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# The anisotropy of 2D or 3D Gaussian random fields through their Lipschitz-Killing curvature densities

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**Abstract:** We are interested here in modeling and estimating the anisotropy of Gaussian random fields through the geometry of their excursion sets. In order to do this, we use Lipschitz-Killing curvatures of the level sets as functions of the levels and see them as generalized processes for which we are able to obtain a joint functional Central Limit Theorem. For 2D and 3D stationary Gaussian fields we provide explicit formulas for the Lipschitz-Killing curvature densities. Then, we can deduce geometrical equivalent of second spectral moments and anisotropy ratios that allow the estimation of the anisotropy of the underlying Gaussian field.

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## 1. Introduction

Since the seminal works of Robert Adler on the geometry of random fields [1] and of Keith Worsley on topological inference in neuroimaging [44], the study of the geometry of excursion sets of random fields has known a growing interest, with important developments especially for the Euler characteristic, used as a good approximation of the tail distribution of the supremum of stationary smooth Gaussian fields [45]. In particular the so-called Gaussian kinematic formula [2] nicely links mean geometries of an excursion set with observation window's ones with respect to the underlying statistical properties of the Gaussian field. However, this formula strongly depends on the metric induced by the stationary Gaussian field that is required for computing Lipschitz-Killing curvatures. The isotropy assumption is then crucial for this metric being the Euclidean one multiplied by the so-called second spectral moment. In this isotropic Gaussian setting lots of theoretical results have been obtained these last years, especially for the study of nodal lines, corresponding to the boundary of excursion set at threshold 0 (see [43] for a review). Numerical investigations, motivated by a wide range of applications (not only on the plane but also on the sphere, as in [21] with spherical eigenfunctions), are also fastly developing, leading for instance to new unbiased estimators as in [15].

In sharp contrast, in this paper we consider anisotropic Gaussian random fields and Lipschitz-Killing curvatures of excursion sets computed with the usual Euclidean metric in dimension  $d$  for both  $d = 2$  and  $d = 3$ , corresponding to the usual dimensions of medical images. This allows us to rely on numerous results and algorithms developed in stochastic geometry for intrinsic volumes or Minkowski measures closely related to Lipschitz-Killing curvatures [35].

Moreover, as in our previous paper [9], we will have here a "weak" point of view. Instead of fixing a threshold level and considering the Lipschitz-Killing curvatures of the excursion set above this level, we will consider simultaneously all the levels and thanks to a change of variable formula (the coarea formula), we will be able to have a representation of these Lipschitz-Killing curvatures as integrals over the function domain. This point of view also allows us to work on the fine functional framework of generalized random processes [23]. In this setting, we are able to propose consistent and asymptotic (as the size of the observation window grows) Gaussian estimators of the Lipschitz-Killing curvature densities. Note that in sharp contrast with previous results, we do not assume isotropy of the Gaussian field.

Actually, a main point of interest is: how to "read" and "estimate" the anisotropy of a random field from the geometry of (some of) its excursion sets? We provide explicit parametric expressions for Lipschitz-Killing densities in both dimension  $d = 2$  and  $d = 3$ , as well as numerical evaluations. It allows us to define new geometrical equivalent of spectral moments, related through elegant isoperimetric inequalities, as well as robust anisotropy ratios in the sense that they do not depend on the mean, nor on the standard deviation of the field, a point that is crucial for image comparisons. As illustrated on Figure 1 we will summarize the geometry of an excursion set as a (2D here) point and this will allow us to visualize and estimate the anisotropy of the underlying Gaussian field.

The paper is organized as follows. In Section 2 we recall the main definitions of Lipschitz-Killing curvatures following [41], and introduce the Integral Lipschitz-Killing curvatures of a smooth function that allow us to consider Lipschitz-Killing curvatures of its excursion sets as a tempered distribution (with respect to the levels). Considering a smooth stationary random field we are therefore able to define Lipschitz-Killing curvatures for which a functional joint Central Limit Theorem is established in dimension  $d = 2$  or  $d = 3$  under the additional assumption that the

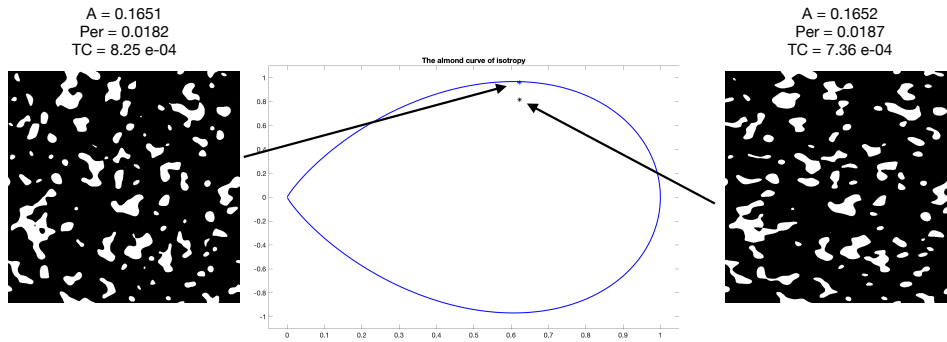


FIG 1. Two excursion sets of Gaussian random fields. The left one is isotropic and the right one is not. They have the same area, the same perimeter but different total curvature. Their geometry can be summarized as a point that is on the almond curve in the isotropic case and that is inside the almond domain in the anisotropic case.

underlying field is Gaussian. In contrast with previous Gaussian results [19, 30] based on Itô-Wiener chaos expansion, our proof relies on preliminary general results stated under a quasi-association assumption closer to [13], and which are postponed to the Appendix A. In Section 3, we focus on dimension  $d = 2$  and extend the Gaussian isotropic results of [10] to the anisotropic framework in order to get a general result. We define two important ordered geometrical equivalents of second spectral moments to state explicit formulas for the Lipschitz-Killing curvature densities. This allows us to define an almond curve of anisotropy and a first ratio of anisotropy  $R$  with values in  $(0, 1]$ . We proceed in a similar way in Section 4 for the dimension  $d = 3$ . Lipschitz-Killing curvature densities are also expressed with respect to now three important geometrical equivalent of second spectral moments. The ordering of these quantities is much more difficult to obtain than in the 2D case and strongly relies on some isoperimetric inequalities. In addition to the almond curve of anisotropy we define the heart curve of anisotropy involving the third index and get two respective ratios of anisotropy  $R$  and  $R_G \leq R$ , both with values in  $(0, 1]$ . We illustrate all this in both dimensions  $d = 2$  and  $d = 3$  through numerical experiments, checking the formulas for the Lipschitz-Killing curvatures, the Central Limit behavior, and the statistical estimation of the anisotropy from the geometry of one excursion set at a single level.

## 2. Lipschitz-Killing curvatures

### 2.1. Lipschitz-Killing curvatures : the general framework

Let us introduce some notations. In the following we will denote by  $\mathcal{L}^d$  or simply by  $\mathcal{L}$  (when there is no ambiguity) the Lebesgue measure on  $\mathbb{R}^d$  and by  $\mathcal{H}^s$  the  $s$ -dimensional Hausdorff measure on  $\mathbb{R}^d$ . We have in particular  $\mathcal{H}^d = \mathcal{L}^d$  on  $\mathbb{R}^d$ .

We now recall the definition of Lipschitz-Killing curvatures measures in the framework of smooth manifolds of  $\mathbb{R}^d$ , as they are presented in the paper of Thäle [41]. Let  $M$  be a compact  $d$ -dimensional submanifold in  $\mathbb{R}^d$  with a  $C^2$  smooth boundary  $\partial M$  (that is then  $(d-1)$ -dimensional). For  $x \in \partial M$ , let  $\kappa_i(x)$ ,  $i = 1, \dots, d-1$ , denote the  $d-1$  principal curvatures of  $\partial M$  at  $x$ . Then the Lipschitz-Killing curvatures measures of  $M$  are defined by, for all Borel set  $B \subset \mathbb{R}^d$

$$\forall j = 0, \dots, d-1, \quad C_j(M, B) = \frac{1}{\alpha_{d-1-j}} \int_{\partial M \cap B} \sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x)) \mathcal{H}^{d-1}(dx), \quad (1)$$

$$\text{and} \quad C_d(M, B) = \mathcal{L}^d(M \cap B),$$

where  $\sigma_k$  is the symmetric elementary function of order  $k$ , that is given for  $1 \leq k \leq d-1$  by

$$\sigma_k(\kappa_1(x), \dots, \kappa_{d-1}(x)) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq d-1} \kappa_{i_1}(x) \dots \kappa_{i_k}(x),$$

with the convention that  $\sigma_0 = 1$ , and where  $\alpha_k$ ,  $k \geq 0$  is the  $k$ -dimensional surface area of the unit ball of  $\mathbb{R}^{k+1}$ , that is :

$$\alpha_0 = 2, \quad \alpha_1 = 2\pi, \quad \alpha_2 = 4\pi, \quad \text{etc.}$$

The Lipschitz-Killing curvatures measures have therefore the following interpretations:

- $C_0(M, B) = \frac{1}{\alpha_{d-1}} \int_{\partial M \cap B} \kappa_1(x) \dots \kappa_{d-1}(x) \mathcal{H}^{d-1}(dx)$  is the integral of the Gaussian curvature on  $\partial M \cap B$  divided by  $\alpha_{d-1}$ .
- $C_{d-2}(M, B) = \frac{1}{\alpha_1} \int_{\partial M \cap B} (\kappa_1(x) + \dots + \kappa_{d-1}(x)) \mathcal{H}^{d-1}(dx)$  is the integral of the mean curvature on  $\partial M \cap B$  divided by  $\alpha_1$ .
- $C_{d-1}(M, B) = \frac{1}{\alpha_0} \int_{\partial M \cap B} \mathcal{H}^{d-1}(dx) = \frac{1}{2} \mathcal{H}^{d-1}(\partial M \cap B)$  is half the surface area (perimeter when  $d = 2$ ) of  $\partial M$  in  $B$ .
- $C_d(M, B) = \mathcal{L}^d(M \cap B)$  is the volume of  $M$  in  $B$ .

As soon as  $M \subset B$  (called the "global case"), these quantities do not depend on  $B$ . In particular, one has

$$C_0(M, B) = \frac{1}{\alpha_{d-1}} \int_{\partial M} \kappa_1(x) \dots \kappa_{d-1}(x) \mathcal{H}^{d-1}(dx) = \chi(M),$$

where  $\chi(M)$  is the Euler Characteristic of  $M$ , by the Gauss-Bonnet Theorem. Note also that in this "global case", for  $d \geq 3$ , and  $M$  a convex body,

$$C_{d-2}(M, B) = \frac{1}{\alpha_1} \int_{\partial M} (\kappa_1(x) + \dots + \kappa_{d-1}(x)) \mathcal{H}^{d-1}(dx) = \frac{d-1}{2\pi} b(M),$$

where  $b(M)$  is the so-called mean breadth of  $M$ .

These curvature measures may be generalized for sets  $M$  with positive reach as done in Theorem 3 in [46] and satisfy the *local property*: if  $M'$  is another positive reach set such that  $M \cap M'$  has positive reach, then for all bounded Borel set  $B \subset \overset{\circ}{M}'$ , one has

$$C_j(M \cap M', B) = C_j(M, B).$$

In view of the local property of curvature measures, when  $U$  is an open bounded set such that  $M \cap \bar{U}$  is a positive reach set (let say  $U = (0, T)^d$  for instance), one has for all  $0 \leq j \leq d$ ,

$$C_j(M \cap \bar{U}, U) = C_j(M, U).$$

Moreover, in view of the measure property one has

$$\begin{aligned} C_j(M \cap \bar{U}, \bar{U}) &= C_j(M \cap \bar{U}, U) + C_j(M \cap \bar{U}, \partial U) \\ &= C_j(M, U) + C_j(M \cap \bar{U}, \partial U). \end{aligned}$$

In particular, for all Borel set  $B$  such that  $\bar{U} \subset B$ , one has  $C_d(M \cap \bar{U}, B) = C_d(M \cap \bar{U}, \bar{U}) = C_d(M, U)$  but  $C_{d-1}(M \cap \bar{U}, B) = C_{d-1}(M \cap \bar{U}, \bar{U}) = C_{d-1}(M, U) + \mathcal{H}^{d-1}(M \cap \partial U)$  and the Euler Characteristic of  $M \cap \bar{U}$  is given by

$$\chi(M \cap \bar{U}) = C_0(M, U) + C_0(M \cap \bar{U}, \partial U). \quad (2)$$

The Lipschitz-Killing curvatures measures are also related to the volume of the  $r$ -parallel set to  $M$  through the Weyl's tube formula. This point will be discussed in the further remarks of Section 5.2.

## 2.2. The framework of smooth functions

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^d$  function defined on  $\mathbb{R}^d$ , with here  $d \geq 2$ . For  $t \in \mathbb{R}$ , the excursion set of  $f$  above the level  $t$  is given by

$$E_f(t) := \{x \in \mathbb{R}^d; f(x) \geq t\}.$$

By continuity of  $f$ , the boundary of  $E_f(t)$  is the level line

$$\partial E_f(t) := \{x \in \mathbb{R}^d; f(x) = t\}.$$

The level  $t$  is called a regular value of  $f$  on  $\mathbb{R}^d$  if

$$\forall x \in \mathbb{R}^d, f(x) = t \implies \|\nabla f(x)\| \neq 0.$$

For  $t$  a regular value, the unit inner normal vector of  $E_f(t)$  at  $x \in \partial E_f(t)$  is given by

$$\nu_f(x) := \frac{\nabla f(x)}{\|\nabla f(x)\|} \in S^{d-1},$$

and one can find a local  $C^2$  parametrization of  $\partial E_f(t)$  by the implicit function Theorem. The Gauss map is defined by  $\nu_f : \partial E_f(t) \mapsto S^{d-1}$ . At a point  $x \in \mathbb{R}^d$  such that  $\nabla f(x) \neq 0$ , we can compute  $D\nu_f(x)$  using  $D^2f(x)$ , the Hessian matrix of  $f$  at  $x$ , and have

$$D\nu_f(x) = \frac{1}{\|\nabla f(x)\|} D^2f(x) - \frac{1}{\|\nabla f(x)\|^3} \nabla f(x) (D^2f(x) \nabla f(x))^t.$$

As the principal curvatures are the eigenvalues of the second fundamental form, it follows that the  $d$  symmetric functions of the curvatures  $\sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x))$  may be written as measurable functions (explicitly in some cases) of

$$\mathbf{f}(x) := (f(x), \nabla f(x), D^2f(x)).$$

Then we propose the following definition of Lipschitz-Killing curvatures on a bounded open set  $U$  of  $\mathbb{R}^d$  (for instance  $U = (0, T)^d$  with  $T > 0$ ) for the excursion set  $E_f(t)$ .

**Definition 2.1.** Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^d$  function and let  $t \in \mathbb{R}$  be a regular value of  $f$  on  $\mathbb{R}^d$ . Let  $U$  be an open bounded set of  $\mathbb{R}^d$ . The Lipschitz-Killing curvatures of  $E_f(t)$  on  $U$  are defined by  $C_d(E_f(t), U) = \mathcal{L}^d(E_f(t) \cap U)$  and, for all  $0 \leq j \leq d-1$ , by

$$C_j(E_f(t), U) = \frac{1}{\alpha_{d-1-j}} \int_{\partial E_f(t) \cap U} \sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x)) \mathcal{H}^{d-1}(dx), \quad (3)$$

where  $\kappa_1(x), \dots, \kappa_{d-1}(x)$  are the eigenvalues of the second fundamental form given by the restriction of  $D\nu_f(x)$  on the tangent space to  $\partial E_f(t)$  at  $x$ .

We can therefore define the Lipschitz-Killing curvatures of the excursion sets of  $f$  for levels  $t$  that are not critical values, in the sense that  $\nabla f(x) \neq 0$  when  $f(x) = t$ . Now, what happens with critical values? By Morse-Sard's Theorem (see [27] p.69 for instance), since  $f$  is  $C^d$ , the set of critical values of  $f$  has Lebesgue measure 0 in  $\mathbb{R}$ .

Moreover, let us recall the coarea formula for Lipschitz mappings (see [20] p.117 for instance), that we will use several times in this paper: for any non-negative  $\mathcal{L}^d$ -measurable function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ , the function  $t \mapsto \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^{d-1}(dx)$  is  $\mathcal{L}^1$ -measurable and

$$\int_U g(x) \|\nabla f(x)\| dx = \int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} g(x) \mathcal{H}^{d-1}(dx) dt. \quad (4)$$

Then we introduce the following auxiliary functions.

**Definition 2.2.** For  $x \in \mathbb{R}^d$ , we denote  $\mathbf{f}(x) := (f(x), \nabla f(x), D^2f(x))$  and define the weighted symmetric elementary function of curvatures for  $j = 0, \dots, d-1$  as the functions

$$F_{d-1-j}(\mathbf{f}(x)) := \|\nabla f(x)\| \sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x)), \quad \text{when } \nabla f(x) \neq 0, \quad (5)$$

and by convention we set  $F_{d-1-j}(\mathbf{f}(x)) := 0$  when  $\nabla f(x) = 0$ .

Some of these weighted symmetric elementary functions of curvatures are explicit. In particular, we have that (see [38] or [24] for instance)

$$F_0(\mathbf{f}(x)) = \|\nabla f(x)\|, \quad (6)$$

$$F_1(\mathbf{f}(x)) = \left[ -\text{Tr}(D^2 f(x)) + \frac{\nabla f(x)^t D^2 f(x) \nabla f(x)}{\|\nabla f(x)\|^2} \right] \mathbf{1}_{\|\nabla f(x)\| > 0}, \quad (7)$$

and for  $d \geq 3$

$$F_{d-1}(\mathbf{f}(x)) = \left[ (-1)^{d-1} \frac{\nabla f(x)^t \text{adj}(D^2 f(x)) \nabla f(x)}{\|\nabla f(x)\|^d} \right] \mathbf{1}_{\|\nabla f(x)\| > 0}, \quad (8)$$

where  $\text{adj}(D^2 f(x))$  is the adjugate matrix of  $D^2 f(x)$  (that is the transpose of its cofactor matrix). Let us remark that for  $d = 2$ , one can also express  $F_1$  as

$$F_1(\mathbf{f}(x)) = \left[ -\frac{(\nabla f(x)^\perp)^t D^2 f(x) \nabla f(x)^\perp}{\|\nabla f(x)\|^2} \right] \mathbf{1}_{\|\nabla f(x)\| > 0}, \quad (9)$$

with  $\nabla f(x)^\perp = (-\partial_2 f(x), \partial_1 f(x))$ .

Now, by convention, for  $j \leq d-1$ , we set  $C_j(E_f(t), U) = 0$  if  $t$  is a critical value and will be able to consider the function  $t \mapsto C_j(E_f(t), U)$  as an integrable function, by neglecting critical values. This is a huge difference with the framework of Adler [1] that focuses on critical points (points where  $\nabla f(x) = 0$ ). However, since  $t \mapsto C_d(E_f(t), U)$  is not an integrable function, it will be more convenient to consider all of them as tempered distributions. More precisely, let us recall that the *Schwartz space*  $\mathcal{S} = \mathcal{S}(\mathbb{R})$  consists of infinitely differentiable functions  $h : \mathbb{R} \rightarrow \mathbb{R}$  that are rapidly decreasing, that is, for all  $q \in \mathbb{N}$  and  $k \in \mathbb{N}$ ,

$$\|h\|_{q,k} = \sup_{t \in \mathbb{R}} (1 + |t|)^q |h^{(k)}(t)| < \infty,$$

where  $h^{(k)}$  denotes the derivative of order  $k$  of  $h$ . It is a real vector space and it is equipped with the topology given by the family of semi-norms  $\|\cdot\|_{q,k}$ ,  $q \in \mathbb{N}$  and  $k \in \mathbb{N}$ . It is well known that  $\mathcal{S}$  is a Fréchet space (i.e. a locally convex metrizable complete space). The *space of tempered distributions*  $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$  is the topological dual of  $\mathcal{S}$ , that is the space of continuous linear functionals on  $\mathcal{S}$ .

We give in the following proposition conditions to ensure the validity of this functional point of view.

**Proposition 2.1.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^d$  function. Let  $U$  be a bounded open set of  $\mathbb{R}^d$  and assume that, for  $j \leq d-1$ ,*

$$\int_U |F_{d-1-j}(\mathbf{f}(x))| dx < +\infty, \quad (10)$$

with  $F_{d-1-j}(\mathbf{f}(x))$  given by (5). Then for  $j \leq d-1$ , one has  $t \mapsto C_j(E_f(t), U) \in L^1(\mathbb{R})$ . Therefore, for  $j \in \{0, \dots, d\}$ , one has  $t \mapsto C_j(E_f(t), U) \in \mathcal{S}'$  with for all  $h \in \mathcal{S}$ , the Integral Lipschitz-Killing Curvatures of  $f$  defined by

$$\text{LC}_j^f(h, U) := \int_{\mathbb{R}} h(t) C_j(E_f(t), U) dt = \int_U L_j(h, \mathbf{f}(x)) dx, \quad (11)$$

where  $h \in \mathcal{S} \mapsto L_j(h, \mathbf{f}(x)) \in \mathbb{R}$  is linear continuous for all  $x \in U$ , and given by

$$L_j(h, \mathbf{f}(x)) = \frac{1}{\alpha_{d-1-j}} h(f(x)) F_{d-1-j}(\mathbf{f}(x)) \quad \text{for } j \neq d,$$

and

$$L_d(h, \mathbf{f}(x)) = \int_{\mathbb{R}} h(t) \mathbf{1}_{f(x) \geq t} dt.$$

*Proof.* For  $j \leq d-1$ , and  $h$  a continuous bounded function we can define measurable functions on  $\mathbb{R}^d$  by

$$\begin{aligned} g_{d-1-j}(x) &= h(f(x)) \frac{F_{d-1-j}(\mathbf{f}(x))}{\|\nabla f(x)\|} \mathbf{1}_{\|\nabla f(x)\|>0} \\ &= h(f(x)) \sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x)) \mathbf{1}_{\|\nabla f(x)\|>0} \quad \text{for } x \in \mathbb{R}^d. \end{aligned}$$

Under the assumption (10) we obtain by the coarea formula (Eq. (4)) that

$$\int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} |g_{d-1-j}(x)| \mathcal{H}^{d-1}(dx) dt \leq \|h\|_{0,0} \int_U |F_{d-1-j}(\mathbf{f}(x))| dx < +\infty.$$

Taking  $h = 1$ , it follows that for a.e.  $t \in \mathbb{R}$

$$\int_{\partial E_f(t) \cap U} |\sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x))| \mathbf{1}_{\|\nabla f(x)\|>0} \mathcal{H}^{d-1}(dx) < +\infty$$

and  $t \mapsto C_j(E_f(t), U)$  is  $\mathcal{L}^1$ -integrable. We can define for a general continuous bounded function  $h$

$$\text{LC}_j^f(h, U) := \int_{\mathbb{R}} h(t) C_j(E_f(t), U) dt.$$

Moreover, using again the coarea formula with  $\max(g_{d-1-j}, 0)$  and  $-\min(g_{d-1-j}, 0)$  and subtracting we get

$$\int_{\mathbb{R}} \int_{\partial E_f(t) \cap U} g_{d-1-j}(x) \mathcal{H}^{d-1}(dx) dt = \int_U h(f(x)) F_{d-1-j}(\mathbf{f}(x)) dx,$$

with for all regular values  $t \in \mathbb{R}$  (and therefore a.e.  $t \in \mathbb{R}$ ),

$$\begin{aligned} \int_{\partial E_f(t) \cap U} g_{d-1-j}(x) \mathcal{H}^{d-1}(dx) &= h(t) \int_{\partial E_f(t) \cap U} \sigma_{d-1-j}(\kappa_1(x), \dots, \kappa_{d-1}(x)) \mathbf{1}_{\|\nabla f(x)\|>0} \mathcal{H}^{d-1}(dx) \\ &= h(t) \alpha_{d-1-j} C_j(E_f(t), U). \end{aligned}$$

Then, we get

$$\text{LC}_j^f(h, U) = \int_U L_j(h, \mathbf{f}(x)) dx,$$

with  $L_j(h, \mathbf{f}(x)) = \frac{1}{\alpha_{d-1-j}} h(f(x)) F_{d-1-j}(\mathbf{f}(x))$ . Moreover, for  $h \in \mathcal{S}$  and  $x \in U$ ,

$$|L_j(h, \mathbf{f}(x))| \leq \frac{1}{\alpha_{d-1-j}} \|h\|_{0,0} |F_{d-1-j}(\mathbf{f}(x))|,$$

and  $h \mapsto L_j(h, \mathbf{f}(x)) \in \mathcal{S}'$ . Under Assumption (10), it allows us to conclude that  $h \mapsto \text{LC}_j^f(h, U) \in \mathcal{S}'$  with

$$|\text{LC}_j^f(h, U)| \leq \left( \int_U \frac{1}{\alpha_{d-1-j}} |F_{d-1-j}(\mathbf{f}(x))| dx \right) \|h\|_{0,0}.$$

Finally, for  $j = d$  we can also define

$$L_d(h, \mathbf{f}(x)) = \int_{\mathbb{R}} h(t) \mathbf{1}_{f(x) \geq t} dt,$$

satisfying  $|L_d(h, \mathbf{f}(x))| \leq 2\|h\|_{2,0}$ , using the fact that  $\int_{\mathbb{R}} (1+|t|)^{-2} dt = 2$ . Since  $U$  is bounded we also have  $h \mapsto \text{LC}_d^f(h, U) = \int_U L_d(h, \mathbf{f}(x)) dx \in \mathcal{S}'$ , with, by Fubini's Theorem,

$$\text{LC}_d^f(h, U) = \int_{\mathbb{R}} h(t) C_d(E_f(t), U) dt \quad \text{and} \quad |\text{LC}_d^f(h, U)| \leq 2\|h\|_{2,0} \mathcal{L}^d(U).$$

□



Let us remark that, in view of (11), several papers start their study by considering at some fixed level  $t \in \mathbb{R}$

$$C_j(E_f(t), U) := \int_U L_j(\delta_t, \mathbf{f}(x)) dx \quad (= LC_d^f(\delta_t, U)),$$

where  $\delta_t$  has to be understood as the limit of a shifted approximation of identity. This can of course be justified as soon as  $s \mapsto C_j(E_f(s), U)$  is continuous in a neighborhood of  $t$  for example. We refer to [7] for detailed further assumptions needed to go from results for a.e. level to results for one level, and to Section 2.5 for a discussion on this point.

