{\frac{\theta}{2}}$ in (10) (with $\rho = \frac{\theta}{2}$ in our case) and n_* in (11). We obtain the following error estimate for $X_t^{\mathcal{P},M_{\mathcal{P}}}$, which represents the main result in our paper.

Theorem 4.1. *Assume that **Hypothesis 2.1∼2.5** hold and the partition $\mathcal{P}$ satisfies **Hypothesis 2.6**. Then an invariant probability measure ν exists and is unique, and for $n > \max\{n_{\frac{\theta}{2}} + 3, n_* + 3\}$, for any $\varepsilon > 0$ there exists a constant C_ε such that*

$$d_{TV}(\mathcal{L}(X^{\mathcal{P},M_{\mathcal{P}}}_{\Gamma_n}), \nu) \leq C_\varepsilon(\gamma_n^{1-\varepsilon} + \int_{\mathbb{R}^d} |x-y|\, d\nu(y) e^{-\frac{\theta}{2}\Gamma_n}). \tag{62}$$

The proof of this theorem will be given in Section 6 by using some Malliavin integration by parts techniques introduced in Section 5.

In order to apply the Malliavin framework which will be presented in Section 5, we introduce additionally an auxiliary equation as follows (see (64) below).

For $\Gamma_n < t \leq \Gamma_{n+1}$, we define

$$a_t^{\mathcal{P}} = (\sum_{1\leq i\leq n} \gamma_i \int_{\{|z|\geq M(\gamma_i)\}} \underline{c}(z)\mu(dz) + (t-\Gamma_n)\int_{\{|z|\geq M(\gamma_{n+1})\}} \underline{c}(z)\mu(dz))^{\frac{1}{2}}, \tag{63}$$

where $\underline{c}$ is given in **Hypothesis 2.3**. We notice that $|a_t^{\mathcal{P}}| \leq \sqrt{t \times \varepsilon_{M(|\mathcal{P}|)}} \leq \sqrt{t} \times |\mathcal{P}|$.

Now we cancel the big jumps in equation (49) and replace them by a ($d-$dimensional) Gaussian random variable Δ which is independent of the Poisson point measure $N(dz, ds)$:

$$X_t^{M_{\mathcal{P}}} = x + a_t^{\mathcal{P}}\Delta + \int_0^t b(X_s^{M_{\mathcal{P}}})ds + \int_0^t \int_{B_{M_{\mathcal{P}}(s)}} c(z, X_{s-}^{M_{\mathcal{P}}})N(dz, ds). \tag{64}$$

We remark that Δ is necessary in order to obtain the non degeneracy of the covariance matrix (see Section 5.2 for details).

Following the same idea as above, we represent the jump's parts of the equation (64) by compound Poisson processes:

$$X_t^{M_{\mathcal{P}}} = x + a_t^{\mathcal{P}}\Delta + \int_0^t b(X_s^{M_{\mathcal{P}}})ds + \sum_{n=0}^{N(t)} \sum_{k=1}^{M(\gamma_{n+1})} \sum_{i=1}^{J_i^k} \mathbb{1}_{\{\Gamma_n < T_i^k \leq \Gamma_{n+1}\wedge t\}} c(Z_i^k, X_{T_i^k-}^{M_{\mathcal{P}}}). \tag{65}$$

We sometimes write $X_t^{\mathcal{P},M_{\mathcal{P}}}(x)$ (resp. $X_t^{M_{\mathcal{P}}}(x)$, $X_t(x)$) instead of $X_t^{\mathcal{P},M_{\mathcal{P}}}$ (resp. $X_t^{M_{\mathcal{P}}}$, X_t) to stress the dependence on the initial value x.

4.4 Some examples

We give some typical examples to illustrate our main results.

Example 1 We take $h = 1$ so the measure μ is the Lebesgue measure. We consider two types of behaviour for c.

i) Exponential decay We assume that $|\bar{c}(z)|^2 = e^{-a_1|z|^p}$ and $\underline{c}(z) = e^{-a_2|z|^p}$ with some constants $0 < a_1 \leq a_2$, $p > 0$. We only check **Hypothesis 2.4** here. We have

$$\overline{\mu}\{\underline{c} > \frac{1}{u}\} = \overline{\mu}\{|z| < (\frac{\ln u}{a_2})^{\frac{1}{p}}\} \geq \frac{r_d}{2}(\frac{\ln(u-1)}{a_2})^{\frac{d}{p}},$$

with r_d the volume of the unit ball in $\mathbb{R}^d$, so that

$$\frac{1}{\ln u}\overline{\mu}\{\underline{c} > \frac{1}{u}\} \geq \frac{r_d}{2(a_2)^{\frac{d}{p}}}\frac{(\ln(u-1))^{\frac{d}{p}}}{\ln u}.$$

We notice that $\underline{\lim}_{u\to+\infty} \frac{1}{\ln u}\overline{\mu}\{\underline{c} \geq \frac{1}{u}\} = \infty$ when $0 < p < d$. Therefore, when $p \geq d$, we can say nothing; when $0 < p < d$, the results in **Theorem 4.1** are true.

ii) Polynomial decay We assume that $|\bar{c}(z)|^2 = \frac{a_1}{1+|z|^p}$ and $\underline{c}(z) = \frac{a_2}{1+|z|^p}$ for some constants $0 < a_2 \leq a_1$ and $p > d$. Then

$$\overline{\mu}\{\underline{c} > \frac{1}{u}\} = \overline{\mu}\{|z| < (a_2 u - 1)^{\frac{1}{p}}\} \geq \frac{r_d}{2}(a_2(u-1)-1)^{\frac{d}{p}},$$

so that

$$\frac{1}{\ln u}\overline{\mu}\{\underline{c} > \frac{1}{u}\} \geq \frac{r_d}{2}\frac{(a_2(u-1)-1)^{\frac{d}{p}}}{\ln u}.$$

We notice that in this case, $\underline{\lim}_{u\to+\infty}\frac{1}{\ln u}\overline{\mu}\{\underline{c} \geq \frac{1}{u}\} = \infty$. Thus, the results in **Theorem 4.1** hold true.

Example 2 We consider the ($1-$dimensional) truncated $\alpha-$stable process: $X_t = X_0 + \int_0^t \sigma(X_{r-})dU_r$. Here $(U_t)_{t\geq 0}$ is a (pure jump) Lévy process with intensity measure

$$\mathbb{1}_{\{|z|\leq 1\}}\frac{1}{|z|^{1+\alpha}}dz, \quad 0 \leq \alpha < 1.$$

We assume that $\sigma \in C_b^\infty(\mathbb{R})$, $0 < \underline{\sigma} \leq \sigma(x) \leq \bar{\sigma}$ and $-1 < \underline{a} \leq \sigma'(x) \leq \bar{\sigma}$, $\forall x \in \mathbb{R}$, for some universal constants $\bar{\sigma}, \underline{\sigma}, \underline{a}$, where σ' is the differential of σ in x. Then by a change of variable $z \mapsto \frac{1}{z}$, we come back to the setting of this paper with $c(r,v,z,x,\rho) = \sigma(x) \times \frac{1}{z}$ and $\mu(dz) = \mathbb{1}_{\{|z|\geq 1\}}\frac{1}{|z|^{1-\alpha}}dz$. We only check **Hypothesis 2.4** here. In this case, $\underline{c}(z) = \underline{\sigma} \times \frac{1}{|z|^4}$, then

$$\frac{1}{\ln u}\overline{\mu}\{\underline{c} > \frac{1}{u}\} \geq \frac{1}{\ln u}\int_1^{(\underline{\sigma}(u-1))^{\frac{1}{4}}} \frac{1}{|z|^{1-\alpha}}dz = \frac{(\underline{\sigma}(u-1))^{\frac{\alpha}{4}}-1}{\alpha \ln u},$$

so that $\underline{\lim}_{u\to+\infty}\frac{1}{\ln u}\overline{\mu}\{\underline{c} \geq \frac{1}{u}\} = \infty$. Thus we can apply **Theorem 4.1**.

5 Malliavin framework for jump equations

We take time $t \in [0,3]$ throughout this section and we use the notations from Section 4. We recall $(X_t)_{t\in[0,3]}$ in (49), $(X_t^{\mathcal{P},M_\mathcal{P}})_{t\in[0,3]}$ in (60) and $(X_t^{M_\mathcal{P}})_{t\in[0,3]}$ in (64), where $\mathcal{P} = \{0 = \Gamma_0 < \Gamma_1 < \cdots < \Gamma_{N(3)} \leq 3\}$ is a general partition (which is not supposed to verify **Hypothesis 2.6**).

Lemma 5.1. *Suppose that **Hypothesis 2.1** holds true. Then we have the followings.*

i) For every $t \in [0,3]$, we have

$$\mathbb{E}|X_t^{\mathcal{P},M_\mathcal{P}} - X_t| \to 0, \text{ as } |\mathcal{P}| \to 0;$$

ii) For every fixed $t \in [0,3]$ and every $p \geq 2$, we have

$$\mathbb{E}|X_t^{M_\mathcal{P}} - X_t|^p \to 0, \text{ as } |\mathcal{P}| \to 0;$$

iii) For every fixed $t \in [0,3]$ and every multi-index β, we have

$$\mathbb{E}|\partial_x^\beta X_t^{M_\mathcal{P}} - \partial_x^\beta X_t| \to 0, \text{ as } |\mathcal{P}| \to 0.$$

Proof. The proof of this lemma is standard and straightforward by Gronwall lemma and Buckholder inequality. So we leave it out. □

Now we use Malliavin calculus for $X_t^{\mathcal{P},M_\mathcal{P}}$, $X_t^{M_\mathcal{P}}$ and X_t. There are several approaches given in [12], [20], [26], [27], [37], [48] and [50] for example. Here we give a framework analogous to [9].

