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The anisotropy of 2D or 3D Gaussian random fields through their Lipschitz-Killing curvature densities

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Abstract: We are interested here in modeling and estimating the anisotropy of 2D and 3D Gaussian random fields through the geometry of their excursion sets. In order to do this, we use Lipschitz-Killing curvatures of the level sets as functions of the levels and see them as generalized processes for which we are able to obtain a joint functional Central Limit Theorem. For 2D and 3D stationary Gaussian fields we provide explicit formulas for the Lipschitz-Killing curvature densities from which we can deduce geometrical equivalent of second spectral moments and anisotropy ratios that allow the estimation of the anisotropy of the underlying Gaussian field.

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Contents

1. Introduction

Since the seminal works of Robert Adler on the geometry of random fields [1] and of Keith Worsley on topological inference in neuroimaging [36], the study of the geometry of excursion sets of random fields has known a growing interest, with important developments especially for the Euler characteristic, used as a good approximation of the tail distribution of the supremum of stationary smooth Gaussian fields [37]. In particular the so-called Gaussian kinematic formula [2] nicely links mean geometries of an excursion set with observation window's ones with respect to the underlying statistical properties of the Gaussian field. However, this formula strongly depends on the metric induced by the stationary Gaussian field, and only mean Euler characteristic can be effectively computed in practice without any prior knowledge on the field. In contrast, in this paper we consider Lipschitz-Killing curvatures of excursion sets computed with the usual Euclidean metric in dimension d for both $d = 2$ and $d = 3$, corresponding to the usual dimensions of medical images. This allows us to rely on numerous results and algorithms developed in stochastic geometry for intrinsic volumes or Minkowski measures closely related to Lipschitz-Killing curvatures [30].

Moreover, as in our previous paper [7], we will have here a "weak" point of view. Instead of fixing a threshold level and considering the Lipschitz-Killing curvatures of the excursion set above this level, we will consider simultaneously all the levels and thanks to a change of variable formula (the coarea formula), we will be able to have a representation of these Lipschitz-Killing curvatures as integrals over the function domain. This point of view also allows us to work on the fine functional framework of generalized random processes [19]. In this setting, we are able to propose consistent and asymptotic (as the size of the observation window grows) Gaussian estimators of the Lipschitz-Killing curvature densities. Note that in sharp contrast with previous results, we do not assume isotropy of the Gaussian field.

Actually, a main point of interest is: how to "read" and "estimate" the anisotropy of a random field from the geometry of (some of) its excursion sets? We provide explicit parametric expressions for Lipschitz-Killing densities in both dimension $d = 2$ and $d = 3$, as well as numerical evaluations. It allows us to define new geometrical equivalent of spectral moments, related through elegant isoperimetric inequalities, as well as robust anisotropy ratios in the sense that they do not depend on the mean, nor on the standard deviation of the field that is crucial for image comparisons. As illustrated on Figure 1 we will summarize the geometry of an excursion set as a (2D here) point and this will allow us to visualize and estimate the anisotropy of the underlying Gaussian field.

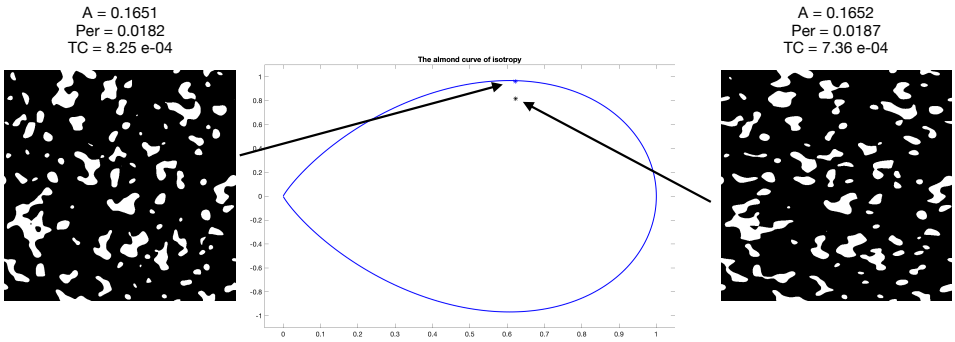

FIG 1. *Two excursion sets of Gaussian random fields. The left one is isotropic and the right one is not. They have the same area, the same perimeter but different total curvature. Their geometry can be summarized as a point that is on the almond curve in the isotropic case and that is inside the almond domain in the anisotropic case.*

The paper is organized as follows. In Section 2 we recall the main definitions of Lipschitz-Killing curvatures following [34], and introduce the Integral Lipschitz-Killing curvatures of a smooth function that allow us to consider Lipschitz-Killing curvatures of its level sets as a tempered distribution (with respect to the levels). Considering a smooth stationary random field we are therefore able to define Lipschitz-Killing curvatures for which a functional joint Central Limit

Theorem is established in dimension $d = 2$ or $d = 3$ under the additional assumption that the underlying field is Gaussian. In contrast with previous Gaussian results [16, 25] based on Itô-Wiener chaos expansion, our proof relies on preliminary general results stated under a quasi-association assumption closer to [11], and which are postponed to the Appendix A. In Section 3, we focus on dimension $d = 2$ and extend the Gaussian isotropic results of [8] to the anisotropic framework. We give explicit formulas for the Lipschitz-Killing curvature densities with respect to two important ordered geometrical equivalent of second spectral moments. This allows us to define an almond curve of anisotropy and a first ratio of anisotropy R with values in $(0, 1]$. We proceed in a simililar way in Section 4 for the dimension $d = 3$. Lipschitz-Killing curvature densities are also expressed with respect to now three important geometrical equivalent of second spectral moments. The ordering of these quantities is much more difficult to obtain than in the 2D case and strongly relies on some isoperimetric inequalities. In addition to the almond curve of anisotropy we define the heart curve of anisotropy involving the third index and get two respective ratios of anisotropy R and $R_G \leq R$, both with values in $(0, 1]$. We illustrate all this in both dimensions $d = 2$ and $d = 3$ through numerical experiments, checking the formulas for the Lipschitz-Killing curvatures, the Central Limit behavior, and the statistical estimation of the anisotropy from the geometry of an excursion set.

2. Lipschitz-Killing curvatures

2.1. Lipschitz-Killing curvatures : the general framework

Let us introduce some notations. In the following we will denote by $\mathcal{L}^d$ or simply by $\mathcal{L}$ (when there is no ambiguity) the Lebesgue measure on $\mathbb{R}^d$ and by $\mathcal{H}^s$ the s-dimensional Hausdorff measure on $\mathbb{R}^d$. We have in particular $\mathcal{H}^d = \mathcal{L}^d$ on $\mathbb{R}^d$.

We now recall the definition of Lipschitz-Killing curvatures in the framework of smooth manifolds of $\mathbb{R}^d$, as they are presented in the paper of Thäle [34]. Let M be a compact d-dimensional submanifold in $\mathbb{R}^d$ with a C^2 smooth boundary ∂M (that is then $(d-1)$-dimensional). For $x \in \partial M$, let $\kappa_i(x)$, $i = 1, \ldots, d-1$, denote the $d-1$ principal curvatures of ∂M at x. Then the Lipschitz-Killing curvatures of M are defined by

$$\forall j = 0, \ldots, d-1, \quad C_j(M) = \frac{1}{\alpha_{d-1-j}} \int_{\partial M} \sigma_{d-1-j}(\kappa_1(x), \ldots, \kappa_{d-1}(x))\, \mathcal{H}^{d-1}(dx), \tag{1}$$

$$\text{and} \quad C_d(M) = \mathcal{L}^d(M),$$

where σ_k is the symmetric elementary function of order k, that is

$$\sigma_k(\kappa_1(x), \ldots, \kappa_{d-1}(x)) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq d-1} \kappa_{i_1}(x) \ldots \kappa_{i_k}(x),$$

with the convention that $\sigma_0 = 1$, and where α_k, $k \geq 0$ is the k-dimensional surface area of the unit ball of $\mathbb{R}^{k+1}$, that is :

$$\alpha_0 = 2, \quad \alpha_1 = 2\pi, \quad \alpha_2 = 4\pi, \quad \text{etc.}$$

The Lipschitz-Killing curvatures have therefore the following interpretations:

- $C_0(M) = \frac{1}{\alpha_{d-1}} \int_{\partial M} \kappa_1(x) \ldots \kappa_{d-1}(x)\, \mathcal{H}^{d-1}(dx) = \chi(M)$ is the Euler Characteristic of M, that is related to the integral of the Gaussian curvature on ∂M by the Gauss-Bonnet Theorem.
- $C_{d-2}(M) = \frac{1}{\alpha_1} \int_{\partial M} (\kappa_1(x) + \ldots + \kappa_{d-1}(x))\, \mathcal{H}^{d-1}(dx)$. When $d \geq 3$ it is equal to $\frac{d-1}{2\pi} b(M)$ where $b(M)$ is the mean breadth of M defined as the integral of the mean curvature on ∂M.
- $C_{d-1}(M) = \frac{1}{\alpha_0} \int_{\partial M} \mathcal{H}^{d-1}(dx) = \frac{1}{2}\mathcal{H}^{d-1}(\partial M)$ is half the surface area (perimeter when $d = 2$) of ∂M in $\mathbb{R}^d$.

- $C_d(M) = \mathcal{L}^d(M)$ is the volume of M in $\mathbb{R}^d$.

Let U be an open bounded set in $\mathbb{R}^d$. Then, the Lipschitz-Killing curvatures can be localized in U: they are then denoted by $C_j(M, U)$ and their formula involving the principal curvatures are analogous to the ones of the $C_j(M)$ given by (1), except that the integrals on ∂M are replaced by integrals on $\partial M \cap U$.

The Lipschitz-Killing curvatures are related to the volume of the r-parallel set to M through the Weyl's tube formula. This point will be discussed in the further remarks of Section 5.2.

2.2. The framework of smooth functions

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function defined on $\mathbb{R}^d$, with here $d \geq 2$. For $t \in \mathbb{R}$, the excursion set (or level set) of f above the level t is given by

$$E_f(t) := \{x \in \mathbb{R}^d ;\, f(x) \geq t\}.$$

Let U be an open bounded subset of $\mathbb{R}^d$, for instance $U = (0, T)^d$ with $T > 0$. Let $t \in \mathbb{R}$, and $x \in U$ such that $f(x) = t$ and $\nabla f(x) \neq 0$. Then $\nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|} \in S^{d-1}$ is the unit inner normal vector of $E_f(t)$ at $x \in \partial E_f(t)$. Now in an open neighborhood of such $x \in U$ we can compute $D\nu_f$ using $D^2 f$, the Hessian matrix of f, and have

$$D\nu_f(x) = \frac{1}{\|\nabla f(x)\|} D^2 f(x) - \frac{1}{\|\nabla f(x)\|^3} \nabla f(x)(D^2 f(x) \nabla f(x))^t.$$

As the principal curvatures are the eigenvalues of the second fundamental form, it follows that the d symmetric functions of the curvatures $\sigma_{d-1-j}(\kappa_1(x), \ldots, \kappa_{d-1}(x))$ may be explicitly computed with formulas involving $\mathbf{f}(x) := \big(f(x), \nabla f(x), D^2 f(x)\big)$ through

$$\sigma_{d-1-j}(\kappa_1(x), \ldots, \kappa_{d-1}(x)) = \frac{1}{\|\nabla f(x)\|} F_{d-1-j}(\mathbf{f}(x)), \tag{2}$$

with in particular (see [33] or [20] for instance)

$$F_0(\mathbf{f}(x)) = \|\nabla f(x)\|, \tag{3}$$

$$F_1(\mathbf{f}(x)) = -\mathrm{Tr}(D^2 f(x)) + \frac{\nabla f(x)^t D^2 f(x) \nabla f(x)}{\|\nabla f(x)\|^2}, \tag{4}$$

and for $d \geq 3$

$$F_{d-1}(\mathbf{f}(x)) = (-1)^{d-1} \frac{\nabla f(x)^t \mathrm{adj}(D^2 f(x)) \nabla f(x)}{\|\nabla f(x)\|^d}, \tag{5}$$

where $\mathrm{adj}(D^2 f(x))$ is the adjugate matrix of $D^2 f(x)$ (that is the transpose of its cofactor matrix). Note that for $d = 2$, one can also express F_1 as

$$F_1(\mathbf{f}(x)) = -\frac{(\nabla f(x)^\perp)^t D^2 f(x) \nabla f(x)^\perp}{\|\nabla f(x)\|^2}, \tag{6}$$

with $\nabla f(x)^\perp = (-\partial_2 f(x), \partial_1 f(x))$.

Now, considering $A = \{x \in U; \|\nabla f(x)\| = 0\}$, the coarea formula (see [17] p.112 for instance) states that

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}\left(A \cap f^{-1}(t)\right) dt = \int_A \|\nabla f(x)\| dx = 0.$$

Hence, for a.e. $t \in \mathbb{R}$ and $\mathcal{H}^{d-1}$ almost every $x \in \partial E_f(t) \cap U$ we have $f(x) = t$ and $\nabla f(x) \neq 0$. Under an additional assumption this will allow us to define for a.e. $t \in \mathbb{R}$

$$C_j(E_f(t), U) = \frac{1}{\alpha_{d-1-j}} \int_{\partial E_f(t) \cap U} \frac{1}{\|\nabla f(x)\|} F_{d-1-j}(\mathbf{f}(x)) \mathcal{H}^{d-1}(dx),$$

such that $t \mapsto C_j(E_f(t), U)$ is $\mathcal{L}^1$-integrable (in the sense that it coincides a.e. with a measurable integrable function on $\mathbb{R}$) and therefore to consider it as a tempered distribution. More precisely, let us recall that the *Schwartz space* $\mathcal{S} = \mathcal{S}(\mathbb{R})$ consists of infinitely differentiable functions $h : \mathbb{R} \to \mathbb{R}$ that are rapidly decreasing, that is, for all $q \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\|h\|_{q,k} = \sup_{t\in\mathbb{R}} (1 + |t|)^q |h^{(k)}(t)| < \infty,$$

where $h^{(k)}$ denotes the derivative of order k. It is a real vector space and it is equipped with the topology given by the family of semi-norms $\| \cdot \|_{q,k}$, $q \in \mathbb{N}$ and $k \in \mathbb{N}$. It is well known that $\mathcal{S}$ is a Fréchet space (i.e. a locally convex metrizable complete space). The *space of tempered distributions* $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ is the topological dual of $\mathcal{S}$, that is the space of continuous linear functionals on $\mathcal{S}$. Now for $h \in \mathcal{S}$ we are interested in the Integral Lipschitz-Killing Curvatures of f defined by

$$\mathrm{LC}_j^f(h, U) := \int_{\mathbb{R}} h(t) C_j(E_f(t), U)\, dt. \tag{7}$$

Proposition 1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function and note $\mathbf{f}(x) := \big(f(x), \nabla f(x), D^2 f(x)\big)$ for $x \in \mathbb{R}^d$. Let U be a bounded open set of $\mathbb{R}^d$ and assume that, for $k \in \{0, 1, d-1\}$,*

$$\int_U |F_k(\mathbf{f}(x))| \mathbb{1}_{\|\nabla f(x)\|>0}\, dx < +\infty, \tag{8}$$

with F_k given by (3), (4) and (5). Then for $j \in \{0, d-2, d-1\}$, one has $C_j(E_f(\cdot), U) \in \mathcal{L}^1(\mathbb{R})$. Moreover, for $j \in \{0, d-2, d-1, d\}$, one has $C_j(E_f(\cdot), U) \in \mathcal{S}'$ with for all $h \in \mathcal{S}$

$$\mathrm{LC}_j^f(h, U) = \langle C_j(E_f(\cdot), U), h\rangle = \int_U L_j(h, \mathbf{f}(x)) dx, \tag{9}$$

where $h \in \mathcal{S} \mapsto L_j(h, \mathbf{f}(x)) \in \mathbb{R}$ is linear continuous for all $x \in U$, and given by

$$L_j(h, \mathbf{f}(x)) = \frac{1}{\alpha_{d-1-j}} h(f(x)) F_{d-1-j}(\mathbf{f}(x)) \mathbb{1}_{\|\nabla f(x)\|>0} \quad \textit{for} j \neq d,$$

and

$$L_d(h, \mathbf{f}(x)) = \int_{\mathbb{R}} h(t) \mathbb{1}_{f(x)\geq t} dt.$$

Proof. Let us recall the coarea formula for Lipschitz mappings (see [17] p.117 for instance): for any non-negative $\mathcal{L}^d$-measurable or any $\mathcal{L}^d$-integrable function $g : \mathbb{R}^d \to \mathbb{R}$, the function $t \mapsto \int_{\partial E_f(t)\cap U} g(x) \mathcal{H}^{d-1}(dx)$ is $\mathcal{L}^1$-measurable and

$$\int_U g(x) \|\nabla f(x)\|\, dx = \int_{\mathbb{R}} \int_{\partial E_f(t)\cap U} g(x) \mathcal{H}^{d-1}(dx)\, dt.$$

Hence for $k \in \{0, 1, d-1\}$, and h a continuous bounded function we can define measurable functions on U by

$$g_k(x) = h(f(x)) \frac{F_k(\mathbf{f}(x))}{\|\nabla f(x)\|} \mathbb{1}_{\|\nabla f(x)\|>0} \quad \text{for } x \in U.$$

Under the assumption (8) we obtain that

$$\int_{\mathbb{R}} \int_{\partial E_f(t)\cap U} |g_k(x)| \mathcal{H}^{d-1}(dx)\, dt \leq \|h\|_{0,0} \int_U |F_k(\mathbf{f}(x))| \mathbb{1}_{\|\nabla f(x)\|>0}\, dx < +\infty.$$