**Illustrating example.** Let us give here a simple example to illustrate the quantities we compute in this paper. We consider the smooth function  $f$  defined on  $\mathbb{R}^2$  by  $f(x) = \exp(-\|x\|^2)$ , and we compute the Lipschitz-Killing curvatures of its excursion sets in the square domain  $U = (-1, 1)^2$ . The level lines of  $f$  are circles and some of them are shown on Figure 2. More precisely for  $t \in \mathbb{R}$ , we have

$$E_f(t) = \{x \in \mathbb{R}^2; f(x) \geq t\} = \{x \in \mathbb{R}^2; \|x\| \leq \sqrt{-\log t}\} \quad \text{for } 0 < t \leq 1,$$

while  $E_f(t) = \mathbb{R}^2$  for  $t \leq 0$  and  $E_f(t) = \emptyset$  for  $t > 1$ . Therefore considering the intersection with  $U$  we obtain  $C_2(E_f(t), U) = \mathcal{L}^2(U \cap E_f(t))$  for all  $t \in \mathbb{R}$ , and this is the orange curve on the right-hand side plot of Figure 2. For the perimeter and the total curvature, we have to take into account the circles that are not entirely contained in  $U$ . Simple computations give that

- for  $\exp(-2) < t \leq \exp(-1)$ , then  $C_1(E_f(t), U) = (\pi - 4 \arccos \frac{1}{\sqrt{-\log t}}) \sqrt{-\log t}$  and  $C_0(E_f(t), U) = 1 - \frac{1}{4\pi} \arccos \frac{1}{\sqrt{-\log t}}$ ,

- for  $\exp(-1) \leq t < 1$ , then  $C_1(E_f(t), U) = \pi \sqrt{-\log t}$  and  $C_0(E_f(t), U) = 1$ ,

- for  $t \leq \exp(-2)$  or  $t \geq 1$  then  $C_1(E_f(t), U) = C_0(E_f(t), U) = 0$ .

The curves for  $C_1$  and  $C_0$  are also shown on the right-hand side plot of Figure 2, respectively in red and blue. We also plot (dotted blue curve) the Euler characteristic of  $E_f(t) \cap \bar{U}$  given by (2) for comparison. The Euler characteristic has value 1 for  $t \leq 1$  and 0 for  $t > 1$ , whereas the quantity  $C_0(E_f(t), U)$  is the integral of the curvature along the boundary of  $E_f(t)$  in  $U$ , and therefore it coincides with the Euler characteristic only when the circles (level lines) are non-degenerate and entirely contained in  $U$ , which is the case for  $\exp(-1) < t < 1$ , or also when  $t > 1$  (since then  $E_f(t) = \emptyset$ ).

This example also illustrates the question of critical points. Indeed here there is only one critical point, at  $x = 0$  for the level  $t = 1$ . At  $t = 1$  the function  $t \rightarrow C_0(E_f(t), U)$  is not continuous, whereas the functions  $C_1$  and  $C_2$  are. The approach we have in this paper allows us to have results for almost every level  $t$ . Obtaining results for all levels  $t$  would require difficult additional results that are beyond the scope of this paper.

### 2.3. Smooth stationary random fields and Lipschitz-Killing curvature densities

We extend now the Lipschitz-Killing curvatures integrals to the case of smooth stationary random fields.

**Proposition 2.2.** *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space and  $(X(x))_{x \in \mathbb{R}^d}$  be a stationary real-valued random field that is almost surely (a.s.) a  $C^d$  function on  $\mathbb{R}^d$  for  $d \geq 2$ . Assume moreover that*

$$\forall j \leq d-1, \quad \mathbb{E}(|F_{d-1-j}(\mathbf{X}(x))|) < +\infty, \quad (12)$$

for  $F_{d-1-j}$  given by (6), (7) and (8), and  $\mathbf{X}(x) = (X(x), \nabla X(x), D^2 X(x))$ ,  $x \in \mathbb{R}^d$ . Let  $U$  be a bounded open set of  $\mathbb{R}^d$ . Then, for  $j \in \{0, d-2, d-1, d\}$ , we can define  $C_j(E_X(\cdot), U)$  as a generalized integrable process, in the sense that

$$C_j(E_X(\cdot), U) : (\Omega, \mathcal{A}) \rightarrow (\mathcal{S}', \mathcal{B}(\mathcal{S}'))$$

is measurable, where  $\mathcal{B}(\mathcal{S}')$  is the Borel  $\sigma$ -field of the strong topology on  $\mathcal{S}'$  and, for all  $h \in \mathcal{S}$ ,

$$\mathbb{E}(|LC_j^X(h, U)|) \leq \int_{\mathbb{R}} |h(t)| \mathbb{E}(|C_j(E_X(t), U)|) dt < +\infty.$$

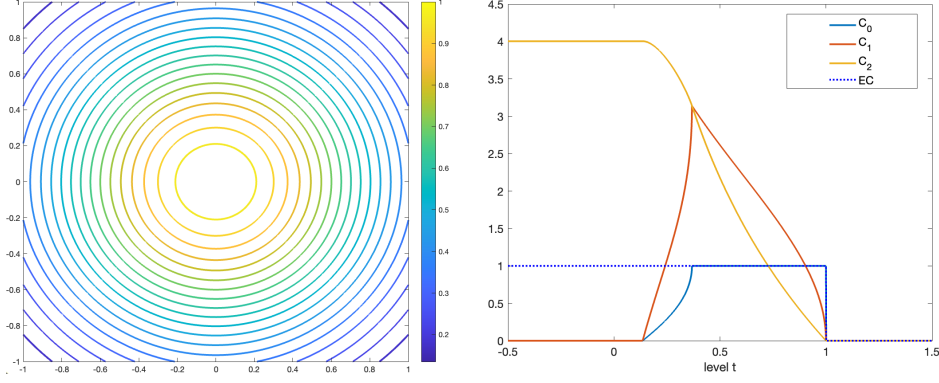


FIG 2. Simple example to illustrate what we compute here. On the left, some of the level sets of the function  $x \rightarrow \exp(-\|x\|^2)$  in  $U = (-1, 1)^2$ . On the right, the Lipschitz-Killing curvatures  $C_j(E_f(t), U)$  for  $j = 0, 1, 2$ , as a function of the level  $t$ . We notice in particular that  $C_0$  (blue curve) is not continuous at  $t = 1$  and that it coincides with the Euler characteristic (dotted blue curve) only when  $\exp(-1) < t < 1$ , which correspond to cases where the level set is included in  $U$  (or when  $t > 1$ ).

*Proof.* First note that since  $(\Omega, \mathcal{A}, \mathbb{P})$  is a complete probability space, in view of the a.s. continuity, up to setting the values of  $X$  at 0 on a negligible set, the field  $X : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a  $\mathbb{P} \otimes \mathcal{L}^d$ -measurable function, as well as its first and second order partial derivatives. By stationarity, we deduce from (12) that for  $j \in \{0, d-2, d-1\}$

$$\int_U |F_{d-1-j}(\mathbf{X}(x))| dx < +\infty \quad \text{a.s.},$$

so that (10) is satisfied. Hence, there exists a negligible set  $N$  of  $\mathcal{A}$  such that for  $j \in \{0, d-2, d-1\}$ ,  $C_j(E_{X(\omega)}(\cdot), U) \in L^1(\mathbb{R}) \subset \mathcal{S}'$  for  $\omega \notin N$  and we can set  $C_j(E_{X(\omega)}(\cdot), U) = 0$  for  $\omega \in N$ . We can therefore define  $\text{LC}_j^{X(\omega)}(h, U)$ , for  $h \in \mathcal{S}$ , satisfying (11) for  $\omega \notin N$  and  $\text{LC}_j^{X(\omega)}(h, U) = 0$  for  $\omega \in N$ . Using the fact that  $(\omega, x) \mapsto L_j(h, \mathbf{X}(\omega, x))$  is  $\mathbb{P} \otimes \mathcal{L}^d$ -measurable, it follows by Fubini's theorem that  $\text{LC}_j^X(h, U) : \omega \in \Omega \mapsto \text{LC}_j^{X(\omega)}(h, U) \in \mathbb{R}$  is a well-defined real-valued random variable. Therefore,  $(\text{LC}_j^X(h_1, U), \dots, \text{LC}_j^X(h_m, U))$  is a random vector for all  $m \geq 1$  and  $h_1, \dots, h_m \in \mathcal{S}$ . Since the cylinder  $\sigma$ -field coincides with the Borel  $\sigma$ -field of the weak topology but also of the strong topology on  $\mathcal{S}'$  denoted by  $\mathcal{B}(\mathcal{S}')$  (see Corollary 3.9 of [11]) we can deduce that  $C_j(E_X(\cdot), U) : (\Omega, \mathcal{A}) \rightarrow (\mathcal{S}', \mathcal{B}(\mathcal{S}'))$  is measurable and therefore defines a generalized random process. To conclude for integrability, it is enough to remark that by Fubini's theorem and the coarea formula

$$\begin{aligned} \mathbb{E}(|\text{LC}_j^X(h, U)|) &\leq \int_{\mathbb{R}} |h(t)| \mathbb{E}(|C_j(E_X(t), U)|) dt \\ &\leq \frac{\|h\|_{0,0}}{\alpha_{d-1-j}} \int_U \mathbb{E}(|F_{d-1-j}(\mathbf{X}(x))|) dx < +\infty, \end{aligned}$$

by (12), using the stationarity of  $X$  and the boundedness of  $U$ . The result for  $j = d$  follows the same lines as  $(\omega, x) \mapsto L_d(h, \mathbf{X}(\omega, x))$  is  $\mathbb{P} \otimes \mathcal{L}^d$ -measurable and bounded by  $2\|h\|_{2,0}$ .  $\square$

Thanks to the stationarity we will focus on Lipschitz-Killing curvature densities defined for  $j \in \{0, \dots, d\}$  by

$$\langle \bar{C}_j^X, h \rangle = \frac{\mathbb{E}(\text{LC}_j^X(h, U))}{\mathcal{L}^d(U)}.$$

Note that we can define for all  $t \in \mathbb{R}$ ,  $\bar{C}_d^X(t) := \frac{\mathbb{E}(C_d(E_X(t), U))}{\mathcal{L}^d(U)} = \mathbb{P}(X(0) \geq t)$  that does not depend

on  $U$  and is such that  $\bar{C}_d^X \in \mathcal{S}'$  (but not in  $L^1(\mathbb{R})$ ) with

$$\langle \bar{C}_d^X, h \rangle = \int_{\mathbb{R}} h(t) \bar{C}_d^X(t) dt = \frac{\mathbb{E}(\text{LC}_d^X(h, U))}{\mathcal{L}^d(U)}.$$

For  $j \leq d-1$ , in view of Fubini's Theorem, we can define for a.e.  $t \in \mathbb{R}$

$$\bar{C}_j^X(t) = \frac{\mathbb{E}(C_j(E_X(t), U))}{\mathcal{L}^d(U)},$$

such that  $\bar{C}_j^X \in L^1(\mathbb{R}) \subset \mathcal{S}'$  and

$$\langle \bar{C}_j^X, h \rangle = \int_{\mathbb{R}} h(t) \bar{C}_j^X(t) dt = \frac{1}{\alpha_{d-1-j}} \mathbb{E}(h(X(0)) F_{d-1-j}(\mathbf{X}(0))). \quad (13)$$

It follows that also  $\bar{C}_j^X$  does not depend on  $U$ .

#### 2.4. Smooth stationary Gaussian random fields and Central Limit Theorem

To ensure sample paths smoothness, a simple assumption for stationary Gaussian random fields may be given in term of covariance functions. More precisely, when  $X$  is a stationary Gaussian field with mean  $m \in \mathbb{R}$  and variance  $\sigma^2 > 0$ , we denote by  $\rho$  its covariance function, given by

$$\rho(x) = \text{Cov}(X(x), X(0)) = \text{Cov}(X(x+y), X(y)),$$

for all  $x, y \in \mathbb{R}^d$  by stationarity. Then it is sufficient to make the following hypothesis, denoted by **H** :

$$\begin{aligned} \rho : \mathbb{R}^d \rightarrow \mathbb{R} \text{ is a } C^{2d} \text{ positive definite function, with } \det(D^2\rho(0)) \neq 0 \\ \text{and there exist } C > 0 \text{ and } \delta > 0 \text{ with } |\partial^{\mathbf{k}}\rho(x) - \partial^{\mathbf{k}}\rho(0)| \leq C\|x\|^\delta, \\ \text{for all } \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d \text{ with } |\mathbf{k}| = \sum_{l=1}^d k_l = 2d. \quad (\mathbf{H}) \end{aligned}$$

**Corollary 2.1.** *Let  $d \in \{2, 3\}$ . Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space and  $(X(x))_{x \in \mathbb{R}^d}$  be a stationary Gaussian random field whose covariance function satisfies **H**. Then, there exists a modification of  $X$  such that the assumptions of Proposition 2.2 are satisfied. Moreover, for  $0 \leq j \leq d$  and  $h \in \mathcal{S}$ , the random variable  $\text{LC}_j^X(h, U)$  is square integrable and*

$$\begin{aligned} \frac{\mathbb{E}(\text{LC}_j^X(h, U))}{\mathcal{L}^d(U)} &= \int_{\mathbb{R}} h(t) \bar{C}_j^X(t) dt \quad \text{with, for } j \leq d-1, \\ \bar{C}_j^X(t) &= \frac{1}{\alpha_{d-1-j}} \mathbb{E}(F_{d-1-j}(\mathbf{X}(0)) | X(0) = t) p_{X(0)}(t) \text{ for a.e. } t \in \mathbb{R}. \end{aligned} \quad (14)$$

*Proof.* By Proposition 2.1 of [14], under **H**, there exists a modification of  $X$  such that  $X$  is  $C^{2d}$  a.s. Note that in this case  $\nabla X(x)$  is an  $\mathbb{R}^d$ -valued Gaussian vector of covariance given by  $\Gamma = -D^2\rho(0)$  and therefore  $\|\nabla X(x)\| > 0$  a.s. with  $\mathbb{E}(\|\nabla X(x)\|^p) < +\infty$  for all  $p \geq 1$  such that  $p(d-2) < d$ . Since  $d \in \{2, 3\}$ , in view of (6), (7), or (8) we have therefore for all  $j \leq d-1$ ,

$$\mathbb{E}(|F_{d-1-j}(\mathbf{X}(x))|) < +\infty,$$

and (12) is satisfied. Moreover, we also have  $\mathbb{E}(F_{d-1-j}(\mathbf{X}(x))^2) < +\infty$ , and  $\text{LC}_j^X(h, U)$  is a square integrable random variable. The result for  $\text{LC}_d^X(h, U)$  simply comes from the fact that  $C_d(E_X(t), U) \leq \mathcal{L}^d(U)$  for all  $t \in \mathbb{R}$ . Finally in view of (13) we simply compute the conditional expectation with respect to  $X(0)$  to identify  $\bar{C}_j^X(t)$  for a.e.  $t \in \mathbb{R}$ .  $\square$

In this Gaussian setting, we obtain the following result, whose technical proof is postponed to Appendix A.

**Theorem 2.1.** *Let  $d \in \{2, 3\}$ . Let  $X$  be a stationary Gaussian field defined on  $\mathbb{R}^d$  whose covariance function satisfies **H** and assume moreover that there exists  $\beta > 19d$  such that*

$$\max_{|\mathbf{k}| \leq 4} |\partial^{\mathbf{k}} \rho(x)| \leq C(1 + |x|)^{-\beta}. \quad (15)$$

Then for  $U_n = (0, n)^d$  one has

$$\left( \sqrt{\mathcal{L}^d(U_n)} \left( \frac{C_j(E_X(\cdot), U_n)}{\mathcal{L}^d(U_n)} - \bar{C}_j^X \right) \right)_{0 \leq j \leq d} \xrightarrow[n \rightarrow +\infty]{\text{distrib.}} (B_j)_{0 \leq j \leq d},$$

where  $(B_j)_{0 \leq j \leq d}$  are centered Gaussian generalized random processes whose covariance functions are given for all  $h, \tilde{h} \in \mathcal{S}$  and  $0 \leq j, k \leq d$ , by

$$\text{Cov} \left( \langle B_j, h \rangle, \langle B_k, \tilde{h} \rangle \right) = \int_{\mathbb{R}^d} \text{Cov} \left( L_j(h, \mathbf{X}(z)), L_k(\tilde{h}, \mathbf{X}(0)) \right) dz.$$

Here the convergence holds in distribution with respect to the strong topology of  $\mathcal{S}'(\mathbb{R})$ .

Let us mention that the decay assumption (15) could be relaxed into  $\max_{|\mathbf{k}| \leq 4} |\partial^{\mathbf{k}} \rho| \in L^1(\mathbb{R}^d)$  as assumed in [19], but here the proof of our theorem does not rely on chaos decomposition. Instead we prove an intermediate result given under a more general assumption of quasi-associated field (not necessarily Gaussian), see Appendix A.

### 2.5. From almost every level to one level

When we consider a single level  $t \in \mathbb{R}$ , under the assumption that  $X$  is a stationary Gaussian field defined on  $\mathbb{R}^d$  for  $d \in \{2, 3\}$ , whose covariance function satisfies **H**, Proposition 6.12 of [5] allows us to get that

$$\mathbb{P}(\exists x \in \mathbb{R}^d; X(x) = t, \|\nabla X(x)\| = 0) = \lim_{n \rightarrow +\infty} \mathbb{P}(\exists x \in U_n; X(x) = t, \|\nabla X(x)\| = 0) = 0,$$

which is a generalization of Bulinskaya's Lemma. It follows that a.s. the level  $t$  is a regular level for which there is no critical points. In particular a.s., for all  $n \geq 1$  one has  $C_j(E_X(t), U_n)$  given by (3) at this level  $t$ .

Assuming that there exists an open interval  $V$  with  $t \in V$  and such that  $s \in V \mapsto \mathbb{E}(C_j(E_X(s), U_n))$  is continuous ensures that (14) holds at level  $t$ . We refer to Theorem 7.1 of [4] for similar concerns and weak Bulinskaya condition.

Now, let us be more explicit about the covariances involved in Theorem 2.1. Note first that for all  $h, \tilde{h} \in \mathcal{S}$ ,

$$\text{Cov} \left( L_d(h, \mathbf{X}(z)), L_d(\tilde{h}, \mathbf{X}(0)) \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(s) \tilde{h}(t) \text{Cov}(\mathbf{1}_{X(z) \geq s}, \mathbf{1}_{X(0) \geq t}) ds dt.$$

Then, introducing for  $z \in \mathbb{R}^d$ ,

$$\Sigma_{dd}^{st}(z) := \text{Cov}(\mathbf{1}_{X(z) \geq s}, \mathbf{1}_{X(0) \geq t}), \quad (16)$$

we obtain

$$\text{Cov} \left( L_d(h, \mathbf{X}(z)), L_d(\tilde{h}, \mathbf{X}(0)) \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(s) \tilde{h}(t) \Sigma_{dd}^{st}(z) ds dt.$$

Furthermore, for the other covariances, we will also be able to write them as

$$\text{Cov} \left( L_j(h, \mathbf{X}(z)), L_k(\tilde{h}, \mathbf{X}(0)) \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(s) \tilde{h}(t) \Sigma_{jk}^{st}(z) ds dt.$$

Actually, as already noticed, by stationarity, for  $j < d$ ,

$$\mathbb{E}(L_j(h, \mathbf{X}(z))) = \mathbb{E}(L_j(h, \mathbf{X}(0))) = \frac{1}{\alpha_{d-1-j}} \int_{\mathbb{R}} h(t) \mathbb{E}(F_j(\mathbf{X}(0)) | X(0) = t) p_{X(0)}(t) dt,$$

where  $p_{X(0)}$  denotes the density of the Gaussian variable  $X(0)$ , and

$$\begin{aligned} \mathbb{E}\left(L_j(h, \mathbf{X}(z)) L_d(\tilde{h}, \mathbf{X}(0))\right) &= \int_{\mathbb{R}} \tilde{h}(t) \mathbb{E}\left(h(X(z)) \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(z))}{\alpha_{d-1-j}} \mathbf{1}_{X(0) \geq t} | X(z)\right)\right) dt \\ &= \int_{\mathbb{R}^2} \tilde{h}(t) h(s) \left[ \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(z))}{\alpha_{d-1-j}} \mathbf{1}_{X(0) \geq t} | X(z) = s\right) p_{X(z)}(s) \right] ds dt. \end{aligned}$$

Then we can set

$$\Sigma_{jd}^{st}(z) := \left[ \mathbb{E}(F_{d-1-j}(\mathbf{X}(z)) \mathbf{1}_{X(0) \geq t} | X(z) = s) - \mathbb{E}(F_{d-1-j}(\mathbf{X}(0)) | X(0) = s) \mathbb{P}(X(0) \geq t) \right] \frac{p_{X(0)}(s)}{\alpha_{d-1-j}}. \quad (17)$$

And similarly, for  $k < d$  and  $z \neq 0$ , we set

$$\begin{aligned} \Sigma_{jk}^{st}(z) &:= \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(z))}{\alpha_{d-1-j}} \frac{F_{d-1-k}(\mathbf{X}(0))}{\alpha_{d-1-k}} | X(0) = t, X(z) = s\right) p_{(X(0), X(z))}(t, s) \\ &\quad - \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(0))}{\alpha_{d-1-j}} | X(0) = s\right) \mathbb{E}\left(\frac{F_{d-1-k}(\mathbf{X}(0))}{\alpha_{d-1-k}} | X(0) = t\right) p_{X(0)}(t) p_{X(0)}(s), \end{aligned} \quad (18)$$

denoting by  $p_{(X(0), X(z))}$  the joint density of  $(X(0), X(z))$ . Note that (15) implies that  $\rho$  is an integrable function and therefore  $X$  admits a spectral density, thus implying  $|\rho(z)| < \rho(0) = \sigma^2$  for all  $z \neq 0$ . This ensures the existence of  $p_{(X(0), X(z))}$  as soon as  $z \neq 0$ . Now, if we could justify the interchange of integrals we should also obtain

$$\text{Cov}\left(\langle B_j, h \rangle, \langle B_k, \tilde{h} \rangle\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{h}(t) h(s) \Sigma_{jk}(s, t) ds dt,$$

with

$$\Sigma_{jk}(s, t) = \int_{\mathbb{R}^d} \Sigma_{jk}^{st}(z) dz, \quad (19)$$

where  $\Sigma_{jk}^{st}(z)$  are given by (16), (17) or (18) depending on the values of  $j, k$ . This allows us to give a pointwise representation of the Gaussian generalized processes obtained at the limit. This is precisely the purpose of the following corollary, whose proof is given in Appendix A.3.

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, assuming moreover that for  $J \subset \{0, \dots, d\}$  and  $V$  an open interval of  $\mathbb{R}$ , we have:*

- (A1) *The map  $t \in V \mapsto C_j(E_X(t), U_n) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$  is continuous for all  $n \geq 1$  and  $j \in J$ ;*
- (A2) *For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $t, s \in V$  with  $|t - s| \leq \delta$ , for all  $n \geq 1$  and  $j \in J$ ,*

$$\text{Var}(C_j(E_X(t), U_n) - C_j(E_X(s), U_n)) \leq \varepsilon \mathcal{L}^d(U_n);$$

- (A3) *For all  $j, k \in J$ , for all  $(s, t) \in V \times V$  we have  $\int_{\mathbb{R}^d} |\Sigma_{jk}^{st}(z)| dz < +\infty$ , where  $\Sigma_{jk}^{st}$  is given by (16), (17) or (18), depending on the values of  $j, k$ , and the function  $(s, t) \in V \times V \mapsto \Sigma_{jk}(s, t) = \int_{\mathbb{R}^d} \Sigma_{jk}^{st}(z) dz$  given by (19) is continuous.*

Then

$$\left( \sqrt{\mathcal{L}^d(U_n)} \left( \frac{C_j(E_X(t), U_n)}{\mathcal{L}^d(U_n)} - \bar{C}_j^X(t) \right) \right)_{j \in J, t \in V} \xrightarrow[n \rightarrow +\infty]{fdd} \left( \tilde{B}_j(t) \right)_{j \in J, t \in V},$$

where  $\left( \tilde{B}_j(t) \right)_{j \in J, t \in V}$  is a centered Gaussian process with covariance given by

$$\text{Cov}(\tilde{B}_j(s), \tilde{B}_k(t)) = \Sigma_{jk}(s, t).$$

Note that for  $J = \{d\}$ , under the assumptions of Theorem 2.1, the random field  $X$  satisfies also the assumptions of Corollary 2.2 on  $V = \mathbb{R}$  and we recover results of Theorem 2 of [13]. Actually,  $z \mapsto \Sigma_{dd}^{st}(z)$  is continuous on  $\mathbb{R}^d$  and for all  $z \in \mathbb{R}^d$ , by Lemma 2 of [13], we also have

$$|\Sigma_{dd}^{st}(z)| \leq \frac{1}{4\sigma^2} |\rho(z)|,$$

such that  $\Sigma_{dd}(s, t) = \int_{\mathbb{R}^d} \Sigma_{dd}^{st}(z) dz$  is well-defined and Lebesgue's theorem allows to check (A3) on  $\mathbb{R}$ .

For  $J = \{d-1\}$  and  $d = 2$ , our result corresponds to the one obtained in Theorem 1 of [28], under a general mixing condition, in Theorem 3 of [31] or in Theorem 4.7 of [6].

For a general  $j \in \{0, \dots, d\}$ , we also refer to [30] who proved marginal Central limit theorems for each  $C_j(E_X(t) \cap \overline{U_n}, \overline{U_n})$ , by means of Crofton formula, under an assumption of isotropy that we have relaxed here. Note also that we obtain a stronger result with a joint Central limit theorem. Actually, since

$$C_j(E_X(t) \cap \overline{U_n}, \overline{U_n}) = C_j(E_X(t), U_n) + C_j(E_X(t) \cap \overline{U_n}, \partial U_n),$$

the results of [30] are closely related to ours (see [10] for details in dimension  $d = 2$ ). In particular we have

$$\frac{\mathbb{E}(C_j(E_X(t) \cap \overline{U_n}, \overline{U_n}))}{\mathcal{L}^d(U_n)} = \frac{\mathbb{E}(C_j(E_X(t), U_n))}{\mathcal{L}^d(U_n)} + o_{n \rightarrow +\infty}(1) = \overline{C}_j^X(t) + o_{n \rightarrow +\infty}(1), \quad (20)$$

since  $\mathcal{H}^{d-1}(\partial U_n) = o_{n \rightarrow +\infty}(\mathcal{L}^d(U_n))$ .

In the sequel we will compute explicitly Lipschitz-Killing densities  $\overline{C}_j^X(t)$  at some level  $t \in \mathbb{R}$  and will estimate them using  $\frac{C_j(E_X(t, U))}{\mathcal{L}^d(U)}$  for an observation window  $U$  large enough in view of Corollary 2.2. This is a very sparse setting, since we only rely on one observation of the set  $E_X(t) \cap U$ . In particular we do not need a sample of the whole field in contrast with [39], nor several level values of excursion sets as in [40] who both focus on the estimation of the Euler characteristic of  $E_X(t) \cap \overline{U}$  as given by (2). This enables us to introduce geometrical equivalent to second spectral moments that are strongly related through some kind of isoperimetric inequalities, in sharp contrast with second spectral moments.