To begin we define a regularization function.

$$a(y) = 1 - \frac{1}{1-(4y-1)^2} \quad for \quad y \in [\tfrac{1}{4}, \tfrac{1}{2}), \tag{66}$$

$$\psi(y) = \mathbb{1}_{\{|y|\leq\frac{1}{4}\}} + \mathbb{1}_{\{\frac{1}{4}<|y|\leq\frac{1}{2}\}}e^{a(|y|)}. \tag{67}$$

We notice that $\psi \in C_b^\infty(\mathbb{R})$ and that its support is included in $[-\frac{1}{2}, \frac{1}{2}]$. We denote

$$\Psi_k(y) = \psi(|y| - (k - \tfrac{1}{2})),\ \forall k \in \mathbb{N}. \tag{68}$$

Then for any $l \in \mathbb{N}$, there exists a constant C_l such that

$$\sup_{k\in\mathbb{N}} \|\Psi_k\|_{l,\infty} \leq C_l < \infty. \tag{69}$$

We focus on $X_t^{\mathcal{P},M_\mathcal{P}}(x)$ and $X_t^{M_\mathcal{P}}(x)$ (solutions of (61) and (65)) which are functions of random variables T_i^k, Z_i^k and Δ .

Now we introduce the space of simple functionals $\mathcal{S}$. We take $\mathcal{G} = \sigma(T_i^k : k, i \in \mathbb{N})$ to be the $\sigma-$algebra associated to the noises which will not be involved in our calculus. In the following, we will do the calculus based on $Z_i^k = (Z_{i,1}^k, \cdots, Z_{i,d}^k), k, i \in \mathbb{N}$ and $\Delta = (\Delta_1, \cdots, \Delta_d)$. We denote by $C_{\mathcal{G},p}$ the space of the functions $f : \Omega \times \mathbb{R}^{m\times m'\times d+d} \to \mathbb{R}$ such that for each ω, the function $(z_{1,1}^1, ..., z_{m,d}^{m'}, \delta_1, \cdots, \delta_d) \mapsto f(\omega, z_{1,1}^1, ..., z_{m,d}^{m'}, \delta_1, \cdots, \delta_d)$ belongs to $C_p^\infty(\mathbb{R}^{m\times m'\times d+d})$ (the space of smooth functions which, together with all the derivatives, have polynomial growth), and for each $(z_{1,1}^1, ..., z_{m,d}^{m'}, \delta_1, \cdots, \delta_d)$, the function $\omega \mapsto f(\omega, z_{1,1}^1, ..., z_{m,d}^{m'}, \delta_1, \cdots, \delta_d)$ is $\mathcal{G}$-measurable. And we consider the weights

$$\xi_i^k = \Psi_k(Z_i^k). \tag{70}$$

Then we define the space of simple functionals

$$\mathcal{S} = \{F = f(\omega, (Z_i^k)_{\substack{1\leq k\leq m'\\ 1\leq i\leq m}}, \Delta) : f \in C_{\mathcal{G},p}, m, m' \in \mathbb{N}\}.$$

Remark. Take $m' = \max_{t\leq 3} M_\mathcal{P}(t)$ and $m = \max_{k\leq m'} J_t^k$. Then $X_t^{M_\mathcal{P}}(x)$ (solution of (65)) is a function of T_i^k, Z_i^k and of Δ, with $k \leq m'$ and $i \leq m$. So it is a simple functional (the same for ${X_t}^{\mathcal{P},M_\mathcal{P}}(x)$ (solution of (61))).

On the space $\mathcal{S}$, for $t \geq 1$, we define the derivative operator $DF = (D^Z F, D^\Delta F)$, where

$$\begin{aligned} D^Z_{(\bar{k},\bar{i},\bar{j})}F &= \xi_{\bar{i}}^{\bar{k}} \frac{\partial f}{\partial z_{\bar{i},\bar{j}}^{\bar{k}}}(\omega, (Z_i^k)_{\substack{1\leq k\leq m'\\ 1\leq i\leq m}}, \Delta), \quad \bar{k}, \bar{i} \in \mathbb{N}, \bar{j} \in \{1, \cdots, d\}, \\ D^\Delta_{\tilde{j}}F &= \frac{\partial f}{\partial \delta_{\tilde{j}}}(\omega, (Z_i^k)_{\substack{1\leq k\leq m'\\ 1\leq i\leq m}}, \Delta), \quad \tilde{j} \in \{1, \cdots, d\}. \end{aligned} \tag{71}$$

We regard $D^Z F$ as an element of the Hilbert space l_2 (the space of the sequences $u = (u_{k,i,j})_{k,i\in\mathbb{N},j\in\{1,\cdots,d\}}$ with $|u|_{l_2}^2 := \sum_{k=1}^\infty \sum_{i=1}^\infty \sum_{j=1}^d |u_{k,i,j}|^2 < \infty$) and DF as an element of $l_2 \times \mathbb{R}^d$, so we have

$$\langle DF, DG\rangle_{l_2\times\mathbb{R}^d} = \sum_{j=1}^d D_j^\Delta F \times D_j^\Delta G + \sum_{k=1}^\infty \sum_{i=1}^\infty \sum_{j=1}^d D^Z_{(k,i,j)}F \times D^Z_{(k,i,j)}G. \tag{72}$$

We also denote $D^1 F = DF$, and we define the derivatives of order $q \in \mathbb{N}$ recursively: $D^q F := DD^{q-1}F$. And we denote $D^{Z,q}$ (respectively $D^{\Delta,q}$) as the derivative D^Z (respectively D^Δ) of order q.

We recall that $\mu(dz) = h(z)dz$ with $h \in C^\infty(\mathbb{R}^d)$ (see **Hypothesis 2.4** b)). We define the Ornstein-Uhlenbeck operator $LF = L^Z F + L^\Delta F$ with

$$\begin{aligned} L^Z F &= -\sum_{k=1}^{m'} \sum_{i=1}^m \sum_{j=1}^d (\partial_{z_{i,j}^k}(\xi_i^k D^Z_{(k,i,j)}F) + D^Z_{(k,i,j)}F \times D^Z_{(k,i,j)} \ln[h(Z_i^k)]), \\ L^\Delta F &= \sum_{j=1}^d D_j^\Delta F \times \Delta_j - \sum_{j=1}^d D_j^\Delta D_j^\Delta F. \end{aligned} \tag{73}$$

One can check that the triplet $(\mathcal{S}, D, L)$ is consistent with the IbP framework given in Section 3.1. In particular the duality formula (28) holds true. We refer to [10](Appendix 5.3). We say that F is a "Malliavin smooth functional" if $F \in \mathcal{D}_\infty$ (with the definition given in (32)).

We recall $X_t^{\mathcal{P},M_\mathcal{P}}(x)$ in (61), $X_t^{M_\mathcal{P}}(x)$ in (65) and $X_t(x)$ in (49). We denote

$$F_t^{\mathcal{P},M_\mathcal{P}}(x) = X_t^{\mathcal{P},M_\mathcal{P}}(x) - x, F_t^{M_\mathcal{P}}(x) = X_t^{M_\mathcal{P}}(x) - x \text{ and } F_t(x) = X_t(x) - x \tag{74}$$

In the following subsections, we will give some lemmas concerning the Sobolev norms and the covariance matrices. We recall (see (26)) that σ_F denotes the covariance matrix of F, and recall the Sobolev norms defined in (29) and (30).

5.1 Sobolev norms

We recall the notations $F_t^{\mathcal{P},M_\mathcal{P}}(x)$, $F_t^{M_\mathcal{P}}(x)$ and $F_t(x)$ in (74).

Lemma 5.2. *Assuming **Hypothesis 2.1** and **Hypothesis 2.4** b), for all $p \geq 1, l \geq 0$, there exists a constant $C_{l,p}$ depending on l, p, d, such that for any $t \in [0,3]$,*

$$i) \quad \sup_{\mathcal{P}} \sup_x (\|{F_t}^{\mathcal{P},M_\mathcal{P}}(x)\|_{L,l,p} + \|F_t^{M_\mathcal{P}}(x)\|_{L,l,p}) \leq C_{l,p}.$$

Moreover, $F_t(x)$ belongs to $\mathcal{D}_\infty^d$ and

$$ii) \quad \sup_x \|F_t(x)\|_{L,l,p} \leq C_{l,p}.$$

For all $p, q \geq 1, l \geq 0$, there exists a constant $C_{l,p,q}$ depending on l, p, q, d, such that for every multi-index β with $|\beta| = q$, we also have

$$iii) \quad \sup_x \|\partial_x^\beta (X_t(x))\|_{l,p} \leq C_{l,p,q}.$$

Remark. Since $Dx = 0, \ \forall x \in \mathbb{R}^d$, we also have

$$\sup_{\mathcal{P}} \sup_x (\mathbb{E}|X_t^{\mathcal{P},M_\mathcal{P}}(x)|_{1,l}^p + \mathbb{E}|X_t^{M_\mathcal{P}}(x)|_{1,l}^p + \mathbb{E}|X_t(x)|_{1,l}^p) \leq C_{l,p}.$$

Proof. We first notice that for any l, p, $\sup_{\mathcal{P}} \sup_x (\|{F_t}^{\mathcal{P},M_\mathcal{P}}(x)\|_{L,l,p} + \|F_t^{M_\mathcal{P}}(x)\|_{L,l,p}) \leq C_{l,p}$ This is a slight variant of the proof of Lemma 3.7 $i)$ in [44]. The difference in that the truncation function M is constant in [44] while here it depends on the time. But this does not change anything. In a similar way, for every multi-index β with $|\beta| = q$, we have $\sup_{\mathcal{P}} \sup_x \|\partial_x^\beta (X_t^{M_\mathcal{P}}(x))\|_{l,p} \leq C_{l,p,q}$.