Moreover, using again the coarea formula with $\max(g_k, 0)$ and $-\min(g_k, 0)$ and subtracting we get

$$\int_{\mathbb{R}} \int_{\partial E_f(t)\cap U} g_k(x) \mathcal{H}^{d-1}(dx)\, dt = \int_U h(f(x)) F_k(\mathbf{f}(x)) \mathbb{1}_{\|\nabla f(x)\|>0}\, dx.$$

It follows that $t \mapsto \int_{\partial E_f(t)\cap U} g_k(x)\mathcal{H}^{d-1}(dx)$ is $\mathcal{L}^1$-integrable. Then, using the definition of the F_k in (2), we have for a.e. $t \in \mathbb{R}$

$$\begin{aligned}\int_{\partial E_f(t)\cap U} g_k(x)\mathcal{H}^{d-1}(dx) &= h(t)\int_{\partial E_f(t)\cap U} \sigma_k(\kappa_1(x),\dots,\kappa_{d-1}(x))\mathbb{1}_{\|\nabla f(x)\|>0}\mathcal{H}^{d-1}(dx)\,dt \\ &= h(t)\int_{\partial E_f(t)\cap U} \sigma_k(\kappa_1(x),\dots,\kappa_{d-1}(x))\mathcal{H}^{d-1}(dx)\,dt \\ &= \alpha_{d-1-k}C_{d-1-k}(E_f(t),U)\,h(t),\end{aligned}$$

since for a.e. $t \in \mathbb{R}$ and $\mathcal{H}^{d-1}$ a.e. $x \in \partial E_f(t)\cap U$ we have $f(x) = t$ and $\nabla f(x) \neq 0$. This implies (taking $h = 1$) that $t \mapsto C_j(E_f(t), U)$ is $\mathcal{L}^1$-integrable for $j \in \{0, d-2, d-1\}$ and in view of (7),

$$\mathrm{LC}_j^f(h,U) = \int_U L_j(h,\mathbf{f}(x))dx,$$

with $L_j(h,\mathbf{f}(x)) = \frac{1}{\alpha_{d-1-j}}h(f(x))F_{d-1-j}(\mathbf{f}(x))\mathbb{1}_{\|\nabla f(x)\|>0}$. Hence we have also $h \mapsto L_j(h,\mathbf{f}(x)) \in \mathcal{S}'$ with

$$|L_j(h,\mathbf{f}(x))| \leq \frac{1}{\alpha_{d-1-j}}\|h\|_{0,0}|F_{d-1-j}(\mathbf{f}(x))|\mathbb{1}_{\|\nabla f(x)\|>0}.$$

Finally, for $j = d$ we can also define

$$L_d(h,\mathbf{f}(x)) = \int_{\mathbb{R}} h(t)\mathbb{1}_{f(x)\geq t}\,dt,$$

satisfying $|L_d(h,\mathbf{f}(x))| \leq 2\|h\|_{2,0}$, using the fact that $\int_{\mathbb{R}}(1+|t|)^{-2}dt = 2$. Since U is bounded we also have $h \mapsto \mathrm{LC}_d^f(h,U) = \int_U L_d(h,\mathbf{f}(x))\,dx \ \in \mathcal{S}'$, with, by Fubini's Theorem,

$$\mathrm{LC}_d^f(h,U) = \int_{\mathbb{R}} h(t)C_d(E_f(t),U)\,dt.$$

□

2.3. Smooth stationary random fields and Lipschitz-Killing curvature densities

We extend now the Lipschitz-Killing curvatures integrals to the case of smooth stationary random fields.

Proposition 2. *Let $(\Omega,\mathcal{A},\mathbb{P})$ be a complete probability space and $(X(x))_{x\in\mathbb{R}^d}$ be a stationary real-valued random field that is almost surely (a.s.) a C^2 function on $\mathbb{R}^d$. Assume moreover that*

$$\forall k \in \{0,1,d-1\}, \quad \mathbb{E}\left(|F_k(\mathbf{X}(x))|\mathbb{1}_{\|\nabla X(x)\|>0}\right) < +\infty, \tag{10}$$

for F_k given by (3), (4) and (5), and $\mathbf{X}(x) = (X(x),\nabla X(x),D^2X(x))$, $x \in \mathbb{R}^d$. Let U be a bounded open set of $\mathbb{R}^d$. Then we can define $C_j(E_X(\cdot),U)$ as a generalized integrable process for $j \in \{0,d-2,d-1,d\}$, in the sense that

$$C_j(E_X(\cdot),U) : (\Omega,\mathcal{A}) \to (\mathcal{S}',\mathcal{B}(\mathcal{S}'))$$

is measurable, where $\mathcal{B}(\mathcal{S}')$ is the Borel σ-field of the strong topology on $\mathcal{S}'$ and, for all $h \in \mathcal{S}$,

$$\mathbb{E}\left(|\mathrm{LC}_j^X(h,U)|\right) \leq \int_{\mathbb{R}} |h(t)|\mathbb{E}\left(|C_j(E_X(t),U)|\right)dt < +\infty.$$

Proof. First note that since $(\Omega,\mathcal{A},\mathbb{P})$ is a complete probability space, in view of the a.s. continuity, up to setting the values of X at 0 on a negligible set, the field $X : \Omega\times\mathbb{R}^d \to \mathbb{R}$ is a $\mathbb{P}\otimes\mathcal{L}^d$-measurable function, as well as its first and second order partial derivatives. By stationarity,

we deduce from (10) that $\int_U |F_k(\mathbf{X}(x))|\mathbb{I}_{\|\nabla X(x))\|>0}\,dx < +\infty$ a.s. and (8) is satisfied for $k \in \{0,1,d-1\}$. Hence, there exists a negligible set N of $\mathcal{A}$ such that for $j \in \{0,d-2,d-1\}$, $C_j(E_{X(\omega)}(\cdot),U) \in L^1(\mathbb{R}) \subset \mathcal{S}'$ for $\omega \notin N$ and we can set $C_j(E_{X(\omega)}(\cdot),U) = 0$ for $\omega \in N$. We can therefore define $\mathrm{LC}_j^{X(\omega)}(h,U)$, for $h \in \mathcal{S}$, satisfying (9) for $\omega \notin N$ and $\mathrm{LC}_j^{X(\omega)}(h,U) = 0$ for $\omega \in N$. Using the fact that $(\omega,x) \mapsto L_j(h,\mathbf{X}(\omega,x))$ is $\mathbb{P}\otimes\mathcal{L}^d$-measurable, it follows by Fubini's theorem that $\mathrm{LC}_j^X(h,U) : \omega \in \Omega \mapsto \mathrm{LC}_j^{X(\omega)}(h,U) \in \mathbb{R}$ is a well-defined real valued random variable. Therefore, $(\mathrm{LC}_j^X(h_1,U),\dots,\mathrm{LC}_j^X(h_m,U))$ is a random vector for all $m \geq 1$ and $h_1,\dots,h_m \in \mathcal{S}$. Since the cylinder σ-field coincides with the Borel σ-field of the weak topology but also of the strong topology on $\mathcal{S}'$ denoted by $\mathcal{B}(\mathcal{S}')$ (see Corollary 3.9 of [9]) we can deduce that $C_j(E_X(\cdot),U) : (\Omega,\mathcal{A}) \to (\mathcal{S}',\mathcal{B}(\mathcal{S}'))$ is measurable and therefore defines a generalized random process. To conclude for integrability, it is enough to remark that by Fubini's theorem and the coarea formula

$$\begin{aligned}\mathbb{E}\left(|\mathrm{LC}_j^X(h,U)|\right) &\leq \int_{\mathbb{R}} |h(t)|\mathbb{E}\left(|C_j(E_X(t),U)|\right)dt \\ &\leq \frac{\|h\|_{0,0}}{\alpha_{d-1-j}}\int_U \mathbb{E}\left(|F_{d-1-j}(\mathbf{X}(x))|\mathbb{I}_{\|\nabla X(x)\|>0}\right)dx < +\infty,\end{aligned}$$

by (10), using the stationarity of X and the boundedness of U. The result for $j = d$ follows the same lines as $(\omega,x) \mapsto L_d(h,\mathbf{X}(\omega,x))$ is $\mathbb{P}\otimes\mathcal{L}^d$-measurable and bounded by $2\|h\|_{2,0}$. □

Thanks to the stationarity we will focus on Lipschitz-Killing curvature densities defined for $j \in \{0,d-2,d-1\}$ by

$$\begin{aligned}\langle \overline{C}_j^X, h\rangle = \int_{\mathbb{R}} h(t)\overline{C}_j^X(t)dt = \frac{\mathbb{E}\left(\mathrm{LC}_j^X(h,U)\right)}{\mathcal{L}^d(U)} \\ = \frac{1}{\alpha_{d-1-j}}\mathbb{E}\left(h(X(x))F_{d-1-j}(\mathbf{X}(x))\mathbb{I}_{\|\nabla X(x)\|>0}\right). \quad (11)\end{aligned}$$

Note that $\overline{C}_j^X \in L^1(\mathbb{R}) \subset \mathcal{S}'$ and that for a.e. $t \in \mathbb{R}$ we have

$$\overline{C}_j^X(t) = \frac{\mathbb{E}\left(C_j(E_X(t),U)\right)}{\mathcal{L}^d(U)}.$$

We can define in a simpler way $\overline{C}_d^X(t) = \frac{\mathbb{E}(C_d(E_X(t),U))}{\mathcal{L}^d(U)} = \mathbb{P}(X(x) \geq t)$ for all $t \in \mathbb{R}$, such that $\overline{C}_d^X \in \mathcal{S}'$ (but not in $L^1(\mathbb{R})$) with

$$\langle \overline{C}_d^X, h\rangle = \int_{\mathbb{R}} h(t)\overline{C}_d^X(t)dt = \frac{\mathbb{E}\left(\mathrm{LC}_d^X(h,U)\right)}{\mathcal{L}^d(U)}.$$

To ensure sample paths smoothness, a simple assumption for stationary Gaussian random fields may be given in term of covariance functions. More precisely, when X is a stationary Gaussian field with mean m and variance σ^2, we denote by ρ its covariance function, given by

$$\rho(x) = \mathrm{Cov}(X(x),X(0)) = \mathrm{Cov}(X(x+y),X(y)),$$

for all $x,y \in \mathbb{R}^d$ by stationarity. Then it is sufficient to make the following hypothesis, denoted by **H** :

$$\begin{aligned}&\rho : \mathbb{R}^d \to \mathbb{R} \text{ is a } C^4 \text{ positive definite function, with } \det(D^2\rho(0)) \neq 0 \\ &\quad \text{and such that there exist } C > 0 \text{ and } \delta > 0 \text{ with } \left|\partial^{\mathbf{k}}\rho(x) - \partial^{\mathbf{k}}\rho(0)\right| \leq C\|x\|^\delta, \\ &\quad \text{for all } \mathbf{k} = (k_1,\dots,k_d) \in \mathbb{N}^d \text{ with } |\mathbf{k}| = \sum_{l=1}^d k_l = 4. \quad \textbf{(H)}\end{aligned}$$

Corollary 1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and $(X(x))_{x\in\mathbb{R}^d}$ be a stationary Gaussian random field whose covariance function satisfies* **H**. *Then, there exists a modification of X such that the assumptions of Proposition 2 are satisfied. Moreover for $d \in \{2,3\}$, $0 \le j \le d$ and $h \in \mathcal{S}$, the random variable $\mathrm{LC}_j^X(h,U)$ is square integrable.*

Proof. By Proposition 2.1 of [12], under **H**, there exists a modification of X such that X is C^2 a.s. Note that in this case $\nabla X(x)$ is an $\mathbb{R}^d$-valued Gaussian vector of covariance given by $\Gamma = -D^2\rho(0)$ and therefore $\|\nabla X(x)\| > 0$ a.s. with $\mathbb{E}(\|\nabla X(x)\|^{p(2-d)}) < +\infty$ for all $p \ge 1$ such that $p(d-2) < d$. In view of (3), (4), or (5) we have therefore for all $k \in \{0, 1, d-1\}$,

$$\mathbb{E}\left(|F_k(\mathbf{X}(x))|\right) < +\infty,$$

and (10) is satisfied. Moreover, for $d \in \{2,3\}$, we also have $\mathbb{E}\left(F_k(\mathbf{X}(x))^2\right) < +\infty$, and $\mathrm{LC}_{d-1-k}^X(h,U)$ is a square integrable random variable. The result for $\mathrm{LC}_d^X(h,U)$ simply comes from the fact that $C_d(E_X(t),U)$ is a bounded random variable. □

In this Gaussian setting, we obtain the following result, whose technical proof is postponed to Appendix A.

Theorem 2.1. *Let X be a stationary Gaussian field defined on $\mathbb{R}^d$ for $d \in \{2,3\}$, whose covariance function satisfies* **H** *and assume moreover that there exists $\beta > 12 + d$ such that*

$$\max_{|\mathbf{k}|\le 4} \left|\partial^{\mathbf{k}}\rho(x)\right| \le C(1+|x|)^{-\beta}. \tag{12}$$

Then for $U_n = (0,n)^d$ one has

$$\left(\sqrt{\mathcal{L}^d(U_n)}\left(\frac{C_j(E_X(\cdot),U_n)}{\mathcal{L}^d(U_n)} - \overline{C}_j^X\right)\right)_{0\le j\le d} \overset{distrib.}{\underset{n\to+\infty}{\longrightarrow}} (B_j)_{0\le j\le d},$$

where $(B_j)_{0\le j\le d}$ are centered Gaussian generalized random processes whose covariance functions are given for all $h, \tilde{h} \in \mathcal{S}$ and $0 \le j,k \le d$, by

$$\mathrm{Cov}\left(\langle B_j, h\rangle, \langle B_k, \tilde{h}\rangle\right) = \int_{\mathbb{R}^d} \mathrm{Cov}\left(L_j(h,\mathbf{X}(z)), L_k(\tilde{h},\mathbf{X}(0))\right) dz.$$

Here the convergence holds in distribution with respect to the strong topology of $\mathcal{S}'(\mathbb{R})$.

Let us mention that the decay assumption (12) could be relaxed into $\max_{|\mathbf{k}|\le 4}\left|\partial^{\mathbf{k}}\rho\right| \in L^1(\mathbb{R}^d)$ for Gaussian fields, as assumed in [16], but here the proof of our theorem relies on an intermediate result given under a more general assumption of quasi-associated field (not necessarily Gaussian).

Now, let us be more explicit about the covariances involved in Theorem 2.1. Note first that for all $h, \tilde{h} \in \mathcal{S}$,

$$\mathrm{Cov}\left(L_d(h,\mathbf{X}(z)), L_d(\tilde{h},\mathbf{X}(0))\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(s)\tilde{h}(t)\mathrm{Cov}(\mathbb{1}_{X(z)\ge s}, \mathbb{1}_{X(0)\ge t})\, ds\, dt.$$

Then, introducing for $z \in \mathbb{R}^d$,

$$\Sigma_{dd}^{st}(z) := \mathrm{Cov}(\mathbb{1}_{X(z)\ge s}, \mathbb{1}_{X(0)\ge t}), \tag{13}$$

we obtain

$$\mathrm{Cov}\left(L_d(h,\mathbf{X}(z)), L_d(\tilde{h},\mathbf{X}(0))\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(s)\tilde{h}(t)\Sigma_{dd}^{st}(z) ds\, dt.$$

Furthermore, for the other covariances, we will also be able to write them as

$$\mathrm{Cov}\left(L_j(h,\mathbf{X}(z)), L_k(\tilde{h},\mathbf{X}(0))\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} h(s)\tilde{h}(t)\Sigma_{jk}^{st}(z) ds\, dt.$$

Actually, by stationarity, for $j < d$,

$$\mathbb{E}\left(L_j(h, \mathbf{X}(z))\right) = \mathbb{E}\left(L_j(h, \mathbf{X}(0)\right) = \frac{1}{\alpha_{d-1-j}} \int_{\mathbb{R}} h(t) \mathbb{E}\left(F_j(\mathbf{X}(0))|X(0) = t\right) p_{X(0)}(t) dt,$$

where $p_{X(0)}$ denotes the density of the Gaussian variable $X(0)$, and

$$\begin{aligned}\mathbb{E}\left(L_j(h, \mathbf{X}(z)) L_d(\tilde{h}, \mathbf{X}(0))\right) &= \int_{\mathbb{R}} \tilde{h}(t) \mathbb{E}\left(h(X(z)) \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(z))}{\alpha_{d-1-j}} \mathbb{I}_{X(0)\geq t}|X(z)\right)\right) dt \\ &= \int_{\mathbb{R}^2} \tilde{h}(t) h(s) \left[\mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(z))}{\alpha_{d-1-j}} \mathbb{I}_{X(0)\geq t}|X(z) = s\right) p_{X(z)}(s)\right] ds\, dt.\end{aligned}$$

Then we can set

$$\Sigma_{jd}^{st}(z) := \left[\mathbb{E}\left(F_{d-1-j}(\mathbf{X}(z)) \mathbb{I}_{X(0)\geq t}|X(z) = s\right) - \mathbb{E}\left(F_{d-1-j}(\mathbf{X}(0))|X(0) = s\right) \mathbb{P}(X(0) \geq t)\right] \frac{p_{X(0)}(s)}{\alpha_{d-1-j}}. \tag{14}$$

And similarly, for $k < d$ and $z \neq 0$, we set

$$\begin{aligned}\Sigma_{jk}^{st}(z) := \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(z))}{\alpha_{d-1-j}} \frac{F_{d-1-k}(\mathbf{X}(0))}{\alpha_{d-1-k}}|X(0) = t, X(z) = s\right) p_{(X(0),X(z))}(t, s) \\ - \mathbb{E}\left(\frac{F_{d-1-j}(\mathbf{X}(0))}{\alpha_{d-1-j}}|X(0) = s\right) \mathbb{E}\left(\frac{F_{d-1-k}(\mathbf{X}(0))}{\alpha_{d-1-k}}|X(0) = t\right) p_{X(0)}(t) p_{X(0)}(s),\end{aligned} \tag{15}$$

denoting by $p_{(X(0),X(z))}$ the joint density of $(X(0), X(z))$. Note that (12) implies that ρ is an integrable function and therefore X admits a spectral density, thus implying $|\rho(z)| < \rho(0) = \sigma^2$ for all $z \neq 0$. This ensures the existence of $p_{(X(0),X(z))}$ as soon as $z \neq 0$. Now, if we could justify the interchange of integrals we should also obtain

$$\mathrm{Cov}\left(\langle B_j, h\rangle, \langle B_k, \tilde{h}\rangle\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{h}(t) h(s) \Sigma_{jk}(s, t)\, ds\, dt,$$

with

$$\Sigma_{jk}(s, t) = \int_{\mathbb{R}^d} \Sigma_{jk}^{st}(z) dz,$$

allowing us to give a pointwise representation of the Gaussian generalized processes obtained at the limit. This is precisely the purpose of the following corollary.