### 3. Smooth stationary 2D Gaussian fields

#### 3.1. Lipschitz-Killing curvature densities

When the dimension is  $d = 2$ , the quantities of interest are the Lipschitz-Killing curvatures  $C_0$ ,  $C_1$  and  $C_2$  that are here, up to multiplicative constants, the Total Curvature (related to the Euler Characteristic, but not equal because of the boundary terms that we don't take into account in our framework in view of (2)), the Perimeter, and the Area of the excursion sets of  $X$  in an open bounded domain  $U$ . More precisely we define for a smooth stationary field  $X$  defined on  $\mathbb{R}^2$  and  $h \in \mathcal{S}$

$$\begin{aligned} \text{LA}_X(h, U) &:= \text{LC}_2^X(h, U), & \text{LP}_X(h, U) &:= 2\text{LC}_1^X(h, U) \\ &\text{and} & \text{LTC}_X(h, U) &:= 2\pi \text{LC}_0^X(h, U). \end{aligned}$$

Now we give explicit formulas for the Lipschitz-Killing curvature densities of a stationary  $C^2$  Gaussian field. To the best of our knowledge, our result for  $\overline{C}_0^X$  in this non-isotropic setting is completely new, even if it is not surprising in view of the Gaussian kinematic formula for the mean value of Euler characteristics. The main interesting fact is the introduction of ordered geometrical spectral moments.

**Theorem 3.1.** *Let  $X$  be a stationary Gaussian field of mean  $m \in \mathbb{R}$ , variance  $\sigma^2 > 0$  and whose covariance function  $\rho$  satisfies Hypothesis **H**. We write  $\gamma_1, \gamma_2 > 0$  the eigenvalues of  $\Gamma = -D^2\rho(0)$  and denote by  $\Phi$  the standard Gaussian cumulative distribution. Then the Lipschitz-Killing curvature densities of  $X$  are given for a.e.  $t \in \mathbb{R}$  by*

$$\begin{aligned}\bar{A}_X(t) &:= \bar{C}_2^X(t) &= 1 - \Phi\left(\frac{t-m}{\sigma}\right), \\ \bar{\text{Per}}_X(t) &:= 2\bar{C}_1^X(t) &= \sqrt{\frac{\bar{\gamma}_{\text{Per}}}{\sigma^2}} \frac{1}{2} e^{-(t-m)^2/2\sigma^2} \\ \bar{\text{TC}}_X(t) &:= 2\pi\bar{C}_0^X(t) &= \frac{\bar{\gamma}_{\text{TC}}}{\sigma^2} \frac{1}{\sqrt{2\pi}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2}\end{aligned}$$

where

$$\bar{\gamma}_{\text{Per}} = \left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta \right)^2, \quad (21)$$

and

$$\bar{\gamma}_{\text{TC}} = \sqrt{\gamma_1 \gamma_2}. \quad (22)$$

Therefore, the Gaussian random field  $X$  has same Perimeter density as a (weakly) isotropic<sup>1</sup> Gaussian field with second spectral moment  $\bar{\gamma}_{\text{Per}}$  and same Total Curvature density as a (weakly) isotropic Gaussian field with second spectral moment  $\bar{\gamma}_{\text{TC}}$ . Moreover, we have

$$\min(\gamma_1, \gamma_2) \leq \bar{\gamma}_{\text{TC}} \leq \bar{\gamma}_{\text{Per}} \leq \max(\gamma_1, \gamma_2),$$

where inequalities are strict if and only if  $\gamma_1 \neq \gamma_2$ .

*Proof.* Let  $h \in \mathcal{S}$ . For  $j = 2$ , the result simply comes from the fact that  $\bar{C}_2^X(t) = \mathbb{P}(X(0) \geq t)$  with  $X(0) \sim \mathcal{N}(m, \sigma^2)$ . For  $j \in \{0, 1\}$ , by (13), since  $\|\nabla X(0)\| > 0$  a.s. by **H** and since the random variable  $X(0)$  admits a probability density given by the function  $p_{X(0)}(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-m)^2/2\sigma^2}$ , we have

$$\alpha_{1-j} \langle \bar{C}_j^X, h \rangle = \mathbb{E}(h(X(0)) F_{1-j}(\mathbf{X}(0))) = \int_{\mathbb{R}} h(t) \mathbb{E}(F_{1-j}(\mathbf{X}(0)) | X(0) = t) p_{X(0)}(t) dt.$$

Hence, for a.e.  $t \in \mathbb{R}$ , using (6) and (9),

$$\begin{aligned}\bar{C}_1^X(t) &= \frac{1}{2} \mathbb{E}(\|\nabla X(0)\| | X(0) = t) p_{X(0)}(t) \\ \bar{C}_0^X(t) &= -\frac{1}{2\pi} \mathbb{E} \left( \frac{(\nabla X(0)^\perp)^t D^2 X(0) \nabla X(0)^\perp}{\|\nabla X(0)\|^2} | X(0) = t \right) p_{X(0)}(t).\end{aligned}$$

For sake of completeness we postponed to Appendix B.1 the detailed computations for conditional expectations in order to obtain the announced formulas.

The remark on the "equivalent" isotropic fields that have same Perimeter density or same Total curvature density as  $X$  comes from the fact that when  $\gamma_1 = \gamma_2 = \bar{\gamma}$ , then (21) and (22) yield  $\bar{\gamma}_{\text{Per}} = \bar{\gamma}_{\text{TC}} = \bar{\gamma}$ .

To prove the inequality between  $\bar{\gamma}_{\text{TC}}$  and  $\bar{\gamma}_{\text{Per}}$ , let us first notice that

$$\sqrt{\bar{\gamma}_{\text{Per}}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left( \sqrt{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} + \sqrt{\gamma_1 \sin^2 \theta + \gamma_2 \cos^2 \theta} \right) d\theta.$$

<sup>1</sup>Usually, as in [6] for instance, isotropy means that the covariance function  $\rho$  is invariant by rotation. It is a strong isotropy notion, and it implies in particular that the Hessian  $D^2\rho(0)$  is proportional to the Identity matrix, which is what we call (weak) isotropy. Of course the converse is false in general.



Then, for any  $s \in [0, 1]$ , we have

$$\begin{aligned} \frac{1}{2} \left( \sqrt{\gamma_1 s + \gamma_2(1-s)} + \sqrt{\gamma_1(1-s) + \gamma_2 s} \right) &\geq ((\gamma_1 s + \gamma_2(1-s))(\gamma_1(1-s) + \gamma_2 s))^{1/4} \\ &= (\gamma_1^2 s(1-s) + \gamma_2^2 s(1-s) + \gamma_1 \gamma_2 (s^2 + (1-s)^2))^{1/4} \\ &= (\gamma_1 \gamma_2 + s(1-s)(\gamma_1 - \gamma_2)^2)^{1/4} \geq (\gamma_1 \gamma_2)^{1/4}, \end{aligned}$$

and this proves the inequality, by simply setting  $s = \cos^2 \theta$  and then integrating over  $\theta$ . We also remark that this inequality is strict if and only if  $\gamma_1 \neq \gamma_2$ .  $\square$

Note that we recover our previous results of [9] in the (weakly) isotropic case. Moreover, in the anisotropic case, assuming that  $\gamma_2 > \gamma_1$  and considering  $k = \sqrt{1 - \gamma_1/\gamma_2}$ , one can write

$$\sqrt{\bar{\gamma}_{\text{Per}}} = \sqrt{\gamma_2} \frac{2}{\pi} \int_0^{\pi/2} \sqrt{(1-k^2)\cos^2 \theta + \sin^2 \theta} d\theta = \sqrt{\gamma_2} \frac{2}{\pi} \mathcal{E}(k),$$

where  $\mathcal{E}(k)$  is the elliptic integral of the first kind, such that  $\bar{\text{Per}}_X(t)$  corresponds to the formula of Theorem 11.3. of [5]. We also refer to Proposition 3.4 of [18] for planar Gaussian waves.

### 3.2. Visualization and estimation of the anisotropy

In this section, we are interested in visualizing and estimating the anisotropy of a Gaussian field from the Lipschitz-Killing curvatures of its excursion sets. Here the anisotropy is summarized through the anisotropy ratio  $R$  defined by

$$R := \frac{\bar{\gamma}_{\text{TC}}}{\bar{\gamma}_{\text{Per}}}.$$

Thanks to Theorem 3.1, we have  $R \in [0, 1]$ , and  $R = 1$  if and only if the spectral moments  $\gamma_1$  and  $\gamma_2$  are equal, meaning that the field  $X$  is (weakly) isotropic. More precisely, we can see  $R$  as a function of  $\alpha := \frac{\min(\gamma_1, \gamma_2)}{\max(\gamma_1, \gamma_2)} \in [0, 1]$  since

$$R = \frac{\bar{\gamma}_{\text{TC}}}{\bar{\gamma}_{\text{Per}}} = \frac{\sqrt{\gamma_1 \gamma_2}}{\left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta \right)^2} = \frac{\sqrt{\alpha}}{\left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\alpha \cos^2 \theta + \sin^2 \theta} d\theta \right)^2}.$$

This is illustrated on the left of Figure 3 where we show the graph of the function  $\alpha \mapsto R$ . It is an increasing function that maps  $[0, 1]$  to  $[0, 1]$ .

#### 3.2.1. The Almond curve of anisotropy

Inspired by the paper of Klatt et al. [29] for the visualization of anisotropy, from the formulas for the Perimeter density and the Total Curvature density (Theorem 3.1), we see that if we set

$$\tilde{x}(t) = \frac{\bar{\text{Per}}_X(t)}{\bar{\text{Per}}_X(m)} = e^{-(t-m)^2/2\sigma^2} \quad \text{and} \quad \tilde{y}(t) = \frac{\bar{\text{TC}}_X(t)}{\bar{\text{Per}}_X(m)^2} = \frac{\bar{\gamma}_{\text{TC}}}{\bar{\gamma}_{\text{Per}}} \frac{4}{\sqrt{2\pi}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2}$$

then the point  $(\tilde{x}(t), \tilde{y}(t))$  is on the curve

$$\mathcal{C}_R = \{(x, y) \in (0, 1] \times \mathbb{R}; y^2 + \frac{16}{\pi} R^2 x^2 \log x = 0\}. \quad (23)$$

Notice that, thanks to the fact that  $R \leq 1$ , this curve is inside the domain defined by the closed curve

$$\mathcal{C}_1 = \{(x, y) \in (0, 1] \times \mathbb{R}; y^2 + \frac{16}{\pi} x^2 \log x = 0\}.$$

This curve is "canonical" in the sense that it is independent of the mean  $m$  and of the variance  $\sigma^2$  of the field. We call this curve the *almond curve* of anisotropy because of its shape, see Figure 3 right. Points on this curve correspond to  $R = 1$ , that is  $\bar{\gamma}_{\text{TC}} = \bar{\gamma}_{\text{Per}}$  and thus  $\gamma_1 = \gamma_2$ , meaning the field  $X$  is (weakly) isotropic.



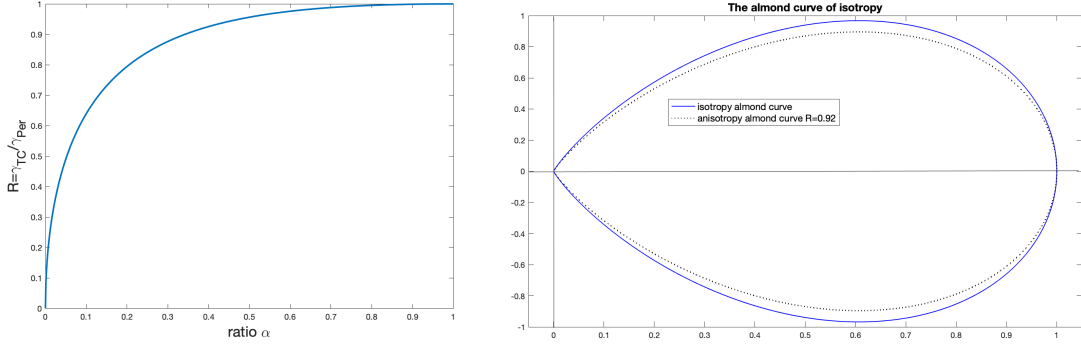


FIG 3. On the left, plot of the anisotropy ratio  $R = \frac{\widehat{\gamma}_{\text{TC}}}{\widehat{\gamma}_{\text{Per}}} \in [0, 1]$  as a function of  $\alpha = \frac{\min(\gamma_1, \gamma_2)}{\max(\gamma_1, \gamma_2)} \in [0, 1]$ . The ratio  $R$  is an increasing function of  $\alpha$  and  $R = 1$  if and only if  $\gamma_1 = \gamma_2$ . On the right, the almond curves of anisotropy, given by Eq. (23): the blue plain curve is the curve  $\mathcal{C}_1$  corresponding to isotropy, and the black dotted curve is a curve  $\mathcal{C}_R$ , with here  $R \simeq 0.92$ .

### 3.2.2. Estimating the anisotropy from one excursion set

Assume we observe an excursion set  $\hat{E}$  of a sample of a Gaussian stationary random field  $X$ , in a window  $U$ . The question is then: how to estimate the anisotropy ratio  $R$  from this set  $\hat{E}$  only, that is without having access to the whole field (nor its increments for instance). We ignore also at which level  $t$  the field  $X$  was thresholded, and also what was its mean  $m$  or its variance  $\sigma^2$ . Now from the three observed Lipschitz-Killing curvatures densities (that are  $\hat{A}(E) = A(\hat{E})/\mathcal{L}^2(U)$  the density Area of  $\hat{E}$ ,  $\widehat{\text{Per}}(E) = \text{Per}(\hat{E})/\mathcal{L}^2(U)$  the density Perimeter of  $\hat{E}$ , and  $\widehat{\text{TC}}(E) = \text{TC}(\hat{E})/\mathcal{L}^2(U)$  the density Total Curvature of  $E$ ), we can proceed the following way:

- Compute an "effective" threshold  $\hat{t}_{\text{eff}}$ , that is an empirical estimate of the effective level  $t_{\text{eff}} := \frac{t-m}{\sigma}$ , using the density area of  $\hat{E}$ :

$$\hat{t}_{\text{eff}} = \Phi^{-1}(1 - \hat{A}(E)).$$

- Then, for  $\hat{t}_{\text{eff}} \neq 0$  (or equivalently  $\hat{A}(E) \neq \frac{1}{2}$ ), estimate the "effective" spectral moments:

$$\frac{\widehat{\gamma}_{\text{Per}}}{\sigma^2} = 4\widehat{\text{Per}}(E)^2 e^{\hat{t}_{\text{eff}}^2}, \quad \frac{\widehat{\gamma}_{\text{TC}}}{\sigma^2} = \widehat{\text{TC}}(E) \frac{\sqrt{2\pi}}{\hat{t}_{\text{eff}}} e^{\hat{t}_{\text{eff}}^2/2}.$$

- Finally, for  $\hat{t}_{\text{eff}} \neq 0$ , estimate the anisotropy ratio with

$$\hat{R} = \frac{\widehat{\gamma}_{\text{TC}}}{\widehat{\gamma}_{\text{Per}}} = \frac{\widehat{\text{TC}}(E) \sqrt{2\pi}}{\widehat{\text{Per}}(E)^2 4\hat{t}_{\text{eff}}} e^{-\hat{t}_{\text{eff}}^2/2}.$$

This estimation procedure is inspired from the one developed by Elena Di Bernardino and Céline Duval in [16], where they introduce the notion of "effective" threshold, but where the framework is quite different since they assume the Gaussian field is isotropic.

Note that only "effective" spectral moments  $\frac{\widehat{\gamma}_{\text{Per}}}{\sigma^2}$  and  $\frac{\widehat{\gamma}_{\text{TC}}}{\sigma^2}$  are empirically accessible in practice when  $X$  is not assumed standard (i.e. so that  $\sigma = 1$ ) but our anisotropic ratio  $R$  itself is empirically accessible. Moreover we can also compute a point on the almond curve (see Figure 4) by setting

$$\hat{x} = e^{-\hat{t}_{\text{eff}}^2/2} \quad \text{and} \quad \hat{y} = \hat{R} \frac{4}{\sqrt{2\pi}} \hat{t}_{\text{eff}} e^{-\hat{t}_{\text{eff}}^2/2}.$$

Let us remark that, assuming  $\hat{E}$  is the realization of  $E_X(t) \cap U_n$  for some  $t \in \mathbb{R}$  and  $n$  large, under the assumptions of Corollary 2.2 on an open neighborhood  $V$  of  $t$  and  $J = \{0, 1, 2\}$ ,  $\hat{A}(E)$ ,  $\widehat{\text{Per}}(E)$  and  $\widehat{\text{TC}}(E)$  are consistent and asymptotically Gaussian. When moreover  $t \neq m$ , by the delta

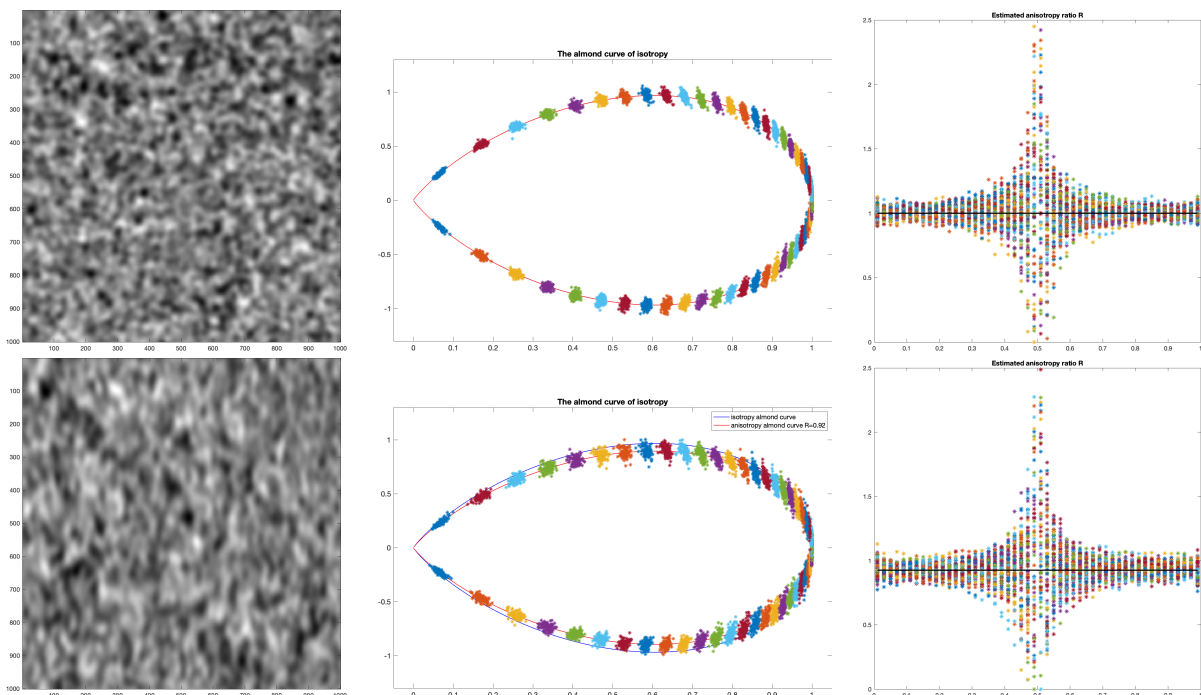


FIG 4. *First line: left, a sample of a centered isotropic Gaussian field  $X$ , size  $1000 \times 1000$  pixels, with variance  $\sigma^2 = 1$  and covariance of the form  $\rho(x_1, x_2) = \exp(-\gamma_1 x_1^2 + \gamma_2 x_2^2)$  with  $\gamma_1 = \gamma_2 = 0.005$ . Middle: the almond curve and some points from the isotropic field (100 samples). Right: corresponding estimations of the anisotropy ratio  $R$  ( $= 1$  here) as a function of the quantiles. Second line: same experiments but with now  $\gamma_1 = 0.005$  and  $\gamma_2 = 0.002$ , that corresponds to an anisotropic field with  $R \simeq 0.92$ .*

method, it should imply the same properties for the effective spectral moments and anisotropy ratio (see Figure 9 in the additional experiments section C in the Appendix, where we also numerically check the central limit behavior of the anisotropy ratio). This restriction on  $t$  means that we shall not have  $\hat{A}(E)$  close to  $1/2 = \Phi(0)$ .

On Figures 4 and 5 (and also on Figure 10 in the additional experiments section C in the Appendix), we illustrate all the theoretical results of this section. The numerical computations of the Lipschitz-Killing curvatures of the excursion sets are done thanks to the MATLAB "imMinkowski" library<sup>2</sup> developed by David Legland and described in [32]. As far as binary images are concerned, it will be more convenient to work with "quantiles"  $q \in [0, 1]$  so that the corresponding threshold levels are given by  $t = m + \sigma\Phi^{-1}(q)$ .

## 4. Smooth stationary 3D Gaussian fields

### 4.1. Lipschitz-Killing curvature densities

We are here interested in the 3D framework, and we consider a real-valued random field  $X$  defined on  $\mathbb{R}^3$ . The quantities of interest are the Lipschitz-Killing curvatures  $C_0$ ,  $C_1$ ,  $C_2$  and  $C_3$  that are now related to the Total Gaussian Curvature (also related to the Euler Characteristic by the Gauss-Bonnet Theorem, but different because we don't take into account what happens on the boundary of the domain  $U$  in view of (2)), the Total Mean Curvature (related to the mean breadth), the Surface Area and the Volume of an excursion set. More precisely, for a smooth stationary field  $X$  defined on  $\mathbb{R}^3$  and  $h \in \mathcal{S}$  we denote

$$LV_X(h, U) := LC_3^X(h, U), \quad LSA_X(h, U) := 2LC_2^X(h, U),$$

<sup>2</sup>available at <https://www.mathworks.com/matlabcentral/fileexchange/33690-imminkowski>

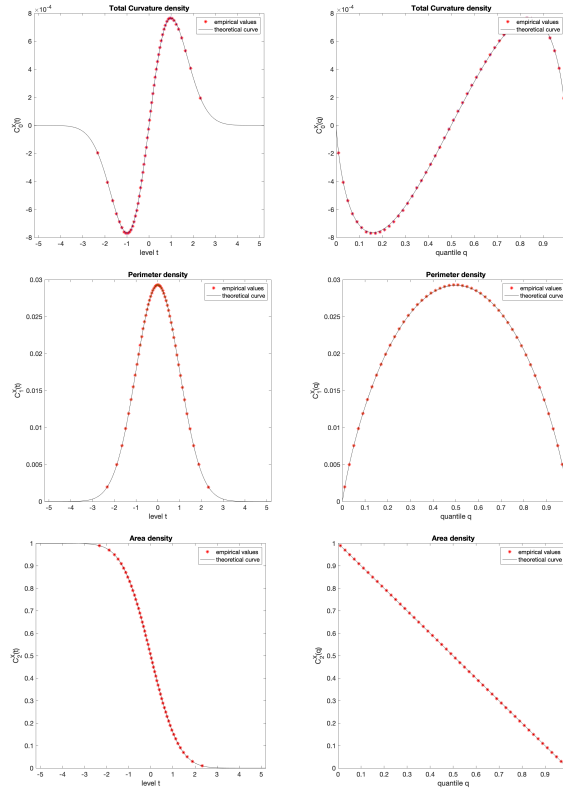


FIG 5. Densities  $\overline{C}_0^X$  (first line),  $\overline{C}_1^X$  (second line) and  $\overline{C}_2^X$  (third line), as a function of the level (left column) or the quantile (right column) in the anisotropic case. The plots would be the same in the isotropic case, up to a scaling factor of the axis. The anisotropic Gaussian field considered here is the same as the one of the bottom of Figure 4, with one sample of size  $1000 \times 1000$  (that is on  $U_n$  with  $n = 1000$ ).

$$\text{LTMC}_X(h, U) := \pi \text{LC}_1^X(h, U) \quad \text{and} \quad \text{LTGC}_X(h, U) := 4\pi \text{LC}_0^X(h, U).$$

To the best of our knowledge, the explicit formulas of the related Lipschitz-Killing densities linked with the introduced geometrical spectral moments, stated in the following theorem are completely new.

**Theorem 4.1.** *Let  $X$  be a stationary Gaussian field of mean  $m \in \mathbb{R}$ , variance  $\sigma^2 > 0$  and whose covariance function  $\rho$  satisfies Hypothesis H. We write  $\gamma_1, \gamma_2, \gamma_3 > 0$  the eigenvalues of  $\Gamma = -D^2\rho(0)$  and denote by  $\Phi$  the standard Gaussian cumulative distribution. Then the Lipschitz-Killing curvature densities of  $X$  are given for a.e.  $t \in \mathbb{R}$  by*

$$\begin{aligned} \overline{V}_X(t) &:= \overline{C}_3^X(t) = 1 - \Phi\left(\frac{t-m}{\sigma}\right), \\ \overline{\text{SA}}_X(t) &:= 2\overline{C}_2^X(t) = \sqrt{\frac{\overline{\gamma}_{\text{SA}}}{\sigma^2}} \frac{2}{\pi} e^{-(t-m)^2/2\sigma^2}, \\ \overline{\text{TMC}}_X(t) &:= \pi\overline{C}_1^X(t) = \frac{\overline{\gamma}_{\text{TMC}}}{\sigma^2} \frac{1}{\sqrt{2\pi}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2}, \\ \overline{\text{TGC}}_X(t) &:= 4\pi\overline{C}_0^X(t) = \left(\frac{\overline{\gamma}_{\text{TGC}}}{\sigma^2}\right)^{3/2} \frac{1}{\pi} \left(\frac{(t-m)^2}{\sigma^2} - 1\right) e^{-(t-m)^2/2\sigma^2}, \end{aligned}$$

where

$$\overline{\gamma}_{\text{SA}} = \left(\frac{1}{4\pi} \int_{S^2} \sqrt{\gamma_1 u_1^2 + \gamma_2 u_2^2 + \gamma_3 u_3^2} \mathcal{H}^2(du)\right)^2, \quad (24)$$

$$\bar{\gamma}_{\text{TMC}} = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3) - \frac{1}{8\pi} \int_{S^2} \frac{\gamma_1^2 u_1^2 + \gamma_2^2 u_2^2 + \gamma_3^2 u_3^2}{\gamma_1 u_1^2 + \gamma_2 u_2^2 + \gamma_3 u_3^2} \mathcal{H}^2(du), \quad (25)$$

$$\text{and } \bar{\gamma}_{\text{TGC}} = (\gamma_1 \gamma_2 \gamma_3)^{1/3}. \quad (26)$$

Therefore, the Gaussian random field  $X$  has same Surface Area density as a (weakly) isotropic Gaussian field with second spectral moment  $\bar{\gamma}_{\text{SA}}$ , same Total Mean Curvature density as a (weakly) isotropic Gaussian field with second spectral moment  $\bar{\gamma}_{\text{TMC}}$  and same Total Gaussian Curvature density as a (weakly) isotropic Gaussian field with second spectral moment  $\bar{\gamma}_{\text{TGC}}$ . Moreover, we have

$$\min(\gamma_1, \gamma_2, \gamma_3) \leq (\gamma_1 \gamma_2 \gamma_3)^{1/3} = \bar{\gamma}_{\text{TGC}} \leq \bar{\gamma}_{\text{TMC}} \leq \bar{\gamma}_{\text{SA}} \leq \frac{\gamma_1 + \gamma_2 + \gamma_3}{3} \leq \max(\gamma_1, \gamma_2, \gamma_3).$$

where the inequalities are strict if and only if  $\min(\gamma_1, \gamma_2, \gamma_3) < \max(\gamma_1, \gamma_2, \gamma_3)$ .