Afterwards, we consider an increasing sequence of partition $\mathcal{P}_n, \ n \in \mathbb{N}$, ($\mathcal{P}_n \subset \mathcal{P}_{n+1}$), such that $|\mathcal{P}_n| \downarrow 0$. In particular, $\forall t, \ M_{\mathcal{P}_n}(t) \uparrow \infty$. Noticing by **Lemma 5.1** $ii)$ that $\mathbb{E}|F_t^{M_{\mathcal{P}_n}} - F_t| \to 0$ as $n \to 0$, and applying **Lemma 3.3 (A)** with $F_n = F_t^{M_{\mathcal{P}_n}}$ and $F = F_t$, we get that F_t belongs to $\mathcal{D}_\infty^d$ and $\sup_x \|F_t(x)\|_{L,l,p} \leq C_{l,p}$.

Furthermore, noticing by **Lemma 5.1** $iii)$ that $\mathbb{E}|\partial_x^\beta X_t^{M_{\mathcal{P}_n}} - \partial_x^\beta X_t| \to 0$ as $n \to 0$, and applying **Lemma 3.3 (A)** with $F_n = \partial_x^\beta X_t^{M_{\mathcal{P}_n}}$ and $F = \partial_x^\beta X_t$, we obtain that $\partial_x^\beta X_t$ belongs to $\mathcal{D}_\infty^d$ and $\sup_x \|\partial_x^\beta (X_t(x))\|_{l,p} \leq C_{l,p,q}$.

□

5.2 Covariance matrix

Lemma 5.3. *Assume that **Hypothesis 2.1, 2.2, 2.3** and **2.4** hold true. We denote the lowest eigenvalue of the Malliavin covariance matrix $\sigma_{X_t^{M_\mathcal{P}}}$ by $\lambda_t^{M_\mathcal{P}}$. Then for every $p \geq 1$, $1 \leq t \leq 3$, we have*

$$i) \quad \sup_{\mathcal{P}} \sup_x \mathbb{E}(1/\det \sigma_{X_t^{M_\mathcal{P}}(x)})^p \leq \sup_{\mathcal{P}} \sup_x \mathbb{E}(|\lambda_t^{M_\mathcal{P}}|^{-dp}) \leq C_p,$$
$$ii) \quad \sup_x \mathbb{E}(1/\det \sigma_{X_t(x)})^p \leq C_p,$$

with C_p a constant depending on p, d.

Remark. We recall the notations $F_t^{M_\mathcal{P}}(x) = X_t^{M_\mathcal{P}}(x) - x$ and $F_t(x) = X_t(x) - x$. Since $Dx = 0$, $\forall x \in \mathbb{R}^d$, the above results are equivalent to

$$i) \quad \sup_{\mathcal{P}} \sup_x \mathbb{E}(1/\det \sigma_{F_t^{M_\mathcal{P}}(x)})^p \leq \sup_{\mathcal{P}} \sup_x \mathbb{E}(|\lambda_t^{M_\mathcal{P}}|^{-dp}) \leq C_p,$$
$$ii) \quad \sup_x \mathbb{E}(1/\det \sigma_{F_t(x)})^p \leq C_p.$$

Proof of $i)$ We proceed in 4 steps.
Step 1 We notice by the definition (71) that for any $k_0, i_0 \in \mathbb{N}, j \in \{1, \cdots, d\}$,

$$\begin{aligned} D^Z_{(k_0,i_0,j)} X_t^{M_\mathcal{P}} &= \int_{T^{k_0}_{i_0}}^t \nabla_x b(X_r^{M_\mathcal{P}}) D^Z_{(k_0,i_0,j)} X_r^{M_\mathcal{P}} dr \\ &+ \sum_{n=0}^{N(t)} \mathbb{1}_{\{\Gamma_n < T^{k_0}_{i_0} \leq \Gamma_{n+1} \wedge t\}} \mathbb{1}_{\{1 \leq k_0 \leq M(\gamma_{n+1})\}} \xi^{k_0}_{i_0} \partial_{z^{k_0}_{i_0,j}} c(Z^{k_0}_{i_0}, X^{M_\mathcal{P}}_{T^{k_0}_{i_0}-}) \\ &+ \sum_{n=0}^{N(t)} \sum_{k=1}^{M(\gamma_{n+1})} \sum_{i=1}^{J^k_t} \mathbb{1}_{\{\Gamma_n \vee T^{k_0}_{i_0} < T^k_i \leq \Gamma_{n+1} \wedge t\}} \nabla_x c(Z^k_i, X^{M_\mathcal{P}}_{T^k_i-}) D^Z_{(k_0,i_0,j)} X^{M_\mathcal{P}}_{T^k_i-}, \end{aligned} \tag{75}$$

$$D^\Delta_j X_t^{M_\mathcal{P}} = a_t^\mathcal{P} \boldsymbol{e_j} + \int_0^t \nabla_x b(X_r^{M_\mathcal{P}}) D^\Delta_j X_r^{M_\mathcal{P}} dr + \sum_{n=0}^{N(t)} \sum_{k=1}^{M(\gamma_{n+1})} \sum_{i=1}^{J^k_t} \mathbb{1}_{\{\Gamma_n < T^k_i \leq \Gamma_{n+1} \wedge t\}} \nabla_x c(Z^k_i, X^{M_\mathcal{P}}_{T^k_i-}) D^\Delta_j X^{M_\mathcal{P}}_{T^k_i-}, \tag{76}$$

where $\boldsymbol{e_j} = (0, \cdots, 0, 1, 0, \cdots, 0)$ with value 1 at the $j-$th component.

Now we introduce $(Y_t^{M_\mathcal{P}})_{t \geq 0}$ (this is so-called the tangent flow) which is the matrix solution of the linear equation

$$Y_t^{M_\mathcal{P}} = I_d \quad + \quad \int_0^t \nabla_x b(X_r^{M_\mathcal{P}}) Y_r^{M_\mathcal{P}} dr + \sum_{n=0}^{N(t)} \sum_{k=1}^{M(\gamma_{n+1})} \sum_{i=1}^{J^k_t} \mathbb{1}_{\{\Gamma_n < T^k_i \leq \Gamma_{n+1} \wedge t\}} \nabla_x c(Z^k_i, X^{M_\mathcal{P}}_{T^k_i-}) Y^{M_\mathcal{P}}_{T^k_i-}.$$

And using Itô's formula, the inverse matrix $\widetilde{Y}_t^{M_\mathcal{P}} = (Y_t^{M_\mathcal{P}})^{-1}$ verifies the equation

$$\widetilde{Y}_t^{M_\mathcal{P}} = I_d \quad - \quad \int_o^t \widetilde{Y}_r^{M_\mathcal{P}} \nabla_x b(X_r^{M_\mathcal{P}}) dr - \sum_{n=0}^{N(t)} \sum_{k=1}^{M(\gamma_{n+1})} \sum_{i=1}^{J^k_t} \mathbb{1}_{\{\Gamma_n < T^k_i \leq \Gamma_{n+1} \wedge t\}} \widetilde{Y}^{M_\mathcal{P}}_{T^k_i-} \nabla_x c (I_d + \nabla_x c)^{-1} (Z^k_i, X^{M_\mathcal{P}}_{T^k_i-}). \tag{77}$$

Remark. We notice that $Y_t^{M_\mathcal{P}} = \nabla_x (X_t^{M_\mathcal{P}}(x))$. If instead we consider the gradient of the Euler scheme $Y_t^{\mathcal{P}, M_\mathcal{P}} = \nabla_x (X_t^{\mathcal{P}, M_\mathcal{P}}(x))$, the matrix $Y_t^{\mathcal{P}, M_\mathcal{P}}$ is not invertible, and this is a specific difficulty when we deal with the Euler scheme. This is why we have to work with $X_t^{M_\mathcal{P}}$ only.

Applying **Hypothesis 2.1** and **Hypothesis 2.2**, one also has

$$\mathbb{E}(\sup_{0<t\leq 2}(\left\|Y_t^{M_{\mathcal{P}}}\right\|^p+\left\|\widetilde{Y}_t^{M_{\mathcal{P}}}\right\|^p))\leq C_p<\infty. \tag{78}$$

The proof of (78) is straightforward and we leave it out.

Then using the uniqueness of solution to the equation (75) and (76), one obtains

$$D^Z_{(k,i,j)}X_t^{M_{\mathcal{P}}}=\sum_{n=0}^{N(t)}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}\mathbb{1}_{\{1\leq k\leq M(\gamma_{n+1})\}}\xi_i^k Y_t^{M_{\mathcal{P}}}\widetilde{Y}_{T_i^k}^{M_{\mathcal{P}}}\partial_{z^k_{i,j}}c(Z_i^k,X_{T_i^k-}^{M_{\mathcal{P}}}), \tag{79}$$

and $D^{\Delta}_j X_t^{M_{\mathcal{P}}}=a_t^{\mathcal{P}}\,Y_t^{M_{\mathcal{P}}}\boldsymbol{e_j}$.