Corollary 2. *Under the assumptions of Theorem 2.1, assuming moreover that for $J \subset \{0, \ldots, d\}$ and V an open interval of $\mathbb{R}$, we have:*

(**A**1) *The map $t \in V \mapsto C_j^X(t, U_n) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ is continuous for all $n \geq 1$ and $j \in J$;*

(**A**2) *For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t, s \in V$ with $|t - s| \leq \delta$, for all $n \geq 1$ and $j \in J$,*

$$Var\left(C_j^X(t, U_n) - C_j^X(s, U_n)\right) \leq \varepsilon \mathcal{L}^d(U_n);$$

(**A**3) *For all $j, k \in J$, for all $(s, t) \in V \times V$ we have $\int_{\mathbb{R}^d} |\Sigma_{jk}^{st}(z)| dz < +\infty$, and the function $(s, t) \in V \times V \mapsto \Sigma_{jk}(s, t) := \int_{\mathbb{R}^d} \Sigma_{jk}^{st}(z) dz$ is continuous.*

Then

$$\left(\sqrt{\mathcal{L}^d(U_n)} \left(\frac{C_j(E_X(t, U_n))}{\mathcal{L}^d(U_n)} - \overline{C}_j^X(t)\right)\right)_{j\in J, t\in V} \xrightarrow[n\to+\infty]{fdd} \left(\tilde{B}_j(t)\right)_{j\in J, t\in V},$$

where $\left(\tilde{B}_j(t)\right)_{j\in J, t\in V}$ is a centered Gaussian process with covariance given by

$$\mathrm{Cov}(\tilde{B}_j(s), \tilde{B}_k(t)) = \Sigma_{jk}(s, t).$$

Proof. For sake of simplicity, we only sketch the proof for $k = j \in J$ and $s = t \in V$. Let us first choose $h \in \mathcal{S}$ non-negative with compact support such that $\int_{\mathbb{R}} h(u)du = 1$, as well as $(h_m)_{m\geq 1}$ its associated approximation of the identity, given by $h_m(u) = mh(mu)$. We set $\tau_t\check{h}(u) = h(t-u)$ and may assume that $\tau_t\check{h}_m$ has support in V for all $m \geq 1$. Then for all $m, p \geq 1$,

$$\mathrm{Cov}\left(\langle B_j, \tau_t\check{h}_m\rangle, \langle B_j, \tau_t\check{h}_p\rangle\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_t\check{h}_m(u)\tau_t\check{h}_p(v)\Sigma_{jj}(u,v)\,du\,dv \underset{m,p\to+\infty}{\longrightarrow} \Sigma_{jj}(t,t),$$

using Fubini's theorem and continuity, thanks to (**A**3). Then $\left(\langle B_j, \tau_t\check{h}_m\rangle\right)_m$ is a Cauchy sequence of Gaussian variables in L^2 and we can set $\tilde{B}_j(t)$ its Gaussian limit. Note that therefore we also have $\mathrm{Var}\left(\tilde{B}_j(t)\right) = \Sigma_{jj}(t,t)$. Moreover, since $C_j(E_X(t,U_n)) \in L^2(\Omega,\mathcal{A},\mathbb{P})$ by (**A**1), we have

$$\mathbb{E}\left(\left[\sqrt{\mathcal{L}^d(U_n)}\left(\frac{C_j(E_X(t,U_n))}{\mathcal{L}^d(U_n)} - \overline{C}_j^X(t)\right) - \sqrt{\mathcal{L}^d(U_n)}\left(\frac{\mathrm{LC}_j^X(\tau_t\check{h}_m,U_n)}{\mathcal{L}^d(U_n)} - \langle\overline{C}_j^X, \tau_t\check{h}_m\rangle\right)\right]^2\right)$$
$$= \frac{1}{\mathcal{L}^d(U_n)}\mathrm{Var}\left(C_j(E_X(t),U_n)) - \mathrm{LC}_j^X(\tau_t\check{h}_m,U_n)\right).$$

But

$$C_j(E_X(t),U_n) - \mathrm{LC}_j^X(\tau_t\check{h}_m,U_n) = \int_{\mathbb{R}} h(u)\left(C_j(E_X(t),U_n) - C_j\left(E_X\left(t-\frac{u}{m}\right),U_n\right)\right)du.$$

Therefore,

$$\mathrm{Var}\left(C_j(E_X(t),U_n)) - \mathrm{LC}_j^X(\tau_t\check{h}_m,U_n)\right) = \int_{\mathbb{R}^2} h(u)h(v)...$$
$$\mathrm{Cov}\left(C_j(E_X(t),U_n) - C_j\left(E_X\left(t-\frac{u}{m}\right),U_n\right), C_j(E_X(t),U_n) - C_j\left(E_X\left(t-\frac{v}{m}\right),U_n\right)\right)dudv.$$

By (**A**2), for $\varepsilon > 0$ and u, v with $|u| \leq \delta m$ and $|v| \leq \delta m$, and by Cauchy-Schwarz inequality, one has

$$\left|\mathrm{Cov}\left(C_j(E_X(t),U_n) - C_j\left(E_X\left(t-\frac{u}{m}\right),U_n\right), C_j(E_X(t),U_n) - C_j\left(E_X\left(t-\frac{v}{m}\right),U_n\right)\right)\right| \leq \varepsilon\mathcal{L}^d(U_n).$$

Hence, for m large enough such that h_m has compact support in $[-\delta,\delta]$ one has,

$$\frac{1}{\mathcal{L}^d(U_n)}\mathrm{Var}\left(C_j(E_X(t),U_n)) - \mathrm{LC}_j^X(\tau_t\check{h}_m,U_n)\right) \leq \varepsilon.$$

Hence, choosing m such that we also have

$$\mathrm{Var}\left(\langle B_j, \tau_t\check{h}_m\rangle - \tilde{B}_j(u)\right) \leq \varepsilon,$$

the results follows from Theorem 2.1, using the fact that we have

$$\sqrt{\mathcal{L}(U_n)}\left(\frac{\mathrm{LC}_j^X(\tau_t\check{h}_m,U_n)}{\mathcal{L}^d(U_n)} - \langle\overline{C}_j^X, \tau_t\check{h}_m\rangle\right) \underset{n\to+\infty}{\overset{d}{\longrightarrow}} \langle B_j, \tau_t\check{h}_m\rangle.$$

□

Note that for $J = \{d\}$, under the assumptions of Theorem 2.1, the random field X satisfies also the assumptions of Corollary 2 on $V = \mathbb{R}$ and we recover results of Theorem 2 of [11]. Actually, $z \mapsto \Sigma_{dd}^{st}(z)$ is continuous on $\mathbb{R}^d$ and for all $z \in \mathbb{R}^d$, by Lemma 2 of [11], we also have

$$\left|\Sigma_{dd}^{st}(z)\right| \leq \frac{1}{4\sigma^2}|\rho(z)|,$$

such that $\Sigma_{dd}(s,t) = \int_{\mathbb{R}^d} \Sigma_{dd}^{st}(z)\,dz$ is well-defined and Lebesgue's theorem allows to check (**A**3) on $\mathbb{R}$.

For $J = \{d-1\}$ and $d = 2$, our result corresponds to the one obtained in Theorem 1 of [23], under a general mixing condition, in Theorem 3 of [26] or in Theorem 4.7 of [5]. For a general $j \in \{0,\dots,d\}$, we also refer to [25] who proved marginal Central limit theorems under an assumption of isotropy that we have relaxed here. We also obtain a stronger result with a joint Central limit theorem.

In the sequel we will compute explicitly Lipschitz-Killing densities $\overline{C}_j^X(t)$ at some level $t \in \mathbb{R}$ and will estimate them using $\frac{C_j(E_X(t,U))}{\mathcal{L}^d(U)}$ for an observation window U large enough.

3. Smooth stationary 2D Gaussian fields

3.1. Lipschitz-Killing curvature densities

When the dimension is $d = 2$, the quantities of interest are the Lipschitz-Killing curvatures C_0, C_1 and C_2 that are here, up to multiplicative constants, the Total Curvature (related to the Euler Characteristic), the Perimeter, and the Area of the excursion sets of X in an open bounded domain U. More precisely we define for a smooth stationary field X defined on $\mathbb{R}^2$ and $h \in \mathcal{S}$

$$\mathrm{LA}_X(h,U) := \mathrm{LC}_2^X(h,U), \quad \mathrm{LP}_X(h,U) := 2\,\mathrm{LC}_1^X(h,U)$$

$$\text{and} \quad \mathrm{LTC}_X(h,U) := 2\pi\,\mathrm{LC}_0^X(h,U).$$

Now we give explicit formulas for the Lipschitz-Killing curvature densities of a stationary C^2 Gaussian field.

Theorem 3.1. *Let X be a stationary Gaussian field of mean $m \in \mathbb{R}$, variance $\sigma^2 > 0$ and whose covariance function ρ satisfies Hypothesis* **H**. *We write $\gamma_1, \gamma_2 > 0$ the eigenvalues of $\Gamma = -D^2\rho(0)$ and denote by Φ the standard Gaussian cumulative distribution. Then the Lipschitz-Killing curvature densities of X are given for a.e. $t \in \mathbb{R}$ by*

$$\begin{aligned}
\overline{\mathrm{A}}_X(t) := \overline{C}_2^X(t) &= 1 - \Phi\Big(\frac{t-m}{\sigma}\Big),\\
\overline{\mathrm{Per}}_X(t) := 2\overline{C}_1^X(t) &= \sqrt{\frac{\overline{\gamma}_{\mathrm{Per}}}{\sigma^2}}\frac{1}{2}e^{-(t-m)^2/2\sigma^2}\\
\overline{\mathrm{TC}}_X(t) := 2\pi\overline{C}_0^X(t) &= \frac{\overline{\gamma}_{\mathrm{TC}}}{\sigma^2}\frac{1}{\sqrt{2\pi}}\frac{t-m}{\sigma}e^{-(t-m)^2/2\sigma^2}
\end{aligned}$$

where

$$\overline{\gamma}_{\mathrm{Per}} = \left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\,d\theta\right)^2, \tag{16}$$

and

$$\overline{\gamma}_{\mathrm{TC}} = \sqrt{\gamma_1\gamma_2}. \tag{17}$$

Therefore, the Gaussian random field X has same Perimeter density as a (weakly) isotropic Gaussian field with second spectral moment $\overline{\gamma}_{\mathrm{Per}}$ and same Total Curvature density as a (weakly) isotropic Gaussian field with second spectral moment $\overline{\gamma}_{\mathrm{TC}}$. Moreover, we have

$$\min(\gamma_1,\gamma_2) \le \overline{\gamma}_{\mathrm{TC}} \le \overline{\gamma}_{\mathrm{Per}} \le \max(\gamma_1,\gamma_2),$$

where inequalities are strict if and only if $\gamma_1 \neq \gamma_2$.

Proof. Let $h \in \mathcal{S}$. For $j = 2$, the result simply comes from the fact that $\overline{C}_2^X(t) = \mathbb{P}(X(0) \ge t)$ with $X(0) \sim \mathcal{N}(m,\sigma^2)$. For $j \in \{0,1\}$, by (11), since $\|\nabla X(0)\| > 0$ a.s. by **H** and since the random

variable $X(0)$ admits a probability density given by the function $p_{X(0)}(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-m)^2/2\sigma^2}$, we have

$$\alpha_{1-j}\langle \overline{C}_j^X, h\rangle = \mathbb{E}\left(h(X(0))F_{1-j}(\mathbf{X}(0))\right) = \int_{\mathbb{R}} h(t)\mathbb{E}\left(F_{1-j}(\mathbf{X}(0))|X(0)=t\right)p_{X(0)}(t)dt.$$

Hence, for a.e. $t \in \mathbb{R}$, using (3) and (6),

$$\begin{aligned}
\overline{C}_1^X(t) &= \frac{1}{2}\mathbb{E}(\|\nabla X(0)\||X(0)=t)p_{X(0)}(t)\\
\overline{C}_0^X(t) &= -\frac{1}{2\pi}\mathbb{E}\left(\frac{(\nabla X(0)^\perp)^t D^2X(0)\nabla X(0)^\perp}{\|\nabla X(0)\|^2}|X(0)=t\right)p_{X(0)}(t).
\end{aligned}$$

In order to go ahead in the computations, we first remark that writing $X = m + \sigma Y \circ P$, for P orthogonal such that $\Gamma = P^t\Delta P$ with $\Delta = \mathrm{diag}(\gamma_1, \gamma_2)$, one has therefore $\overline{C}_j^X(t) = \overline{C}_j^Y(\frac{t-m}{\sigma})$ and Y is centered with unit variance and admits $\frac{\gamma_1}{\sigma^2}, \frac{\gamma_2}{\sigma^2}$ for spectral moments with (denoting by Y_i and Y_{ij} the first and second order partial derivatives of Y):

$$\frac{\gamma_1}{\sigma^2} = \mathbb{E}(Y_1(0)^2), \quad \frac{\gamma_2}{\sigma^2} = \mathbb{E}(Y_2(0)^2), \quad \text{and} \quad \mathbb{E}(Y_1(0)Y_2(0)) = 0.$$

Since Y is stationary Gaussian, we also have that $\nabla Y(0)$ is independent from $Y(0)$ and $D^2Y(0)$, and that $Y(0)$ and $D^2Y(0)$ are correlated with covariance

$$\mathbb{E}(Y(0)Y_{11}(0)) = -\frac{\gamma_1}{\sigma^2}, \quad \mathbb{E}(Y(0)Y_{22}(0)) = -\frac{\gamma_2}{\sigma^2}, \quad \text{and} \quad \mathbb{E}(Y(0)Y_{12}(0)) = 0.$$

Therefore the conditional expectations are given by

$$\mathbb{E}(Y_{11}(0)|Y(0)=t) = -\frac{\gamma_1}{\sigma^2}t, \quad \mathbb{E}(Y_{22}(0)|Y(0)=t) = -\frac{\gamma_2}{\sigma^2}t, \quad \text{and} \quad \mathbb{E}(Y_{12}(0)|Y(0)=t) = 0.$$

Now, using that for a vector $v \in \mathbb{R}^2$, we have, denoting $e_\theta = (\cos\theta, \sin\theta)$,

$$\|v\| = \frac{1}{4}\int_0^{2\pi} |\langle v, e_\theta\rangle|\, d\theta,$$

and using that for a 1D Gaussian random variable Z of mean 0 and variance σ_Z^2, we have $\mathbb{E}(|Z|) = \sqrt{\frac{2}{\pi}}\sigma_Z$, we get

$$\overline{C}_1^Y(t) = \frac{1}{2}\mathbb{E}(\|\nabla Y(0)\|)p_{Y(0)}(t) = \frac{1}{8\pi\sigma}e^{-t^2/2}\int_0^{2\pi}\sqrt{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta.$$

To compute $\overline{C}_0^Y(t)$, we first need to compute $\mathbb{E}(\cos^2\Theta)$ where Θ is the angle of $\nabla Y(0)$, i.e. $\nabla Y(0) = (\|\nabla Y(0)\|\cos\Theta, \|\nabla Y(0)\|\sin\Theta)$. Since Y is not assumed isotropic, the law of Θ is not necessarily uniform on $[0, 2\pi)$. Indeed we have

$$\begin{aligned}
\mathbb{E}(\cos^2\Theta) &= \frac{\sigma^2}{2\pi\sqrt{\gamma_1\gamma_2}}\iint \frac{y_1^2}{y_1^2+y_2^2}e^{-(\sigma y_1)^2/2\gamma_1}e^{-(y_2\sigma)^2/2\gamma_2}\, dy_1dy_2\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{\gamma_1\cos^2\theta}{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta. \quad (18)
\end{aligned}$$

It follows that

$$\mathbb{E}(\sin^2\Theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\gamma_2\sin^2\theta}{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta.$$

Finally,

$$\overline{C}_0^Y(t) = -\frac{1}{2\pi}\left(\mathbb{E}(\sin^2\Theta)\mathbb{E}(Y_{11}(0)|Y(0)=t) + \mathbb{E}(\cos^2\Theta)\mathbb{E}(Y_{22}(0)|Y(0)=t)\right)p_{Y(0)}(t),$$

and combining everything, we get

$$\overline{C}_0^Y(t) = \frac{t}{\sigma^2(\sqrt{2\pi})^5} e^{-t^2/2} \int_0^{2\pi} \frac{\gamma_1\gamma_2}{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta = \frac{t}{\sigma^2(\sqrt{2\pi})^3} e^{-t^2/2}\sqrt{\gamma_1\gamma_2}.$$

The last equality comes from the following computation:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma_1\gamma_2}{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \frac{\gamma_1\gamma_2}{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta$$
$$= \frac{2}{\pi}\int_0^{+\infty} \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 s^2}\, ds = \sqrt{\gamma_1\gamma_2}, \quad (19)$$

where we have used a change of variable $s = \tan\theta$, and then recognized the derivative of the function $s \mapsto \arctan(\sqrt{\frac{\gamma_2}{\gamma_1}}s)$.