*Proof.* As in the 2D case, we postpone the computations of the  $\bar{C}_j^X(t)$  with the conditional expectations  $\mathbb{E}(F_{2-j}(\mathbf{X}(0)) | X(0) = t)$  to Appendix B.2.

To prove the inequalities, let us first rewrite things using the random variable  $U = (U_1, U_2, U_3) \in \mathbb{R}^3$  that is assumed to be uniformly distributed on the unit sphere  $S^2$ . Then

$$\bar{\gamma}_{\text{SA}} = \mathbb{E} \left( \sqrt{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2} \right)^2,$$

$$\bar{\gamma}_{\text{TMC}} = \frac{1}{2} \mathbb{E} \left( \frac{\gamma_1(\gamma_2 + \gamma_3)U_1^2 + \gamma_2(\gamma_1 + \gamma_3)U_2^2 + \gamma_3(\gamma_1 + \gamma_2)U_3^2}{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2} \right).$$

By concavity of the square root function, we first have that

$$\bar{\gamma}_{\text{SA}} \leq \mathbb{E}(\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2) = \frac{1}{3}(\gamma_1 + \gamma_2 + \gamma_3),$$

because  $\mathbb{E}(U_i^2) = 1/3$  (this comes from the symmetry of the  $U_i$ 's and the fact that  $U_1^2 + U_2^2 + U_3^2 = 1$  a.s.). Note also that the inequality is strict for  $\min(\gamma_1, \gamma_2, \gamma_3) < \max(\gamma_1, \gamma_2, \gamma_3)$  by strict concavity of the square root function.

The inequality between  $\bar{\gamma}_{\text{SA}}$  and  $\bar{\gamma}_{\text{TGC}} = (\gamma_1 \gamma_2 \gamma_3)^{1/3}$  can also be obtained in a straightforward way. Indeed, using again the concavity of the square root function, and the fact that  $\sum_i U_i^2 = 1$ , we have

$$\sqrt{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2} \geq \sqrt{\gamma_1} U_1^2 + \sqrt{\gamma_2} U_2^2 + \sqrt{\gamma_3} U_3^2.$$

And then taking the expectation on both sides we have

$$\mathbb{E} \left( \sqrt{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2} \right) \geq \frac{1}{3}(\sqrt{\gamma_1} + \sqrt{\gamma_2} + \sqrt{\gamma_3}) \geq (\gamma_1 \gamma_2 \gamma_3)^{1/6},$$

thanks to the arithmetic-geometric inequality. Therefore

$$\bar{\gamma}_{\text{SA}} \geq \bar{\gamma}_{\text{TGC}}.$$

The inequalities for  $\bar{\gamma}_{\text{TMC}}$  are more difficult to obtain. We first introduce the function

$$g_\gamma : x \in \mathbb{R}^3 \mapsto \sqrt{\gamma_1 x_1^2 + \gamma_2 x_2^2 + \gamma_3 x_3^2}$$

and note that it is a twice continuously differentiable positive function on  $\mathbb{R}^3 \setminus \{0\}$  with gradient given by  $\nabla g_\gamma(x) = \frac{1}{g_\gamma(x)}(\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3)$ . It follows that its directional gradient on the sphere at a point  $x \in S^2$  is given by the orthogonal projection of  $\nabla g_\gamma(x)$  onto the tangent space at  $x$  that is precisely  $\langle x \rangle^\perp$ , namely

$$\nabla_{S^2} g_\gamma(x) = \nabla g_\gamma(x) - \langle \nabla g_\gamma(x), x \rangle x = \nabla g_\gamma(x) - g_\gamma(x)x.$$

According to Poincaré-Wirtinger Inequality (Theorem 5.4.1 of [25]) we have

$$\text{Var}(g_\gamma(U)) \leq \frac{1}{2} \mathbb{E} \left( \|\nabla_{S^2} g_\gamma(U)\|^2 \right).$$

But we remark that  $\bar{\gamma}_{\text{SA}} = \mathbb{E}(g_\gamma(U))^2$  and that  $\mathbb{E}(g_\gamma(U)^2) = \frac{1}{3}(\gamma_1 + \gamma_2 + \gamma_3)$ , while

$$\mathbb{E} \left( \|\nabla_{S^2} g_\gamma(U)\|^2 \right) = \mathbb{E} \left( \|\nabla g_\gamma(U)\|^2 \right) - \mathbb{E}(g_\gamma(U)^2).$$

Then Poincaré-Wirtinger Inequality rewrites as

$$\frac{3}{2} \mathbb{E}(g_\gamma(U)^2) - \bar{\gamma}_{\text{SA}} \leq \frac{1}{2} \mathbb{E} \left( \|\nabla g_\gamma(U)\|^2 \right).$$

But

$$\frac{3}{2} \mathbb{E}(g_\gamma(U)^2) - \frac{1}{2} \mathbb{E} \left( \|\nabla g_\gamma(U)\|^2 \right) = \frac{1}{2} \mathbb{E} \left( \gamma_1 + \gamma_2 + \gamma_3 - \frac{\gamma_1^2 U_1^2 + \gamma_2^2 U_2^2 + \gamma_3^2 U_3^2}{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2} \right) = \bar{\gamma}_{\text{TMC}}.$$

Hence we have obtained  $\bar{\gamma}_{\text{TMC}} \leq \bar{\gamma}_{\text{SA}}$ . Note also that the case of equality in Poincaré-Wirtinger Inequality may only be achieved when  $g_\gamma$  is constant on  $S^2$ . Actually by Theorem 5.4.1 of [25], the equality holds if and only if  $g_\gamma - \mathbb{E}(g_\gamma(U))$  is a spherical harmonic of degree 1. This implies that  $\tilde{g}_\gamma : \mathbb{R}^3 \mapsto g_\gamma(x) - \mathbb{E}(g_\gamma(U))\|x\|$  is an homogeneous polynomial of degree 1 with constant gradient. But since, for  $(e_i)_{1 \leq i \leq 3}$  the canonical basis, we have  $\nabla \tilde{g}_\gamma(e_i) = (\sqrt{\gamma_i} - \mathbb{E}(g_\gamma(U)))e_i$ , this implies that  $\sqrt{\gamma_i} = \mathbb{E}(g_\gamma(U))$  and therefore  $\min(\gamma_1, \gamma_2, \gamma_3) = \max(\gamma_1, \gamma_2, \gamma_3)$  with  $g_\gamma(x) = \mathbb{E}(g_\gamma(U))\|x\|$ .

For the other inequality involving  $\bar{\gamma}_{\text{TMC}}$ , we will use the isoperimetric inequality for 3D convex bodies (see [33] for instance), that states that

$$\text{SA}^3 \geq 36\pi V^2,$$

where SA is the surface area and  $V$  is the volume. Now, taking a 3D ellipsoid of semi-axes  $\sqrt{\gamma_1}$ ,  $\sqrt{\gamma_2}$ ,  $\sqrt{\gamma_3}$ , we have that its volume is

$$V = \frac{4}{3}\pi\sqrt{\gamma_1}\sqrt{\gamma_2}\sqrt{\gamma_3} = \frac{4}{3}\pi\bar{\gamma}_{\text{TGC}}^{3/2},$$

and that its surface area is (see [26])

$$\text{SA} = 2\pi(\gamma_1 + \gamma_2 + \gamma_3) - 2\pi\mathbb{E} \left( \frac{\gamma_1^2 U_1^2 + \gamma_2^2 U_2^2 + \gamma_3^2 U_3^2}{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2} \right) = 4\pi\bar{\gamma}_{\text{TMC}}.$$

Therefore the isoperimetric inequality implies that

$$4^3\pi^3\bar{\gamma}_{\text{TMC}}^3 \geq 36\pi\frac{4^2}{3^2}\pi^2\bar{\gamma}_{\text{TGC}}^3,$$

which exactly means  $\bar{\gamma}_{\text{TMC}} \geq \bar{\gamma}_{\text{TGC}}$ . Note also that equality in the isoperimetric inequality may only be achieved when the body is a sphere (see [33] p.1190, result (2.7) for instance), that is when  $\min(\gamma_1, \gamma_2, \gamma_3) = \max(\gamma_1, \gamma_2, \gamma_3)$ .  $\square$

#### 4.2. Visualization and estimation of the anisotropy

We are here interested in the visualization and in the estimation of the anisotropy from an excursion set of a sample of a 3D Gaussian field, in a way similar to what was done in the 2D framework.

We define now two anisotropy ratios:

$$R := \frac{\bar{\gamma}_{\text{TMC}}}{\bar{\gamma}_{\text{SA}}} \quad \text{and} \quad R_G := \frac{\bar{\gamma}_{\text{TGC}}}{\bar{\gamma}_{\text{SA}}}.$$

We have  $0 \leq R_G \leq R \leq 1$  because of Theorem 4.1, and since they are symmetric functions of the  $\gamma_i$ 's, we can assume  $\gamma_3 = \max(\{\gamma_i\})$ , and see  $R$  and  $R_G$  as functions of  $\alpha_1 := \frac{\gamma_1}{\gamma_3}$  and  $\alpha_2 := \frac{\gamma_2}{\gamma_3}$ . This is illustrated on Figure 6.

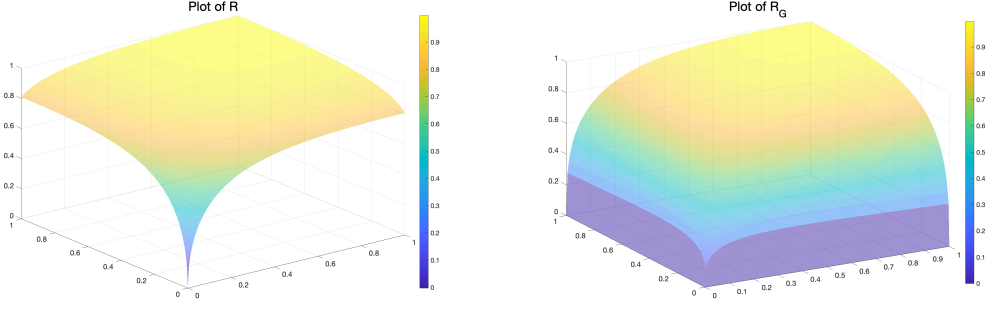


FIG 6. The anisotropy ratios  $R$  (on the left) and  $R_G$  (on the right) as functions of  $\alpha_1 := \frac{\gamma_1}{\gamma_3}$  and  $\alpha_2 := \frac{\gamma_2}{\gamma_3}$ .

#### 4.2.1. The Almond and Heart curves of anisotropy

Again, as in the 2D case, we can see the Lipschitz-Killing curvature densities as points on a curve, that is now in  $\mathbb{R}^3$ . More precisely, considering

$$\tilde{x}(t) = \frac{\overline{\text{SA}}_X(t)}{\overline{\text{SA}}_X(m)} = e^{-(t-m)^2/2\sigma^2},$$

$$\begin{aligned} \tilde{y}(t) &= \frac{\overline{\text{TMC}}_X(t)}{\overline{\text{SA}}_X(m)^2} = \frac{\bar{\gamma}_{\text{TMC}}}{\bar{\gamma}_{\text{SA}}} \frac{\pi\sqrt{\pi}}{4\sqrt{2}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2} \\ &= R \frac{\pi\sqrt{\pi}}{4\sqrt{2}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2}, \end{aligned}$$

$$\begin{aligned} \text{and } \tilde{z}(t) &= \frac{\overline{\text{TGC}}_X(t)}{\overline{\text{SA}}_X(m)^3} = \left( \frac{\bar{\gamma}_{\text{TGC}}}{\bar{\gamma}_{\text{SA}}} \right)^{3/2} \frac{\pi^2}{8} \left( \frac{(t-m)^2}{\sigma^2} - 1 \right) e^{-(t-m)^2/2\sigma^2} \\ &= R_G^{3/2} \frac{\pi^2}{8} \left( \frac{(t-m)^2}{\sigma^2} - 1 \right) e^{-(t-m)^2/2\sigma^2}, \end{aligned}$$

then this defines a parametric curve in  $\mathbb{R}^3$ , that has as a projection on the first two coordinates an almond curve of cartesian equation

$$\mathcal{C}_R^{(3d)} = \{(x, y) \in (0, 1] \times \mathbb{R}; y^2 + \frac{\pi^3}{16} R^2 x^2 \log x = 0\},$$

and on the last two coordinates, the projection is a parametric curve of the form

$$\tilde{\mathcal{C}}_{R, R_G}^{(3d)} = \{(y(s), z(s)) \in \mathbb{R}^2; y(s) = R \frac{\pi\sqrt{\pi}}{4\sqrt{2}} s e^{-s^2/2}, z(s) = R_G^{3/2} \frac{\pi^2}{8} (s^2 - 1) e^{-s^2/2}, s \in \mathbb{R}\}.$$

These two curves are shown on the first line of Figure 8, where the blue curves correspond to the isotropic case ( $R = R_G = 1$ ) and the black ones correspond on the figures to  $R \simeq 0.91$  and  $R_G \simeq 0.84$ . These black curves are inside the domains defined by the isotropic (blue) curves.

#### 4.2.2. Estimating the anisotropy from one excursion set

Again, as in the 2D case, assume we observe an excursion set  $\hat{E}$  of a sample of a Gaussian stationary random field  $X$ , in a window  $U$ . The question is then: how to estimate the anisotropy of  $X$  from this set  $\hat{E}$  only, that is without having access to the whole field (nor its increments for instance). We ignore also at which level  $t$  the field  $X$  was thresholded, and also what was its

mean  $m$  or its variance  $\sigma^2$ . Now, from the four observed Lipschitz-Killing curvatures densities (that are  $\widehat{V}(E) = V(\widehat{E})/\mathcal{L}^3(U)$  the density Volume of  $\widehat{E}$ ,  $\widehat{SA}(E) = SA(\widehat{E})/\mathcal{L}^3(U)$  the density Surface Area of  $\widehat{E}$ ,  $\widehat{TMC}(E) = TMC(\widehat{E})/\mathcal{L}^3(U)$  the density Total Mean Curvature of  $\widehat{E}$  and  $\widehat{TGC}(E) = TGC(\widehat{E})/\mathcal{L}^3(U)$  the density Total Gaussian Curvature of  $\widehat{E}$ ), we can proceed the following way:

- Compute an "effective" threshold  $\widehat{t}_{\text{eff}}$ , that is an empirical estimate of  $\frac{t-m}{\sigma}$ , using the volume of  $\widehat{E}$ :

$$\widehat{t}_{\text{eff}} = \Phi^{-1}(1 - \widehat{V}(E)).$$

- Then, for  $\widehat{t}_{\text{eff}} \notin \{0, \pm 1\}$ , estimate the "effective" spectral moments:

$$\frac{\widehat{\gamma}_{\text{SA}}}{\sigma^2} = \frac{\pi^2}{4} \widehat{SA}(E)^2 e^{\widehat{t}_{\text{eff}}^2}, \quad \frac{\widehat{\gamma}_{\text{TMC}}}{\sigma^2} = \widehat{TMC}(E) \frac{\sqrt{2\pi}}{\widehat{t}_{\text{eff}}} e^{\widehat{t}_{\text{eff}}^2/2},$$

$$\text{and } \frac{\widehat{\gamma}_{\text{TGC}}}{\sigma^2} = \left( \widehat{TGC}(E) \frac{\pi}{\widehat{t}_{\text{eff}}^2 - 1} e^{\widehat{t}_{\text{eff}}^2/2} \right)^{2/3}.$$

- Finally, for  $\widehat{t}_{\text{eff}} \notin \{0, \pm 1\}$ , estimate the anisotropy ratios via

$$\widehat{R} = \frac{\widehat{TMC}(E)}{\widehat{SA}(E)^2} \frac{4\sqrt{2}}{\pi\sqrt{\pi}} \frac{1}{\widehat{t}_{\text{eff}}} e^{-\widehat{t}_{\text{eff}}^2/2} \quad \text{and} \quad \widehat{R}_G^{3/2} = \frac{\widehat{TGC}(E)}{\widehat{SA}(E)^3} \frac{8}{\pi^2} \frac{1}{(\widehat{t}_{\text{eff}}^2 - 1)} e^{-\widehat{t}_{\text{eff}}^2}.$$

As in the 2D case, we have performed some numerical experiments to illustrate all this. Let us mention that such numerical experiments are much more difficult to conduct than in the 2D case. The memory size and the computation time needed to proceed with a 3D volume are very high. Only "small" volumes were considered, with size  $200^3$  voxels. Therefore the variances in the estimation of the Lipschitz-Killing curvatures is quite high, see Figure 7. The estimation of the anisotropy ratios from one excursion set is illustrated on Figure 8 (anisotropic case). See the caption of the figures for more details. Additional experiments are also provided in the experimental section C of the Appendix.

## 5. Further remarks

### 5.1. Link with the kinematic formula

For sake of simplicity, let us assume that  $X = (X(x))_{x \in \mathbb{R}^d}$  is a stationary smooth centered ( $m = 0$ ) Gaussian random field with unit variance ( $\sigma^2 = 1$ ), whose excursion set is given by  $E_X(t)$  for some regular level  $t \in \mathbb{R}$  and such that Lipschitz-Killing curvature densities are continuous in a neighborhood of  $t$ . Then, the random set  $Z := E_X(t)$  is a closed stationary random set and we explore here links between the classical kinematic formula of stochastic geometry (we rely here on [36]) and the Gaussian's one of [2]. The kinematic formula of Theorem 9.4.1 of [36] p.414 holds under the strong assumption that  $Z$  is a standard set (see Definition 9.2.1), meaning that one can write  $Z \cap [0, 1]^d$  as an union of  $N$  convex bodies with  $\mathbb{E}(2^N) < +\infty$ . Under the additional assumption that  $Z$  is isotropic (implied by the isotropy of  $X$ ), this result states that for any convex body  $W$  and any Borel set  $U \subset \mathbb{R}^d$ ,

$$\mathbb{E}(C_j(Z \cap W, U)) = \sum_{k=j}^d c_{j,d}^{k,d-k+j} \overline{V}_k(Z) C_{d-k+j}(W, U),$$

where  $(\overline{V}_k(Z))_{0 \leq k \leq d}$  are the so-called intrinsic volumes' densities of  $Z$  and, following the notations of [36],  $c_{j,d}^{k,d-k+j} = \frac{\gamma(\frac{k+1}{2})\gamma(\frac{d-k+j+1}{2})}{\gamma(\frac{j+1}{2})\gamma(\frac{d+1}{2})}$ , with  $\gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt$  for  $a > 0$ . But, as noticed at the

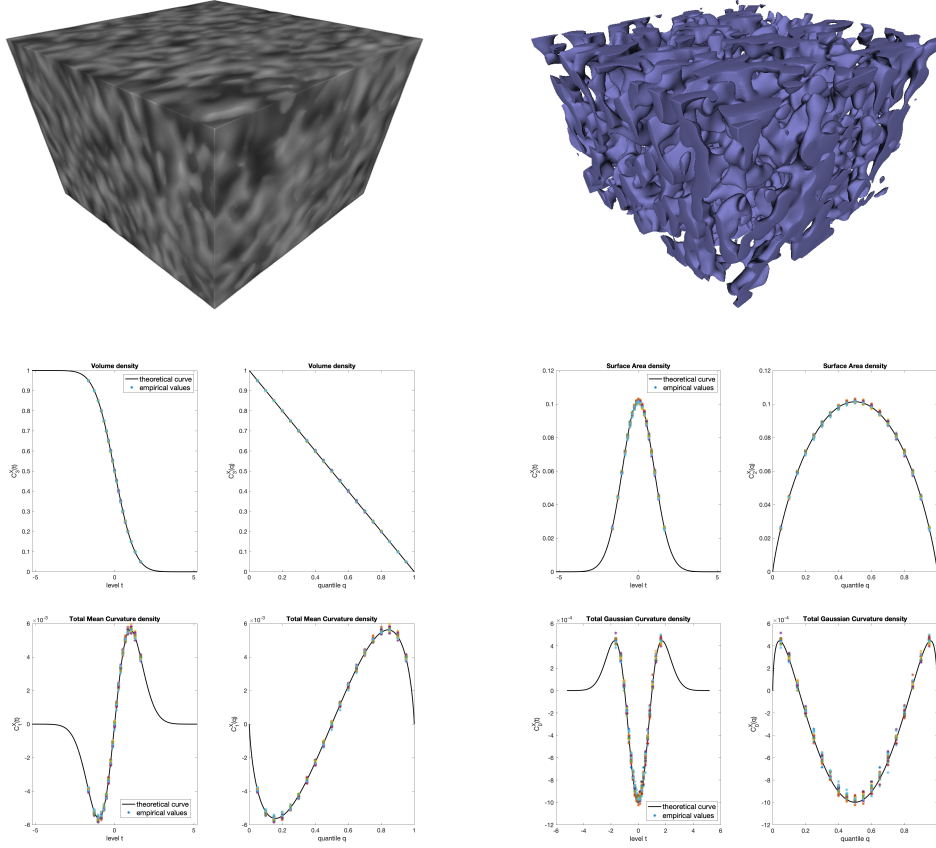


FIG 7. Lipschitz-Killing curvatures densities of the excursion sets of an isotropic Gaussian volume. Here we have taken 20 samples of a 3D volume of size  $200^3$  voxels, of a Gaussian field with mean  $m = 0$ , variance  $\sigma^2 = 1$  and covariance of the form  $\exp(-\gamma_1 x_1^2 - \gamma_2 x_2^2 - \gamma_3 x_3^2)$  with  $\gamma_1 = 0.01$ ,  $\gamma_2 = 0.02$  and  $\gamma_3 = 0.05$ . An example of such a Gaussian volume is shown on the left of the first line, with one excursion set on the right. Using the Matlab toolbox *ImMinkowski* [32], we have estimated the volume densities (second line, left), the surface area densities (second line, right), the total mean curvature densities (third line, left) and the total Gaussian curvature densities (third line, right). The densities are plotted else as functions of the level  $t$  or as functions of the quantile  $q$ . The stars are the values for the 20 different samples.

end of p.416 for the "global case" where  $W \subset U$ , the isotropy assumption on  $Z$  may be removed when considering  $W$  as an isotropic ball. Hence, for any  $r > 0$ , one has

$$\mathbb{E}(C_j(Z \cap \overline{B}(0, r), U)) = \sum_{k=j}^d c_{j,d}^{k,d-k+j} \overline{V}_k(Z) C_{d-k+j}(\overline{B}(0, r), U),$$

as soon as  $\overline{B}(0, r) \subset U$ . Now, following Corollary 9.4.1, we have for any bounded Borel set  $U$  and  $k \in \{0, \dots, d\}$

$$\mathbb{E}(C_k(Z, U)) = \overline{V}_k(Z) \mathcal{L}^d(U),$$

and therefore  $\overline{V}_k(Z) = \overline{C}_k^X(t)$  (identifying the two terms when  $U$  is open bounded). Since we have for  $\overline{B}(0, r) \subset U$ ,

$$C_{d-k+j}(\overline{B}(0, r), U) = r^{d-k+j} V_{d-k+j}(\overline{B}(0, 1)),$$



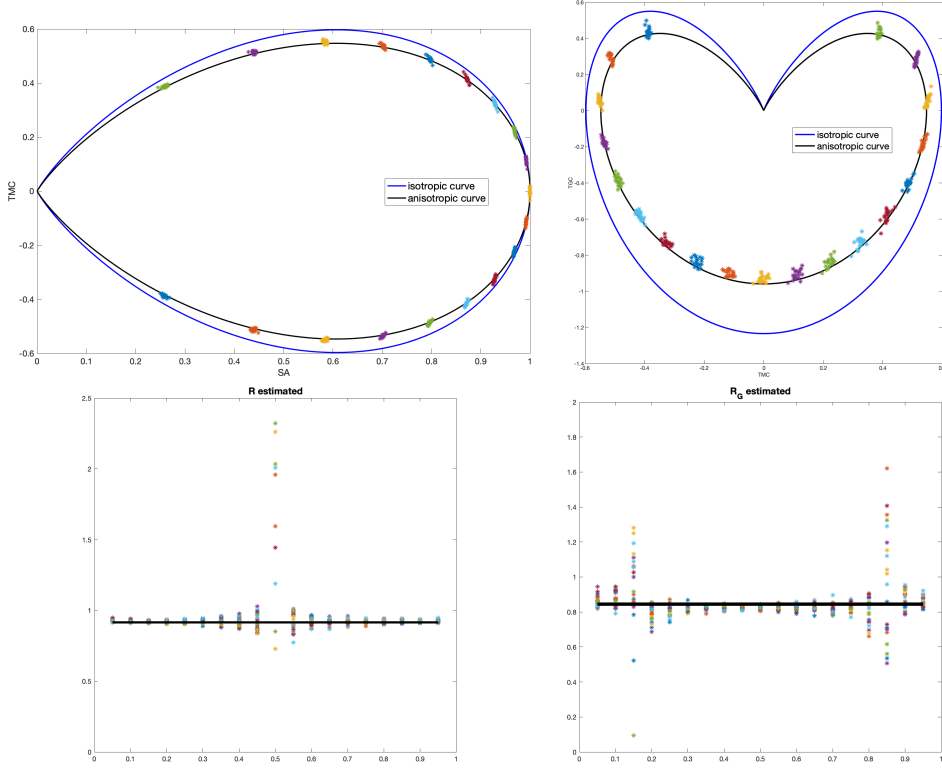


FIG 8. Estimation of  $R$  and  $R_G$  from the Lipschitz-Killing curvatures densities of an excursion set. The samples are the same as the ones of Figure 7. Notice that, because of the term  $1/\hat{t}_{\text{eff}}$ , the estimation of  $R$  is inaccurate for values around  $\hat{t}_{\text{eff}} = 0$ , whereas for  $R_G$  the inaccuracy of estimation occurs around  $\hat{t}_{\text{eff}} = \pm 1$ .