We recall that we denote the lowest eigenvalue of the Malliavin covariance matrix $\sigma_{X_t^{M_{\mathcal{P}}}}$ by $\lambda_t^{M_{\mathcal{P}}}$. Then we have (recalling the definitions (26) and (72))

$$\lambda_t^{M_{\mathcal{P}}}=\inf_{|\zeta|=1}\langle\sigma_{X_t^{M_{\mathcal{P}}}}\zeta,\zeta\rangle\geq\inf_{|\zeta|=1}\sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}\sum_{j=1}^{d}\langle D^Z_{(k,i,j)}\,X_t^{M_{\mathcal{P}}},\zeta\rangle^2+\inf_{|\zeta|=1}\sum_{j=1}^{d}\langle D^{\Delta}_j X_t^{M_{\mathcal{P}}},\zeta\rangle^2.$$

By (79),

$$\begin{aligned}\lambda_t^{M_{\mathcal{P}}}&\geq\inf_{|\zeta|=1}\sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}\sum_{j=1}^{d}|\xi_i^k|^2\langle\partial_{z^k_{i,j}}c(Z_i^k,X_{T_i^k-}^{M_{\mathcal{P}}}),(Y_t^{M_{\mathcal{P}}}\widetilde{Y}_{T_i^k}^{M_{\mathcal{P}}})^*\zeta\rangle^2\\&+\inf_{|\zeta|=1}\sum_{j=1}^{d}|a_t^{\mathcal{P}}|^2\langle\boldsymbol{e_j},(Y_t^{M_{\mathcal{P}}})^*\zeta\rangle^2,\end{aligned}$$

where Y^* denotes the transposition of a matrix Y.

We recall the ellipticity hypothesis (**Hypothesis 2.3**): there exists a non-negative function $\underline{c}(z)$ such that

$$\sum_{j=1}^{d}\langle\partial_{z_j}c(z,x),\zeta\rangle^2\geq\underline{c}(z)|\zeta|^2.$$

So we deduce that

$$\lambda_t^{M_{\mathcal{P}}}\geq\inf_{|\zeta|=1}(\sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}|\xi_i^k|^2\underline{c}(Z_i^k)|(Y_t^{M_{\mathcal{P}}}\widetilde{Y}_{T_i^k}^{M\ \mathcal{P}})^*\zeta|^2)+|a_t^{\mathcal{P}}|^2\inf_{|\zeta|=1}|(Y_t^{M_{\mathcal{P}}})^*\zeta|^2.$$

For every invertible matrix A and every vector y, one has $|Ay|\geq\frac{1}{\|A^{-1}\|}|y|$, so that

$$\begin{aligned}\lambda_t^{M_{\mathcal{P}}}&\geq(\sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}|\xi_i^k|^2\underline{c}(Z_i^k)\|\widetilde{Y}_t^{M_{\mathcal{P}}}\|^{-2}\|Y_{T_i^k}^{M_{\mathcal{P}}}\|^{-2})+|a_t^{\mathcal{P}}|^2\|\widetilde{Y}_t^{M_{\mathcal{P}}}\|^{-2}\\&\geq(\inf_{0<t\leq 2}\|\widetilde{Y}_t^{M_{\mathcal{P}}}\|^{-2}\|Y_t^{M_{\mathcal{P}}}\|^{-2})((\sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}|\xi_i^k|^2\underline{c}(Z_i^k))+|a_t^{\mathcal{P}}|^2).\end{aligned}$$

We denote

$$\chi_t^{M_{\mathcal{P}}}=\sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\sum_{i=1}^{J_t^k}\mathbb{1}_{\{\Gamma_n<T_i^k\leq\Gamma_{n+1}\wedge t\}}|\xi_i^k|^2\underline{c}(Z_i^k). \tag{80}$$

By (78), $(\mathbb{E}\sup_{0\le t\le 2}\|\widetilde{Y}_t^{M_{\mathcal{P}}}\|^{4dp}\|Y_t^{M_{\mathcal{P}}}\|^{4dp})^{1/2}\le C_{d,p}<\infty$, so that using Schwartz inequality, we have

$$\mathbb{E}|\frac{1}{\det\sigma_{X_t^{M_{\mathcal{P}}}}}|^p\le\mathbb{E}(|\lambda_t^{M_{\mathcal{P}}}|^{-dp})\le C(\mathbb{E}(|\chi_t^{M_{\mathcal{P}}}+|a_t^{\mathcal{P}}|^2|^{-2dp}))^{\frac{1}{2}}. \tag{81}$$

Step 2 Since it is not easy to compute $\mathbb{E}(|\chi_t^{M_{\mathcal{P}}}+|a_t^{\mathcal{P}}|^2|^{-2dp}))$ directly, we make the following argument where the idea comes originally from [12]. Let $\Gamma(p)=\int_0^\infty s^{p-1}e^{-s}ds$ be the Gamma function. By a change of variables, we have the numerical equality

$$\frac{1}{|\chi_t^{M_{\mathcal{P}}}+|a_t^{\mathcal{P}}|^2|^{2dp}}=\frac{1}{\Gamma(2dp)}\int_0^\infty s^{2dp-1}e^{-s(\chi_t^{M_{\mathcal{P}}}+|a_t^{\mathcal{P}}|^2)}ds,$$

which, by taking expectation, gives

$$\mathbb{E}(\frac{1}{|\chi_t^{M_{\mathcal{P}}}+|a_t{}^{\mathcal{P}}|^2|^{2dp}})=\frac{1}{\Gamma(2dp)}\int_0^\infty s^{2dp-1}\mathbb{E}(e^{-s(\chi_t^{M_{\mathcal{P}}}+|a_t^{\mathcal{P}}|^2)})ds. \tag{82}$$

Step 3 Now we compute $\mathbb{E}(e^{-s(\chi_t^{M\ \mathcal{P}}+|a_t^{\mathcal{P}}|^2)})$ for any $s>0$. We recall that $I_1=B_1$, $I_k=B_k-B_{k-1}, k\ge 2$ (given in Section 4.3), and $\xi_i^k=\Psi_k(Z_i^k)$ (see (70)). Then

$$\chi_t^{M_{\mathcal{P}}} = \sum_{n=0}^{N(t)}\sum_{k=1}^{M(\gamma_{n+1})}\int_{\Gamma_n}^{\Gamma_{n+1}\wedge t}\int_{I_k}|\Psi_k(z)|^2\underline{c}(z)N(dz,dr)=\int_0^t\int_{B_{M_{\mathcal{P}}(r)}}\Psi(z)\underline{c}(z)N(dz,dr),$$

with

$$\Psi(z)=\sum_{k=1}^\infty|\Psi_k(z)|^2\mathbb{1}_{I_k}(z)\ge\sum_{k=1}^\infty\mathbb{1}_{[k-\frac{3}{4},k-\frac{1}{4}]}(|z|)\mathbb{1}_{I_k}(z).$$

Using Itô formula,

$$\begin{aligned}\mathbb{E}(e^{-s\chi_t^{M_{\mathcal{P}}}}) &= 1+\mathbb{E}\int_0^t\int_{B_{M_{\mathcal{P}}(r)}}(e^{-s(\chi_{r-}^{M\ \mathcal{P}}+\Psi(z)\underline{c}(z))}-e^{-s\chi_{r-}^{M_{\mathcal{P}}}})\widehat{N}(dz,dr)\\ &= 1-\sum_{n=0}^{N(t)}(\int_{\Gamma_n}^{\Gamma_{n+1}\wedge t}\mathbb{E}(e^{-s\chi_r^{M_{\mathcal{P}}}})dr\sum_{k=1}^{M(\gamma_{n+1})}\int_{I_k}(1-e^{-s|\Psi_k(z)|^2\underline{c}(z)})\mu(dz)).\end{aligned}$$

Solving the above equation we obtain

$$\begin{aligned}\mathbb{E}(e^{-s\chi_t^{M_{\mathcal{P}}}}) &= \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\sum_{k=1}^{M(\gamma_{n+1})}\int_{I_k}(1-e^{-s|\Psi_k(z)|^2\underline{c}(z)})\mu(dz)))\\ &\le \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\sum_{k=1}^{M(\gamma_{n+1})}\int_{I_k}(1-e^{-s\mathbb{1}_{[k-\frac{3}{4},k-\frac{1}{4}]}(|z|)\underline{c}(z)})\mu(dz)))\\ &= \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\sum_{k=1}^{M(\gamma_{n+1})}\int_{I_k}(1-e^{-s\underline{c}(z)})\mathbb{1}_{[k-\frac{3}{4},k-\frac{1}{4}]}(|z|)\mu(dz)))\\ &= \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\int_{B_{M(\gamma_{n+1})}}(1-e^{-s\underline{c}(z)})\overline{\mu}(dz)),\end{aligned}$$

with

$$\overline{\mu}(dz)=\sum_{k=1}^\infty\mathbb{1}_{[k-\frac{3}{4},k-\frac{1}{4}]}(|z|)\mu(dz).$$

On the other hand, we denote

$$\bar{\chi}_t^{M_{\mathcal{P}}} = \int_0^t \int_{B^c_{M_{\mathcal{P}}(r)}} \Psi(z)\underline{c}(z)N(dz,dr),$$

where B_m^c denote the complementary set of B_m. Then in the same way,

$$\mathbb{E}(e^{-s\bar{\chi}_t^{M_{\mathcal{P}}}}) \leq \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\int_{B^c_{M(\gamma_{n+1})}}(1-e^{-s\underline{c}(z)})\overline{\mu}(dz))).$$

We recall by (63) that for $\Gamma_n < t \leq \Gamma_{n+1}$,

$$a_t^{\mathcal{P}} = (\sum_{1\leq i\leq n}\gamma_i\int_{\{|z|\geq M(\gamma_i)\}}\underline{c}(z)\mu(dz) + (t-\Gamma_n)\int_{\{|z|\geq M(\gamma_{n+1})\}}\underline{c}(z)\mu(dz))^{\frac{1}{2}}.$$

Then

$$a_t^{\mathcal{P}} \geq \sqrt{\mathbb{E}\bar{\chi}_t^{M_{\mathcal{P}}}}.$$

Using Jensen inequality for the convex function $f(x) = e^{-sx}$, $s, x > 0$, we have

$$e^{-s|a_t^{M_{\mathcal{P}}}|^2} \leq e^{-s\mathbb{E}\bar{\chi}_t^{M_{\mathcal{P}}}} \leq \mathbb{E}(e^{-s\bar{\chi}_t^{M_{\mathcal{P}}}}) \leq \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\int_{B^c_{M(\gamma_{n+1})}}(1-e^{-s\underline{c}(z)})\overline{\mu}(dz))).$$

So we deduce that

$$\begin{aligned}
&\mathbb{E}(e^{-s(\chi_t^{M_{\mathcal{P}}}+|a_t{}^{\mathcal{P}}|^2)}) = \mathbb{E}(e^{-s\chi_t^{M_{\mathcal{P}}}})\times e^{-s|a_t^{\mathcal{P}}|^2}\\
&\leq \exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\int_{B_{M(\gamma_{n+1})}}(1-e^{-s\underline{c}(z)})\overline{\mu}(dz)))\\
&\times\exp(-\sum_{n=0}^{N(t)}(((\Gamma_{n+1}\wedge t)-\Gamma_n)\int_{B^c_{M(\gamma_{n+1})}}(1-e^{-s\underline{c}(z)})\overline{\mu}(dz)))\\
&= \exp(-t\int_{\mathbb{R}^d}(1-e^{-s\underline{c}(z)})\overline{\mu}(dz)),
\end{aligned} \tag{83}$$

and the last term does not depend on $M_{\mathcal{P}}(t)$.