The remark on the "equivalent" isotropic fields that have same Perimeter density or same Total curvature density as X comes from the fact that when $\gamma_1 = \gamma_2 = \overline{\gamma}$, then the above computations yield $\overline{\gamma}_{\mathrm{Per}} = \overline{\gamma}_{\mathrm{TC}} = \overline{\gamma}$.

To prove the inequality between $\overline{\gamma}_{\mathrm{TC}}$ and $\overline{\gamma}_{\mathrm{Per}}$, let us first notice that

$$\sqrt{\overline{\gamma}_{\mathrm{Per}}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\left(\sqrt{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta} + \sqrt{\gamma_1\sin^2\theta + \gamma_2\cos^2\theta}\right) d\theta.$$

Then, for any $s \in [0,1]$, we have

$$\begin{aligned}\frac{1}{2}\left(\sqrt{\gamma_1 s + \gamma_2(1-s)} + \sqrt{\gamma_1(1-s) + \gamma_2 s}\right) &\geq ((\gamma_1 s + \gamma_2(1-s))(\gamma_1(1-s) + \gamma_2 s))^{1/4} \\ &= \left(\gamma_1^2 s(1-s) + \gamma_2^2 s(1-s) + \gamma_1\gamma_2(s^2 + (1-s)^2)\right)^{1/4} \\ &= \left(\gamma_1\gamma_2 + s(1-s)(\gamma_1-\gamma_2)^2\right)^{1/4} \geq (\gamma_1\gamma_2)^{1/4},\end{aligned}$$

and this proves the inequality, by simply setting $s = \cos^2\theta$ and then integrating over θ. We also remark that this inequality is strict if and only if $\gamma_1 \neq \gamma_2$. □

Note that we recover our previous results of [7] in the (weakly) isotropic case. Moreover, in the anisotropic case, assuming that $\gamma_2 > \gamma_1$ and considering $k = \sqrt{1 - \gamma_1/\gamma_2}$, one can write

$$\sqrt{\overline{\gamma}_{\mathrm{Per}}} = \sqrt{\gamma_2}\frac{2}{\pi}\int_0^{\pi/2} \sqrt{(1-k^2)\cos^2\theta + \sin^2\theta}\, d\theta = \sqrt{\gamma_2}\frac{2}{\pi}\mathcal{E}(k),$$

where $\mathcal{E}(k)$ is the elliptic integral of the first kind, such that $\overline{\mathrm{Per}}_X(t)$ corresponds to the formula of Theorem 11.3. of [4]. We also refer to Proposition 3.4 of [15] for planar Gaussian waves.

3.2. Visualization and estimation of the anisotropy

In this section, we are interested in visualizing and estimating the anisotropy of a Gaussian field from the Lipschitz-Killing curvatures of its excursion sets. Here the anisotropy is understood in a weak way, and it is summarized through the anisotropy ratio R defined by

$$R := \frac{\overline{\gamma}_{\mathrm{TC}}}{\overline{\gamma}_{\mathrm{Per}}}.$$

Thanks to Theorem 3.1, we have $R \in [0,1]$, and $R = 1$ if and only if the spectral moments γ_1 and γ_2 are equal, meaning that the field X is (weakly) isotropic. More precisely, we can see R as a function of $\alpha := \frac{\min(\gamma_1,\gamma_2)}{\max(\gamma_1,\gamma_2)} \in [0,1]$ since

$$R = \frac{\overline{\gamma}_{\mathrm{TC}}}{\overline{\gamma}_{\mathrm{Per}}} = \frac{\sqrt{\gamma_1\gamma_2}}{\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\gamma_1\cos^2\theta + \gamma_2\sin^2\theta}\, d\theta\right)^2} = \frac{\sqrt{\alpha}}{\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\alpha\cos^2\theta + \sin^2\theta}\, d\theta\right)^2}.$$

This is illustrated on the left of Figure 2 where we show the graph of the function $\alpha \mapsto R$. It is an increasing function that maps $[0,1]$ to $[0,1]$.

3.2.1. *The Almond curve of anisotropy*

Inspired by the paper of Klatt et al. [24] for the visualization of anisotropy, from the formulas for the Perimeter density and the Total Curvature density (Theorem 3.1), we see that if we set

$$\tilde{x}(t) = \frac{\overline{\mathrm{Per}}_X(t)}{\overline{\mathrm{Per}}_X(m)} = e^{-(t-m)^2/2\sigma^2} \quad \text{and} \quad \tilde{y}(t) = \frac{\overline{\mathrm{TC}}_X(t)}{\overline{\mathrm{Per}}_X(m)^2} = \frac{\overline{\gamma}_{\mathrm{TC}}}{\overline{\gamma}_{\mathrm{Per}}} \frac{4}{\sqrt{2\pi}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2},$$

then the point $(\tilde{x}(t), \tilde{y}(t))$ is on the curve

$$\mathcal{C}_R = \{(x,y) \in \mathbb{R}^2 \,;\, y^2 + \frac{16}{\pi} R^2 x^2 \log x = 0\}.$$

Notice that this curve is in the sub-domain $x \in (0,1]$ and that, thanks to the fact that $R \leq 1$, it is inside the domain defined by the closed curve

$$\mathcal{C}_1 = \{(x,y) \in \mathbb{R}^2 \,;\, y^2 + \frac{16}{\pi} x^2 \log x = 0\}.$$

This curve is "canonical" in the sense that it is independent of the mean and of the variance of the field. We call this curve the *almond curve* of anisotropy because of its shape, see Figure 2 right. Points on this curve correspond to $R = 1$, that is $\overline{\gamma}_{\mathrm{TC}} = \overline{\gamma}_{\mathrm{Per}}$ and thus $\gamma_1 = \gamma_2$, meaning the field X is (weakly) isotropic.

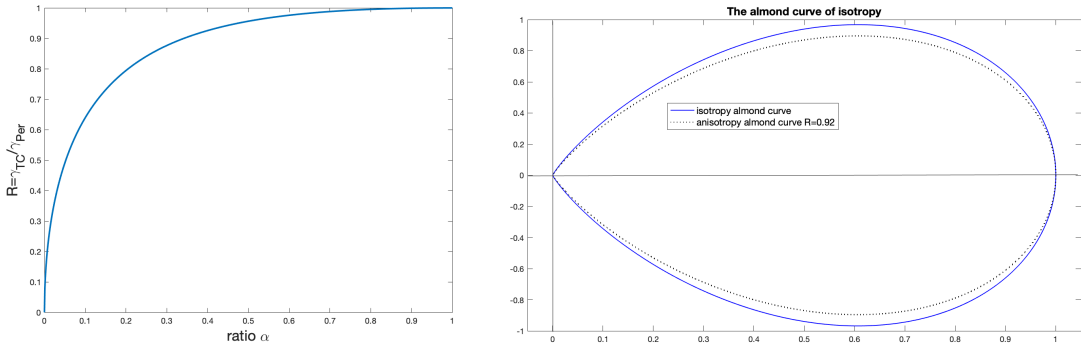

FIG 2. *On the left, plot of the function $\alpha \mapsto R$. On the right, the almond curves of anisotropy: the blue plain curve is the curve $\mathcal{C}_1$ corresponding to isotropy, and the black dotted curve is a curve $\mathcal{C}_R$, with here $R \simeq 0.92$.*

3.2.2. *Estimating the anisotropy from one excursion set*

Assume we observe an excursion set $\hat{E}$ of a sample of a Gaussian stationary random field X, in a window U. The question is then: how to estimate the anisotropy ratio R from this set $\hat{E}$ only, that is without having access to the whole field (nor its increments for instance). We ignore also at which level t the field X was thresholded, and also what was its mean m or its variance σ^2. Now from the three observed Lipschitz-Killing curvatures densities (that are $\hat{\mathrm{A}}(E) = \mathrm{A}(\hat{E})/\mathcal{L}^2(U)$ the normalized Area of $\hat{E}$, $\widehat{\mathrm{Per}}(E) = \mathrm{Per}(\hat{E})/\mathcal{L}^2(U)$ the normalized Perimeter of $\hat{E}$, and $\widehat{\mathrm{TC}}(E) = \mathrm{TC}(\hat{E})/\mathcal{L}^2(U)$ the normalized Total Curvature of E), we can proceed the following way:
- Compute an "effective" threshold $\widehat{t}$, that is an empirical estimate of $\frac{t-m}{\sigma}$, using the density area of $\hat{E}$:

$$\widehat{t} = \Phi^{-1}(1 - \hat{\mathrm{A}}(E)).$$

- Then estimate $\overline{\mathrm{Per}}_X(m)$ by

$$\widehat{p_m} := \widehat{\mathrm{Per}}(E) e^{\widehat{t}^2/2}.$$

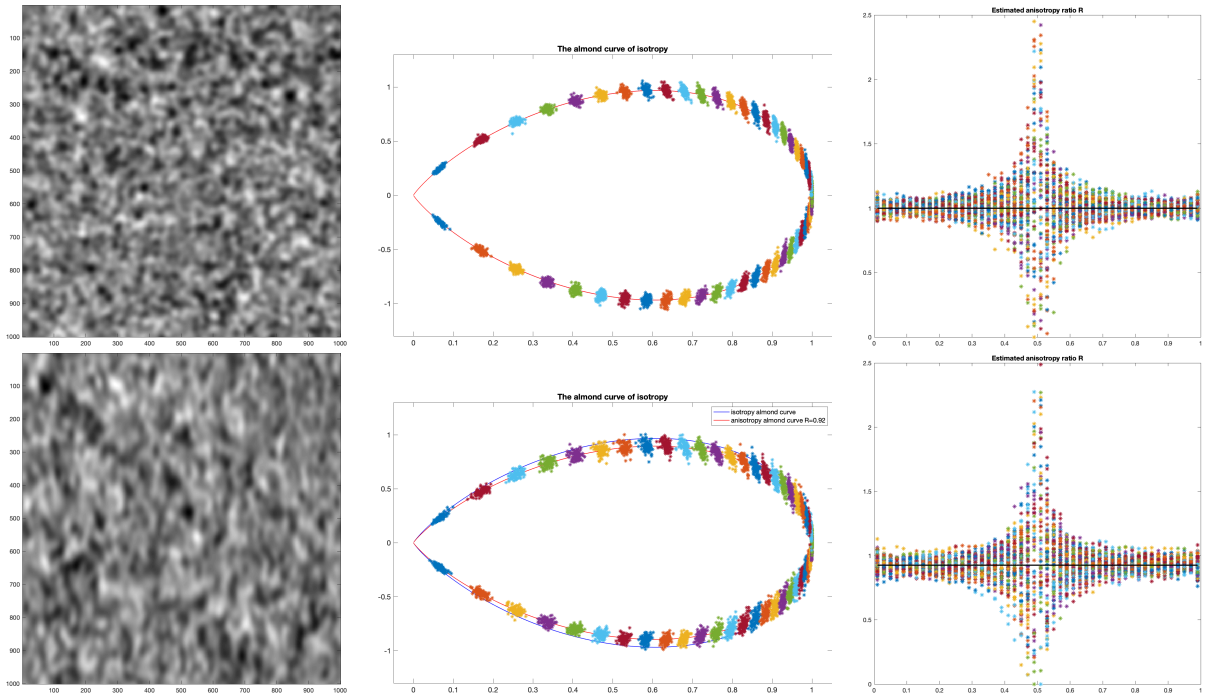

FIG 3. *First line: left, a sample of a centered isotropic Gaussian field X, size 1000×1000 pixels, with variance $\sigma^2 = 1$ and covariance of the form $\rho(x_1, x_2) = \exp(-\gamma_1 x_1^2 + \gamma_2 x_2^2)$ with $\gamma_1 = \gamma_2 = 0.005$. Middle: the almond curve and some points from the isotropic field (100 samples). Right: corresponding estimations of the anisotropy ratio R ($= 1$ here) as a function of the quantiles. Second line: same experiments but with now $\gamma_1 = 0.005$ and $\gamma_2 = 0.002$, that corresponds to an anisotropic field with $R \simeq 0.92$.*

- And the "effective" spectral moments via

$$\frac{\widehat{\gamma}_{\mathrm{Per}}}{\sigma^2} = (2\widehat{p_m})^2, \quad \frac{\widehat{\gamma}_{\mathrm{TC}}}{\sigma^2} = \widehat{\mathrm{TC}}(E)\frac{\sqrt{2\pi}}{\widehat{t}}e^{\widehat{t}^2/2}.$$

Finally estimate the anisotropy ratio with

$$\widehat{R} = \frac{\widehat{\gamma}_{\mathrm{TC}}}{\widehat{\gamma}_{\mathrm{Per}}} = \frac{\widehat{\mathrm{TC}}(E)}{\widehat{p_m}^2}\frac{\sqrt{2\pi}}{4\widehat{t}}e^{\widehat{t}^2/2}.$$

This estimation procedure is inspired from the one developed by Elena Di Bernardino and Céline Duval in [13], where they introduce the notion of "effective" threshold, but where the framework is quite different since they assume the Gaussian field is isotropic.

Note that only "effective" spectral moments $\frac{\widehat{\gamma}_{\mathrm{Per}}}{\sigma^2}$ and $\frac{\widehat{\gamma}_{\mathrm{TC}}}{\sigma^2}$ are empirically accessible in practice when X is not assumed standard (i.e. so that $\sigma = 1$) but our anisotropic ratio R itself is empirically accessible. Moreover we can also compute a point on the almond curve (see Figure 3) by setting

$$\hat{x} = \frac{\widehat{\mathrm{Per}}(E)}{\widehat{p_m}} = e^{-\widehat{t}^2/2} \quad \text{and} \quad \hat{y} = \frac{\widehat{\mathrm{TC}}(E)}{\widehat{p_m}^2} = \widehat{R}\frac{4}{\sqrt{2\pi}}\widehat{t}e^{-\widehat{t}^2/2}.$$

Let us remark that, assuming $\hat{E}$ is the realization of $E_X(t) \cap U_n$ for some $t \in \mathbb{R}$ and n large, under the assumptions of Corollary 2 on an open neigborhood V of t and $J = \{0, 1, 2\}$, $\hat{\mathrm{A}}(E)$, $\widehat{\mathrm{Per}}(E)$ and $\widehat{\mathrm{TC}}(E)$ are consistent and asymptotically Gaussian. When moreover $t \neq m$, by the delta method, it should imply the same properties for the effective spectral moments and anisotropy ratio (see Figure 8 in the additional experiments section B in the Appendix, where we also numerically check

the central limit behavior of the anisotropy ratio). This restriction on t means that we shall not have $\hat{A}(E)$ close to $1/2 = \Phi(0)$.

On Figures 3 and 4 (and also on Figure 9 in the additional experiments section B in the Appendix), we illustrate all the theoretical results of this section. The numerical computations of the Lipschitz-Killing curvatures of the excursion sets are done thanks to the MATLAB "imMinkowski" library[1] developed by David Legland and described in [27]. As far as binary images are concerned, it will be more convenient to work with "quantiles" $q \in [0, 1]$ so that the corresponding threshold levels are given by $t = m + \sigma\Phi^{-1}(q)$.