we end with the following kinematic formula, under the assumption that  $Z = E_X(t)$  is a standard set,

$$\mathbb{E}(C_j(Z \cap \bar{B}(0, r), U)) = \sum_{k=j}^d \mathcal{L}_{j,d}^{k,d-k+j} V_{d-k+j}(\bar{B}(0, 1)) \bar{C}_k^X(t) r^{d-k+j}. \quad (27)$$

Meanwhile, under the additional assumption that the joint distribution of the Gaussian vector  $\mathbf{X}(0)$  is non degenerate, the stationary random field  $X$  fulfills the assumptions of Theorem 4.2.1 of [3] and the Gaussian kinematic formula states that (see (4.0.1) therein)

$$\begin{aligned} \mathbb{E}(\mathcal{L}_j^\nabla(Z \cap \bar{B}(0, r))) &= \sum_{i=0}^{d-j} \begin{bmatrix} i+j \\ i \end{bmatrix} \mathcal{L}_{i+j}^\nabla(\bar{B}(0, r)) (2\pi)^{-i/2} h_{i-1}(t) \\ &= \sum_{k=j}^d \begin{bmatrix} d-k+j \\ d-k \end{bmatrix} \mathcal{L}_{d-k+j}^\nabla(\bar{B}(0, 1)) (2\pi)^{-(d-k)/2} h_{d-k-1}(t) r^{d-k+j} \end{aligned} \quad (28)$$

with the change of variable  $k = d - i$ , where  $\mathcal{L}_k^\nabla$  denotes the Lipschitz-Killing curvatures under the Riemannian metric given by  $\Gamma$ , the covariance matrix of  $\nabla X(0)$ ,

$$\begin{bmatrix} k \\ k-j \end{bmatrix} = \binom{k}{k-j} \frac{\omega_k}{\omega_{k-j}\omega_j},$$

with  $\omega_k$  the volume of the  $k$ -dimensional unit ball of  $\mathbb{R}^k$ , that is

$$\omega_0 = 1, \quad \omega_1 = 2, \quad \omega_2 = \pi, \quad \omega_3 = \frac{4}{3}\pi, \quad \text{etc..}$$

and  $h_{-1}(t) = \Phi(t)$ , whereas  $h_k(t) = \frac{1}{\sqrt{2\pi}} H_k(t) e^{-t^2/2}$  are the Hermite functions of order  $k \in \mathbb{N}$ . Note that without knowing  $\Gamma$ , we won't be able to compute  $\mathcal{L}_j^\nabla$  for  $j \neq 0$ . However, for  $j = 0$ , the Lipschitz-Killing curvature  $\mathcal{L}_0^\nabla$  does not depend on  $\Gamma$  and we obtain the mean Euler characteristic of  $E_X(t) \cap \overline{B}(0, r)$  as

$$\mathbb{E}(\chi(E_X(t) \cap \overline{B}(0, r))) = \sum_{k=0}^d \mathcal{L}_{d-k}^\nabla(\overline{B}(0, 1)) (2\pi)^{-(d-k)/2} h_{d-k-1}(t) r^{d-k}. \quad (29)$$

But recall that, in the "global case" for  $U$  open bounded set such that  $\overline{B}(0, r) \subset U$ , one has

$$\mathbb{E}(\chi(E_X(t) \cap \overline{B}(0, r))) = \mathbb{E}(C_0(E_X(t) \cap \overline{B}(0, r), U)).$$

Then (27) rewrites as

$$\mathbb{E}(\chi(E_X(t) \cap \overline{B}(0, r))) = \sum_{k=0}^d c_{0,d}^{k,d-k} V_{d-k}(\overline{B}(0, 1)) \overline{C}_k^X(t) r^{d-k}.$$

Hence, identifying with (29), we get for all  $0 \leq k \leq d$ ,

$$\overline{C}_k^X(t) = \frac{\mathcal{L}_{d-k}^\nabla(\overline{B}(0, 1))}{V_{d-k}(\overline{B}(0, 1))} c_{d,k} h_{d-k-1}(t),$$

where we set  $c_{d,k} = \frac{(2\pi)^{-(d-k)/2}}{c_{0,d}^{k,d-k}}$ . Note that  $c_{d,d} = 1$ ,  $c_{d,0} = (2\pi)^{-d/2}$ , and

$$c_{2,1} = (2\pi)^{-1/2} \frac{\gamma(1/2)^2}{2} = \frac{1}{2} \sqrt{\frac{\pi}{2}},$$

while  $c_{3,1} = (2\pi)^{-1} (\frac{1}{2})^{-1} = \frac{1}{\pi}$  and  $c_{3,2} = (2\pi)^{-1/2} (\frac{1}{2})^{-1} = \sqrt{\frac{2}{\pi}}$ . This allows us to state the following result in general dimension  $d \geq 1$ .

**Proposition 5.1.** *Let  $d \geq 1$  and let  $\Gamma$  be the covariance matrix of  $\nabla X(0)$  assumed to be positive definite. For  $0 \leq k \leq d-1$ , we can define some geometrical second spectral moments of the smooth centered unit variance field  $X$  by*

$$\overline{\lambda}_{d,k} = \left( \frac{\mathcal{L}_{d-k}^\nabla(\overline{B}(0, 1))}{V_{d-k}(\overline{B}(0, 1))} \right)^{2/(d-k)},$$

where  $\mathcal{L}_{d-k}^\nabla$  denotes the Lipschitz-Killing curvatures under the Riemannian metric given by  $\Gamma$ . Moreover,

$$\forall 0 \leq k \leq d-1, \quad \overline{\lambda}_{d,0} \leq \overline{\lambda}_{d,k} \leq \overline{\lambda}_{d,d-1}.$$

*Proof.* Since  $\Gamma$  is symmetric, there exists an orthogonal matrix  $P$  such that  $\Gamma = P^t \text{diag}(\gamma_1, \dots, \gamma_d) P$  and for  $0 \leq k \leq d-1$  one has  $\overline{\lambda}_{d,k} = \left( \frac{V_{d-k}(\mathcal{E}\ell(\Gamma))}{V_{d-k}(\overline{B}(0, 1))} \right)^{2/(d-k)}$ , where  $\mathcal{E}\ell(\Gamma)$  is the Ellipsoid of semi-axes  $\sqrt{\gamma_1}, \dots, \sqrt{\gamma_d}$  (see [26] for explicit computations). Then, extended isoperimetric inequalities (see (1.1) of [34]) rewrite as

$$\forall 0 \leq k \leq d-1, \quad \overline{\lambda}_{d,0} \leq \overline{\lambda}_{d,k},$$

while the generalized Urysohn inequality (see (1.2) of [34]) states that

$$\forall 0 \leq k \leq d-1, \quad \overline{\lambda}_{d,k} \leq \overline{\lambda}_{d,d-1}.$$

□

This result is in accordance with our results for dimension  $d \in \{2, 3\}$  with  $\bar{\lambda}_{2,0} = \bar{\gamma}_{\text{TC}}$ ,  $\bar{\lambda}_{2,1} = \bar{\gamma}_{\text{Per}}$  in dimension  $d = 2$  and  $\bar{\lambda}_{3,0} = \bar{\gamma}_{\text{TGC}}$ ,  $\bar{\lambda}_{3,1} = \bar{\gamma}_{\text{TMC}}$ ,  $\bar{\lambda}_{3,2} = \bar{\gamma}_{\text{SA}}$  in dimension  $d = 3$ .

Finally, let us end by remarking that when  $\Gamma$  is diagonal the mean Euler characteristic of  $E_X(t) \cap [0, T]^d$  for some  $T > 0$  is given according to Theorem 4.4.1 of [3] by

$$\begin{aligned} \mathbb{E}(\chi(E_X(t) \cap [0, T]^d)) &= \sum_{k=0}^d \mathcal{L}_{d-k}^\nabla([0, T]^d) (2\pi)^{-(d-k)/2} h_{d-k-1}(t) \\ &= \mathcal{L}_d^\nabla([0, T]^d) (2\pi)^{-d/2} h_{d-1}(t) + o_{T \rightarrow +\infty}(T^d), \end{aligned}$$

with  $\mathcal{L}_d^\nabla([0, T]^d) = T^d \left( \prod_{i=1}^d \gamma_i \right)^{1/2} = T^d (\bar{\lambda}_{d,0})^{d/2}$  and therefore, the Euler characteristic density of  $E_X(t) \cap [0, T]^d$  satisfies

$$\lim_{T \rightarrow +\infty} \frac{1}{\mathcal{L}_d([0, T]^d)} \mathbb{E}(\chi(E_X(t) \cap [0, T]^d)) = (\bar{\lambda}_{d,0})^{d/2} (2\pi)^{-d/2} h_{d-1}(t).$$

Hence in view of (20) we should obtain that

$$\bar{C}_0^X(t) = (\bar{\lambda}_{d,0})^{d/2} (2\pi)^{-d/2} h_{d-1}(t),$$

generalizing our results in dimension  $d = 2$  and  $d = 3$  when  $m = 0$  and  $\sigma^2 = 1$ .

## 5.2. A functional Weyl's tube formula

Considering a compact  $d$ -dimensional submanifold in  $\mathbb{R}^d$  with a  $C^2$  smooth boundary, according to Weyl's tube formula [42] also called Steiner's formula [22], the Lipschitz-Killing curvatures are related to the volume of the  $r$ -parallel set (also called  $r$ -extension or  $r$ -dilation) of  $M$  by

$$\mathcal{L}^d(M_r) = \sum_{k=0}^d \omega_k C_{d-k}(M, \mathbb{R}^d) r^k,$$

where  $M_r = M \oplus B(0, r) = \{x; \text{dist}(x, M) \leq r\}$  is the  $r$ -parallel set to  $M$ , and where  $\omega_k$  is still the volume of the  $k$ -dimensional unit ball of  $\mathbb{R}^k$ . The above tube formula is valid for  $r$  small enough (more precisely smaller than the so-called reach of  $M$ , denoted by  $\text{reach}(M)$ , and that is strictly positive when  $M$  is a smooth compact submanifold). Now, this formula can be localized in the sense that if we consider an open bounded subset  $U \subset \mathbb{R}^d$ , the above tube formula becomes (see [22] or [41] Theorem 26), for  $0 < r < \text{reach}(M)$ ,

$$\mathcal{L}^d(M_{r,U}) = \sum_{k=0}^d \omega_k C_{d-k}(M, U) r^k, \quad (30)$$

where here  $M_{r,U} = \{x; \text{dist}(x, M) \leq r \text{ and } \Pi_M(x) \in U\}$ , with  $\Pi_M(x)$  being the projection of  $x$  on  $M$ , that is the closest point to  $x$  in  $M$  which is unique when  $\text{dist}(x, M) \leq r < \text{reach}(M)$ . Notice that generally  $M_{r,U}$  is not equal to  $M_r \cap U$ . Here the Lipschitz-Killing curvatures  $C_{d-k}(M, U)$  are localized in  $U$  and their formula involving the principal curvatures are the ones given in Section 2.

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth function, and let  $U$  be a bounded open subset of  $\mathbb{R}^d$ . For almost every  $t \in \mathbb{R}$ ,  $E_f(t) = \{x \in \mathbb{R}^d; f(x) \geq t\}$  is a smooth manifold and we can consider its Lipschitz-Killing curvatures in  $U$ . We denote them by  $C_k(E_f(t), U)$  for  $k = 0, \dots, d$ . As recalled by Weyl's tube formula (30), we can obtain these Lipschitz-Killing curvatures as the coefficients of the powers of  $r$  in the volume of the  $r$ -parallel set to  $E_f(t)$ . More precisely we have, for  $r$  small enough,

$$\mathcal{L}^d((E_f(t))_{r,U}) = \sum_{k=0}^d \omega_k C_{d-k}(E_f(t), U) r^k.$$

In Section 2.2, we have 1) used the formulas for the  $C_k$  involving the symmetric functions of the curvatures, and the explicit formulas for these curvatures in 2D or 3D; 2) integrate these with a test function  $h$  to have formulas for  $\int_t h(t)C_k(E_f(t), U) dt$ ; 3) and finally these integrals were written as an integral on  $x \in U$  thanks to the coarea formula [20].

But we could have done it the other way round. Indeed, we can use directly the coarea formula on  $\int_t h(t)\mathcal{L}^d((E_f(t))_{r,U}) dt$ , without knowing the formulas for the curvatures, and in fact recovering them.

More precisely, let  $h$  be a test function, and for  $z = (x, s) \in U \times \mathbb{R}_+ \subset \mathbb{R}^{d+1}$ , let us define the functions  $\tilde{h} : U \times [0, r] \rightarrow \mathbb{R}$  and  $F : U \times [0, r] \rightarrow \mathbb{R}^{d+1}$  by  $\tilde{h}(z) = \tilde{h}(x, s) = h(f(x))$  and

$$F(x, s) = (f(x), x - s\nu_f(x)),$$

where we denote

$$\nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|} \in S^{d-1} \quad \text{if } \nabla f(x) \neq 0, \quad \text{and } \nu_f(x) = 0 \quad \text{if } \nabla f(x) = 0.$$

A simple computation shows that  $\nu_f$  is differentiable at all  $x$  such that  $\nabla f(x) \neq 0$  and that

$$D\nu_f(x) = \frac{1}{\|\nabla f(x)\|} D^2 f(x) - \frac{1}{\|\nabla f(x)\|^3} \nabla f(x) (D^2 f(x) \nabla f(x))^t.$$

Now, let  $K$  be a compact ball containing  $\bar{U}_1$  (1-dilation of  $\bar{U}$ ). Then for  $\varepsilon > 0$ , let us consider the following set of values

$$V_\varepsilon = \{t \in \mathbb{R}; \text{reach}(E_f(t) \cap K) \geq \varepsilon\}.$$

Then, since by Morse-Sard's theorem the set of critical values has Lebesgue measure 0 in  $\mathbb{R}$ , we have that  $\mathcal{L}^1(\mathbb{R} \setminus \cup_{\varepsilon>0} V_\varepsilon) = 0$ . Then, let us denote  $G_f^\varepsilon = f^{-1}(V_\varepsilon)$ . For  $r < \varepsilon$  and  $t \in V_\varepsilon$ , we have that

$$E_f(t)_{r,U} \setminus (E_f(t) \cap U) = \{x - s\nu_f(x); x \in U \text{ and } f(x) = t \text{ and } s \in (0, r)\}.$$

Therefore, by the coarea formula:

$$\int_{(U \cap G_f^\varepsilon) \times [0, r]} h(f(x)) |\det(DF(x, s))| dx ds = \int_{V_\varepsilon} h(t) (\mathcal{L}^d((E_f(t))_{r,U}) - \mathcal{L}^d(E_f(t) \cap U)) dt.$$

The right-hand term is equal to  $\sum_{k=1}^d \omega_k r^k \int_{V_\varepsilon} h(t) C_{d-k}(E_f(t), U) dt$ , whereas in the left-hand term we have

$$\det(DF(x, s)) = \det \begin{pmatrix} \nabla f(x)^t & 0 \\ \mathbf{I}_d - sD\nu_f(x) & -\nu_f(x) \end{pmatrix} = \|\nabla f(x)\| \det \begin{pmatrix} \nu_f(x)^t & 0 \\ \mathbf{I}_d - sD\nu_f(x) & -\nu_f(x) \end{pmatrix}. \quad (31)$$

This determinant is a polynomial in  $s$ , with degree  $d-1$  and with constant term (corresponding to  $s=0$ ) equal to  $\pm 1$ .

To identify the values of  $\int_{\mathbb{R}} h(t) C_{d-k}(E_f(t), U) dt$  for each  $k$  we just need to develop the above determinant, and then let  $\varepsilon$  goes to 0.

When the dimension is  $d=2$ , we have that

$$\det(DF(x, s)) = \|\nabla f(x)\| (1 - s(\text{Tr}(D\nu_f(x)) - \nu_f(x)^t D\nu_f(x) \nu_f(x))),$$

which is positive for  $s$  small enough. Therefore, computing the integral on  $s$  between 0 and  $r$ , we get a polynomial in  $r$ , that allows us to identify its coefficients, and we get

$$\omega_1 \int h(t) C_1(E_f(t), U) dt = \int_U h(f(x)) \|\nabla f(x)\| dx,$$

and

$$\begin{aligned} \omega_2 \int h(t) C_0(E_f(t), U) dt &= -\frac{1}{2} \int_U h(f(x)) \|\nabla f(x)\| (\text{Tr}(D\nu_f(x)) - \nu_f(x)^t D\nu_f(x) \nu_f(x)) dx \\ &= -\frac{1}{2} \int_U h(f(x)) \left( \text{Tr}(D^2 f(x)) - \frac{\nabla f(x)^t D^2 f(x) \nabla f(x)}{\|\nabla f(x)\|^2} \right) dx. \end{aligned}$$

We have thus recovered the formula for the curvature in dimension  $d = 2$ .

When the dimension is  $d = 3$ , considering a direct orthonormal basis of  $\mathbb{R}^3$  of the form  $(\nu_f(x), T_1(x), T_2(x))$ , developing the above determinant in this basis, we get

$$\det(DF(x, s)) = \|\nabla f(x)\| (1 - s \operatorname{Tr}(D\nu_f(x)) + s^2 \det(\nu_f(x), D\nu_f(x)^t T_1(x), D\nu_f(x)^t T_2(x))),$$

which is positive for  $s$  small enough. Therefore, computing the integral on  $s$  between 0 and  $r$ , and identifying the coefficients, we get

$$\omega_1 \int h(t) C_2(E_f(t), U) dt = \int_U h(f(x)) \|\nabla f(x)\| dx,$$

$$\begin{aligned} \omega_2 \int h(t) C_1(E_f(t), U) dt &= -\frac{1}{2} \int_U h(f(x)) \|\nabla f(x)\| \operatorname{Tr}(D\nu_f(x)) dx \\ &= -\frac{1}{2} \int_U h(f(x)) \left( \operatorname{Tr}(D^2 f(x)) - \frac{\nabla f(x)^t D^2 f(x) \nabla f(x)}{\|\nabla f(x)\|^2} \right) dx, \end{aligned}$$

and

$$\begin{aligned} \omega_3 \int h(t) C_0(E_f(t), U) dt &= \frac{1}{3} \int_U h(f(x)) \|\nabla f(x)\| \det(\nu_f(x), D\nu_f(x)^t T_1(x), D\nu_f(x)^t T_2(x)) dx \\ &= \frac{1}{3} \int_U h(f(x)) \frac{\nabla f(x)^t \operatorname{adj}(D^2 f(x)) \nabla f(x)}{\|\nabla f(x)\|^3} dx, \end{aligned}$$

where  $\operatorname{adj}(D^2 f(x))$  is the adjugate matrix of  $D^2 f(x)$  (that is the matrix of its cofactors). We have thus recovered the formulas for the mean curvature and the Gaussian curvature for implicit surfaces in  $\mathbb{R}^3$ .

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## Appendix A: Central Limit Theorem

### A.1. General results

We assume that  $X$  is a  $C^2$  stationary second order random field (not necessarily Gaussian) and write as before  $\mathbf{X} = (X, \nabla X, D^2 X)$  the vector-valued field with values in  $\mathbb{R}^s = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2}$ , following [19]. We recall that since  $X$  is  $C^2$  stationary second order, the covariance function  $x \mapsto \rho(x) = \text{Cov}(X(x), X(0))$  is  $C^4$  and for  $0 \leq i, j, k, l \leq d$

$$\text{Cov}(X_{ij}(x), X_{kl}(0)) = (-1)^{\varepsilon(k)+\varepsilon(l)} \partial_{ijkl}^{\varepsilon(i)+\varepsilon(j)+\varepsilon(k)+\varepsilon(l)} \rho(x),$$

where  $\varepsilon(0) = 0$ ,  $\varepsilon(i) = 1$  if  $i \neq 0$  and  $X_{0j}(x) = X_{j0}(x) = X_j(x)$  while  $X_{00}(x) = X(x)$ . Hence we introduce

$$\tilde{\rho}(x) = \max_{|\mathbf{k}| \leq 4} |\partial^{\mathbf{k}} \rho(x)|,$$

where  $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$  and  $|\mathbf{k}| = \sum_{i=1}^d k_i$ . In order to prove a general central limit theorem, we will work in this section under a strong quasi-association assumption, namely, we assume that there exists  $C \geq 1$  such that, for all finite subsets  $I$  and  $J$  of  $\mathbb{Z}^d$ , for all  $p \geq 1$  and all  $\vec{x} = (x_1, \dots, x_p) \in ([0, 1]^d)^p$ , for all Lipschitz functions  $f : \mathbb{R}^{|I|p} \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^{|J|p} \rightarrow \mathbb{R}$ , we have

$$|\text{Cov}(f(\mathbf{X}_{\vec{x}, I}), g(\mathbf{X}_{\vec{x}, J}))| \leq C \text{Lip}(f) \text{Lip}(g) \sum_{\mathbf{i} \in I} \sum_{\mathbf{j} \in J} \sum_{l=1}^p \sum_{l'=1}^p \tilde{\rho}(x_l + \mathbf{i} - (x_{l'} + \mathbf{j})), \quad (32)$$

where  $\mathbf{X}_{\vec{x}, I} = (\mathbf{X}(x_l + \mathbf{i}), \mathbf{i} \in I, 1 \leq l \leq p)$ . For  $h \in \mathcal{S}$  and  $q \in \mathbb{N}$ , we write  $\|h\|_q = \|h\|_{q,0} + \|h\|_{q,1}$ . Note that  $h$  is a Lipschitz bounded function with  $\text{Lip}(h) = \|h\|_{0,1} \leq \|h\|_{q,1}$  and that  $\|h\|_{0,0} \leq \|h\|_{q,0}$ . We consider a sequence  $(\mathcal{T}_n)_n$  of generalized random processes given, for  $n \in \mathbb{N}$ ,  $U_n = (0, n)^d$  and  $h \in \mathcal{S}$ , by

$$\begin{aligned} \langle \mathcal{T}_n, h \rangle &= \sqrt{\mathcal{L}^d(U_n)} \left( \frac{1}{\mathcal{L}^d(U_n)} \int_{U_n} L(h, \mathbf{X}(x)) dx - \mathbb{E}(L(h, \mathbf{X}(0))) \right) \\ &= \int_{[0,1]^d} S_n(h, x) dx \end{aligned} \quad (33)$$

with, for  $V = (0, 1]^d$ ,

$$S_n(h, x) = \frac{1}{n^{d/2}} \sum_{\mathbf{k} \in nV \cap \mathbb{Z}^d} (L(h, \mathbf{X}(x + \mathbf{k})) - \mathbb{E}(L(h, \mathbf{X}(0))))$$

and  $h \in \mathcal{S} \mapsto L(h, \mathbf{x}) \in \mathbb{R}$  is linear continuous for all  $\mathbf{x} \in \mathbb{R}^s$ ,  $x \in \mathbb{R}^s \mapsto L(h, \mathbf{x}) \in \mathbb{R}$  is measurable for all  $h \in \mathcal{S}$  and  $\mathbb{E}(L(h, \mathbf{X}(0))^2) < +\infty$ . Now let  $p \geq 1$ ,  $\vec{x} = (x_1, \dots, x_p) \in ([0, 1]^d)^p$  and  $h \in \mathcal{S}$  be fixed. We consider the stationary vector-valued random field

$Y = (Y_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  with values in  $\mathbb{R}^p$  where

$$Y_{\mathbf{k}} = (L(h, \mathbf{X}(x_1 + \mathbf{k})), \dots, L(h, \mathbf{X}(x_p + \mathbf{k}))) := L(h, \mathbf{X}_{\vec{x} + \mathbf{k}}). \quad (34)$$

The quasi-association assumption (32) will imply some dependence properties on the stationary vector-valued random field  $Y = (Y_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ .

In the following, we will use the notation  $BL(q')$  for  $q' \geq 1$  to denote the set of functions from  $\mathbb{R}^{q'}$  to  $\mathbb{R}$  that are Bounded and Lipschitz.

**Definition A.1** (Definition 5.14 p.94 of [12]). *The random field  $Y$  given by (34) is called  $(BL, \psi, \theta)$  dependent if there exists a non-increasing sequence  $\theta = (\theta_r)_{r \in \mathbb{N}}$  with  $\lim_{r \rightarrow +\infty} \theta_r = 0$  such that for any disjoint finite sets  $I, J \subset \mathbb{Z}^d$  with  $\text{dist}(I, J) \geq r$ , and any bounded Lipschitz functions  $f \in BL(p|I|)$ ,  $g \in BL(p|J|)$ , one has*

$$|\text{Cov}(f(Y_I), g(Y_J))| \leq \psi(|I|, |J|, f, g) \theta_r.$$

The case where  $\psi(|I|, |J|, f, g) = \text{Lip}(f) \text{Lip}(g) \min(|I|, |J|)$  is simply called  $(BL, \theta)$  dependence.