Step 4 Now we use the Lemma 14 from [9], which states the following.

Lemma 5.4. *We consider an abstract measurable space B, a σ-finite measure $\mathcal{M}$ on this space and a non-negative measurable function $f : B \to \mathbb{R}_+$ such that $\int_B f d\mathcal{M} < \infty$. For $t > 0$ and $p \geq 1$, we note*

$$\beta_f(s) = \int_B (1-e^{-sf(x)})\mathcal{M}(dx) \quad \textit{and} \quad I_t^p(f) = \int_0^\infty s^{p-1}e^{-t\beta_f(s)}ds.$$

We suppose that for some $t > 0$ and $p \geq 1$,

$$\underline{\lim}_{u\to\infty}\frac{1}{\ln u}\mathcal{M}(f\geq\frac{1}{u}) > \frac{p}{t}, \tag{84}$$

then $I_t^p(f) < \infty$.

We will use the above lemma for $\mathcal{M}(dz) = \overline{\mu}(dz)$, $f(z) = \underline{c}(z)$ and $B = \mathbb{R}^d$. Thanks to (54) in **Hypothesis 2.4**,

$$\underline{\lim}_{u\to\infty}\frac{1}{\ln u}\overline{\mu}(\underline{c}\geq\frac{1}{u}) = \infty. \tag{85}$$

Then for every $p \geq 1, 1 \leq t \leq 3$, we deduce from (81),(82),(83) and **Lemma 5.4** that

$$\begin{aligned}
\mathbb{E}|\frac{1}{\det \sigma_{X_t^{M_{\mathcal{P}}}}}|^p &\leq \mathbb{E}(|\lambda_t^{M_{\mathcal{P}}}|^{-dp}) \leq C(\mathbb{E}(|\chi_t^{M_{\mathcal{P}}} + |a_t^{\mathcal{P}}|^2|^{-2dp}))^{\frac{1}{2}} \\
&\leq C(\frac{1}{\Gamma(2dp)} \int_0^\infty s^{2dp-1} \mathbb{E}(e^{-s(\chi_t^{M_{\mathcal{P}}} + |a_t^{\mathcal{P}}|^2)}) ds)^{\frac{1}{2}} \\
&\leq C(\frac{1}{\Gamma(2dp)} \int_0^\infty s^{2dp-1} \exp(-t \int_{\mathbb{R}^d} (1 - e^{-s\underline{c}(z)}) \overline{\mu}(dz) ds)^{\frac{1}{2}} < \infty. \qquad (86)
\end{aligned}$$

Proof of $ii)$ We consider an increasing sequence of partition $\mathcal{P}_n,\ n \in \mathbb{N}$, $(\mathcal{P}_n \subset \mathcal{P}_{n+1})$, such that $|\mathcal{P}_n| \downarrow 0$. In particular, $\forall t,\ M_{\mathcal{P}_n}(t) \uparrow \infty$.

We recall the notations $F_t^{M_{\mathcal{P}}}(x) = X_t^{M_{\mathcal{P}}}(x) - x$ and $F_t(x) = X_t(x) - x$. We notice by **Lemma 5.1** $ii)$ that $\mathbb{E}|F_t^{M_{\mathcal{P}_n}} - F_t| \to 0$ as $n \to \infty$, and by **Lemma 5.2** that $\sup_n \sup_x \|F_t^{M_{\mathcal{P}_n}}(x)\|_{L,l,p} \leq C_{l,p}$.

Moreover, by **Lemma 5.5** $ii)$ (given immediately below), we know that $(DF_t^{M_{\mathcal{P}_n}})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega; l^2 \times \mathbb{R}^d)$. Then applying **Lemma 3.3 (B)** with $F_n = F_t^{M_{\mathcal{P}_n}}$ and $F = F_t$, **Lemma 5.3** $i)$ implies **Lemma 5.3** $ii)$.

□

5.3 Auxiliary results

Besides the lemmas concerning the Sobolev norms and covariance matrices, we establish an auxiliary result. We recall ε_m given in (58).

Lemma 5.5. *We assume that* ***Hypothesis 2.1*** *and* ***Hypothesis 2.4*** $b)$ *hold true.*

$i)$ *Then for any $\varepsilon_0 > 0$, there exists a constant C dependent on d, ε_0 such that for every $t \in [0,3]$ and every stating point $x \in \mathbb{R}^d$, we have*

$$\mathbb{E}|\det \sigma_{X_t^{\mathcal{P},M_{\mathcal{P}}}} - \det \sigma_{X_t}|^{\frac{2}{1+\varepsilon_0}} \leq C\|DX_t^{\mathcal{P},M_{\mathcal{P}}} - DX_t\|^{\frac{2}{1+\varepsilon_0}}_{L^2(\Omega;l_2\times\mathbb{R}^d)} \leq C(|\mathcal{P}| + \varepsilon_{M(|\mathcal{P}|)})^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}.$$

$ii)$ *We consider an increasing sequence of partition $\mathcal{P}_n,\ n \in \mathbb{N}$, $(\mathcal{P}_n \subset \mathcal{P}_{n+1})$, such that $|\mathcal{P}_n| \downarrow 0$. In particular, $\forall t,\ M_{\mathcal{P}_n}(t) \uparrow \infty$. We denote*

$$F_n(x) = X_t^{M_{\mathcal{P}_n}}(x).$$

Then for each $x \in \mathbb{R}^d$, the sequence $DF_n(x),\ n \in \mathbb{N}$ is Cauchy in $L^2(\Omega; l_2 \times \mathbb{R}^d)$, uniformly with respect to x :

$$\sup_x \|DF_n(x) - DF_m(x)\|_{L^2(\Omega;l_2\times\mathbb{R}^d)} \to 0, \text{ as } n, m \to \infty.$$

Proof. **Proof of** $i)$

By **Lemma 5.2**, we know that $\|DX_t^{\mathcal{P},M_{\mathcal{P}}}\|_{L^2(\Omega;l_2\times\mathbb{R}^d)}$ and $\|DX_t\|_{L^2(\Omega;l_2\times\mathbb{R}^d)}$ are bounded, uniformly with respect to x. Then using Hölder's inequality with conjugates $1+\varepsilon_0$ and $\frac{1+\varepsilon_0}{\varepsilon_0}$, we get

$$\mathbb{E}(|\det \sigma_{X_t^{\mathcal{P},M_{\mathcal{P}}}} - \det \sigma_{X_t}|^{\frac{2}{1+\varepsilon_0}}) \leq C\|DX_t^{\mathcal{P},M\ \mathcal{P}} - DX_t\|^{\frac{2}{1+\varepsilon_0}}_{L^2(\Omega;l_2\times\mathbb{R}^d)}. \qquad (87)$$

Now we only need to prove that

$$\|DX_t^{\mathcal{P},M_{\mathcal{P}}} - DX_t\|^{\frac{2}{1+\varepsilon_0}}_{L^2(\Omega;l_2\times\mathbb{R}^d)} \leq C(|\mathcal{P}| + \varepsilon_{M(|\mathcal{P}|)})^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}. \qquad (88)$$

The proof of (88) is a slight variant of the proof of Lemma 3.9 $iii)$ in the paper [44]. The difference in that the truncation function M is constant in [44] while here it may vary on different time intervals. We do not discuss in detail here. So we conclude that **Lemma 5.5** $i)$ holds.

Proof of $ii)$
We consider an increasing sequence of partition $\mathcal{P}_n,\ n \in \mathbb{N}$, $(\mathcal{P}_n \subset \mathcal{P}_{n+1})$, such that $|\mathcal{P}_n| \downarrow 0$. In particular, $\forall t,\ M_{\mathcal{P}_n}(t) \uparrow \infty$. We need to prove that

$$\|DX_t^{M_{\mathcal{P}_n}} - DX_t^{M_{\mathcal{P}_m}}\|_{L^2(\Omega; l_2 \times \mathbb{R}^d)} \to 0, \text{ as } n, m \to \infty. \tag{89}$$

The proof of (89) is a slight variant of the proof of (148) p.47-49 in [44], so we omit it.

□

6 Proof of Theorem 4.1

In this section, we give the proof of **Theorem 4.1**. We apply **Proposition 2.1.1** in Section 2. For a measurable function f, we denote $\overline{P}_t f(x) = \mathbb{E} f(X_t^{\mathcal{P}, M_{\mathcal{P}}}(x))$ and $P_t f(x) = \mathbb{E} f(X_t(x))$. In the following subsections, we will check the conditions of **Proposition 2.1.1**.