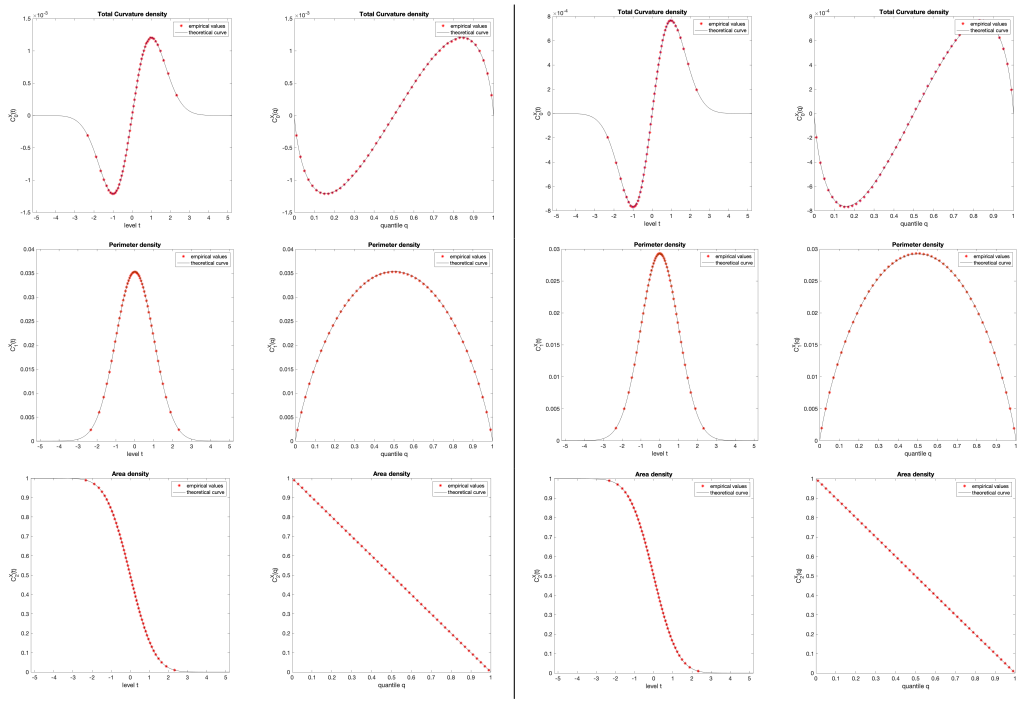

FIG 4. *Densities $\overline{C}_0^X$ (first line), $\overline{C}_1^X$ (second line) and $\overline{C}_2^X$ (third line), as a function of the level or the quantile in the isotropic case (left part of the figure) and in the anisotropic one (right part of the figure). The Gaussian fields are the same as the ones of Figure 3, here with one sample of size* 1000×1000 *($n = 1000$).*

4. Smooth stationary 3D Gaussian fields

4.1. Lipschitz-Killing curvature densities

We are here interested in the 3D framework, and we consider a real-valued random field X defined on $\mathbb{R}^3$. The quantities of interest are the Lipschitz-Killing curvatures C_0, C_1, C_2 and C_3 that are now related to the Total Gaussian Curvature (also related to the Euler Characteristic by the Gauss-Bonnet Theorem), the Total Mean Curvature (related to the mean breadth), the Surface Area and the Volume of an excursion set. More precisely, for a smooth stationary field X defined on $\mathbb{R}^3$ and $h \in \mathcal{S}$ we denote

$$\mathrm{LV}_X(h, U) := \mathrm{LC}_3^X(h, U), \quad \mathrm{LSA}_X(h, U) := 2\mathrm{LC}_2^X(h, U),$$

$$\mathrm{LTMC}_X(h, U) := \pi\mathrm{LC}_1^X(h, U) \quad \text{and} \quad \mathrm{LTGC}_X(h, U) := 4\pi\mathrm{LC}_0^X(h, U).$$

[1] available at https://www.mathworks.com/matlabcentral/fileexchange/33690-imminkowski

Theorem 4.1. *Let X be a stationary Gaussian field of mean $m \in \mathbb{R}$, variance $\sigma^2 > 0$ and whose covariance function ρ satisfies Hypothesis* **H**. *We write $\gamma_1, \gamma_2, \gamma_3 > 0$ the eigenvalues of $\Gamma = -D^2\rho(0)$ and denote by Φ the standard Gaussian cumulative distribution. Then the Lipschitz-Killing curvature densities of X are given for a.e. $t \in \mathbb{R}$ by*

$$\begin{aligned}
\overline{\mathrm{V}}_X(t) := \overline{C}_3^X(t) &= 1 - \Phi\Big(\frac{t-m}{\sigma}\Big),\\
\overline{\mathrm{SA}}_X(t) := 2\overline{C}_2^X(t) &= \sqrt{\frac{\overline{\gamma}_{\mathrm{SA}}}{\sigma^2}}\frac{2}{\pi}e^{-(t-m)^2/2\sigma^2},\\
\overline{\mathrm{TMC}}_X(t) := \pi\overline{C}_1^X(t) &= \frac{\overline{\gamma}_{\mathrm{TMC}}}{\sigma^2}\frac{1}{\sqrt{2\pi}}\frac{t-m}{\sigma}e^{-(t-m)^2/2\sigma^2},\\
\overline{\mathrm{TGC}}_X(t) := 4\pi\overline{C}_0^X(t) &= \left(\frac{\overline{\gamma}_{\mathrm{TGC}}}{\sigma^2}\right)^{3/2}\frac{1}{\pi}\left(\frac{(t-m)^2}{\sigma^2} - 1\right)e^{-(t-m)^2/2\sigma^2},
\end{aligned}$$

where

$$\overline{\gamma}_{\mathrm{SA}} = \left(\frac{1}{4\pi}\int_{S^2}\sqrt{\gamma_1 u_1^2 + \gamma_2 u_2^2 + \gamma_3 u_3^2}\,\mathcal{H}^2(du)\right)^2, \tag{20}$$

$$\overline{\gamma}_{\mathrm{TMC}} = \frac{1}{2}(\gamma_1+\gamma_2+\gamma_3) - \frac{1}{8\pi}\int_{S^2}\frac{\gamma_1^2u_1^2 + \gamma_2^2u_2^2 + \gamma_3^2u_3^2}{\gamma_1u_1^2 + \gamma_2u_2^2 + \gamma_3u_3^2}\,\mathcal{H}^2(du), \tag{21}$$

$$\textit{and}\quad \overline{\gamma}_{\mathrm{TGC}} = (\gamma_1\gamma_2\gamma_3)^{1/3}. \tag{22}$$

Therefore, the Gaussian random field X has same Surface Area density as a (weakly) isotropic Gaussian field with second spectral moment $\overline{\gamma}_{\mathrm{SA}}$, same Total Mean Curvature density as a (weakly) isotropic Gaussian field with second spectral moment $\overline{\gamma}_{\mathrm{TMC}}$ and same Total Gaussian Curvature density as a (weakly) isotropic Gaussian field with second spectral moment $\overline{\gamma}_{\mathrm{TGC}}$. Moreover, we have

$$\min(\gamma_1,\gamma_2,\gamma_3) \le (\gamma_1\gamma_2\gamma_3)^{1/3} = \overline{\gamma}_{\mathrm{TGC}} \le \overline{\gamma}_{\mathrm{TMC}} \le \overline{\gamma}_{\mathrm{SA}} \le \frac{\gamma_1+\gamma_2+\gamma_3}{3} \le \max(\gamma_1,\gamma_2,\gamma_3).$$

where the inequalities are strict if and only if $\min(\gamma_1,\gamma_2,\gamma_3) < \max(\gamma_1,\gamma_2,\gamma_3)$.

Proof. As in the 2D case, we may assume that $m = 0$, $\sigma^2 = 1$ and $\Gamma = \mathrm{Diag}(\gamma_1,\gamma_2,\gamma_3)$ meaning that

$$\forall j = 1,2,3, \quad \gamma_j := \Gamma_{jj} = \mathbb{E}(X_j(0)^2), \quad \text{and} \quad \mathbb{E}(X_i(0)X_j(0)) = 0 \quad \text{when} \quad i \neq j,$$

denoting $X_i := \frac{\partial X}{\partial x_i}, X_{ij} := \frac{\partial^2 X}{\partial x_i \partial x_j}$ for $1 \le i,j \le d$. The formula for $\overline{\mathrm{V}}_X$ is straightforward. Now for $j < 3$, by Hypothesis **H**, we have $\|\nabla X(0)\| > 0$ a.s. and $\mathbb{E}(\|\nabla X(0)\|^{-1}) < +\infty$. Since $D^2X(0)$ is independent from $\nabla X(0)$, recalling (3), (4) (5), it follows that the assumption (10) is satisfied. Then, for $h \in \mathcal{S}$, by (11), since the standard Gaussian random variable $X(0)$ admits a probability density given by a function $p_{X(0)}$, we have

$$\alpha_{2-j}\langle\overline{C}_j^X, h\rangle = \mathbb{E}\left(h(X(0))F_{2-j}(\mathbf{X}(0))\right) = \int_{\mathbb{R}} h(t)\mathbb{E}\left(F_{2-j}(\mathbf{X}(0))|X(0)=t\right)p_{X(0)}(t)dt.$$

Hence, for a.e. $t \in \mathbb{R}$,

$$\begin{aligned}
\overline{C}_2^X(t) &= \frac{1}{2}\mathbb{E}(\|\nabla X(0)\|\big|X(0)=t)p_{X(0)}(t)\\
\overline{C}_1^X(t) &= \frac{1}{2\pi}\mathbb{E}\left(\frac{\nabla X(0)^t D^2X(0)\nabla X(0)}{\|\nabla X(0)\|^2} - \mathrm{Tr}\left(D^2X(0)\right)\big|X(0)=t\right)p_{X(0)}(t).\\
\overline{C}_0^X(t) &= \frac{1}{4\pi}\mathbb{E}\left(\frac{\nabla X(0)^t \mathrm{adj}(D^2X(0))\nabla X(0)}{\|\nabla X(0)\|^3}\big|X(0)=t\right)p_{X(0)}(t).
\end{aligned}$$

Note that since X is Gaussian, we have that $\nabla X(0)$ is independent from $X(0)$ and $D^2X(0)$, and that $X(0)$ and $D^2X(0)$ are correlated with covariance

$$\forall j = 1,2,3, \quad \mathbb{E}(X(0)X_{jj}(0)) = -\gamma_j, \quad \text{and} \quad \mathbb{E}(X(0)X_{ij}(0)) = 0 \quad \text{when} \quad i \neq j.$$

$$\forall j = 1,2,3, \quad \mathbb{E}(X(0)X_{jj}(0)) = -\gamma_j, \quad \text{and} \quad \mathbb{E}(X(0)X_{ij}(0)) = 0 \quad \text{when} \quad i \neq j.$$

Therefore the conditional expectations are given by

$$\mathbb{E}(X_{jj}(0)|X(0) = t) = -\gamma_j t, \quad \text{and} \quad \mathbb{E}(X_{ij}(0)|X(0) = t) = 0.$$

Moreover, we have

$$\partial^4_{iijj}\rho(0) = \mathbb{E}(X_{ii}(0)X_{jj}(0)) = \mathbb{E}(X_{ij}(0)^2).$$

Since by assumption $D^2\rho(0) = -\Gamma$ is diagonal, we have $\mathbb{E}(X_{ij}(0)X(0)) = 0$ and therefore

$$\mathbb{E}(X_{ij}(0)^2|X(0) = t) = \partial^4_{iijj}\rho(0).$$

Using the formulas for the conditional distribution of Gaussian vectors, the covariance of $(X_{ii}(0), X_{jj}(0))$ knowing $X(0) = t$ is given by

$$\mathbb{E}(X_{ii}(0)X_{jj}(0)|X(0) = t) = \gamma_i\gamma_j t^2 + \partial^4_{iijj}\rho(0) - \gamma_i\gamma_j.$$

Therefore, putting all together, we get

$$\mathbb{E}(X_{ii}(0)X_{jj}(0) - X_{ij}(0)^2|X(0) = t) = \gamma_i\gamma_j(t^2 - 1),$$

which doesn't require the fourth-order spectral moment anymore. We are now in position to compute the different densities. We start with $\overline{C}_2^X(t)$, which is equal to

$$\overline{C}_2^X(t) = \frac{1}{2}\mathbb{E}(\|\nabla X(0)\||X(0) = t)p_X(t) = \frac{1}{2}\mathbb{E}(\|\nabla X(0)\|)p_{X(0)}(t).$$

We use here the property that if $v \in \mathbb{R}^3$, then

$$\|v\| = \frac{1}{2\pi}\int_{S^2} |\langle v, u\rangle| \, \mathcal{H}^2(du).$$

Now, as in the 2D case, using the fact that the expectation of the absolute value of a centered Gaussian real random variable is $\sqrt{\frac{2}{\pi}}$ times its standard deviation, we get, denoting $u = (u_1, u_2, u_3) \in S^2$,

$$\overline{C}_2^X(t) = \frac{1}{4\pi^2}e^{-t^2/2}\int_{S^2}\sqrt{\gamma_1u_1^2 + \gamma_2u_2^2 + \gamma_3u_3^2}\,\mathcal{H}^2(du) = \frac{1}{\pi}\sqrt{\overline{\gamma}_{\mathrm{SA}}}e^{-t^2/2}.$$

Then, for $\overline{C}_1^X$, using the conditional expectations and the fact that $\nabla X(0)$ is independant from $(X(0), D^2X(0))$, we have

$$2\pi\overline{C}_1^X(t) = tp_{X(0)}(t)\left[\gamma_1 + \gamma_2 + \gamma_3 - \mathbb{E}\left(\frac{\gamma_1X_1(0)^2 + \gamma_2X_2(0)^2 + \gamma_3X_3(0)^2}{X_1(0)^2 + X_2(0)^2 + X_3(0)^2}\right)\right].$$

Using the law of the $X_i(0)$ and a change of variable with spherical coordinates, we get that

$$\mathbb{E}\left(\frac{\gamma_1X_1(0)^2 + \gamma_2X_2(0)^2 + \gamma_3X_3(0)^2}{X_1(0)^2 + X_2(0)^2 + X_3(0)^2}\right) = \frac{1}{4\pi}\int_{S^2}\frac{\gamma_1^2u_1^2 + \gamma_2^2u_2^2 + \gamma_3^2u_3^2}{\gamma_1u_1^2 + \gamma_2u_2^2 + \gamma_3u_3^2}\,\mathcal{H}^2(du).$$

Hence,

$$2\pi\overline{C}_1^X(t) = 2\overline{\gamma}_{\mathrm{TMC}}tp_{X(0)}(t).$$

Finally, by independence of $\nabla X(0)$ from $(X(0), D^2X(0))$, and the conditional expectations computed above, we get (where the indices i, j, k below are all distincts):

$$\begin{aligned}
4\pi\overline{C}_0^X(t) &= p_{X(0)}(t)\sum_{i=1}^3 \mathbb{E}\left(\frac{X_i(0)^2}{(X_1(0)^2+X_2(0)^2+X_3(0))^{3/2}}\right)\mathbb{E}(X_{jj}(0)X_{kk}(0)-X_{jk}(0)^2|X(0)=t)\\
&= (t^2-1)p_{X(0)}(t)\sum_{i=1}^3 \mathbb{E}\left(\frac{X_i(0)^2}{(X_1(0)^2+X_2(0)^2+X_3(0))^{3/2}}\right)\gamma_j\gamma_k\\
&= (t^2-1)p_{X(0)}(t)\times\frac{1}{(2\pi)^{3/2}}\int_{S^2}\frac{\gamma_1\gamma_2\gamma_3}{(\gamma_1u_1^2+\gamma_2u_2^2+\gamma_3u_3^2)^{3/2}}\,\mathcal{H}^2(du)\\
&= \overline{\gamma}_{\mathrm{TGC}}^{3/2}\frac{1}{\pi}(t^2-1)e^{-t^2/2}.
\end{aligned}$$

The last line above comes from the following computation. Let J denote the above integral on S^2, that can be also written as

$$J := 2\int_0^{\pi/2}\int_0^{2\pi}\frac{\gamma_1\gamma_2\gamma_3}{(\gamma_1\sin^2\varphi\cos^2\theta+\gamma_2\sin^2\varphi\sin^2\theta+\gamma_3\cos^2\varphi)^{3/2}}\,\sin\varphi\,d\theta d\varphi.$$

Then by a change of variable $s=\tan\varphi$, we have $ds=(1+s^2)d\varphi$, $\cos^2\varphi=\frac{1}{1+s^2}$ and $\sin\varphi=\frac{s}{\sqrt{1+s^2}}$. Therefore

$$\begin{aligned}
J = 2\int_0^{+\infty}\int_0^{2\pi}\frac{\gamma_1\gamma_2\gamma_3\,s}{((\gamma_1\cos^2\theta+\gamma_2\sin^2\theta)s^2+\gamma_3)^{3/2}}\,d\theta\,ds&\\
= 2\sqrt{\gamma_3}\int_0^{2\pi}\frac{\gamma_1\gamma_2}{\gamma_1\cos^2\theta+\gamma_2\sin^2\theta}\,d\theta = 4\pi\sqrt{\gamma_1\gamma_2\gamma_3},&
\end{aligned}$$

where we have used the computation made in the 2D case.

To prove the inequalities, let us first rewrite things using the random variable $U=(U_1,U_2,U_3)\in\mathbb{R}^3$ that is assumed to be uniformly distributed on the unit sphere S^2. Then

$$\overline{\gamma}_{\mathrm{SA}} = \mathbb{E}\left(\sqrt{\gamma_1U_1^2+\gamma_2U_2^2+\gamma_3U_3^2}\right)^2,$$

$$\overline{\gamma}_{\mathrm{TMC}} = \frac{1}{2}\mathbb{E}\left(\frac{\gamma_1(\gamma_2+\gamma_3)U_1^2+\gamma_2(\gamma_1+\gamma_3)U_2^2+\gamma_3(\gamma_1+\gamma_2)U_3^2}{\gamma_1U_1^2+\gamma_2U_2^2+\gamma_3U_3^2}\right).$$

By concavity of the square root function, we first have that

$$\overline{\gamma}_{\mathrm{SA}} \le \mathbb{E}(\gamma_1U_1^2+\gamma_2U_2^2+\gamma_3U_3^2) = \frac{1}{3}(\gamma_1+\gamma_2+\gamma_3),$$

because $\mathbb{E}(U_i^2)=1/3$ (this comes from the symmetry of the U_i's and the fact that $U_1^2+U_2^2+U_3^2=1$ a.s.). Note also that the inequality is strict for $\min(\gamma_1,\gamma_2,\gamma_3)<\max(\gamma_1,\gamma_2,\gamma_3)$ by strict concavity of the square root function.