Note that when  $\mathbf{x} \in \mathbb{R}^s \mapsto L(h, \mathbf{x})$  is a Lipschitz function, assumption (32) implies that  $Y = (Y_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$  is a stationary  $(BL, \theta)$  dependent field. Then, by Theorem 1.12 p. 178 of [12], the sequence  $(S_n(h, x))_n$  will have a Gaussian limit with variance  $\sigma^2$  as soon as  $(\text{Var}(S_n(h, x)))_n$  tends to  $\sigma^2$ . However, for our specific application, we can not assume that  $\mathbf{x} \in \mathbb{R}^s \mapsto L(h, \mathbf{x})$  is Lipschitz but we can assume that  $\mathbf{x} \in \mathbb{R}^s \mapsto L(h, \mathbf{x})$  may be approximated by a Lipschitz function according to the following hypothesis:

There exist  $\alpha > 0$  and  $C \geq 1$  such that  $\forall \varepsilon \in (0, 1]$ , there is  $T_\varepsilon : \mathbb{R}^s \rightarrow [0, 1]$  such that

$$\begin{aligned} F_{h,\varepsilon} : \mathbf{x} \in \mathbb{R}^s \mapsto L(h, \mathbf{x})T_\varepsilon(\mathbf{x}) \in \mathbb{R} \text{ is Lipschitz with } \text{Lip}(F_{h,\varepsilon}) \leq C\|h\|_q \varepsilon^{-1}, \\ \mathbb{E}(L(h, \mathbf{X}(x))^2) \leq C\|h\|_{q,0}^2 \quad \text{and} \quad \mathbb{E}(L(h, \mathbf{X}(x))^2(1 - T_\varepsilon(\mathbf{X}(x)))) \leq C\|h\|_{q,0}^2 \varepsilon^{2\alpha}. \end{aligned} \quad (\mathbf{LA})$$

**Proposition A.1.** *Assuming that (32) and (LA) hold, if*

$$\sum_{\mathbf{i} \in \mathbb{Z}^d} \sup_{\|y\|_\infty \leq 1} |\tilde{\rho}(y + \mathbf{i})| < +\infty,$$

then the random field  $Y$  given by (34) is  $(BL, \psi, \theta)$  dependent with

$$\psi(|I|, |J|, f, g) = \text{Lip}(f)\|g\|_\infty|I| + \text{Lip}(g)\|f\|_\infty|J| + \min(|I|, |J|)\text{Lip}(f)\text{Lip}(g) \quad (35)$$

and for  $C_p(h) = 5C^3 p^2 \|h\|_q (4 + \|h\|_q)$ ,

$$\theta_r = C_p(h) \max \left( \left( \tilde{\theta}_r \right)^{\frac{\alpha}{\alpha+2}}, \tilde{\theta}_r \right), \quad \text{with } \tilde{\theta}_r = \sum_{\mathbf{i} \in \mathbb{Z}^d; |\mathbf{i}| \geq r} \sup_{\|y\|_\infty \leq 1} |\tilde{\rho}(y + \mathbf{i})| \quad (36)$$

*Proof.* Let  $r \in \mathbb{N}$ , we set  $I, J \subset \mathbb{Z}^d$  with  $\text{dist}(I, J) \geq r$  and  $|J| \leq |I|$  such that for any  $y \in [-1, 1]^d$

$$\sum_{\mathbf{i} \in I} \tilde{\rho}(y + \mathbf{i} - \mathbf{j}) \leq \tilde{\theta}(r), \quad \text{for all } \mathbf{j} \in J.$$

Let  $\varepsilon \in (0, 1]$  to be fixed later. In view of (34), we also write  $F_{h,\varepsilon}(\mathbf{X}_{\bar{x},I}) := (F_{h,\varepsilon}(\mathbf{X}(x_l + \mathbf{i})))_{\mathbf{i} \in I, 1 \leq l \leq p}$ . For  $f \in BL(p|I|)$  and  $g$ , we then write  $f \circ F_{h,\varepsilon}$  with a slight abuse of notation to denote the function that is in  $BL(sp|I|)$  with  $\text{Lip}(f \circ F_{h,\varepsilon}) \leq \text{Lip}(f)\text{Lip}(F_{h,\varepsilon})$ . Hence for  $g \in BL(p|J|)$ , we get by (32)

$$\begin{aligned} |\text{Cov}(f \circ F_{h,\varepsilon}(\mathbf{X}_{\bar{x},I}), g \circ F_{h,\varepsilon}(\mathbf{X}_{\bar{x},J}))| &\leq C \text{Lip}(f)\text{Lip}(g)\text{Lip}(F_{h,\varepsilon})^2 \sum_{\mathbf{i} \in I} \sum_{\mathbf{j} \in J} \sum_{l=1}^p \sum_{l'=1}^p \tilde{\rho}(x_l + \mathbf{i} - (x_{l'} + \mathbf{j})) \\ &\leq C \text{Lip}(f)\text{Lip}(g)\text{Lip}(F_{h,\varepsilon})^2 |J| p^2 \tilde{\theta}_r \\ &\leq C^3 \|h\|_q^2 \text{Lip}(f)\text{Lip}(g) |J| p^2 \varepsilon^{-2} \tilde{\theta}_r, \end{aligned}$$

by assumption (LA). Now, let us remark that since  $Y_I = L(h, \mathbf{X}_{\bar{x},I})$ ,

$$\begin{aligned} |\text{Cov}(f(Y_I) - f \circ F_{h,\varepsilon}(\mathbf{X}_{\bar{x},I}), g(Y_J))| &\leq 2\|g\|_\infty \mathbb{E}(|f(Y_I) - f \circ F_{h,\varepsilon}(\mathbf{X}_{\bar{x},I})|) \\ &\leq 2\|g\|_\infty \text{Lip}(f) \mathbb{E}(\|L(h, \mathbf{X}_{\bar{x},I}) - F_{h,\varepsilon}(\mathbf{X}_{\bar{x},I})\|_\infty) \\ &\leq 2\|g\|_\infty \text{Lip}(f) \sum_{\mathbf{k} \in I} \sum_{l=1}^p \mathbb{E}(\|L(h, \mathbf{X}_{x_l + \mathbf{k}})\| (1 - T_\varepsilon(\mathbf{X}_{x_l + \mathbf{k}}))) \\ &\leq 2C^{1/2} p |I| \|g\|_\infty \text{Lip}(f) \|h\|_{q,0} \varepsilon^\alpha, \end{aligned}$$

by assumption (LA), using Cauchy-Schwarz inequality and the fact that  $T_\varepsilon$  has values in  $[0, 1]$ . Similarly, one has

$$|\text{Cov}(f \circ F_{h,\varepsilon}(\mathbf{X}_{\bar{x},I}), g(Y_J) - g \circ F_{h,\varepsilon}(\mathbf{X}_{\bar{x},J}))| \leq 2C^{1/2} p |J| \|f\|_\infty \text{Lip}(g) \|h\|_{q,0} \varepsilon^\alpha.$$

Since  $C \geq 1$ , by definition of (35), it follows that

$$|\text{Cov}(f(Y_I), g(Y_J))| \leq \frac{C_p(h)}{5} \psi(|I|, |J|, f, g) \left( \tilde{\theta}_r \varepsilon^{-2} + 4\varepsilon^\alpha \right),$$

with  $C_p(h) = 5C^3 p^2 \|h\|_q (4 + \|h\|_q)$ . Now, when  $\tilde{\theta}_r \in (0, 1]$  we can choose  $\varepsilon = (\tilde{\theta}_r)^{\frac{1}{\alpha+2}}$  to get

$$|\text{Cov}(f(Y_I), g(Y_J))| \leq C_p(h) \psi(|I|, |J|, f, g) \tilde{\theta}_r^{\frac{\alpha}{\alpha+2}}.$$

Otherwise, if  $\tilde{\theta}_r = 0$  the result comes from letting  $\varepsilon$  tend to 0 and if  $\tilde{\theta}_r \geq 1$  by choosing  $\varepsilon = 1$ .  $\square$

However, we can in fact be more precise in term of covariance control and will need it in the sequel.

**Proposition A.2.** *Assuming that (32) and (LA) hold, one has for all  $x, x' \in \mathbb{R}^d$*

$$|\text{Cov}(L(h, \mathbf{X}(x)), L(h, \mathbf{X}(x')))| \leq 3C^3 \|h\|_q^2 \max(\tilde{\rho}(x - x')^{\frac{\alpha}{\alpha+2}}, \tilde{\rho}(x - x')).$$

*Proof.* As previously, let us choose  $\varepsilon \in (0, 1]$  to be fixed later and write for  $x \in \mathbb{R}^d$

$$L(h, \mathbf{X}(x)) = F_{h,\varepsilon}(\mathbf{X}(x)) + L(h, \mathbf{X}(x)) (1 - T_\varepsilon(\mathbf{X}(x))).$$

Noting that  $x$  may be written as  $x = \tilde{x} + \mathbf{k}$  for some  $\tilde{x} \in [0, 1]^d$  and  $\mathbf{k} \in \mathbb{Z}^d$ , by (32) we simply get for  $x' \in \mathbb{R}^d$ ,

$$\begin{aligned} |\text{Cov}(F_{h,\varepsilon}(\mathbf{X}(x)), F_{h,\varepsilon}(\mathbf{X}(x')))| &\leq C \text{Lip}(F_{h,\varepsilon})^2 \tilde{\rho}(x - x') \\ &\leq C^3 \|h\|_q^2 \varepsilon^{-2} \tilde{\rho}(x - x'). \end{aligned}$$

Now, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &|\text{Cov}(L(h, \mathbf{X}(x)) (1 - T_\varepsilon(\mathbf{X}(x))), L(h, \mathbf{X}(x')))| \\ &\leq \text{Var}(L(h, \mathbf{X}(x)) (1 - T_\varepsilon(\mathbf{X}(x))))^{1/2} \text{Var}(L(h, \mathbf{X}(x')))^{1/2} \\ &\leq C \|h\|_{q,0}^2 \varepsilon^\alpha, \end{aligned}$$

by (LA) using that the variance is smaller than the second order moment and the fact that  $T_\varepsilon$  has values in  $[0, 1]$ . Moreover, we obtain similarly

$$\begin{aligned} &|\text{Cov}(F_{h,\varepsilon}(\mathbf{X}(x)), L(h, \mathbf{X}(x')) (1 - T_\varepsilon(\mathbf{X}(x'))))| \\ &\leq \text{Var}(F_{h,\varepsilon}(\mathbf{X}(x)))^{1/2} \text{Var}(L(h, \mathbf{X}(x')) (1 - T_\varepsilon(\mathbf{X}(x'))))^{1/2} \\ &\leq C \|h\|_{q,0}^2 \varepsilon^\alpha, \end{aligned}$$

since  $\mathbb{E}(F_{h,\varepsilon}(\mathbf{X}(x))^2) \leq \mathbb{E}(L(h, \mathbf{X}(x))^2)$ . Hence,

$$|\text{Cov}(L(h, \mathbf{X}(x)), L(h, \mathbf{X}(x')))| \leq C^3 \|h\|_q^2 (\varepsilon^{-2} \tilde{\rho}(x - x') + 2\varepsilon^\alpha).$$

Note that if  $\tilde{\rho}(x - x') = 0$  we get the upper-bound by letting  $\varepsilon$  tends to 0. If  $\tilde{\rho}(x - x') \in (0, 1]$  we choose  $\varepsilon = \tilde{\rho}(x - x')^{\frac{1}{\alpha+2}}$  and otherwise we choose  $\varepsilon = 1$  to get the result.  $\square$

Then, under a natural decay assumption, we can define the asymptotic covariances of the sequence  $(S_n(h, x))_n$ . In particular this ensures the so-called finite susceptibility assumption (see (5.3) p.90 of [12]).

**Corollary A.1.** *Assuming that (32) and (LA) hold, we also assume that there exists  $\beta > \frac{d(\alpha+2)}{\alpha}$  such that, for all  $x \in \mathbb{R}^d$ ,*

$$\tilde{\rho}(x) \leq C(1 + |x|)^{-\beta}. \quad (37)$$

Then, there exists  $C_{\alpha,\beta,d} > 0$  such that, for all  $x, y \in \mathbb{R}^d$ ,

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} |\text{Cov}(L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y)))| \leq C_{\alpha,\beta,d} \|h\|_q^2 < +\infty.$$

Hence, there exists a real-valued stationary Gaussian random field  $(B(h, x))_{x \in \mathbb{R}^d}$  with covariance function given by

$$\text{Cov}(B(h, x), B(h, y)) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov}(L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y))), \quad (38)$$

such that

$$\text{Cov}(S_n(h, x), S_n(h, y)) \xrightarrow{n \rightarrow +\infty} \text{Cov}(B(h, x), B(h, y)).$$

*Proof.* Note that by Proposition A.2, under assumption 37, one has for all  $x, y \in \mathbb{R}^d$ ,

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} |\text{Cov}(L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y)))| \leq 3C^4 \|h\|_q^2 \sum_{\mathbf{k} \in \mathbb{Z}^d} (1 + |\mathbf{k} + x - y|)^{-\frac{\beta\alpha}{\alpha+2}},$$

and one can choose  $C_{\alpha,\beta,d} = 3C^4 \int_{\mathbb{R}^d} (1 + |z|)^{-\frac{\beta\alpha}{\alpha+2}} dz < +\infty$  since  $\beta > \frac{d(\alpha+2)}{\alpha}$ . Moreover, by stationarity of  $(L(h, \mathbf{X}(x)))_{x \in \mathbb{R}^d}$  one has

$$\begin{aligned} \text{Cov}(S_n(h, x), S_n(h, y)) &= \sum_{\mathbf{k} \in (-n, n)^d \cap \mathbb{Z}^d} \prod_{j=1}^d \left(1 - \frac{|k_j|}{n}\right) \text{Cov}(L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y))) \\ &\xrightarrow{n \rightarrow +\infty} \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov}(L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y))). \end{aligned}$$

Hence one can define the stationary Gaussian field  $(B(h, x))_{x \in \mathbb{R}^d}$  with covariance function given by (38) since this function is of non-negative type as a limit of covariance functions.  $\square$

Note that when  $\mathbf{x} \mapsto L(h, \mathbf{x})$  is Lipschitz with  $\mathbb{E}(L(h, \mathbf{X}(x))^2) < +\infty$ , assumption (LA) is satisfied for all  $\alpha > 0$  since one can choose  $T_\varepsilon = 1$ . Then, as soon as there exists  $\beta > d$  such that (37) holds true, we obtain the asymptotic normality of the  $\mathbb{R}^p$ -valued vector  $(S_n(h, \vec{x}))_n$  for any  $p \geq 1$  and  $\vec{x} = (x_1, \dots, x_p) \in ([0, 1]^d)^p$ , by Corollary 1.13 p.180 of [12], using the  $(BL, \theta)$  dependence of  $Y$ . This could be directly used for the volume of the excursion set. In the general case, noting that the  $\psi$ -dependency coefficient in (35) satisfies

$$\psi(|I|, |J|, f, g) \leq \text{Lip}(f) \|g\|_\infty |I| + \text{Lip}(g) \|f\|_\infty |J| + |I| |J| \text{Lip}(f) \text{Lip}(g),$$

the field  $Y$  is also  $\lambda$ -dependent as defined in [17]. By (37) we obtain that  $\lambda_Y(r) (= \theta_r) = \mathcal{O}_{r \rightarrow +\infty}(r^{-\lambda})$  with  $\lambda = \frac{\alpha}{\alpha+2}(\beta - d)$  and  $\theta_r$  given by (36). Then, using Cramer-Wold device and Theorem 2 p.219 in [17] we obtain the asymptotic normality under stronger assumptions (MA<sub>1</sub>) and (MA<sub>2</sub>) as stated in the following corollary.

**Corollary A.2.** *Assuming that (32), (LA) and (37) hold, assuming moreover that*

(MA<sub>1</sub>) *there exists  $\eta > 0$  such that  $\mathbb{E}(|L(h, \mathbf{X}(0))|^{2+\eta}) < +\infty$ ,*

(MA<sub>2</sub>)  $\frac{\alpha}{\alpha+2}(\beta - d) > \max\left(2d, d\left(1 + \frac{1}{\eta}\right)\right)$ .

Then

$$(S_n(h, x))_{x \in [0, 1]^d} \xrightarrow[n \rightarrow +\infty]{fdd} (B(h, x))_{x \in [0, 1]^d}.$$

This result will allow us to get a central limit theorem for the sequence of generalized processes given by (33), by approximating integrals by Riemann sums, under an appropriate continuity assumption.

**Proposition A.3.** *Assuming that (32) and (LA) hold, if there exists  $\delta > 0$  such that for all  $x \in \mathbb{R}^d$*

$$\|x\| \leq 1 \Rightarrow \omega_X(x) := \mathbb{E} [\|\mathbf{X}(x) - \mathbf{X}(0)\|_\infty^2]^{1/2} \leq C\|x\|^\delta. \quad (39)$$

*Then, there exists  $C_\alpha > 0$  such that, for all  $x \in \mathbb{R}^d$  with  $\|x\| \leq 1$ ,*

$$\mathbb{E} [|L(h, \mathbf{X}(x)) - L(h, \mathbf{X}(0))|^2]^{1/2} \leq C_\alpha \|h\|_q \omega_X(x)^{\frac{\alpha}{1+\alpha}}. \quad (40)$$

*Assuming moreover (37) for  $\beta > \frac{d(\alpha+2)}{\alpha}$ , one can choose a version of  $(B(h, x))_{x \in \mathbb{R}^d}$ , given in Corollary A.1, that is  $\gamma$ -Hölder continuous on any compact sets of  $\mathbb{R}^d$  for all  $\gamma < \frac{\delta}{2} \frac{\alpha}{1+\alpha} \left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)$ .*

*Proof.* Let  $x \in \mathbb{R}^d$  with  $\|x\| \leq 1$ . First note that we may assume that  $\omega_X(x) \in (0, C]$ . Let us remark that, as previously, using the fact that for  $\varepsilon \in (0, 1]$

$$L(h, \mathbf{X}(x)) = F_{h,\varepsilon}(X(x)) + L(h, \mathbf{X}(x))(1 - T_\varepsilon(\mathbf{X}(x))),$$

we have

$$\begin{aligned} \mathbb{E} [|L(h, \mathbf{X}(x)) - L(h, \mathbf{X}(0))|^2]^{1/2} &\leq \text{Lip}(F_{h,\varepsilon}) \mathbb{E} [\|\mathbf{X}(x) - \mathbf{X}(0)\|_\infty] + 2C^{1/2} \|h\|_{q,0} \varepsilon^\alpha \\ &\leq C \|h\|_q [\varepsilon^{-1} \omega_X(x) + 2\varepsilon^\alpha]. \end{aligned}$$

Taking  $\varepsilon = \left(\frac{\omega_X(x)}{C}\right)^{\frac{1}{1+\alpha}}$  we obtain (40).

Assuming moreover (37) for  $\beta > \frac{d(\alpha+2)}{\alpha}$ , for  $K \geq 1$ ,

$$\begin{aligned} \mathbb{E} [|B(h, x) - B(h, 0)|^2] &= 2 \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov}(L(h, \mathbf{X}(\mathbf{k})), L(h, \mathbf{X}(0)) - L(h, \mathbf{X}(x))), \\ &\leq 2C^{1/2} \|h\|_{q,0} \mathbb{E} [|L(h, \mathbf{X}(x)) - L(h, \mathbf{X}(0))|^2]^{1/2} (2K+3)^d \\ &\quad + 6C^3 \|h\|_q^2 \sum_{|\mathbf{k}| > K+1} [\max(\tilde{\rho}(\mathbf{k})^{\frac{\alpha}{\alpha+2}}, \tilde{\rho}(\mathbf{k})) + \max(\tilde{\rho}(\mathbf{k}-x)^{\frac{\alpha}{\alpha+2}}, \tilde{\rho}(\mathbf{k}-x))], \end{aligned}$$

using Cauchy-Schwarz inequality and (LA) for the first term and Proposition A.2 for the second one. Hence, by (37) and (40), we can find  $\tilde{C}_{\alpha,\beta,d} > 0$  such that

$$\mathbb{E} [|B(h, x) - B(h, 0)|^2] \leq \tilde{C}_{\alpha,\beta,d} \|h\|_q^2 \left( K^d \omega_X(x)^{\frac{\alpha}{1+\alpha}} + K^{-\left(\frac{\beta\alpha}{\alpha+2} - d\right)} \right).$$

Then, choosing  $K = \omega_X(x)^{-\frac{\alpha+2}{\alpha\beta} \frac{\alpha}{1+\alpha}}$ , there exists  $C_{\alpha,\beta,d} > 0$  such that

$$\mathbb{E} [|B(h, x) - B(h, 0)|^2] \leq C_{\alpha,\beta,d} \|h\|_q^2 \omega_X(x)^{\frac{\alpha}{1+\alpha} \left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)}.$$

In view of (39) and by stationarity of  $(B(h, x))_{x \in \mathbb{R}^d}$ , we can choose a version such that  $(B(h, x))_{x \in \mathbb{R}^d}$  is  $\gamma$ -Hölder continuous on any compact set of  $\mathbb{R}^d$  for  $\gamma < \frac{\delta}{2} \frac{\alpha}{1+\alpha} \left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)$  (see Proposition 9 in [8] for instance).  $\square$

We are now in position to prove the main result.

**Theorem A.1.** *Under the assumptions of Corollary A.2, assuming moreover (39), one has for  $h \in \mathcal{S}$ ,*

$$\langle \mathcal{T}_n, h \rangle = \sqrt{\mathcal{L}^d(U_n)} \left( \frac{1}{\mathcal{L}^d(U_n)} \int_{U_n} L(h, \mathbf{X}(x)) dx - \mathbb{E}(L(h, \mathbf{X}(0))) \right) \xrightarrow[n \rightarrow +\infty]{d} \int_{[0,1]^d} B(h, x) dx,$$

*where  $(B(h, x))_{x \in [0,1]^d}$  is the continuous Gaussian random field with covariance function given by (38). The random variable  $\int_{[0,1]^d} B(h, x) dx$  is centered Gaussian with variance given by*

$$\int_{\mathbb{R}^d} \text{Cov}(L(h, \mathbf{X}(z)), L(h, \mathbf{X}(0))) dz. \quad (41)$$

Hence, there exists  $B$  a generalized Gaussian random process on  $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ , with characteristic functional given by

$$\mathcal{L}_B(h) := \mathbb{E} \left( e^{i(B,h)} \right) = \mathbb{E} \left( e^{i \int_{[0,1]^d} B(h,x) dx} \right) = \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} \text{Cov} (L(h, \mathbf{X}(z)), L(h, \mathbf{X}(0))) dz \right)$$

such that  $\mathcal{T}_n \xrightarrow[n \rightarrow +\infty]{\text{distrib.}}$   $B$ , where the convergence holds in distribution with respect to the strong topology of  $\mathcal{S}'(\mathbb{R})$ .

*Proof.* In view of Proposition A.3, by continuity of  $(B(h, x))_{x \in [0,1]^d}$  we have the almost sure convergence of the Riemann sums

$$\int_{[0,1]^d} B(h, x) dx = \lim_{m \rightarrow +\infty} \frac{1}{m^d} \sum_{\mathbf{l} \in mV \cap \mathbb{Z}^d} B \left( h, \frac{\mathbf{l}}{m} \right),$$

where  $V = [0, 1]^d$ . Hence the random variable  $\int_{[0,1]^d} B(h, x) dx$  is centered Gaussian with variance given by

$$\begin{aligned} \int_{[0,1]^d} \int_{[0,1]^d} \text{Cov} (B(h, x), B(h, y)) dx dy &= \int_{[0,1]^d} \int_{[0,1]^d} \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov} (L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y))) dx dy \\ &= \int_{\mathbb{R}^d} \text{Cov} (L(h, \mathbf{X}(z)), L(h, \mathbf{X}(0))) dz, \end{aligned}$$

by change of variables and Fubini's theorem.

Note also that for  $x \in [0, 1]^d$ ,  $m > 0$  and  $\mathbf{l} \in mV \cap \mathbb{Z}^d$  we have

$$S_n \left( h, x + \frac{\mathbf{l}}{m} \right) - S_n \left( h, \frac{\mathbf{l}}{m} \right) = \frac{1}{n^{d/2}} \sum_{\mathbf{k}' \in nV \cap \mathbb{Z}^d} \left( L \left( h, \mathbf{X} \left( x + \frac{\mathbf{l}}{m} + \mathbf{k}' \right) \right) - L \left( h, \mathbf{X} \left( \frac{\mathbf{l}}{m} + \mathbf{k}' \right) \right) \right).$$

Then, by stationarity,

$$\text{Var} \left( S_n \left( h, x + \frac{\mathbf{l}}{m} \right) - S_n \left( h, \frac{\mathbf{l}}{m} \right) \right) \leq \sum_{\|\mathbf{k}\|_\infty \leq n} |r(h, \mathbf{k}, x)|,$$

where

$$r(h, \mathbf{k}, x) = \text{Cov} (L(h, \mathbf{X}(x)) - L(h, \mathbf{X}(0)), L(h, \mathbf{X}(x + \mathbf{k})) - L(h, \mathbf{X}(\mathbf{k})))$$

satisfies, by Proposition A.3 and Proposition A.2 together with (37),

$$|r(h, \mathbf{k}, x)| \leq C_\alpha^2 \|h\|_q^2 \omega_X(x)^{\frac{2\alpha}{1+\alpha}} \quad \text{and} \quad |r(h, \mathbf{k}, x)| \leq 12C^d \|h\|_q^2 |\mathbf{k}|^{-\frac{\beta\alpha}{\alpha+2}}.$$

It follows that we can find  $C_{\alpha,\beta,d} > 0$  such that

$$\begin{aligned} \mathbb{E} \left[ \left( S_n \left( h, x + \frac{\mathbf{l}}{m} \right) - S_n \left( h, \frac{\mathbf{l}}{m} \right) \right)^2 \right] &= \text{Var} \left( S_n \left( h, x + \frac{\mathbf{l}}{m} \right) - S_n \left( h, \frac{\mathbf{l}}{m} \right) \right) \\ &\leq C_{\alpha,\beta,d} \|h\|_q^2 \omega_X(x)^{\frac{2\alpha}{1+\alpha} \left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)}. \end{aligned}$$

Using Cauchy-Schwarz inequality,

$$\mathbb{E} \left( \int_{[0,1/m]^d} \left( S_n \left( h, x + \frac{\mathbf{l}}{m} \right) - S_n \left( h, \frac{\mathbf{l}}{m} \right) \right) dx \right)^2 \leq \frac{1}{m^d} \int_{[0,1/m]^d} \mathbb{E} \left( S_n \left( h, x + \frac{\mathbf{l}}{m} \right) - S_n \left( h, \frac{\mathbf{l}}{m} \right) \right)^2 dx.$$

Then,

$$\begin{aligned}
& \left\| \int_{[0,1]^d} S_n(h,x)dx - \frac{1}{m^d} \sum_{\mathbf{1} \in mV \cap \mathbb{Z}^d} S_n\left(h, \frac{\mathbf{1}}{m}\right) \right\|_{L^2(\Omega)} \\
& \leq \sum_{\mathbf{1} \in mV \cap \mathbb{Z}^d} \left\| \int_{[0,1/m]^d} \left( S_n\left(h, x + \frac{\mathbf{1}}{m}\right) - S_n\left(h, \frac{\mathbf{1}}{m}\right) \right) dx \right\|_{L^2(\Omega)} \\
& \leq C_{\alpha,\beta,d}^{1/2} \|h\|_q m^{d/2} \left( \int_{[0,1/m]^d} \omega_X(x)^{\frac{2\alpha}{1+\alpha} \left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)} dx \right)^{1/2}.
\end{aligned}$$

In view of (39), this implies that

$$\int_{[0,1]^d} S_n(h,x)dx = \lim_{m \rightarrow +\infty} \frac{1}{m^d} \sum_{\mathbf{1} \in mV \cap \mathbb{Z}^d} S_n\left(h, \frac{\mathbf{1}}{m}\right),$$

where the convergence holds in  $L^2(\Omega)$ , uniformly in  $n$ . Since

$$\sqrt{\mathcal{L}^d(U_n)} \left( \frac{1}{\mathcal{L}^d(U_n)} \int_{U_n} L(h, \mathbf{X}(x)) dx - \mathbb{E}(L(h, \mathbf{X}(0))) \right) = \int_{[0,1]^d} S_n(h,x)dx,$$

the stated convergence in distribution will simply follows from the fact that for  $m > 0$ , by Corollary A.2, one has

$$\frac{1}{m^d} \sum_{\mathbf{1} \in mV \cap \mathbb{Z}^d} S_n\left(h, \frac{\mathbf{1}}{m}\right) \xrightarrow[n \rightarrow +\infty]{d} \frac{1}{m^d} \sum_{\mathbf{1} \in mV \cap \mathbb{Z}^d} B\left(h, \frac{\mathbf{1}}{m}\right).$$