6.1 Euler: condition (7)

For every $\gamma > 0$, we recall in (59) that we define $M(\gamma) \in \mathbb{N}$ such that

$$\varepsilon_{M(\gamma)} \leq \gamma^2.$$

We recall the basic equation X_t (see (49)). We denote by $\tilde{X}_t^{M_{\mathcal{P}}}$ the one step truncated Euler scheme:

$$\tilde{X}_t^{M_{\mathcal{P}}}(x) = x + \int_0^t \int_{B_{M(\gamma)}} c(z, x) dN(z, s) + \int_0^t b(x) ds.$$

Then,

$$\begin{aligned}
\mathbb{E}|\tilde{X}_\gamma^{M_{\mathcal{P}}} - X_\gamma| &\leq \mathbb{E} \int_0^\gamma \int_{\{|z| \geq M(\gamma)\}} |c(z, X_s)| dN(z, s) + \mathbb{E} \int_0^\gamma \int_{B_{M(\gamma)}} |c(z, x) - c(z, X_s)| dN(z, s) \\
&+ \mathbb{E} \int_0^\gamma |b(x) - b(X_s)| ds \\
&\leq \gamma \int_{\{|z| \geq M(\gamma)\}} \bar{c}(z) \mu(dz) + C \int_0^\gamma \mathbb{E}|x - X_s| ds \\
&\leq \gamma \sqrt{\varepsilon_{M(\gamma)}} + C \times \gamma^2 \leq C \times \gamma^2
\end{aligned}$$

So

$$W_1(\tilde{X}_\gamma^{M_{\mathcal{P}}}, X_\gamma) \leq \mathbb{E}|\tilde{X}_\gamma^M - X_\gamma| \leq C \times \gamma^2.$$

So we conclude that (7) holds for $\alpha = 1$ and $k_0 = 0$.

6.2 Lipschitz: condition (4) and the existence of an invariant measure

We recall that X is the solution of the equation (49).

Lemma 6.1. *Suppose that* ***Hypothesis 2.5*** *(see (55) and (56)) holds.*
a) Then, for a Lipschitz continuous function f

$$|\mathbb{E}(f(X_t(x)) - \mathbb{E}(f(X_t(y))| \leq L_f e^{-\frac{\theta}{2} t} |x - y|,$$

with L_f the Lipschitz constant of f.
b) Moreover, there exists at least one invariant probability.

Proof a) We fix $x, y \in \mathbb{R}^d$ and we construct on the same probability space, with the same Poisson point measure N the solution $X_t^M(y)$ which starts from y. Then we denote

$$\begin{aligned} Y_t &= X_t(x) - X_t(y), \\ \Delta_t^c(z) &= c(z, X_{s-}(x)) - c(z, X_{s-}(y)) \\ \Delta_t^b &= b(X_{s-}(x)) - b(X_{s-}(y)) \end{aligned}$$

and we have

$$Y_t = x - y + \int_0^t \int_{\mathbb{R}^d} \Delta_s^c(z) dN(z,s) + \int_0^t \Delta_s^b ds.$$

Using Itô's formula for $\Phi(t,u) = e^{\lambda t} |u|^2$ we obtain

$$\begin{aligned} \Phi(t, Y_t) &= |x-y|^2 + \lambda \int_0^t \Phi(s, Y_s) ds + \int_0^t 2e^{\lambda s} \left\langle Y_s, \Delta_s^b \right\rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (\Phi(s, Y_{s-} + \Delta_s^c(z)) - \Phi(s, Y_{s-})) dN(z,s) \\ &= |x-y|^2 + \lambda \int_0^t \Phi(s, Y_s) ds + \int_0^t 2e^{\lambda s} \left\langle Y_s, \Delta_s^b \right\rangle ds \\ &\quad + M_t + \int_0^t \int_{\mathbb{R}^d} (\Phi(s, Y_{s-} + \Delta_s^c(z)) - \Phi(s, Y_{s-})) d\mu(z) ds \end{aligned}$$

with M_t a martingale. Taking the expectation we get

$$e^{\lambda t} \mathbb{E} |Y_t|^2 \leq |x-y|^2 + \int_0^t e^{\lambda s} \mathbb{E}(\Psi_s) ds$$

with

$$\begin{aligned} \Psi_s &= \lambda |Y_s|^2 + 2 \left\langle Y_s, \Delta_s^b \right\rangle + \int_{\mathbb{R}^d} |Y_s + \Delta_s^c(z)|^2 - |Y_s|^2 \mu(dz) \\ &= \lambda |Y_s|^2 + 2 \left\langle Y_s, \Delta_s^b \right\rangle + \int_{\mathbb{R}^d} \left\langle \Delta_s^c(z), 2Y_s + \Delta_s^c(z) \right\rangle \mu(dz). \end{aligned}$$

We need to prove that $\mathbb{E}(\Psi_s) \leq 0$. We recall that we assume **Hypothesis 2.5** $i)ii)$ (see (55)). We also have

$$|\langle \Delta_s^c(z), 2Y_s + \Delta_s^c(z) \rangle| \leq (2\bar{c}(z) + \bar{c}^2(z)) |Y_s|^2,$$

so that

$$\Psi_s \leq |Y_s|^2 (\lambda + \int_{\mathbb{R}^d} (2\bar{c}(z) + \bar{c}^2(z)) \mu(dz) - 2\bar{b}).$$

Thanks to **Hypothesis 2.5** $iii)$ (see (56)), taking $\lambda \leq \theta$, we have

$$e^{\lambda t} \mathbb{E} |Y_t|^2 \leq |x-y|^2 + \int_0^t e^{\lambda s} \mathbb{E}(\Psi_s) ds \leq |x-y|^2$$

so that

$$\mathbb{E} |X_t(x) - X_t(y)|^2 \leq e^{-\theta t} |x-y|^2.$$

Then, for a Lipschitz continuous function f,

$$|\mathbb{E}(f(X_t(x)) - \mathbb{E}(f(X_t(y))| \leq L_f \mathbb{E} |X_t(x) - X_t(y)| \leq L_f e^{-\frac{\theta}{2} t} |x-y|.$$

b) We denote L to be the infinitesimal operator of (49). We take $V(x) = |x|^2$ and we will prove that

$$LV \leq \bar{\beta} - \bar{\alpha} V$$

for some $\bar{\beta}, \bar{\alpha} > 0$ (the Lyapunov mean reverting condition). This implies $LV \leq C$ and $\lim_{|x|\to\infty} LV(x) = -\infty$. Then we use Theorem 9.3 and Lemma 9.7 from [18] (with $\varphi = V$ and $\psi = LV$) which guarantees existence of an invariant distribution. We have

$$\begin{aligned} LV(x) &= 2\langle x, b(x)\rangle + \int_{\mathbb{R}^d} (V(x + c(z,x)) - V(x))\mu(dz) \\ &\leq -2\bar{b}\,|x|^2 + \int_{\mathbb{R}^d} (2\langle x, c(z,x)\rangle + |c(z,x)|^2)\mu(dz) \\ &\leq -2\bar{b}\,|x|^2 + (|x|^2 + 1)\int_{\mathbb{R}^d} \bar{c}(z)\mu(dz) + \int_{\mathbb{R}^d} \bar{c}^2(z)\mu(dz) \\ &= \int_{\mathbb{R}^d} (\bar{c}(z) + \bar{c}^2(z))\mu(dz) - (2\bar{b} - \int_{\mathbb{R}^d} \bar{c}(z)\mu(dz))\,|x|^2 . \end{aligned}$$

□

6.3 Regularity: conditions (14), (15), (16) and (20)

Firstly, we deal with (14). Let $t \in [1, 2]$. For any k and any multi-index β_0 with $|\beta_0| = k$, we write

$$\partial_x^{\beta_0} P_t\varphi(x) = \mathbb{E}[\partial_x^{\beta_0}(\varphi(X_t(x))] = \sum_{|\alpha_0|\leq|\beta_0|} \mathbb{E}[(\partial^{\alpha_0}\varphi)(X_t(x))\mathbf{P}_{\alpha_0}(x)],$$

with $\mathbf{P}_{\alpha_0}(x)$ a polynomial of $\partial_x^{\alpha_1} X_t(x)$, $|\alpha_1| \leq |\beta_0|$.

In the following, we use the results from Section 5. In **Lemma 5.2**, we prove that the Sobolev norms of each $\partial_x^{\alpha_1} X_t(x)$ are bounded, uniformly with respect to x. It follows that this is also true for $\mathbf{P}_{\alpha_0}(x)$.

We denote that $F_t(x) = X_t(x) - x$. In **Lemma 5.2**, we have proved that the Sobolev norms of each $F_t(x)$ are bounded, uniformly with respect to x. Moreover, in **Lemma 5.3**, we prove that $F_t(x)$ is non-degenerated, uniformly with respect to x, that is $\Sigma_p(F_t(x)) < \infty$ for each p (see (26)).

Then we use **Lemma 3.4 (A)** which asserts that $(\boldsymbol{B}_k)$ is true for $F = F_t(x)$ and $G = \mathbf{P}_{\alpha_0}(x)$. By the remark of **Lemma 3.4**, $(\boldsymbol{B}_k)$ is also true for $F = X_t(x) = F_t(x) + x$ and $G = \mathbf{P}_{\alpha_0}(x)$. This reads

$$|\mathbb{E}[(\partial^{\alpha_0}\varphi)(X_t(x))\mathbf{P}_{\alpha_0}(x)]| \leq C\|\varphi\|_\infty,$$

which gives (14).

In a similar way, we can obtain (15).