The inequality between $\overline{\gamma}_{\mathrm{SA}}$ and $\overline{\gamma}_{\mathrm{TGC}}=(\gamma_1\gamma_2\gamma_3)^{1/3}$ can also be obtained in a straightforward way. Indeed, using again the concavity of the square root function, and the fact that $\sum_i U_i^2=1$, we have

$$\sqrt{\gamma_1U_1^2+\gamma_2U_2^2+\gamma_3U_3^2} \ge \sqrt{\gamma_1}U_1^2+\sqrt{\gamma_2}U_2^2+\sqrt{\gamma_3}U_3^2.$$

And then taking the expectation on both sides we have

$$\mathbb{E}\left(\sqrt{\gamma_1U_1^2+\gamma_2U_2^2+\gamma_3U_3^2}\right) \ge \frac{1}{3}(\sqrt{\gamma_1}+\sqrt{\gamma_2}+\sqrt{\gamma_3}) \ge (\gamma_1\gamma_2\gamma_3)^{1/6},$$

thanks to the arithmetic-geometric inequality. Therefore

$$\overline{\gamma}_{\mathrm{SA}} \ge \overline{\gamma}_{\mathrm{TGC}}.$$

The inequalities for $\overline{\gamma}_{\mathrm{TMC}}$ are more difficult to obtain. We first introduce the function

$$g_\gamma : x \in \mathbb{R}^3 \mapsto \sqrt{\gamma_1 x_1^2 + \gamma_2 x_2^2 + \gamma_3 x_3^2}$$

and note that it is a twice continuously differentiable positive function on $\mathbb{R}^3 \setminus \{0\}$ with gradient given by $\nabla g_\gamma(x) = \frac{1}{g_\gamma(x)}(\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3)$. It follows that its directional gradient on the sphere at a point $x \in S^2$ is given by the orthonormal projection of $\nabla g_\gamma(x)$ onto the tangent space at x that is precisely $\langle x \rangle^\perp$, namely

$$\nabla_{S^2} g_\gamma(x) = \nabla g_\gamma(x) - \langle \nabla g_\gamma(x), x \rangle x = \nabla g_\gamma(x) - g_\gamma(x) x.$$

According to Poincaré - Wirtinger inequality (Theorem 5.4.1 of [21]) we have

$$\mathrm{Var}\left(g_\gamma(U)\right) \leq \frac{1}{2}\mathbb{E}\left(\|\nabla_{S^2} g_\gamma(U)\|^2\right).$$

But we remark that $\overline{\gamma}_{\mathrm{SA}} = \mathbb{E}\left(g_\gamma(U)\right)^2$ and that $\mathbb{E}\left(g_\gamma(U)^2\right) = \frac{1}{3}(\gamma_1 + \gamma_2 + \gamma_3)$, while

$$\mathbb{E}\left(\|\nabla_{S^2} g_\gamma(U)\|^2\right) = \mathbb{E}\left(\|\nabla g_\gamma(U)\|^2\right) - \mathbb{E}\left(g_\gamma(U)^2\right).$$

Then Poincaré - Wirtinger inequality rewrites as

$$\frac{3}{2}\mathbb{E}\left(g_\gamma(U)^2\right) - \overline{\gamma}_{\mathrm{SA}} \leq \frac{1}{2}\mathbb{E}\left(\|\nabla g_\gamma(U)\|^2\right).$$

But

$$\frac{3}{2}\mathbb{E}\left(g_\gamma(U)^2\right) - \frac{1}{2}\mathbb{E}\left(\|\nabla g_\gamma(U)\|^2\right) = \frac{1}{2}\mathbb{E}\left(\gamma_1 + \gamma_2 + \gamma_3 - \frac{\gamma_1^2 U_1^2 + \gamma_2^2 U_2^2 + \gamma_3^2 U_3^2}{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2}\right) = \overline{\gamma}_{\mathrm{TMC}}.$$

Hence we have obtained $\overline{\gamma}_{\mathrm{TMC}} \leq \overline{\gamma}_{\mathrm{SA}}$. Note also that the case of equality in Poincaré - Wirtinger inequality may only be achieved when g_γ is constant on S^2. Actually by Theorem 5.4.1 of [21], the equality holds if and only if $g_\gamma - \mathbb{E}(g_\gamma(U))$ is a spherical harmonic of degree 1. This implies that $\tilde{g}_\gamma : \mathbb{R}^3 \mapsto g_\gamma(x) - \mathbb{E}(g_\gamma(U))\|x\|$ is an homogeneous polynomial of degree 1 with constant gradient. But since, for $(e_i)_{1 \leq i \leq 3}$ the canonical basis, we have $\nabla \tilde{g}_\gamma(e_i) = (\sqrt{\gamma_i} - \mathbb{E}(g_\gamma(U)))e_i$, this implies that $\sqrt{\gamma_i} = \mathbb{E}(g_\gamma(U))$ and therefore $\min(\gamma_1, \gamma_2, \gamma_3) = \max(\gamma_1, \gamma_2, \gamma_3)$ with $g_\gamma(x) = \mathbb{E}(g_\gamma(U))\|x\|$.

For the other inequality involving $\overline{\gamma}_{\mathrm{TMC}}$, we will use the isoperimetric inequality for 3D convex bodies (see [28] for instance), that states that

$$\mathrm{SA}^3 \geq 36\pi V^2,$$

where SA is the surface area and V is the volume. Now, taking a 3D ellipsoid of semi-axes $\sqrt{\gamma_1}$, $\sqrt{\gamma_2}$, $\sqrt{\gamma_3}$, we have that its volume is

$$V = \frac{4}{3}\pi\sqrt{\gamma_1}\sqrt{\gamma_2}\sqrt{\gamma_3} = \frac{4}{3}\pi{\overline{\gamma}_{\mathrm{TGC}}}^{3/2},$$

and that its surface area is [22]

$$\mathrm{SA} = 2\pi(\gamma_1 + \gamma_2 + \gamma_3) - 2\pi\mathbb{E}\left(\frac{\gamma_1^2 U_1^2 + \gamma_2^2 U_2^2 + \gamma_3^2 U_3^2}{\gamma_1 U_1^2 + \gamma_2 U_2^2 + \gamma_3 U_3^2}\right) = 4\pi\overline{\gamma}_{\mathrm{TMC}}.$$

Therefore the isoperimetric inequality implies that

$$4^3\pi^3\overline{\gamma}_{\mathrm{TMC}}^3 \geq 36\pi\frac{4^2}{3^2}\pi^2\overline{\gamma}_{\mathrm{TGC}}^3,$$

which exactly means $\overline{\gamma}_{\mathrm{TMC}} \geq \overline{\gamma}_{\mathrm{TGC}}$. Note also that equality in the isoperimetric inequality may only be achieved when the body is a sphere (see [28] p.1190, result (2.7) for instance), that is when $\min(\gamma_1, \gamma_2, \gamma_3) = \max(\gamma_1, \gamma_2, \gamma_3)$. □

4.2. Visualization and estimation of the anisotropy

We are here interested in the visualization and in the estimation of the anisotropy from an excursion set of a sample of a 3D Gaussian field, in a way similar to what was done in the 2D framework.

We define now two anisotropy ratios:

$$R := \frac{\overline{\gamma}_{\mathrm{TMC}}}{\overline{\gamma}_{\mathrm{SA}}} \quad \text{and} \quad R_G := \frac{\overline{\gamma}_{\mathrm{TGC}}}{\overline{\gamma}_{\mathrm{SA}}}.$$

We have $0 \leq R_G \leq R \leq 1$ because of Theorem 4.1, and since they are symmetric functions of the γ_i's, we can assume $\gamma_3 = \max(\{\gamma_i\})$, and see R and R_G as functions of $\alpha_1 := \frac{\gamma_1}{\gamma_3}$ and $\alpha_2 := \frac{\gamma_2}{\gamma_3}$. This is illustrated on Figure 5.

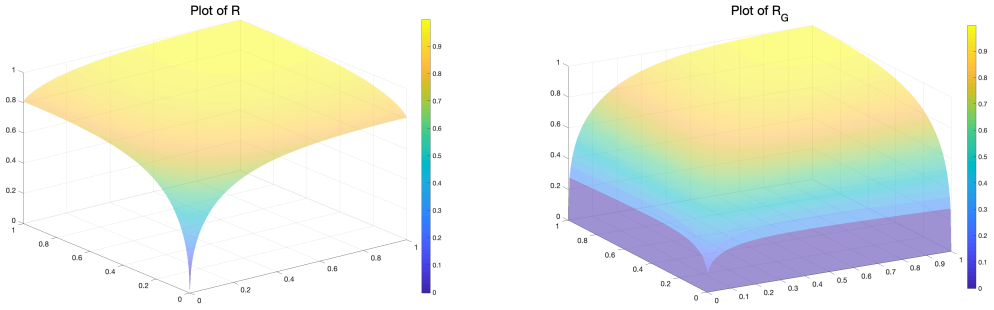

FIG 5. *The anisotropy ratios R (on the left) and R_G (on the right) as functions of $\alpha_1 := \frac{\gamma_1}{\gamma_3}$ and $\alpha_2 := \frac{\gamma_2}{\gamma_3}$.*

4.2.1. The Almond and Heart curves of anisotropy

Again, as in the 2D case, we can see the Lipschitz-Killing curvature densities as points on a curve, that is now in $\mathbb{R}^3$. More precisely, considering

$$\tilde{x}(t) = \frac{\overline{\mathrm{SA}}_X(t)}{\overline{\mathrm{SA}}_X(m)} = e^{-(t-m)^2/2\sigma^2},$$

$$\begin{aligned}\tilde{y}(t) = \frac{\overline{\mathrm{TMC}}_X(t)}{\overline{\mathrm{SA}}_X(m)^2} &= \frac{\overline{\gamma}_{\mathrm{TMC}}}{\overline{\gamma}_{\mathrm{SA}}} \frac{\pi\sqrt{\pi}}{4\sqrt{2}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2} \\ &= R \frac{\pi\sqrt{\pi}}{4\sqrt{2}} \frac{t-m}{\sigma} e^{-(t-m)^2/2\sigma^2},\end{aligned}$$

$$\begin{aligned}\text{and } \tilde{z}(t) = \frac{\overline{\mathrm{TGC}}_X(t)}{\overline{\mathrm{SA}}_X(m)^3} &= \left(\frac{\overline{\gamma}_{\mathrm{TGC}}}{\overline{\gamma}_{\mathrm{SA}}}\right)^{3/2} \frac{\pi^2}{8} \left(\frac{(t-m)^2}{\sigma^2} - 1\right) e^{-(t-m)^2/2\sigma^2} \\ &= R_G^{3/2} \frac{\pi^2}{8} \left(\frac{(t-m)^2}{\sigma^2} - 1\right) e^{-(t-m)^2/2\sigma^2},\end{aligned}$$

then this defines a parametric curve in $\mathbb{R}^3$, that has as a projection on the first two coordinates an almond curve of cartesian equation

$$\mathcal{C}_R^{(3d)} = \{(x, y) \in \mathbb{R}^2; y^2 + \frac{\pi^3}{16} R^2 x^2 \log x = 0\},$$

and on the last two coordinates, the projection is a parametric curve of the form

$$\tilde{\mathcal{C}}_{R,R_G}^{(3d)} = \{(y(s), z(s)) \in \mathbb{R}^2 \,;\, y(s) = R\frac{\pi\sqrt{\pi}}{4\sqrt{2}} s e^{-s^2/2}, z(s) = R_G^{3/2} \frac{\pi^2}{8}(s^2-1)e^{-s^2/2},\ s \in \mathbb{R}\}.$$

These two curves are shown on the first line of Figure 7, where the blue curves correspond to the isotropic case ($R = R_G = 1$) and the black ones correspond on the figures to $R \simeq 0.91$ and $R_G \simeq 0.84$. These black curves are inside the domains defined by the isotropic (blue) curves.

4.2.2. Estimating the anisotropy from one excursion set

Again, as in the 2D case, assume we observe an excursion set $\hat{E}$ of a sample of a Gaussian stationary random field X, in a window U. The question is then: how to estimate the anisotropy of X from this set $\hat{E}$ only, that is without having access to the whole field (nor its increments for instance). We ignore also at which level t the field X was thresholded, and also what was its mean m or its variance σ^2. Now, from the four observed Lipschitz-Killing curvatures densities (that are $\widehat{\mathrm{V}}(E) = \mathrm{V}(\hat{E})/\mathcal{L}^3(U)$ the normalized Volume of $\hat{E}$, $\widehat{\mathrm{SA}}(E) = \mathrm{SA}(\hat{E})/\mathcal{L}^3(U)$ the normalized Surface Area of $\hat{E}$, $\widehat{\mathrm{TMC}}(E) = \mathrm{TMC}(\hat{E})/\mathcal{L}^3(U)$ the normalized Total Mean Curvature of $\hat{E}$ and $\widehat{\mathrm{TGC}}(E) = \mathrm{TGC}(\hat{E})/\mathcal{L}^3(U)$ the normalized Total Gaussian Curvature of $\hat{E}$), we can proceed the following way:
- Compute an "effective" threshold $\widehat{t}$, that is an empirical estimate of $\frac{t-m}{\sigma}$, using the volume of $\hat{E}$:

$$\widehat{t} = \Phi^{-1}(1 - \widehat{\mathrm{V}}(E)).$$

- Then estimate $\overline{\mathrm{SA}}_X(m)$ by

$$\widehat{\overline{\mathrm{SA}}_m} := \widehat{\mathrm{SA}}(E) e^{\widehat{t}^2/2}.$$

- And finally estimate the anisotropy ratios via

$$\widehat{R} = \frac{\widehat{\mathrm{TMC}}(E)}{\widehat{\mathrm{SA}}(E)^2} \frac{4\sqrt{2}}{\pi\sqrt{\pi}} \frac{1}{\widehat{t}} e^{-\widehat{t}^2/2} \quad \text{and} \quad \widehat{R_G}^{3/2} = \frac{\widehat{\mathrm{TGC}}(E)}{\widehat{\mathrm{SA}}(E)^3} \frac{8}{\pi^2} \frac{1}{(\widehat{t}^2 - 1)} e^{-\widehat{t}^2}.$$

As in the 2D case, we have performed some numerical experiments to illustrate all this. Let us mention that such numerical experiments are much more difficult to conduct than in the 2D case. The memory size and the computation time needed to proceed with a 3D volume are very high. Only "small" volumes were considered, with size 200^3 voxels. Therefore the variances in the estimation of the Lipschitz-Killing curvatures is quite high, see Figure 6. The estimation of the anisotropy ratios from one excursion set is illustrated on Figure 7 (anisotropic case). See the caption of the figures for more details. Additional experiments are also provided in the experimental section B of the Appendix.

5. Further remarks

5.1. Link with the kinematic formula

For sake of simplicity, let us assume that $X = (X(x))_{x \in \mathbb{R}^d}$ is a stationary smooth centered Gaussian random field with unit variance, whose excursion set is given by $E_X(t)$ for some $t \in \mathbb{R}$. By the kinematic formula of Theorem 9.4.1 of [31], under the assumptions that $Z := E_X(t)$ is a standard set (see Definition 9.2.1), it follows (see end of p.416) that for $r > 0$,

$$\mathbb{E}\left(V_j\left(Z \cap \overline{B}(0,r)\right)\right) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \overline{V}_k(Z) V_{d-k+j}(\overline{B}(0,r)),$$

where, following the notations of [31], $c_{j,d}^{k,d-k+j} = \frac{\gamma\left(\frac{k+1}{2}\right)\gamma\left(\frac{d-k+j+1}{2}\right)}{\gamma\left(\frac{j+1}{2}\right)\gamma\left(\frac{d+1}{2}\right)}$, with $\gamma(a) = \int_0^{+\infty} t^{a-1}e^{-t}dt$ for $a > 0$, and the intrinsic volumes V_l correspond to Lipschitz-Killing curvatures C_l. Moreover,