Note that the variance of the Gaussian variable  $\int_{[0,1]^d} B(h,x)dx$  is explicitly given by

$$\sigma^2(h) := \int_{[0,1]^2} \int_{[0,1]^2} \text{Cov}(B(h,x), B(h,y)) dx dy,$$

with

$$\text{Cov}(B(h,x), B(h,y)) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \text{Cov}(L(h, \mathbf{X}(x + \mathbf{k})), L(h, \mathbf{X}(y))).$$

It follows by Proposition A.2 that, on the one hand we can use Fubini's theorem to obtain (41), and on the other hand, there exists  $\tilde{C}$  such that  $\sigma^2(h) \leq \tilde{C} \|h\|_q^2$ . Therefore the characteristic functional  $\mathcal{L}_B : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$  given by

$$\mathcal{L}_B(h) = \mathbb{E} \left( e^{i \int_{[0,1]^d} B(h,x) dx} \right) = e^{-\frac{1}{2} \sigma^2(h)},$$

is continuous at 0 and the conclusion comes from Theorem 2.3 of [11]. □

## A.2. Gaussian case and Proof of Theorem 2.1

Assuming that  $X$  is a Gaussian field satisfying the assumptions of Theorem 2.1, by the Cramer-Wold device, it is enough to prove that for  $(a_j)_{0 \leq j \leq d} \in \mathbb{R}^{d+1}$  one has

$$\sqrt{\mathcal{L}^d(U_n)} \sum_{j=0}^d a_j \left( \frac{C_j(E_X(\cdot), U_n)}{\mathcal{L}^d(U_n)} - \bar{C}_j^X \right) \xrightarrow[n \rightarrow +\infty]{\text{distrib.}} \sum_{j=0}^d a_j B_j,$$

where  $(B_j)_{0 \leq j \leq d}$  are the centered Gaussian generalized processes introduced in Theorem 2.1 and the convergence holds in distribution with respect to the strong topology of  $\mathcal{S}'(\mathbb{R})$ . Therefore we let

$$\mathcal{T}_n = \sqrt{\mathcal{L}^d(U_n)} \sum_{j=0}^d a_j \left( \frac{C_j(E_X(\cdot), U_n)}{\mathcal{L}^d(U_n)} - \bar{C}_j^X \right) \quad \text{and} \quad B = \sum_{j=0}^d a_j B_j.$$

Hence we need to check that the assumptions of Theorem A.1 are satisfied for  $h \in \mathcal{S}$  and

$$L(h, \mathbf{X}(x)) := \sum_{j=0}^d a_j L_j(h, \mathbf{X}(x)),$$

since

$$\langle \mathcal{T}_n, h \rangle = \sqrt{\mathcal{L}^d(U_n)} \left( \frac{1}{\mathcal{L}^d(U_n)} \int_{U_n} L(h, \mathbf{X}(x)) dx - \mathbb{E}(L(h, \mathbf{X}(0))) \right).$$

Then we will identify the Gaussian limit. First, since  $X$  is a Gaussian field the quasi-association assumption (32) is satisfied (see [37]). Second, for  $\mathbf{x} = (u, v, w) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2}$  with a slight abuse of notation we write

$$F_0(\mathbf{x}) = \|v\|, \quad F_1(\mathbf{x}) = -\text{Tr}(w) + \frac{v^t w v}{\|v\|^2} \quad \text{and} \quad F_2(\mathbf{x}) = -\frac{v^t \text{adj}(w) v}{\|v\|^3}.$$

Then for  $\mathbf{x} \in R^s$  and  $h \in \mathcal{S}$  we have  $L_d(h, \mathbf{x}) = \int_u^{+\infty} h(t) dt$  and for  $0 \leq j \leq d-1$ ,  $L_j(h, \mathbf{x}) = h(u) F_{d-1-j}(\mathbf{x})$ . First remark that  $\mathbf{x} \mapsto L_d(h, \mathbf{x})$  is bounded by  $2\|h\|_{2,0}$  and Lipschitz. Then it satisfies (LA) for  $q = 2$  and any  $\alpha > 0$  by choosing  $T_\varepsilon = 1$ . Therefore, under (37), one has (MA<sub>1</sub>), (MA<sub>2</sub>) as soon as  $\beta > 3d$ . However, for  $0 \leq j \leq d-1$  we need to conveniently approximate  $\mathbf{x} \mapsto L_j(h, \mathbf{x}) = h(u) F_{d-1-j}(\mathbf{x})$ . For  $\delta > 0$  we can find a  $C^1$  function  $\mathsf{T}_\delta : \mathbb{R}^d \rightarrow [0, 1]$  such that  $\mathsf{T}_\delta(v) = 1$  for  $\|v\| > \delta$  and  $\|\nabla \mathsf{T}_\delta\|_\infty \leq \delta^{-1}$ . Moreover for  $M > 0$  we can find a  $C^1$  function  $\mathsf{T}_M : \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2} \rightarrow [0, 1]$  such that  $\mathsf{T}_M(v, w) = 1$  for  $\|(v, w)\| < M$  and  $\mathsf{T}_M(v, w) = 0$  for  $\|(v, w)\| > 2M$  with  $\|\nabla \mathsf{T}_M\|_\infty \leq 1$ . Then  $\mathbf{x} \mapsto L_j(h, \mathbf{x}) \mathsf{T}_M(v, w) \mathsf{T}_\delta(v)$  is  $C^1$  with gradient bounded by  $C\|h\|_q \left(\frac{M}{\delta}\right)$  for  $j = d-1-k$  and  $k = 0$  or  $k = 1$  but by  $C\|h\|_q \left(\frac{M}{\delta}\right)^2$  for  $d = 3$  and  $j = 0$ , with  $C \geq 1$  that does not depend on  $M, \delta, q = 0$  (and consequently  $q = 2$ ), and  $\|h\|_q = \|h\|_{q,0} + \|h\|_{q,1}$ . Using the fact that  $\nabla X(x)$  admits a bounded density in the neighborhood of 0 one can find  $c > 0$  such that

$$\mathbb{P}(\|\nabla X(x)\| \leq \delta) \leq c\delta^d.$$

Moreover, in view of Gaussian marginal distribution one can find  $c_d > 0$  such that for  $M$  large enough

$$\mathbb{P}(\|(\nabla X(x), D^2 X(x))\| \geq M) \leq e^{-c_d M}.$$

We consider in the sequel conditions on  $\alpha > 0$  for which (LA) can be satisfied and then conditions on  $\beta$  in order to also check (MA<sub>1</sub>), (MA<sub>2</sub>) for each case separately.

• *Case  $j = d-1-k$  for  $k = 0, 1$ .* For  $\varepsilon \in (0, 1)$ , we set  $T_\varepsilon(\mathbf{x}) = \mathsf{T}_\delta(v) \mathsf{T}_M(v, w)$  with  $\delta = \frac{d}{c_d} \varepsilon |\log(\varepsilon)|$  and  $M = \frac{d}{c_d} |\log(\varepsilon)|$  such that  $\frac{\delta}{M} = \varepsilon$ .

Since  $\mathbb{E}(F_0(\mathbf{X}(x))^{2p}) = \mathbb{E}(\|\nabla X(x)\|^{2p}) < +\infty$  and  $\mathbb{E}(F_1(\mathbf{X}(x))^{2p}) \leq \mathbb{E}\left(|2 \sum_{i,j=1}^d X_{ij}(x)|^{2p}\right) < +\infty$  for any  $p \geq 1$ , by Hölder inequality we get

$$\mathbb{E}(L_{d-1-k}(h, \mathbf{X}(x))^2 (1 - T_\varepsilon(\mathbf{X}(x)))) \leq \|h\|_{0,0}^2 \mathbb{E}(F_k(\mathbf{X}(x))^{2p})^{1/p} \mathbb{E}\left((1 - T_\varepsilon(\mathbf{X}(x)))^{p'}\right)^{1/p'},$$

for  $\frac{1}{p} + \frac{1}{p'} = 1$  as soon as  $p > 1$ , with

$$\begin{aligned} \mathbb{E}\left((1 - T_\varepsilon(\mathbf{X}(x)))^{p'}\right) &\leq \mathbb{P}(\|\nabla X(x)\| \leq \delta) + \mathbb{P}(\|(\nabla X(x), D^2 X(x))\| \geq M) \\ &\leq C\varepsilon^d (1 + |\log(\varepsilon)|^d), \end{aligned}$$

choosing  $C \geq \max(1, (cd/c_d)^d)$ , such that

$$\mathbb{E} (L_{d-1-k}(h, \mathbf{X}(x))^2(1 - T_\varepsilon(\mathbf{X}(x)))) \leq C \|h\|_{0,0}^2 \mathbb{E} (F_k(\mathbf{X}(x))^{2p})^{1/p} \varepsilon^{d/p'} (1 + |\log(\varepsilon)|^d)^{1/p'}.$$

We also have  $\mathbb{E} (L_{d-1-k}(h, \mathbf{X}(x))^2) \leq C \|h\|_{0,0}^2$  and  $\mathbb{E} (|L_{d-1-k}(h, \mathbf{X}(x))|^3) < +\infty$ . It follows that for any  $\beta > d(1 + 2(1 + 4/d))$  one can find  $\alpha \in (0, d/2)$  such that  $\beta - d > 2d(1 + 2/\alpha)$  ensuring **(MA<sub>1</sub>)** and **(MA<sub>2</sub>)** (with  $\eta = 1$ ) and then choose  $p$  large enough such that  $d/p' > 2\alpha$  in order to check **(LA)** with  $q = 0$ .

• *Case  $d = 3$  and  $j = 0$  with  $d - 1 - j = 2$ .* In contrast with the previous cases, using the independence between  $\nabla X(x)$  and  $D^2 X(x)$ , one has only

$$\mathbb{E} (F_2(\mathbf{X}(x))^{2+\eta}) \leq \mathbb{E} \left( \left[ \max_{1 \leq i, j \leq d} |\text{adj}(D^2 X(x))| \right]^{2+\eta} \right) \mathbb{E} (\|\nabla X(x)\|^{-2-\eta}) < +\infty,$$

for any  $\eta < 1$ . Moreover  $\mathbf{x} \mapsto L_0(h, \mathbf{x}) \mathbb{T}_M(v, w) \mathbb{T}_\delta(v)$  has now a Lipschitz constant bounded by  $C \|h\|_q \left(\frac{M}{\delta}\right)^2$ . Hence for  $\varepsilon \in (0, 1)$ , we must set  $T_\varepsilon(\mathbf{x}) = \mathbb{T}_\delta(v) \mathbb{T}_M(v, w)$  with  $\delta = \frac{d}{c_d} \varepsilon^{1/2} |\log(\varepsilon^{1/2})|$  and  $M = \frac{d}{c_d} |\log(\varepsilon^{1/2})|$  such that  $\left(\frac{\delta}{M}\right)^2 = \varepsilon$ .

Therefore, for any  $p > 1$  and  $p' > 1$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ ,

$$\begin{aligned} \mathbb{E} (L_0(h, \mathbf{X}(x))^2(1 - T_\varepsilon(\mathbf{X}(x)))) &\leq \|h\|_{0,0}^2 \mathbb{E} \left( \left[ \max_{1 \leq i, j \leq d} |\text{adj}(D^2 X(x))| \right]^2 \right) \mathbb{E} (\|\nabla X(x)\|^{-2} \mathbf{1}_{\|\nabla X(x)\| \leq \delta}) \\ &\quad + \|h\|_{0,0}^2 \delta^{-2} \mathbb{E} \left( \left[ \max_{1 \leq i, j \leq d} |\text{adj}(D^2 X(x))| \right]^{2p} \right)^{1/p} \mathbb{P}(\|(\nabla X(x), D^2 X(x))\| \geq M)^{1/p'}, \end{aligned}$$

by Hölder inequality for the second term. Then we can find  $c_p > 0$  and  $\tilde{c}_p > 0$  such that

$$\begin{aligned} \mathbb{E} (L_0(h, \mathbf{X}(x))^2(1 - T_\varepsilon(\mathbf{X}(x)))) &\leq c_p \|h\|_{0,0}^2 \left( \delta^{d-2} + \delta^{-2} e^{-\frac{c_d}{p'} M} \right) \\ &\leq \tilde{c}_p \|h\|_{0,0}^2 \varepsilon^{\frac{d}{2p'} - 1}. \end{aligned}$$

Then for  $\beta > d \left(1 + 2\left(1 + \frac{8}{d-2}\right)\right)$  one can find  $\eta < 1$  and  $\alpha < \frac{d-2}{4}$  such that  $\beta - d > d \left(1 + \frac{1}{\eta}\right) \left(1 + \frac{2}{\alpha}\right)$  to ensure **(MA<sub>1</sub>)** and **(MA<sub>2</sub>)**. We can then choose  $p$  large enough to ensure that  $\frac{d}{2p'} - 1 > 2\alpha$  in order to verify **(LA)**. Combining both conditions on  $\beta$  in order to get **(LA)**, **(MA<sub>1</sub>)** and **(MA<sub>2</sub>)** satisfied for a general linear combination we must have  $\beta > 19d$  (corresponding to the worst case). Finally, note that for  $\|x\| \leq 1$  one has

$$\mathbb{E} ([X_{ij}(x) - X_{ij}(0)]^2) = 2(-1)^{\varepsilon(i)+\varepsilon(j)} \left( \partial_{ijij}^{2\varepsilon(i)+2\varepsilon(j)} \rho(0) - \partial_{ijij}^{2\varepsilon(i)+2\varepsilon(j)} \rho(x) \right),$$

such that (39) is satisfied by **(H)**.

It follows that all assumptions and therefore conclusions of Theorem 2.1 hold true.

To conclude it only remains to identify the asymptotic covariance. By linearity of  $B$  and  $L$  for  $h, \tilde{h} \in \mathcal{S}$  we obtain that

$$\begin{aligned} \text{Var}(\langle B, h + \tilde{h} \rangle) &= \text{Var}(\langle B, h \rangle) + \text{Var}(\langle B, \tilde{h} \rangle) + 2\text{Cov}(\langle B, h \rangle, \langle B, \tilde{h} \rangle) \\ &= \int_{\mathbb{R}^d} \text{Cov} \left( L(h + \tilde{h}, \mathbf{X}(z)), L(h + \tilde{h}, \mathbf{X}(0)) \right) dz \\ &= \int_{\mathbb{R}^d} \text{Cov} (L(h, \mathbf{X}(z)), L(h, \mathbf{X}(0))) dz + \int_{\mathbb{R}^d} \text{Cov} (L(\tilde{h}, \mathbf{X}(z)), L(\tilde{h}, \mathbf{X}(0))) dz \\ &\quad + 2 \int_{\mathbb{R}^d} \text{Cov} (L(h, \mathbf{X}(z)), L(\tilde{h}, \mathbf{X}(0))) dz, \end{aligned}$$



where we use the fact that, by stationarity and change of variables, one has

$$\begin{aligned} \int_{\mathbb{R}^d} \text{Cov} \left( L(\tilde{h}, \mathbf{X}(z)), L(h, \mathbf{X}(0)) \right) dz &= \int_{\mathbb{R}^d} \text{Cov} \left( L(\tilde{h}, \mathbf{X}(0)), L(h, \mathbf{X}(-z)) \right) dz \\ &= \int_{\mathbb{R}^d} \text{Cov} \left( L(\tilde{h}, \mathbf{X}(0)), L(h, \mathbf{X}(z)) \right) dz. \end{aligned}$$

It follows that

$$\begin{aligned} \text{Cov}(\langle B, h \rangle, \langle B, \tilde{h} \rangle) &= \int_{\mathbb{R}^d} \text{Cov} \left( L(h, \mathbf{X}(z)), L(\tilde{h}, \mathbf{X}(0)) \right) dz \\ &= \sum_{j=0}^d \sum_{k=0}^d a_j a_k \int_{\mathbb{R}^d} \text{Cov} \left( L_j(h, \mathbf{X}(z)), L_k(\tilde{h}, \mathbf{X}(0)) \right) dz, \end{aligned}$$

and  $B = \sum_{j=0}^d a_j B_j$  where  $(B_j)_{0 \leq j \leq d}$  are centered Gaussian generalized processes with covariance given by

$$\forall j, k \in \{0, \dots, d\}, \quad \text{Cov}(\langle B_j, h \rangle, \langle B_k, \tilde{h} \rangle) = \int_{\mathbb{R}^d} \text{Cov} \left( L_j(h, \mathbf{X}(z)), L_k(\tilde{h}, \mathbf{X}(0)) \right) dz.$$

### A.3. Proof of Corollary 2.2

*Proof.* For sake of simplicity, we only sketch the proof for  $k = j \in J$  and  $s = t \in V$ . Let us first choose  $h \in \mathcal{S}$  non-negative with compact support such that  $\int_{\mathbb{R}} h(u) du = 1$ , as well as  $(h_m)_{m \geq 1}$  its associated approximation of the identity, given by  $h_m(u) = mh(mu)$ . We set  $\tau_t \check{h}_m(u) = h(t - u)$  and may assume that  $\tau_t \check{h}_m$  has support in  $V$  for all  $m \geq 1$ . Then for all  $m, p \geq 1$ ,

$$\text{Cov}(\langle B_j, \tau_t \check{h}_m \rangle, \langle B_j, \tau_t \check{h}_p \rangle) = \int_{\mathbb{R}} \int_{\mathbb{R}} \tau_t \check{h}_m(u) \tau_t \check{h}_p(v) \Sigma_{jj}(u, v) du dv \xrightarrow{m, p \rightarrow +\infty} \Sigma_{jj}(t, t),$$

using Fubini's theorem and continuity, thanks to (A3). Then  $(\langle B_j, \tau_t \check{h}_m \rangle)_m$  is a Cauchy sequence of Gaussian variables in  $L^2$  and we can set  $\tilde{B}_j(t)$  its Gaussian limit. Note that therefore we also have  $\text{Var}(\tilde{B}_j(t)) = \Sigma_{jj}(t, t)$ . Moreover, since  $C_j(E_X(t, U_n)) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$  by (A1), we have

$$\begin{aligned} \mathbb{E} \left( \left[ \sqrt{\mathcal{L}^d(U_n)} \left( \frac{C_j(E_X(t, U_n))}{\mathcal{L}^d(U_n)} - \bar{C}_j^X(t) \right) - \sqrt{\mathcal{L}^d(U_n)} \left( \frac{\text{LC}_j^X(\tau_t \check{h}_m, U_n)}{\mathcal{L}^d(U_n)} - \langle \bar{C}_j^X, \tau_t \check{h}_m \rangle \right) \right]^2 \right) \\ = \frac{1}{\mathcal{L}^d(U_n)} \text{Var} \left( C_j(E_X(t, U_n)) - \text{LC}_j^X(\tau_t \check{h}_m, U_n) \right). \end{aligned}$$

But

$$C_j(E_X(t, U_n)) - \text{LC}_j^X(\tau_t \check{h}_m, U_n) = \int_{\mathbb{R}} h(u) \left( C_j(E_X(t, U_n)) - C_j \left( E_X \left( t - \frac{u}{m} \right), U_n \right) \right) du.$$

Therefore,

$$\begin{aligned} \text{Var} \left( C_j(E_X(t, U_n)) - \text{LC}_j^X(\tau_t \check{h}_m, U_n) \right) &= \int_{\mathbb{R}^2} h(u) h(v) \dots \\ \text{Cov} \left( C_j(E_X(t, U_n)) - C_j \left( E_X \left( t - \frac{u}{m} \right), U_n \right), C_j(E_X(t, U_n)) - C_j \left( E_X \left( t - \frac{v}{m} \right), U_n \right) \right) & dudv. \end{aligned}$$

By (A2), for  $\varepsilon > 0$  and  $u, v$  with  $|u| \leq \delta m$  and  $|v| \leq \delta m$ , and by Cauchy-Schwarz inequality, one has

$$\left| \text{Cov} \left( C_j(E_X(t, U_n)) - C_j \left( E_X \left( t - \frac{u}{m} \right), U_n \right), C_j(E_X(t, U_n)) - C_j \left( E_X \left( t - \frac{v}{m} \right), U_n \right) \right) \right| \leq \varepsilon \mathcal{L}^d(U_n).$$

Hence, for  $m$  large enough such that  $h_m$  has compact support in  $[-\delta, \delta]$  one has,

$$\frac{1}{\mathcal{L}^d(U_n)} \text{Var} (C_j(E_X(t), U_n)) - \text{LC}_j^X(\tau_t \check{h}_m, U_n)) \leq \varepsilon.$$

Hence, choosing  $m$  such that we also have

$$\text{Var} \left( \langle B_j, \tau_t \check{h}_m \rangle - \tilde{B}_j(u) \right) \leq \varepsilon,$$

the results follow from Theorem 2.1, using the fact that we have

$$\sqrt{\mathcal{L}(U_n)} \left( \frac{\text{LC}_j^X(\tau_t \check{h}_m, U_n)}{\mathcal{L}^d(U_n)} - \langle \bar{C}_j^X, \tau_t \check{h}_m \rangle \right) \xrightarrow[n \rightarrow +\infty]{d} \langle B_j, \tau_t \check{h}_m \rangle.$$

□

## Appendix B: Additional proofs

### B.1. Additional proof of Theorem 3.1

Recall that, for a.e.  $t \in \mathbb{R}$ , using (6) and (9), we have

$$\begin{aligned} \bar{C}_1^X(t) &= \frac{1}{2} \mathbb{E}(\|\nabla X(0)\| | X(0) = t) p_{X(0)}(t) \quad \text{and} \\ \bar{C}_0^X(t) &= -\frac{1}{2\pi} \mathbb{E} \left( \frac{(\nabla X(0)^\perp)^t D^2 X(0) \nabla X(0)^\perp}{\|\nabla X(0)\|^2} | X(0) = t \right) p_{X(0)}(t). \end{aligned}$$

In order to go ahead in the computations, we first remark that writing  $X = m + \sigma Y \circ P$ , for  $P$  orthogonal such that  $\Gamma = P^t \Delta P$  with  $\Delta = \text{diag}(\gamma_1, \gamma_2)$ , one has therefore  $\bar{C}_j^X(t) = \bar{C}_j^Y\left(\frac{t-m}{\sigma}\right)$  where  $Y$  is centered with unit variance and admits  $\frac{\gamma_1}{\sigma^2}, \frac{\gamma_2}{\sigma^2}$  for spectral moments with (denoting by  $Y_i$  and  $Y_{ij}$  the first and second order partial derivatives of  $Y$ ):

$$\frac{\gamma_1}{\sigma^2} = \mathbb{E}(Y_1(0)^2), \quad \frac{\gamma_2}{\sigma^2} = \mathbb{E}(Y_2(0)^2), \quad \text{and} \quad \mathbb{E}(Y_1(0)Y_2(0)) = 0.$$

Since  $Y$  is stationary Gaussian, we also have that  $\nabla Y(0)$  is independent from  $Y(0)$  and  $D^2 Y(0)$ , and that  $Y(0)$  and  $D^2 Y(0)$  are correlated with covariance

$$\mathbb{E}(Y(0)Y_{11}(0)) = -\frac{\gamma_1}{\sigma^2}, \quad \mathbb{E}(Y(0)Y_{22}(0)) = -\frac{\gamma_2}{\sigma^2}, \quad \text{and} \quad \mathbb{E}(Y(0)Y_{12}(0)) = 0.$$

Therefore the conditional expectations are given by

$$\mathbb{E}(Y_{11}(0)|Y(0) = t) = -\frac{\gamma_1}{\sigma^2} t, \quad \mathbb{E}(Y_{22}(0)|Y(0) = t) = -\frac{\gamma_2}{\sigma^2} t, \quad \text{and} \quad \mathbb{E}(Y_{12}(0)|Y(0) = t) = 0.$$

Now, using that for a vector  $v \in \mathbb{R}^2$ , we have, denoting  $e_\theta = (\cos \theta, \sin \theta)$ ,

$$\|v\| = \frac{1}{4} \int_0^{2\pi} |\langle v, e_\theta \rangle| d\theta,$$

and using that for a 1D Gaussian random variable  $Z$  of mean 0 and variance  $\sigma_Z^2$ , we have  $\mathbb{E}(|Z|) = \sqrt{\frac{2}{\pi}} \sigma_Z$ , we get

$$\bar{C}_1^Y(t) = \frac{1}{2} \mathbb{E}(\|\nabla Y(0)\|) p_{Y(0)}(t) = \frac{1}{8\pi\sigma} e^{-t^2/2} \int_0^{2\pi} \sqrt{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta.$$

To compute  $\overline{C}_0^Y(t)$ , we first need to compute  $\mathbb{E}(\cos^2 \Theta)$  where  $\Theta$  is the angle of  $\nabla Y(0)$ , i.e.  $\nabla Y(0) = (\|\nabla Y(0)\| \cos \Theta, \|\nabla Y(0)\| \sin \Theta)$ . Since  $Y$  is not assumed isotropic, the law of  $\Theta$  is not necessarily uniform on  $[0, 2\pi)$ . Indeed we have

$$\begin{aligned} \mathbb{E}(\cos^2 \Theta) &= \frac{\sigma^2}{2\pi\sqrt{\gamma_1\gamma_2}} \iint \frac{y_1^2}{y_1^2 + y_2^2} e^{-(\sigma y_1)^2/2\gamma_1} e^{-(y_2\sigma)^2/2\gamma_2} dy_1 dy_2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma_1 \cos^2 \theta}{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta. \end{aligned} \quad (42)$$

It follows that

$$\mathbb{E}(\sin^2 \Theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma_2 \sin^2 \theta}{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta.$$

Finally,

$$\overline{C}_0^Y(t) = -\frac{1}{2\pi} (\mathbb{E}(\sin^2 \Theta)\mathbb{E}(Y_{11}(0)|Y(0) = t) + \mathbb{E}(\cos^2 \Theta)\mathbb{E}(Y_{22}(0)|Y(0) = t)) p_{Y(0)}(t),$$

and combining everything, we get

$$\overline{C}_0^Y(t) = \frac{t}{\sigma^2(\sqrt{2\pi})^5} e^{-t^2/2} \int_0^{2\pi} \frac{\gamma_1\gamma_2}{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta = \frac{t}{\sigma^2(\sqrt{2\pi})^3} e^{-t^2/2} \sqrt{\gamma_1\gamma_2}.$$

The last equality comes from the following computation:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma_1\gamma_2}{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta &= \frac{2}{\pi} \int_0^{\pi/2} \frac{\gamma_1\gamma_2}{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 s^2} ds = \sqrt{\gamma_1\gamma_2}, \end{aligned} \quad (43)$$

where we have used a change of variable  $s = \tan \theta$ , and then recognized the derivative of the function  $s \mapsto \arctan(\sqrt{\frac{\gamma_2}{\gamma_1}} s)$ .