For (16), $i)$ is a direct consequence of (4) which has been proved in Section 6.2. For (16) $ii)$, we take $t \in (0, 1]$. For any k and any multi-index β_0 with $|\beta_0| = k$, we notice that

$$|\partial_x^{\beta_0}\nabla P_t\varphi(x)| = |\mathbb{E}[\partial_x^{\beta_0}(\nabla\varphi(X_t(x))]| = |\sum_{|\alpha_0|\leq|\beta_0|} \mathbb{E}[(\partial^{\alpha_0}\nabla\varphi)(X_t(x))\mathbf{P}_{\alpha_0}(x)]| \leq \|\nabla\varphi\|_{k,\infty} \sum_{|\alpha_0|\leq|\beta_0|} \mathbb{E}|\mathbf{P}_{\alpha_0}(x)|,$$

with $\mathbf{P}_{\alpha_0}(x)$ a polynomial of $\partial_x^{\alpha_1} X_t(x)$, $|\alpha_1| \leq |\beta_0|$. In [29], Kunita has shown in Theorem 3.4.1 and Theorem 3.4.2 the regularity of the flow associated with the jump-diffusion. So in our case, we have $\mathbb{E}|\mathbf{P}_{\alpha_0}(x)| < \infty$ and thus (16) $ii)$ holds true.

Now we prove (20). In order to prove (20), we need to represent $\overline{P}_{s,t}\varphi(x)$ and $P_{s,t}\varphi(x)$. So we consider the following equations.

We denote $X_{s,t}^{\mathcal{P},M_\mathcal{P}}$ and $X_{s,t}$ the solutions of the following equations respectively:

$$X_{s,t}^{\mathcal{P},M_\mathcal{P}} = x + \int_s^t b(X_{s,\tau(r)}^{\mathcal{P},M_\mathcal{P}})dr + \int_s^t \int_{B_{M_\mathcal{P}(r)}} c(z, X_{s,\tau(r)-}^{\mathcal{P},M_\mathcal{P}})N(dz, dr); \tag{90}$$

$$X_{s,t} = x + \int_s^t b(X_{s,r})dr + \int_s^t \int_{\mathbb{R}^d} c(z, X_{s,r-})N(dz, dr). \tag{91}$$

We sometimes write $X_{s,t}^{\mathcal{P},M_{\mathcal{P}}}(x)$ (and $X_{s,t}(x)$) instead of $X_{s,t}^{\mathcal{P},M_{\mathcal{P}}}$ (and $X_{s,t}$) to stress the dependence on the initial value x. And we denote $\overline{P}_{s,t}\varphi(x)=\mathbb{E}\varphi(X_{s,t}^{\mathcal{P},M_{\mathcal{P}}}(x))$ and $P_{s,t}\varphi(x)=\mathbb{E}\varphi(X_{s,t}(x))$.

Let $1<t<r<t+2$. We recall that $\mathcal{P}=\{0=\Gamma_0<\Gamma_1<\cdots<\Gamma_{l-1}<\Gamma_l<\cdots\}$, $\gamma_l=\Gamma_l-\Gamma_{l-1}$ and for $\Gamma_l\leq t<\Gamma_{l+1}$, $N(t)=l$. We denote

$$F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)=X_{t,r}\circ X_{t-1,t}^{\mathcal{P},M_{\mathcal{P}}}(x)-x \text{ and } F_{r-t+1}(x)=X_{t,r}\circ X_{t-1,t}(x)-x=X_{t-1,r}(x)-x. \tag{92}$$

We also denote $|\mathcal{P}^{t-1,t}|:=\max\limits_{\substack{l\in\mathbb{N}\ s.t.\\ \Gamma_{l+1}>t-1,\Gamma_l<t}}((\Gamma_{l+1}\wedge t)-(\Gamma_l\vee(t-1)))$. Before we give the proof of (20), we state the following lemma concerning $F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)$ and $F_{r-t+1}(x)$ given in (92).

Lemma 6.2. *Under the **Hypothesis 2.1∼2.4**, we have these results.*

i) For all $p\geq 1, q\geq 0$, there exists a constant $C_{q,p}$ depending on q,p,d, such that $F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)$ and $F_{r-t+1}(x)$ belong to $\mathcal{D}_\infty^d$ and

$$\sup_x\sup_{\mathcal{P}}\|F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)+F_{r-t+1}(x)\|_{L,q,p}\leq C_{q,p}.$$

ii) For every $p\geq 1$, we have

$$\sup_{\mathcal{P}}\sup_x\mathbb{E}(1/\det\sigma_{F_{r-t+1}(x)})^p\leq C_p,$$

with C_p a constant depending on p,d.

iii) For any $\varepsilon_0>0$, there exists a constant C dependent on d,ε_0 such that

$$\sup_x\mathbb{E}|\det\sigma_{F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)}-\det\sigma_{F_{r-t+1}(x)}|^{\frac{2}{1+\varepsilon_0}}\leq C|\mathcal{P}^{t-1,t}|^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}.$$

Proof. Firstly, we will construct an approximation scheme for $X_{t,r}\circ X_{t-1,t}^{\mathcal{P},M_{\mathcal{P}}}(x)$. We take an integer N_0 such that $\frac{1}{2^{N_0}}\leq|\mathcal{P}|$. For $n>N_0$, we take a "mixed partition"

$$\begin{aligned}&\mathcal{P}_n=\{t-1<\Gamma_{N(t-1)+1}<\cdots<\Gamma_{N(t)}\leq t\\&<t+\frac{1}{2^n}(r-t)<t+\frac{2}{2^n}(r-t)<\cdots<t+\frac{l}{2^n}(r-t)<t+\frac{l+1}{2^n}(r-t)<\cdots<r\}\\&:=\{t-1=s_0<s_1<\cdots<s_{n_0}=r\}.\end{aligned}$$

We remark that we take the partition $\{\Gamma_l\}$ on $[t-1,t]$ and take the partition $\{\frac{l}{2^n}\}$ on $[t,r]$. We denote $|\mathcal{P}_n|:=\max\limits_{k\in\{1,\cdots,n_0\}}s_k-s_{k-1}$. We construct $M_{\mathcal{P}_n}(t)=M(s_{l+1}-s_l)$ when $s_l<t\leq s_{l+1}$ with the truncation function $M(\bullet)$ given in (59). And we denote $\tau^{\mathcal{P}_n}(t)=s_l$ when $s_l<t\leq s_{l+1}$. Then we consider the truncated Euler scheme based on $\mathcal{P}_n$, $M_{\mathcal{P}_n}$:

$$X_{t-1,r}^{\mathcal{P}_n,M_{\mathcal{P}_n}}=x+\int_{t-1}^r b(X_{t-1,\tau^{\mathcal{P}_n}(s)}^{\mathcal{P}_n,M_{\mathcal{P}_n}})ds+\int_{t-1}^r\int_{B_{M_{\mathcal{P}_n}(s)}}c(z,X_{t-1,\tau^{\mathcal{P}_n}(s)-}^{\mathcal{P}_n,M_{\mathcal{P}_n}})N(dz,ds).$$

We denote

$$F_{r-t+1}^{\mathcal{P}_n,M_{\mathcal{P}_n}}(x)=X_{t-1,r}^{\mathcal{P}_n,M_{\mathcal{P}_n}}(x)-x. \tag{93}$$

We notice that we can apply the results from Section 5 for $F_{r-t+1}^{\mathcal{P}_n,M_{\mathcal{P}_n}}(x)$, $F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)$ and $F_{r-t+1}(x)$ defined in (92) and (93).

Since $r-t+1<3$, by **Lemma 5.2** i), the Sobolev norms of $F_{r-t+1}^{\mathcal{P}_n,M_{\mathcal{P}_n}}(x)$ and $F_{r-t+1}(x)$ are bounded, uniformly with respect to x. One can check that $F_{r-t+1}^{\mathcal{P}_n,M_{\mathcal{P}_n}}(x)\to F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)$ in $L^1(\Omega)$, as $n\to\infty$ (which is a variant of **Lemma 5.1** i)). So we can apply **Lemma 3.3 (A)** for $F_n=F_{r-t+1}^{\mathcal{P}_n,M_{\mathcal{P}_n}}(x)$ and $F=F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)$ in order to get that $F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)\in\mathcal{D}_\infty^d$ and $\sup\limits_x\sup\limits_{\mathcal{P}}\|F_{r-t+1}^{\mathcal{P},M_{\mathcal{P}}}(x)\|_{L,q,p}\leq C_{q,p}$. Hence, **Lemma 6.2** i) is proved.

Moreover, since $r-t+1>1$, using **Lemma 5.3** $ii)$ we have $\sup_{\mathcal{P}}\sup_x \mathbb{E}(1/\det\sigma_{F^{M_{\mathcal{P}}}_{r-t+1}(x)})^p \le C_p$. So **Lemma 6.2** $ii)$ is proved.