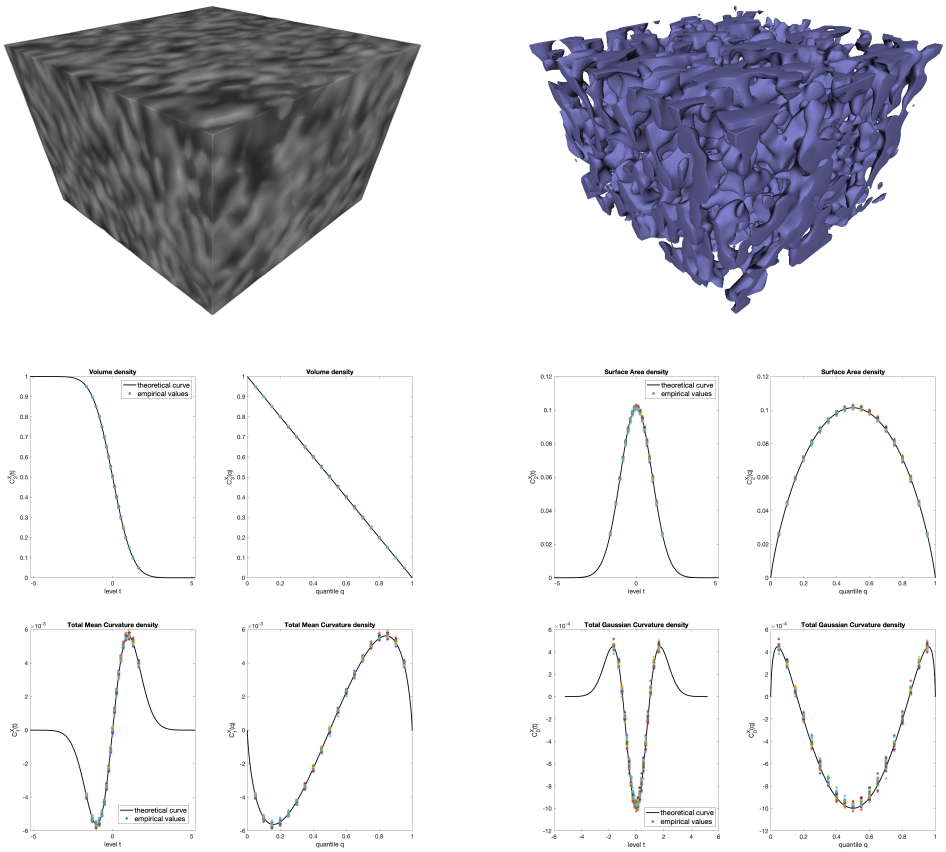

FIG 6. *Lipschitz-Killing curvatures densities of the excursion sets of an isotropic Gaussian volume. Here we have taken* 20 *samples of a 3D volume of size* 200^3 *voxels, of a Gaussian field with mean* $m = 0$*, variance* $\sigma^2 = 1$ *and covariance of the form* $\exp(-\gamma_1 x_1^2 - \gamma_2 x_2^2 - \gamma_3 x_3^2)$ *with* $\gamma_1 = 0.01$*,* $\gamma_2 = 0.02$ *and* $\gamma_3 = 0.05$*. An example of such a Gaussian volume is shown on the left of the first line, with one excursion set on the right. Using the Matlab toolbox ImMinkowski [27], we have estimated the volume densities (second line, left), the surface area densities (second line, right), the total mean curvature densities (third line, left) and the total Gaussian curvature densities (third line, right). The densities are plotted else as functions of the level* t *or as functions of the quantile* q*. The stars are the values for the* 20 *different samples.*

following Corollary 9.4.1, we can identify $\overline{V}_k(Z)$ with $\overline{C}_k^X(t)$ and rewrite the previous formula as

$$\mathbb{E}\left(C_j\left(Z \cap \overline{B}(0,r)\right)\right) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_{d-k+j}(\overline{B}(0,1)) \overline{C}_k^X(t) r^{d-k+j}.$$

Meanwhile, under the additional assumption that the joint distribution of the Gaussian vector $\mathbf{X}(0)$ is non degenerate, the stationary random field X fulfills the assumptions of Theorem 4.2.1 of [3] and the Gaussian kinematic formula states that (see (4.0.1) therein)

$$\begin{aligned}
\mathbb{E}\left(\mathcal{L}_j^{\nabla}\left(Z \cap \overline{B}(0,r)\right)\right) &= \sum_{i=0}^{d-j} \left[\begin{array}{c} i+j \\ i \end{array} \right] \mathcal{L}_{i+j}^{\nabla}(\overline{B}(0,r))(2\pi)^{-i/2} h_{i-1}(t) \\
&= \sum_{k=j}^{d} \left[\begin{array}{c} d-k+j \\ d-k \end{array} \right] \mathcal{L}_{d-k+j}^{\nabla}(\overline{B}(0,1))(2\pi)^{-(d-k)/2} h_{d-k-1}(t) r^{d-k+j},
\end{aligned}$$

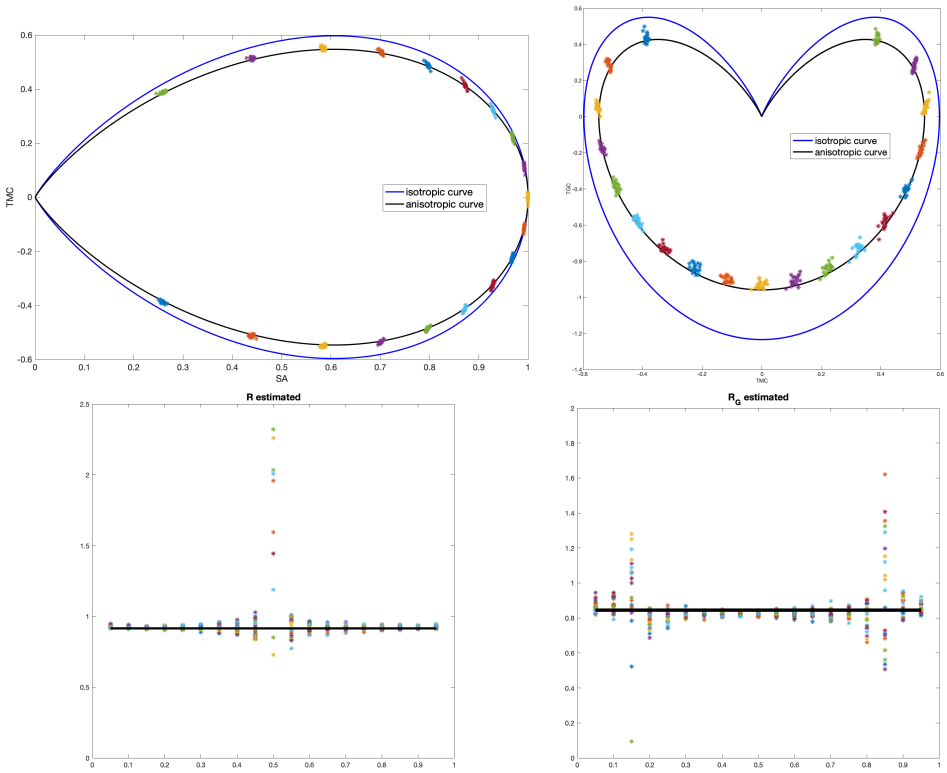

FIG 7. *Estimation of R and R_G from the Lipschitz-Killing curvatures densities of an excursion set. The samples are the same as the ones of Figure 6. Notice that, because of the term $1/\widehat{t}$, the estimation of R is inaccurate for values around $\widehat{t}=0$, whereas for R_G the inaccuracy of estimation occurs around $\widehat{t}=\pm 1$.*

with the change of variable $k = d - i$, where $\mathcal{L}_k^{\nabla}$ denotes the Lipschitz-Killing curvatures under the Riemannian metric given by Γ, the covariance matrix of $\nabla X(0)$,

$$\left[\begin{array}{c} k \\ k-j \end{array} \right] = \binom{k}{k-j} \frac{\omega_k}{\omega_{k-j}\omega_j},$$

with ω_k the volume of the k-dimensional unit ball of $\mathbb{R}^k$, that is

$$\omega_0 = 1, \quad \omega_1 = 2, \quad \omega_2 = \pi, \quad \omega_3 = \frac{4}{3}\pi, \quad \text{etc..}$$

and $h_{-1}(t) = \Phi(t)$, whereas $h_k(t) = \frac{1}{\sqrt{2\pi}} H_k(t) e^{-t^2/2}$ are the Hermite functions of order $k \in \mathbb{N}$.

For $j = 0$ we obtain the mean Euler characteristic of $E_X(t) \cap \overline{B}(0,r)$ as

$$\begin{aligned} \mathbb{E}\left(\chi(E_X(t) \cap \overline{B}(0,r)\right) &= \sum_{k=0}^{d} \mathcal{L}_{d-k}^{\nabla}(\overline{B}(0,1))(2\pi)^{-(d-k)/2} h_{d-k-1}(t) r^{d-k} \\ &= \sum_{k=0}^{d} c_{0,d}^{k,d-k} V_{d-k}(\overline{B}(0,1)) \overline{C}_k^X(t) r^{d-k}. \end{aligned}$$

Hence, identifying, we get for all $0 \leq k \leq d$,

$$\overline{C}_k^X(t) = \frac{\mathcal{L}_{d-k}^{\nabla}(\overline{B}(0,1))}{V_{d-k}(\overline{B}(0,1))} c_{d,k} h_{d-k-1}(t),$$

where we set $c_{d,k} = \frac{(2\pi)^{-(d-k)/2}}{c_{0,d}^{k,d-k}}$. Note that $c_{d,d} = 1$, $c_{d,0} = (2\pi)^{-d/2}$, and $c_{2,1} = (2\pi)^{-1/2}\frac{\gamma(1/2)^2}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$, while $c_{3,1} = (2\pi)^{-1}\left(\frac{1}{2}\right)^{-1} = \frac{1}{\pi}$ and $c_{3,2} = (2\pi)^{-1/2}\left(\frac{1}{2}\right)^{-1} = \sqrt{\frac{2}{\pi}}$. Hence we can define geometrical second spectral moments for $0 \leq k \leq d-1$ by

$$\overline{\lambda}_k = \left(\frac{\mathcal{L}^{\nabla}_{d-k}(\overline{B}(0,1))}{V_{d-k}(\overline{B}(0,1))}\right)^{2/(d-k)}.$$

Note that assuming that $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_d)$ we obtain that $\overline{\lambda}_k = \left(\frac{V_{d-k}(\mathcal{E}\ell\ell(\Gamma))}{V_{d-k}(\overline{B}(0,1))}\right)^{2/(d-k)}$, where $\mathcal{E}\ell\ell(\Gamma)$ is the Ellipsoid of semi-axes $\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_d}$ (see [22] for explicit computations). Then, extended isoperimetric inequalities (see (1.1) of [29]) rewrite as

$$\forall 0 \leq k \leq d-1,\ \overline{\lambda}_0 \leq \overline{\lambda}_k,$$

while the generalized Urysohn inequality (see (1.2) of [29]) states that

$$\forall 0 \leq k \leq d-1,\ \overline{\lambda}_k \leq \overline{\lambda}_{d-1},$$

in accordance with our results for dimension $d \in \{2,3\}$ with $\overline{\lambda}_0 = \overline{\gamma}_{\mathrm{TC}}$, $\overline{\lambda}_1 = \overline{\gamma}_{\mathrm{Per}}$ in dimension $d = 2$ and $\overline{\lambda}_0 = \overline{\gamma}_{\mathrm{TGC}}$, $\overline{\lambda}_1 = \overline{\gamma}_{\mathrm{TMC}}$, $\overline{\lambda}_2 = \overline{\gamma}_{\mathrm{SA}}$ in dimension $d = 3$.

Finally, let us end by remarking that the mean Euler characteristic of $E_X(t) \cap [0,T]^d$ for some $T > 0$ is given according to Theorem 4.4.1 of [3] by

$$\mathbb{E}\left(\chi(E_X(t) \cap [0,T]^d)\right) = \sum_{k=0}^{d} \mathcal{L}^{\nabla}_{d-k+j}([0,T]^d)(2\pi)^{-(d-k)/2} h_{d-k-1}(t).$$

and therefore, the Euler characteristic density of $E_X(t) \cap [0,T]^d$ satisfies

$$\lim_{T\to+\infty} \frac{1}{T^d}\mathbb{E}(\chi(E_X(t) \cap [0,T]^d)) = (\gamma_1 \ldots \gamma_d)^{1/2}(2\pi)^{-d/2} h_{d-1}(t) = \overline{C}_0^X(t).$$

5.2. A functional Weyl's tube formula

Considering a compact d-dimensional submanifold in $\mathbb{R}^d$ with a C^2 smooth boundary, according to Weyl's tube formula [35] also called Steiner's formula [18], the Lipschitz-Killing curvatures are related to the volume of the r-parallel set (also called r-extension or r-dilation) of M by

$$\mathcal{L}^d(M_r) = \sum_{k=0}^{d} \omega_k C_{d-k}(M) r^k,$$

where $M_r = M \oplus B(0,r) = \{x; \mathrm{dist}(x,M) \leq r\}$ is the r-parallel set to M, and where ω_k is still the volume of the k-dimensional unit ball of $\mathbb{R}^k$. The above tube formula is valid for r small enough (more precisely smaller than the so-called reach of M, denoted by $\mathrm{reach}(M)$, and that is strictly positive when M is a smooth compact submanifold). Now, this formula can be localized in the sense that if we consider an open bounded subset $U \subset \mathbb{R}^d$, the above tube formula becomes (see [18] or [34] Theorem 26), for $0 < r < \mathrm{reach}(M)$,

$$\mathcal{L}^d(M_{r,U}) = \sum_{k=0}^{d} \omega_k C_{d-k}(M,U) r^k, \tag{23}$$

where here $M_{r,U} = \{x; \mathrm{dist}(x,M) \leq r \text{ and } \Pi_M(x) \in U\}$, with $\Pi_M(x)$ being the projection of x on M, that is the closest point to x in M which is unique when $\mathrm{dist}(x,M) \leq r < \mathrm{reach}(M)$. Notice that generally $M_{r,U}$ is not equal to $M_r \cap U$. Here the Lipschitz-Killing curvatures $C_{d-k}(M,U)$

are localized in U and their formula involving the principal curvatures are analogous to the ones of the $C_{d-k}(M)$ given in Section 2 except that the integrals on ∂M are replaced by integrals on $\partial M \cap U$. These are the quantities that we have considered in the previous sections.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, and let U be a bounded open subset of $\mathbb{R}^d$. For almost every $t \in \mathbb{R}$, $E_f(t) = \{x \in \mathbb{R}^d\,;\, f(x) \geq t\}$ is a smooth manifold and we can consider its Lipschitz-Killing curvatures in U. We denote them by $C_k(E_f(t), U)$ for $k = 0, \dots, d$. As recalled by Weyl's tube formula (23), we can obtain these Lipschitz-Killing curvatures as the coefficients of the powers of r in the volume of the r-parallel set to $E_f(t)$. More precisely we have, for r small enough,

$$\mathcal{L}^d((E_f(t))_{r,U}) = \sum_{k=0}^{d} \omega_k C_{d-k}(E_f(t), U) r^k.$$

In Section 2.2, we have 1) used the formulas for the C_k involving the symmetric functions of the curvatures, and the explicit formulas for these curvatures in 2D or 3D; 2) integrate these with a test function h to have formulas for $\int_t h(t) C_k(E_f(t), U)\, dt$; 3) and finally these integrals were written as an integral on $x \in U$ thanks to the coarea formula [17].

But we could have done it the other way round. Indeed, we can use directly the coarea formula on $\int_t h(t)\mathcal{L}^d((E_f(t))_{r,U})\, dt$, without knowing the formulas for the curvatures, and in fact recovering them.

More precisely, let h be a test function, and for $z = (x, s) \in U \times \mathbb{R}_+ \subset \mathbb{R}^{d+1}$, let us define the functions $\tilde{h} : U \times [0, r] \to \mathbb{R}$ and $F : U \times [0, r] \to \mathbb{R}^{d+1}$ by $\tilde{h}(z) = \tilde{h}(x, s) = h(f(x))$ and

$$F(x, s) = (f(x), x - s\,\nu_f(x))\,,$$

where we denote

$$\nu_f(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|} \in S^{d-1} \quad \text{if} \quad \nabla f(x) \neq 0, \quad \text{and} \quad \nu_f(x) = 0 \quad \text{if} \quad \nabla f(x) = 0.$$

A simple computation shows that ν_f is differentiable at all x such that $\nabla f(x) \neq 0$ and that

$$D\nu_f(x) = \frac{1}{\|\nabla f(x)\|} D^2 f(x) - \frac{1}{\|\nabla f(x)\|^3} \nabla f(x) (D^2 f(x) \nabla f(x))^t.$$

Now, let K be a compact ball containing $\overline{U}_1$ (1-dilation of $\overline{U}$). Then for $\varepsilon > 0$, let us consider the following set of values

$$V_\varepsilon = \{t \in \mathbb{R}\,;\, \mathrm{reach}(E_f(t) \cap K) \geq \varepsilon\}.$$

Then, since by Morse-Sard's theorem the set of critical values has Lebesgue measure 0 in $\mathbb{R}$, we have that $\mathcal{L}^1(\mathbb{R} \setminus \cup_{\varepsilon > 0} V_\varepsilon) = 0$. Then, let us denote $G_f^\varepsilon = f^{-1}(V_\varepsilon)$. For $r < \varepsilon$ and $t \in V_\varepsilon$, we have that

$$E_f(t)_{r,U} \setminus (E_f(t) \cap U) = \{x - s\,\nu_f(x)\,;\, x \in U \text{ and } f(x) = t \text{ and } s \in (0, r)\}.$$

Therefore, by the coarea formula:

$$\int_{(U \cap G_f^\varepsilon) \times [0,r]} h(f(x)) \big|\det(DF(x, s))\big|\, dx\, ds = \int_{V_\varepsilon} h(t)(\mathcal{L}^d((E_f(t))_{r,U}) - \mathcal{L}^d(E_f(t) \cap U))\, dt.$$

The right-hand term is equal to $\sum_{k=1}^{d} \omega_k r^k \int_{V_\varepsilon} h(t) C_{d-k}(E_f(t), U)\, dt$, whereas in the left-hand term we have

$$\det(DF(x, s)) = \det \begin{pmatrix} \nabla f(x)^t & 0 \\ \mathbf{I}_d - s D\nu_f(x) & -\nu_f(x) \end{pmatrix} = \|\nabla f(x)\| \det \begin{pmatrix} \nu_f(x)^t & 0 \\ \mathbf{I}_d - s\, D\nu_f(x) & -\nu_f(x) \end{pmatrix}. \tag{24}$$

This determinant is a polynomial in s, with degree $d - 1$ and with constant term (corresponding to $s = 0$) equal to ± 1.

To identify the values of $\int_{\mathbb{R}} h(t)C_{d-k}(E_f(t),U)\,dt$ for each k we just need to develop the above determinant, and then let ε goes to 0.