## B.2. Additional proof of Theorem 4.1

As in the 2D case, we may assume that  $m = 0$ ,  $\sigma^2 = 1$  and  $\Gamma = \text{Diag}(\gamma_1, \gamma_2, \gamma_3)$  meaning that

$$\forall j = 1, 2, 3, \quad \gamma_j := \Gamma_{jj} = \mathbb{E}(X_j(0)^2), \quad \text{and} \quad \mathbb{E}(X_i(0)X_j(0)) = 0 \quad \text{when} \quad i \neq j,$$

denoting  $X_i := \frac{\partial X}{\partial x_i}$ ,  $X_{ij} := \frac{\partial^2 X}{\partial x_i \partial x_j}$  for  $1 \leq i, j \leq d$ . The formula for  $\overline{V}_X$  is straightforward. Now for  $j < 3$ , by Hypothesis **H**, we have  $\|\nabla X(0)\| > 0$  a.s. and  $\mathbb{E}(\|\nabla X(0)\|^{-1}) < +\infty$ . Since  $D^2 X(0)$  is independent from  $\nabla X(0)$ , recalling (6), (7) (8), it follows that the assumption (12) is satisfied. Then, for  $h \in \mathcal{S}$ , by (13), since the standard Gaussian random variable  $X(0)$  admits a probability density given by a function  $p_{X(0)}$ , we have

$$\alpha_{2-j} \langle \overline{C}_j^X, h \rangle = \mathbb{E}(h(X(0))F_{2-j}(\mathbf{X}(0))) = \int_{\mathbb{R}} h(t) \mathbb{E}(F_{2-j}(\mathbf{X}(0)) | X(0) = t) p_{X(0)}(t) dt.$$

Hence, for a.e.  $t \in \mathbb{R}$ ,

$$\begin{aligned} \overline{C}_2^X(t) &= \frac{1}{2} \mathbb{E}(\|\nabla X(0)\| | X(0) = t) p_{X(0)}(t) \\ \overline{C}_1^X(t) &= \frac{1}{2\pi} \mathbb{E} \left( \frac{\nabla X(0)^t D^2 X(0) \nabla X(0)}{\|\nabla X(0)\|^2} - \text{Tr}(D^2 X(0)) | X(0) = t \right) p_{X(0)}(t). \\ \overline{C}_0^X(t) &= \frac{1}{4\pi} \mathbb{E} \left( \frac{\nabla X(0)^t \text{adj}(D^2 X(0)) \nabla X(0)}{\|\nabla X(0)\|^3} | X(0) = t \right) p_{X(0)}(t). \end{aligned}$$

Note that since  $X$  is Gaussian, we have that  $\nabla X(0)$  is independent from  $X(0)$  and  $D^2X(0)$ , and that  $X(0)$  and  $D^2X(0)$  are correlated with covariance

$$\forall j = 1, 2, 3, \quad \mathbb{E}(X(0)X_{jj}(0)) = -\gamma_j, \quad \text{and} \quad \mathbb{E}(X(0)X_{ij}(0)) = 0 \quad \text{when} \quad i \neq j.$$

$$\forall j = 1, 2, 3, \quad \mathbb{E}(X(0)X_{jj}(0)) = -\gamma_j, \quad \text{and} \quad \mathbb{E}(X(0)X_{ij}(0)) = 0 \quad \text{when} \quad i \neq j.$$

Therefore the conditional expectations are given by

$$\mathbb{E}(X_{jj}(0)|X(0) = t) = -\gamma_j t, \quad \text{and} \quad \mathbb{E}(X_{ij}(0)|X(0) = t) = 0.$$

Moreover, we have

$$\partial_{iijj}^4 \rho(0) = \mathbb{E}(X_{ii}(0)X_{jj}(0)) = \mathbb{E}(X_{ij}(0)^2).$$

Since by assumption  $D^2\rho(0) = -\Gamma$  is diagonal, we have  $\mathbb{E}(X_{ij}(0)X(0)) = 0$  and therefore

$$\mathbb{E}(X_{ij}(0)^2|X(0) = t) = \partial_{iijj}^4 \rho(0).$$

Using the formulas for the conditional distribution of Gaussian vectors, the covariance of  $(X_{ii}(0), X_{jj}(0))$  knowing  $X(0) = t$  is given by

$$\mathbb{E}(X_{ii}(0)X_{jj}(0)|X(0) = t) = \gamma_i \gamma_j t^2 + \partial_{iijj}^4 \rho(0) - \gamma_i \gamma_j.$$

Therefore, putting all together, we get

$$\mathbb{E}(X_{ii}(0)X_{jj}(0) - X_{ij}(0)^2|X(0) = t) = \gamma_i \gamma_j (t^2 - 1),$$

which doesn't require the fourth-order spectral moment anymore. We are now in position to compute the different densities. We start with  $\bar{C}_2^X(t)$ , which is equal to

$$\bar{C}_2^X(t) = \frac{1}{2} \mathbb{E}(\|\nabla X(0)\| | X(0) = t) p_X(t) = \frac{1}{2} \mathbb{E}(\|\nabla X(0)\|) p_{X(0)}(t).$$

We use here the property that if  $v \in \mathbb{R}^3$ , then

$$\|v\| = \frac{1}{2\pi} \int_{S^2} |\langle v, u \rangle| \mathcal{H}^2(du).$$

Now, as in the 2D case, using the fact that the expectation of the absolute value of a centered Gaussian real random variable is  $\sqrt{\frac{2}{\pi}}$  times its standard deviation, we get, denoting  $u = (u_1, u_2, u_3) \in S^2$ ,

$$\bar{C}_2^X(t) = \frac{1}{4\pi^2} e^{-t^2/2} \int_{S^2} \sqrt{\gamma_1 u_1^2 + \gamma_2 u_2^2 + \gamma_3 u_3^2} \mathcal{H}^2(du) = \frac{1}{\pi} \sqrt{\gamma_{SA}} e^{-t^2/2}.$$

Then, for  $\bar{C}_1^X$ , using the conditional expectations and the fact that  $\nabla X(0)$  is independent from  $(X(0), D^2X(0))$ , we have

$$2\pi \bar{C}_1^X(t) = t p_{X(0)}(t) \left[ \gamma_1 + \gamma_2 + \gamma_3 - \mathbb{E} \left( \frac{\gamma_1 X_1(0)^2 + \gamma_2 X_2(0)^2 + \gamma_3 X_3(0)^2}{X_1(0)^2 + X_2(0)^2 + X_3(0)^2} \right) \right].$$

Using the law of the  $X_i(0)$  and a change of variable with spherical coordinates, we get that

$$\mathbb{E} \left( \frac{\gamma_1 X_1(0)^2 + \gamma_2 X_2(0)^2 + \gamma_3 X_3(0)^2}{X_1(0)^2 + X_2(0)^2 + X_3(0)^2} \right) = \frac{1}{4\pi} \int_{S^2} \frac{\gamma_1^2 u_1^2 + \gamma_2^2 u_2^2 + \gamma_3^2 u_3^2}{\gamma_1 u_1^2 + \gamma_2 u_2^2 + \gamma_3 u_3^2} \mathcal{H}^2(du).$$

Hence,

$$2\pi \bar{C}_1^X(t) = 2\bar{\gamma}_{TMC} t p_{X(0)}(t).$$

Finally, by independence of  $\nabla X(0)$  from  $(X(0), D^2X(0))$ , and the conditional expectations computed above, we get (where the indices  $i, j, k$  below are all distinct):

$$\begin{aligned}
4\pi\bar{C}_0^X(t) &= p_{X(0)}(t) \sum_{i=1}^3 \mathbb{E} \left( \frac{X_i(0)^2}{(X_1(0)^2 + X_2(0)^2 + X_3(0)^2)^{3/2}} \right) \mathbb{E}(X_{jj}(0)X_{kk}(0) - X_{jk}(0)^2 | X(0) = t) \\
&= (t^2 - 1)p_{X(0)}(t) \sum_{i=1}^3 \mathbb{E} \left( \frac{X_i(0)^2}{(X_1(0)^2 + X_2(0)^2 + X_3(0)^2)^{3/2}} \right) \gamma_j \gamma_k \\
&= (t^2 - 1)p_{X(0)}(t) \times \frac{1}{(2\pi)^{3/2}} \int_{S^2} \frac{\gamma_1 \gamma_2 \gamma_3}{(\gamma_1 u_1^2 + \gamma_2 u_2^2 + \gamma_3 u_3^2)^{3/2}} \mathcal{H}^2(du) \\
&= \bar{\gamma}_{\text{TGC}}^{3/2} \frac{1}{\pi} (t^2 - 1) e^{-t^2/2}.
\end{aligned}$$

The last line above comes from the following computation. Let  $J$  denote the above integral on  $S^2$ , that can be also written as

$$J := 2 \int_0^{\pi/2} \int_0^{2\pi} \frac{\gamma_1 \gamma_2 \gamma_3}{(\gamma_1 \sin^2 \varphi \cos^2 \theta + \gamma_2 \sin^2 \varphi \sin^2 \theta + \gamma_3 \cos^2 \varphi)^{3/2}} \sin \varphi \, d\theta d\varphi.$$

Then by a change of variable  $s = \tan \varphi$ , we have  $ds = (1+s^2)d\varphi$ ,  $\cos^2 \varphi = \frac{1}{1+s^2}$  and  $\sin \varphi = \frac{s}{\sqrt{1+s^2}}$ . Therefore

$$\begin{aligned}
J &= 2 \int_0^{+\infty} \int_0^{2\pi} \frac{\gamma_1 \gamma_2 \gamma_3 s}{((\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta) s^2 + \gamma_3)^{3/2}} d\theta ds \\
&= 2\sqrt{\gamma_3} \int_0^{2\pi} \frac{\gamma_1 \gamma_2}{\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta} d\theta = 4\pi \sqrt{\gamma_1 \gamma_2 \gamma_3},
\end{aligned}$$

where we have used the computation made in the 2D case.

## Appendix C: Additional numerical experiments

We provide in this section several additional figures to illustrate, through numerical experiments, the statistical estimation of the anisotropy from the Lipschitz-Killing curvatures of an excursion set.

### C.1. In the 2D case

On Figures 9 and 10 we show error bars on the estimation of the anisotropy from one excursion set of a 2D stationary Gaussian field, and we illustrate the central limit behavior by plotting the standard deviation of the estimation of the anisotropy ratio  $R$  as a function of the observation window size. See the caption of the figures for detailed comments on these numerical experiments.

### C.2. In the 3D case

On Figures 11 and 12, we show some additional 3D experiments in the isotropic case. As in the anisotropic case, the estimation of  $R = R_G = 1$  here is inaccurate when the effective level is close to 0 (for  $R$ ) or close to  $\pm 1$  for  $R_G$ . Now, these two ratios could be combined to accurately estimate the isotropy whatever the effective level.

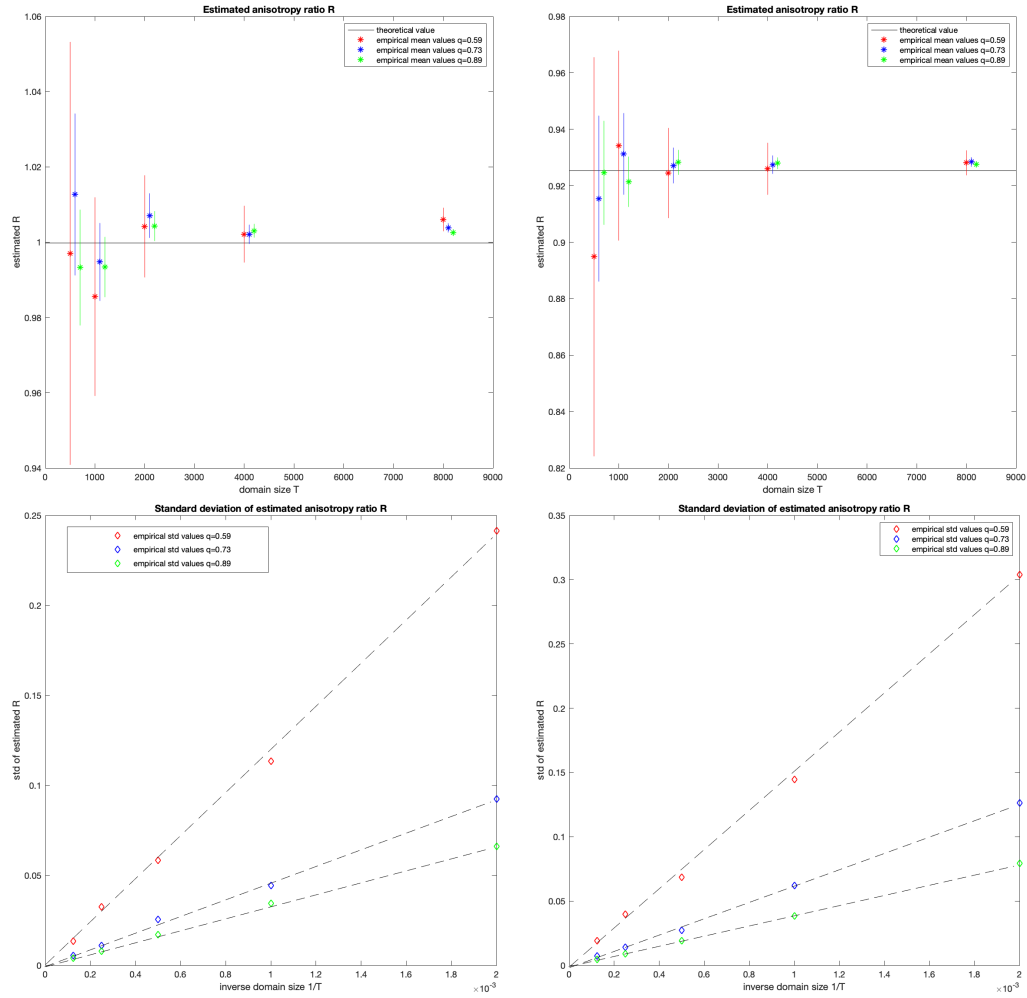


FIG 9. Checking the central limit behavior. First line: estimation of  $R$  in the isotropic case (left) and in the anisotropic one (right, here  $R \simeq 0.92$ ). The estimation is done for three different quantiles (corresponding to the three colors) and on different domain size  $T$  (x-axis of the plot). The confidence intervals have been obtained with 100 samples. Second line: standard deviation of the estimated value of  $R$  in the isotropic case (left) and in the anisotropic one (right), for three different quantiles as a function of the inverse domain size  $1/T$ . This plot shows experimentally that the empirical standard deviation behaves like  $1/T$ , with a slope that depends on the quantile.

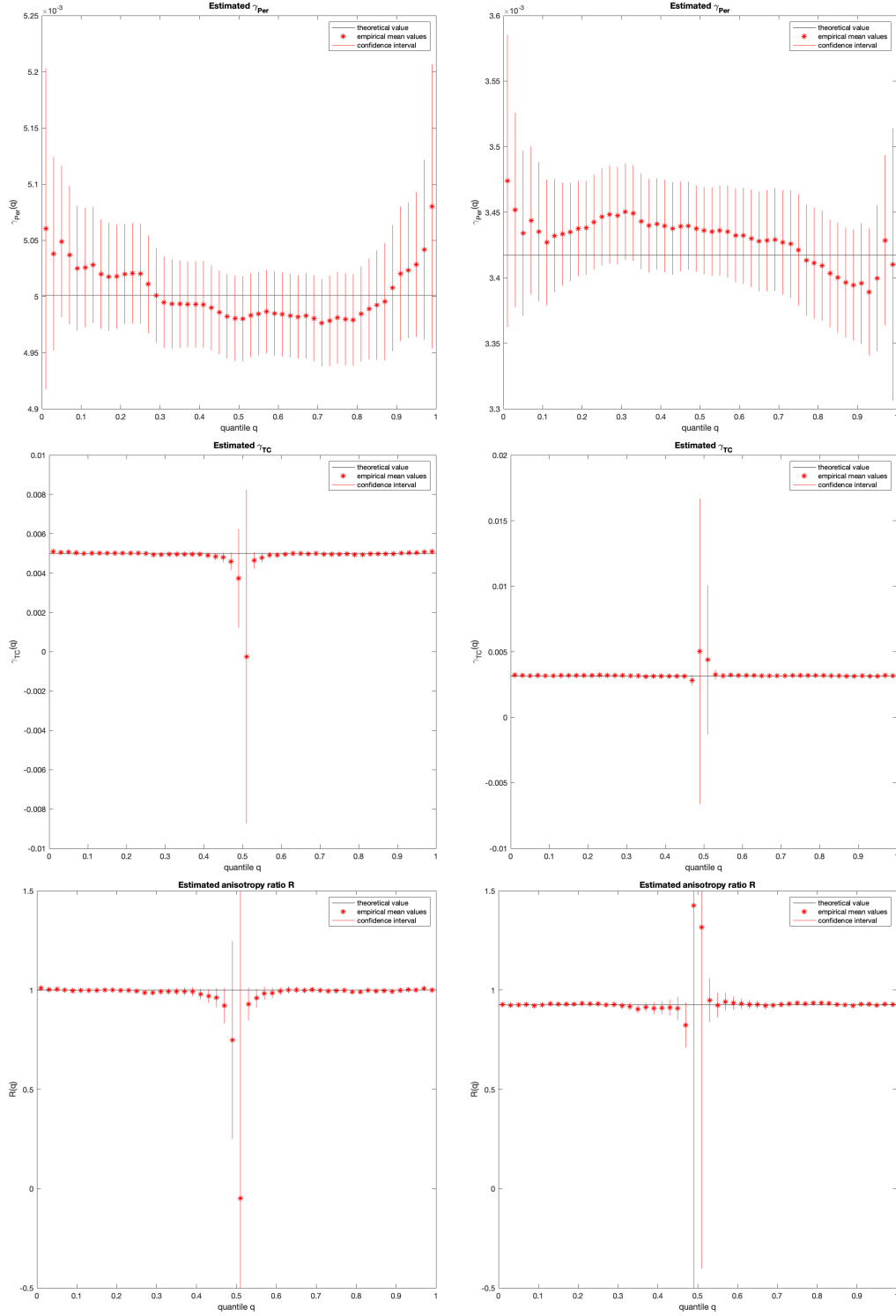


FIG 10. Estimating the anisotropy from one level. We consider here the same two Gaussian random fields as in Figure 4. First line: estimation of  $\bar{\gamma}_{Per}$  from one excursion set of one sample (of size  $1000 \times 1000$ ), as a function of the quantile, in the isotropic case (left figure) and in the anisotropic one (right figure). Second line: same experiment but for the estimation of  $\bar{\gamma}_{TC}$ . Third line: same experiment but for the estimation of the anisotropy ratio  $R$ , that is 1 on the left and 0.92 on the right. The confidence intervals have been obtained with 100 samples. Notice how the estimations of  $\bar{\gamma}_{TC}$  and of  $R$  degenerate around the quantile  $q = 0.5$  since it corresponds to the effective level  $\hat{t}_{eff} = 0$ , and the division by  $\hat{t}_{eff}$  is therefore unstable.

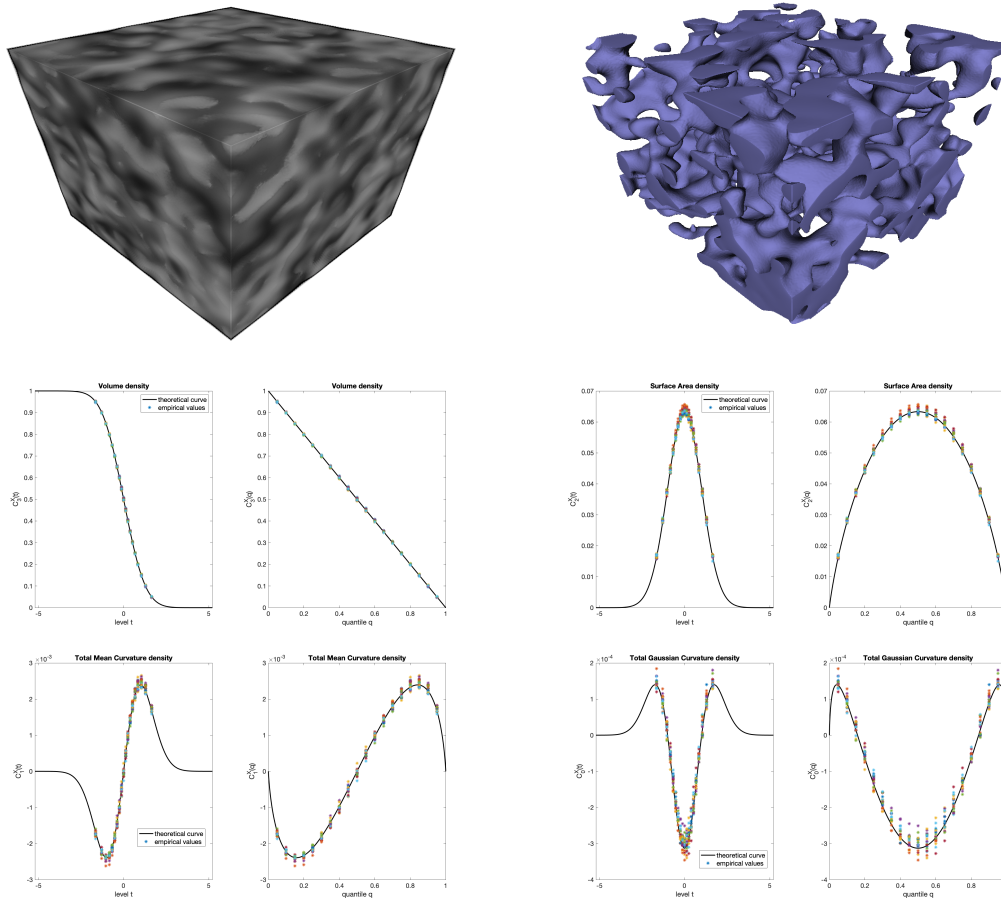


FIG 11. Lipschitz-Killing curvature densities of the excursion sets of an isotropic Gaussian volume. Here we have taken 20 samples of a 3D volume of size  $200^3$  voxels, of a Gaussian field with mean  $m = 0$ , variance  $\sigma^2 = 1$  and covariance of the form  $\exp(-\gamma_1 x_1^2 - \gamma_2 x_2^2 - \gamma_3 x_3^2)$  with here  $\gamma_1 = \gamma_2 = \gamma_3 = 0.01$ . An example of such a Gaussian volume is shown on the left of the first line, with one excursion set on the right. Using the Matlab toolbox *ImMinkowski* [32], we have estimated the volume densities (second line, left), the surface area densities (second line, right), the total mean curvature densities (third line, left) and the total Gaussian curvature densities (third line, right). The densities are plot else as functions of the level  $t$  or as functions of the quantile  $q$ . The stars are the values for the 20 different samples.



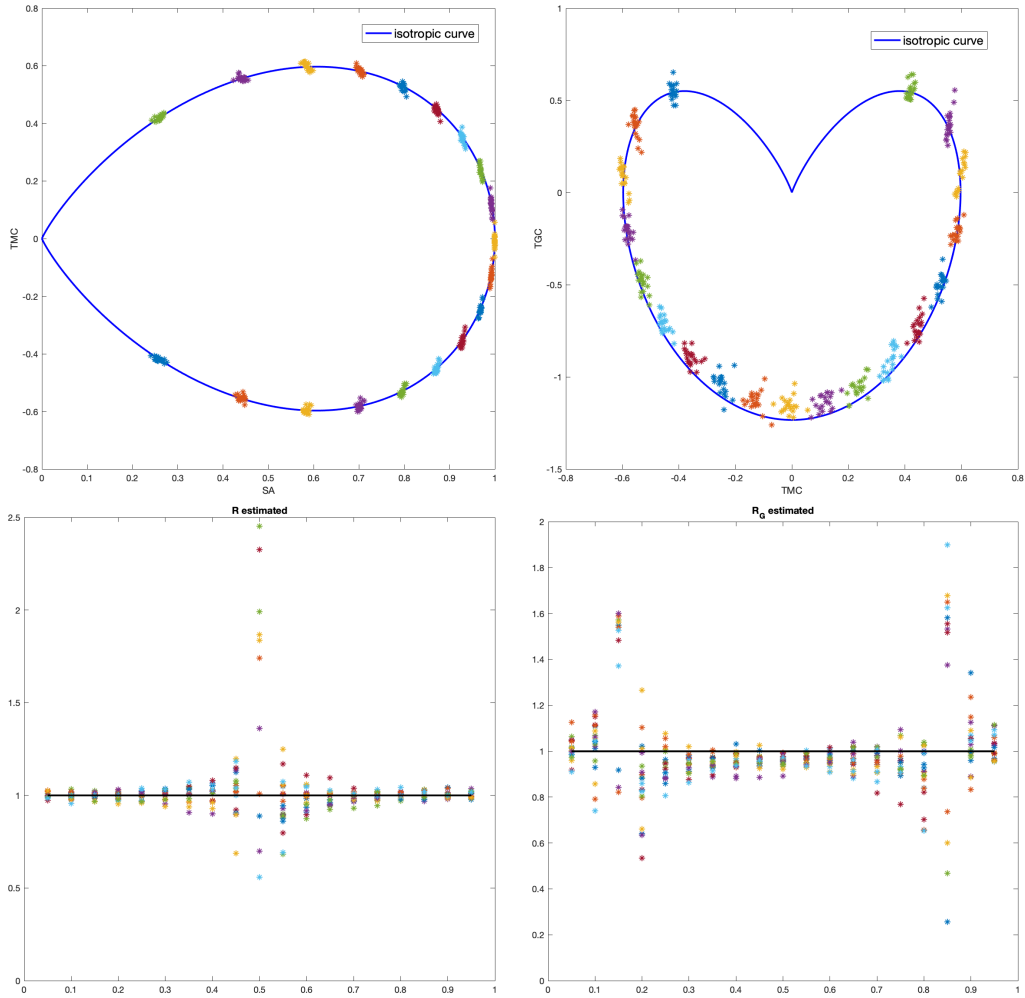


FIG 12. First line: the samples of the previous figure are plotted as points on the almond and heart curves of isotropy. Second line: estimation of  $R$  and  $R_G$  from the Lipschitz-Killing curvature densities of the excursion sets of the samples, as a function of the quantiles.