Finally, by **Lemma 5.5** $i)$ and recalling by (59) that $\varepsilon_{M(\gamma)} \le \gamma^2$, we have

$$\|DF^{\mathcal{P}_n,M_{\mathcal{P}_n}}_{r-t+1}(x) - DF_{r-t+1}(x)\|^{\frac{2}{1+\varepsilon_0}}_{L^2(\Omega;l_2\times\mathbb{R}^d)} \le C(|\mathcal{P}_n| + \varepsilon_{M(|\mathcal{P}_n|)})^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}} \le C|\mathcal{P}^{t-1,t}|^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}},$$

where the last equality is true since $\frac{1}{2^n} \le |\mathcal{P}^{t-1,t}|$ for every $n > N_0$. Then we can apply **Lemma 3.3 (C)** for $(F_n,\bar{F}_n) = (F^{\mathcal{P}_n,M_{\mathcal{P}_n}}_{r-t+1}(x), F_{r-t+1}(x))$ and $(F,\bar{F}) = (F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x), F_{r-t+1}(x))$. So $\sup_x \mathbb{E}|\det\sigma_{F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x)} - \det\sigma_{F_{r-t+1}(x)}|^{\frac{2}{1+\varepsilon_0}} \le C\|DF^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x) - DF_{r-t+1}(x)\|^{\frac{2}{1+\varepsilon_0}}_{L^2(\Omega;l_2\times\mathbb{R}^d)} \le C|\mathcal{P}^{t-1,t}|^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}$ and **Lemma 6.2** $iii)$ is proved. □

Then we can prove (20). By **Lemma 6.2** $i)$, the Sobolev norms of $F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x)$ are bounded, uniformly with respect to x. Using **Lemma 6.2** $ii)$, the covariance matrix of $F_{r-t+1}(x)$ is non-degenerated. Then we are able to apply **Lemma 3.5** for $F = F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x)$ and $Q = F_{r-t+1}(x)$ so (38) holds for $F = F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x)$ and $Q = F_{r-t+1}(x)$. Thanks to the remark of **Lemma 3.5**, (38) also holds for $F = X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x) = F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x) + x,\ Q = X_{t,r}\circ X_{t-1,t}(x) = F_{r-t+1}(x) + x$ and get

$$\begin{aligned}&\left|\mathbb{E}(f(X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x))) - \mathbb{E}(f_\delta(X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x)))\right| \\ &\le C\,\|f\|_\infty \times \Big(\frac{\delta^q}{\eta^{2q}} + \eta^{-p}\mathbb{E}|\det\sigma_{X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x)} - \det\sigma_{X_{t,r}\circ X_{t-1,t}(x)}|^p + \eta^\kappa\Big),\end{aligned} \tag{94}$$

where we have also used the fact that $\sup_{\mathcal{P}}\sup_x \mathbb{E}(1/\det\sigma_{X_{t,r}\circ X_{t-1,t}(x)})^\kappa \le C_\kappa$ from **Lemma 6.2** $ii)$.

We take $p = \frac{2}{1+\varepsilon_0}$ for any small ε_0. Thanks to **Lemma 6.2** $iii)$,

$$\sup_x \mathbb{E}|\det\sigma_{X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x)} - \det\sigma_{X_{t,r}\circ X_{t-1,t}(x)}|^{\frac{2}{1+\varepsilon_0}} = \sup_x \mathbb{E}|\det\sigma_{F^{\mathcal{P},M_{\mathcal{P}}}_{r-t+1}(x)} - \det\sigma_{F_{r-t+1}(x)}|^{\frac{2}{1+\varepsilon_0}} \le C|\mathcal{P}^{t-1,t}|^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}.$$

This implies that

$$\sup_x \mathbb{E}|\det\sigma_{X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x)} - \det\sigma_{X_{t,r}\circ X_{t-1,t}(x)}|^{\frac{2}{1+\varepsilon_0}} \le C\gamma^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}_{N(t-1)}.$$

Substituting into (94), we obtain

$$\sup_x \left|\mathbb{E}(f(X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x))) - \mathbb{E}(f_\delta(X_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x)))\right| \le C\,\|f\|_\infty \times \Big(\frac{\delta^q}{\eta^{2q}} + \eta^{-\frac{2}{1+\varepsilon_0}}\gamma^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}_{N(t-1)} + \eta^\kappa\Big).$$

By a similar argument, we have

$$\sup_x \left|\mathbb{E}(f(X^{\mathcal{P},M_{\mathcal{P}}}_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x))) - \mathbb{E}(f_\delta(X^{\mathcal{P},M_{\mathcal{P}}}_{t,r}\circ X^{\mathcal{P},M_{\mathcal{P}}}_{t-1,t}(x)))\right| \le C\,\|f\|_\infty \times \Big(\frac{\delta^q}{\eta^{2q}} + \eta^{-\frac{2}{1+\varepsilon_0}}\gamma^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}}_{N(t-1)} + \eta^\kappa\Big).$$

So (20) holds for $p = \frac{2}{1+\varepsilon_0}$ and $\beta = \frac{1}{2+\varepsilon_0}$.

Finally, we can apply **Proposition 2.1.1** for $X^{\mathcal{P},M_{\mathcal{P}}}_{0,\Gamma_n}$ and X_{0,Γ_n} with $\alpha = 1,\ k_0 = 0,\ p = \frac{2}{1+\varepsilon_0},\ \beta = \frac{1}{2+\varepsilon_0}$ (for any small ε_0), and obtain the following result: for every $\varepsilon > 0$, there exists a constant C such that

$$d_{TV}(X^{\mathcal{P},M_{\mathcal{P}}}_{0,\Gamma_n}, X_{0,\Gamma_n}) \le C\gamma_n^{\frac{2}{(2+\varepsilon_0)(1+\varepsilon_0)}-\varepsilon} = C\gamma_n^{1-\bar{\varepsilon}}, \tag{95}$$

with $\bar{\varepsilon} = 1 - \frac{2-\varepsilon(2+\varepsilon_0)(1+\varepsilon_0)}{(2+\varepsilon_0)(1+\varepsilon_0)}$.
And moreover, we have

$$d_{TV}(\mathcal{L}(X_{\Gamma_n}^{\mathcal{P},M_{\mathcal{P}}}),\nu) \leq C(\gamma_n^{1-\varepsilon} + \int_{\mathbb{R}^d} |x-y|\, d\nu(y) e^{-\frac{\theta}{2}\Gamma_n}),$$

where ν is the unique invariant probability measure.

7 Appendix

7.1 The numerical lemma

In Section 2, we need to use the following numerical lemma.

Lemma 7.1. ***(A)*** *Take an integer n_*. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a non-increasing positive sequence such that for $n \geq n_*$, we have*

$$\frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2} \leq 2\overline{\omega}. \tag{96}$$

We denote $\Gamma_n = \sum_{i=1}^n \gamma_i$. Then for every $n_* \leq i \leq n$, we have

$$\gamma_i \leq e^{2\overline{\omega}(\Gamma_n - \Gamma_i)} \times \gamma_n. \tag{97}$$

(B) *We assume that $(\gamma_n)_{n\in\mathbb{N}}$ is a non-increasing positive sequence verifying*

$$\frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2} \leq c_* < \frac{\rho}{\alpha}. \tag{98}$$

We denote $\Gamma_n = \sum_{i=1}^n \gamma_i$. Then

$$u_n := \sum_{i=1}^n \gamma_i^{1+\alpha} e^{-\rho(\Gamma_n - \Gamma_i)} \leq C\gamma_n^{\alpha}. \tag{99}$$

Proof of *(A)* Notice that (96) implies

$$\frac{\gamma_n}{\gamma_{n+1}} \leq 1 + 2\overline{\omega}\gamma_{n+1} \leq e^{2\overline{\omega}\gamma_{n+1}}.$$

Then

$$\frac{\gamma_i}{\gamma_n} = \prod_{k=i}^{n-1} \frac{\gamma_k}{\gamma_{k+1}} \leq \prod_{k=i}^{n-1} e^{2\overline{\omega}(\gamma_{k+1})} \leq e^{2\overline{\omega}(\Gamma_n - \Gamma_i)}.$$

Proof of *(B)* Notice that (98) implies

$$\frac{\gamma_n}{\gamma_{n+1}} \leq 1 + c_*\gamma_{n+1} \leq e^{c_*\gamma_{n+1}}.$$

Then we define $v_n = u_n/\gamma_n^{\alpha}$ and we have the recurrence relation

$$v_{n+1} = \theta_n v_n + \gamma_{n+1}, \quad \theta_n = \frac{\gamma_n^{\alpha}}{\gamma_{n+1}^{\alpha}} \times e^{-\rho\gamma_{n+1}}.$$

Using the previous inequality we obtain

$$v_{n+1} \leq e^{(\alpha c_* - \rho)\gamma_{n+1}} v_n + \gamma_{n+1}$$

and further

$$\begin{aligned} e^{(\rho-\alpha c_*)\Gamma_{n+1}} v_{n+1} &\leq e^{(\rho-\alpha c_*)\Gamma_n} v_n + e^{(\rho-\alpha c_*)\Gamma_{n+1}} \gamma_{n+1} \\ &\leq e^{(\rho-\alpha c_*)\Gamma_n} v_n + C' e^{(\rho-\alpha c_*)\Gamma_n} \gamma_{n+1}, \end{aligned}$$

with $C' = \sup_{k\geq 1} e^{(\rho-\alpha c_*)\gamma_k} = e^{(\rho-\alpha c_*)\gamma_1}$. We use recursively this inequality and we obtain

$$\begin{aligned} e^{(\rho-\alpha c_*)\Gamma_{n+1}} v_{n+1} &\leq e^{(\rho-\alpha c_*)\Gamma_1} v_1 + C' \sum_{i=1}^{n} e^{(\rho-\alpha c_*)\Gamma_n} \gamma_{n+1} \\ &\leq e^{(\rho-\alpha c_*)\Gamma_1} v_1 + C' \int_0^{\Gamma_n} e^{(\rho-\alpha c_*)s} ds \\ &\leq e^{(\rho-\alpha c_*)\Gamma_1} v_1 + \frac{C'}{\rho-\alpha c_*} e^{(\rho-\alpha c_*)\Gamma_{n+1}}. \end{aligned}$$

That is

$$v_{n+1} \leq v_1 + \frac{C'}{\rho-\alpha c_*} \leq \gamma_1 + \frac{C'}{\rho-\alpha c_*}$$

which finally gives

$$u_{n+1} \leq (\gamma_1 + \frac{C'}{\rho-\alpha c_*})\gamma_{n+1}^{\alpha}.$$

□

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