When the dimension is $d=2$, we have that

$$\det(DF(x,s)) = \|\nabla f(x)\|\big(1 - s\,(\mathrm{Tr}(D\nu_f(x)) - \nu_f(x)^t D\nu_f(x)\nu_f(x))\big),$$

which is positive for s small enough. Therefore, computing the integral on s between 0 and r, we get a polynomial in r, that allows us to identify its coefficients, and we get

$$\omega_1 \int h(t)C_1(E_f(t),U)\,dt = \int_U h(f(x))\|\nabla f(x)\|\,dx,$$

and

$$\begin{aligned}\omega_2 \int h(t)C_0(E_f(t),U)\,dt &= -\frac{1}{2}\int_U h(f(x))\|\nabla f(x)\|\big(\mathrm{Tr}(D\nu_f(x)) - \nu_f(x)^t D\nu_f(x)\nu_f(x)\big)\,dx \\ &= -\frac{1}{2}\int_U h(f(x))\left(\mathrm{Tr}(D^2 f(x)) - \frac{\nabla f(x)^t D^2 f(x)\nabla f(x)}{\|\nabla f(x)\|^2}\right)dx.\end{aligned}$$

We have thus recovered the formula for the curvature in dimension $d=2$.

When the dimension is $d=3$, considering a direct orthonormal basis of $\mathbb{R}^3$ of the form $(\nu_f(x),T_1(x),T_2(x))$, developing the above determinant in this basis, we get

$$\det(DF(x,s)) = \|\nabla f(x)\|(1 - s\,\mathrm{Tr}(D\nu_f(x)) + s^2\det(\nu_f(x), D\nu_f(x)^t T_1(x), D\nu_f(x)^t T_2(x))),$$

which is positive for s small enough. Therefore, computing the integral on s between 0 and r, and identifying the coefficients, we get

$$\omega_1 \int h(t)C_2(E_f(t),U)\,dt = \int_U h(f(x))\|\nabla f(x)\|\,dx,$$

$$\begin{aligned}\omega_2 \int h(t)C_1(E_f(t),U)\,dt &= -\frac{1}{2}\int_U h(f(x))\|\nabla f(x)\|\mathrm{Tr}(D\nu_f(x))\,dx \\ &= -\frac{1}{2}\int_U h(f(x))\left(\mathrm{Tr}(D^2 f(x)) - \frac{\nabla f(x)^t D^2 f(x)\nabla f(x)}{\|\nabla f(x)\|^2}\right)dx,\end{aligned}$$

and

$$\begin{aligned}\omega_3 \int h(t)C_0(E_f(t),U)\,dt &= \frac{1}{3}\int_U h(f(x))\|\nabla f(x)\|\det(\nu_f(x), D\nu_f(x)^t T_1(x), D\nu_f(x)^t T_2(x)))\,dx \\ &= \frac{1}{3}\int_U h(f(x))\frac{\nabla f(x)^t \mathrm{adj}(D^2 f(x))\nabla f(x)}{\|\nabla f(x)\|^3}\,dx,\end{aligned}$$

where $\mathrm{adj}(D^2 f(x))$ is the adjugate matrix of $D^2 f(x)$ (that is the matrix of its cofactors). We have thus recovered the formulas for the mean curvature and the Gaussian curvature for implicit surfaces in $\mathbb{R}^3$.

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Appendix A: Central Limit Theorem

A.1. General results

We assume that X is a C^2 stationary second order field (not necessarily Gaussian) and write as before $\mathbf{X} = (X, \nabla X, D^2X)$ the vector-valued field with values in $\mathbb{R}^s = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2}$, following [16]. We recall that since X is C^2 stationary second order, the covariance function $x \mapsto \rho(x) = \mathrm{Cov}(X(x), X(0))$ is C^4 and for $0 \leq i, j, k, l \leq d$

$$\mathrm{Cov}(X_{ij}(x), X_{kl}(0)) = (-1)^{\varepsilon(k)+\varepsilon(l)} \partial_{ijkl}^{\varepsilon(i)+\varepsilon(j)+\varepsilon(k)+\varepsilon(l)} \rho(x),$$

where $\varepsilon(i) = 1$ if $i \neq 0$ and $X_{0j}(x) = X_{j0}(x) = X_j(x)$ while $X_{00}(x) = X(x)$. Hence we introduce

$$\tilde{\rho}(x) = \max_{|\mathbf{k}| \leq 4} |\partial^{\mathbf{k}} \rho(x)|,$$

where $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $|\mathbf{k}| = \sum_{i=1}^d k_i$. In order to prove a general central limit theorem, we will work in this section under a strong quasi-association assumption, namely, we assume that there exists $C \geq 1$ such that, for all finite subsets I and J of $\mathbb{Z}^d$, for all $p \geq 1$ and all $\vec{x} = (x_1, \ldots, x_p) \in [0,1]^{dp}$, for all Lipschitz functions $f : \mathbb{R}^{s|I|p} \to \mathbb{R}$ and $g : \mathbb{R}^{s|J|p} \to \mathbb{R}$, we have

$$|\mathrm{Cov}(f(\mathbf{X}_{\vec{x},I}), g(\mathbf{X}_{\vec{x},J})| \leq C \mathrm{Lip}(f) \mathrm{Lip}(g) \sum_{\mathbf{i} \in I} \sum_{\mathbf{j} \in J} \sum_{l=1}^{p} \sum_{l'=1}^{p} \tilde{\rho}(x_l + \mathbf{i} - (x_{l'} + \mathbf{j})), \tag{25}$$

where $\mathbf{X}_{\vec{x},I} = (\mathbf{X}(x_l + \mathbf{i}), \mathbf{i} \in I, 1 \leq l \leq p)$. For $h \in \mathcal{S}$ and $q \in \mathbb{N}$, we write $\|h\|_q = \|h\|_{q,0} + \|h\|_{q,1}$. Note that h is a Lipschitz bounded function with $\mathrm{Lip}(h) = \|h\|_{0,1} \leq \|h\|_{q,1}$ and that $\|h\|_{0,0} \leq \|h\|_{q,0}$. We consider a sequence $(\mathcal{T}_n)_n$ of generalized random processes given, for $n \in \mathbb{N}$, $U_n = (0,n)^d$ and $h \in \mathcal{S}$, by

$$\begin{aligned} \langle \mathcal{T}_n, h \rangle &= \sqrt{\mathcal{L}^d(U_n)} \left(\frac{1}{\mathcal{L}^d(U_n)} \int_{U_n} L(h, \mathbf{X}(x)) dx - \mathbb{E}(L(h, \mathbf{X}(0))) \right) \\ &= \int_{[0,1]^d} S_n(h, x) dx \end{aligned} \tag{26}$$

with, for $B = (0,1]^d$,

$$S_n(h,x) = \frac{1}{n^{d/2}} \sum_{\mathbf{k} \in nB \cap \mathbb{Z}^d} (L(h, \mathbf{X}(x + \mathbf{k})) - \mathbb{E}(L(h, \mathbf{X}(0))))$$

and $h \in \mathcal{S} \mapsto L(h, \mathbf{x}) \in \mathbb{R}$ is linear continuous for all $\mathbf{x} \in \mathbb{R}^s$. Now let $p \geq 1$ and $\vec{x} = (x_1, \ldots, x_p) \in ([0,1]^d)^p$ be fixed. We consider the stationary vector-valued random field $Y = (Y_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ with values in $\mathbb{R}^p$ where

$$Y_{\mathbf{k}} = (L(h, \mathbf{X}(x_1 + \mathbf{k}), \ldots, L(h, \mathbf{X}(x_p + \mathbf{k})))) := L(h, \mathbf{X}_{\vec{x}+\mathbf{k}}).$$

The quasi-association assumption (25) will imply some dependence properties on the stationary vector-valued random field $Y = (Y_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$.

In the following, we will use the notation $BL(q)$ for $q \geq 1$ to denote the set of functions from $\mathbb{R}^q$ to $\mathbb{R}$ that are Lipschitz and bounded.

Definition A.1 (Definition 5.14 p.94 of [10]). The random field Y is called (BL, ψ, θ) dependent if there exists a non-increasing sequence $\theta = (\theta_r)_{r \in \mathbb{N}}$ with $\lim_{r \to +\infty} \theta_r = 0$ such that for any disjoint finite sets $I, J \subset \mathbb{Z}^d$ with $\mathrm{dist}(I,J) \geq r$, and any bounded Lipschitz functions $f \in BL(p|I|)$, $g \in BL(p|J|)$, one has

$$|\mathrm{Cov}\,(f(Y_I), g(Y_J))| \leq \psi(|I|, |J|, f, g) \theta_r.$$

The case where $\psi(|I|, |J|, f, g) = \mathrm{Lip}(f)\mathrm{Lip}(g) \min(|I|, |J|)$ is simply called (BL, θ) dependence.

Note that when $\mathbf{x} \in \mathbb{R}^s \mapsto L(h,\mathbf{x})$ is a Lipschitz function, assumption (25) implies that $Y = (Y_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ is a stationary (BL,θ) dependent field. Then, by Theorem 1.12 p. 178 of [10], the sequence $(S_n(h,x))_n$ will have a Gaussian limit with variance σ^2 as soon as $(\mathrm{Var}(S_n(h,x)))_n$ tends to σ^2. However, for our specific application, we can not assume that the functional L is Lipschitz but we can assume that L may be approximated by a Lipschitz function according to the following hypothesis:

There exist $\alpha > 0$ and $C \geq 1$ such that $\forall \varepsilon \in (0,1]$, there is $T_\varepsilon : \mathbb{R}^s \to [0,1]$ such that

$$F_{h,\varepsilon} : \mathbf{x} \in \mathbb{R}^s \mapsto L(h,\mathbf{x})T_\varepsilon(\mathbf{x}) \in \mathbb{R} \text{ is Lipschitz with } \mathrm{Lip}(F_{h,\varepsilon}) \leq C\|h\|_q \varepsilon^{-1},$$
$$\mathbb{E}\left(L(h,\mathbf{X}(x))^2\right) \leq C\|h\|_{q,0}^2 \quad \text{and} \quad \mathbb{E}\left(L(h,\mathbf{X}(x))^2(1-T_\varepsilon(\mathbf{X}(x)))\right) \leq C\|h\|_{q,0}^2\varepsilon^{2\alpha}. \quad (\mathbf{LA})$$

Proposition 3. *Assuming that (25) and (LA) hold, if*

$$\sum_{\mathbf{i}\in\mathbb{Z}^d} \sup_{\|y\|_\infty \leq 1} |\tilde{\rho}(y+\mathbf{i})| < +\infty,$$

then the random field Y is (BL,ψ,θ) dependent with

$$\psi(|I|,|J|,f,g) = \mathrm{Lip}(f)\|g\|_\infty|I| + \mathrm{Lip}(g)\|f\|_\infty|J| + \min(|I|,|J|)\mathrm{Lip}(f)\mathrm{Lip}(g) \quad (27)$$

and for $C_p(h) = 5C^3p^2\|h\|_q(1+\|h\|_q)$,

$$\theta_r = C_p(h)\max\left(\left(\sum_{\mathbf{i}\in\mathbb{Z}^d;|i|\geq r}\sup_{\|y\|_\infty\leq 1}|\tilde{\rho}(y+\mathbf{i})|\right)^{\frac{\alpha}{\alpha+2}}, \sum_{\mathbf{i}\in\mathbb{Z}^d;|i|\geq r}\sup_{\|y\|_\infty\leq 1}|\tilde{\rho}(y+\mathbf{i})|\right). \quad (28)$$

Proof. Let $r \in \mathbb{N}$ and denote $\tilde{\theta}_r = \sum_{\mathbf{i}\in\mathbb{Z}^d;|i|\geq r}\sup_{\|y\|_\infty\leq 1}|\tilde{\rho}(y+\mathbf{i})|$. We set $I,J \subset \mathbb{Z}^d$ with $\mathrm{dist}(I,J) \geq r$ and $|J| \leq |I|$ such that for any $y \in [-1,1]^d$

$$\sum_{\mathbf{i}\in I}\tilde{\rho}(y+\mathbf{i}-\mathbf{j}) \leq \tilde{\theta}(r), \text{ for all } \mathbf{j} \in J.$$

For $f \in BL(p|I|)$ and $\varepsilon \in (0,1]$ to be fixed later, we write $f \circ F_{h,\varepsilon}$ with a slight abuse of notation to denote the function that is in $BL(sp|I|)$ with $\mathrm{Lip}(f \circ F_{h,\varepsilon}) \leq \mathrm{Lip}(f)\mathrm{Lip}(F_{h,\varepsilon})$. Hence for $g \in BL(p|J|)$, we get by (25)

$$\begin{aligned}|\mathrm{Cov}(f\circ F_{h,\varepsilon}(\mathbf{X}_{\vec{x},I}), g\circ F_{h,\varepsilon}(\mathbf{X}_{\vec{x},J})| &\leq C\mathrm{Lip}(f)\mathrm{Lip}(g)\mathrm{Lip}(F_{h,\varepsilon})^2\sum_{\mathbf{i}\in I}\sum_{\mathbf{j}\in J}\sum_{l=1}^{p}\sum_{l'=1}^{p}\tilde{\rho}(x_l+\mathbf{i}-(x_{l'}+\mathbf{j}))\\ &\leq C\mathrm{Lip}(f)\mathrm{Lip}(g)\mathrm{Lip}(F_{h,\varepsilon})^2|J|p^2\tilde{\theta}_r. \quad (29)\end{aligned}$$

Now, let us remark that

$$\begin{aligned}|\mathrm{Cov}(f(Y_I) - f\circ F_{h,\varepsilon}(\mathbf{X}_{\vec{x},I}), g(Y_J))| &\leq 2\|g\|_\infty\mathbb{E}(|f(Y_I) - f\circ F_{h,\varepsilon}(\mathbf{X}_{\vec{x},I})|)\\ &\leq 2\|g\|_\infty\mathrm{Lip}(f)\mathbb{E}\left(\|L(h,\mathbf{X}_{\vec{x},I}) - F_{h,\varepsilon}(\mathbf{X}_{\vec{x},I}))\|_\infty\right)\\ &\leq 2\|g\|_\infty\mathrm{Lip}(f)\sum_{\mathbf{k}\in I}\sum_{l=1}^{p}\mathbb{E}\left(|L(h,\mathbf{X}_{x_l+\mathbf{k}})|(1-T_\varepsilon(\mathbf{X}_{x_l+\mathbf{k}}))\right)\\ &\leq 2C^{1/2}p|I|\|g\|_\infty\mathrm{Lip}(f)\|h\|_{q,0}\varepsilon^\alpha,\end{aligned}$$

by assumption (**LA**), using Cauchy-Schwarz inequality. Similarly, one has

$$|\mathrm{Cov}(f\circ F_{h,\varepsilon}(\mathbf{X}_{\vec{x},I}), g(Y_J) - g\circ F_{h,\varepsilon}(\mathbf{X}_{\vec{x},J}))| \leq 2C^{1/2}p|J|\|f\|_\infty\mathrm{Lip}(g)\|h\|_{q,0}\varepsilon^\alpha.$$

Since $C \geq 1$, by definition of (27), it follows that

$$|\mathrm{Cov}\left(f(Y_I), g(Y_J)\right)| \leq \frac{C_p(h)}{5}\psi(|I|,|J|,f,g)\left(\tilde{\theta}_r\varepsilon^{-2} + 4\varepsilon^{\alpha}\right).$$

Now, when $\tilde{\theta}_r \in (0,1]$ we can choose $\varepsilon = (\tilde{\theta}_r)^{\frac{1}{\alpha+2}}$ to get

$$|\mathrm{Cov}\left(f(Y_I), g(Y_J)\right)| \leq C_p(h)\psi(|I|,|J|,f,g)\tilde{\theta}_r^{\frac{\alpha}{\alpha+2}}.$$

Otherwise, if $\tilde{\theta}_r = 0$ the result comes from letting ε tend to 0 and if $\tilde{\theta}_r \geq 1$ by choosing $\varepsilon = 1$. □

However, we can in fact be more precise in term of covariance control.

Proposition 4. *Assuming that (25) and (**LA**) hold, one has for all $x, x' \in \mathbb{R}^d$*

$$|\mathrm{Cov}\left(L(h,\mathbf{X}(x)), L(h,\mathbf{X}(x'))\right)| \leq 3C^3\|h\|_q^2 \max\left(\tilde{\rho}(x-x')^{\frac{\alpha}{\alpha+2}}, \tilde{\rho}(x-x')\right).$$

Proof. As previously, let us choose $\varepsilon \in (0,1]$ to be fixed later and write for $x \in \mathbb{R}^d$

$$L(h,\mathbf{X}(x)) = F_{h,\varepsilon}(\mathbf{X}(x)) + L(h,\mathbf{X}(x))\left(1 - T_\varepsilon\left(\mathbf{X}(x)\right)\right).$$

Noting that x may be written as $x = \tilde{x} + \mathbf{k}$ for some $\tilde{x} \in [0,1]^d$ and $\mathbf{k} \in \mathbb{Z}^d$, by (25) we simply get for $x' \in \mathbb{R}^d$,

$$\begin{aligned} |\mathrm{Cov}(F_{h,\varepsilon}(\mathbf{X}(x)), F_{h,\varepsilon}(\mathbf{X}(x'))| &\leq C\mathrm{Lip}(F_{h,\varepsilon})^2\tilde{\rho}(x-x') \\ &\leq C^3\|h\|_q^2\varepsilon^{-2}\tilde{\rho}(x-x'). \end{aligned}$$

Now, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & |\mathrm{Cov}(L(h,\mathbf{X}(x))\left(1 - T_\varepsilon\left(\mathbf{X}(x)\right)\right), L(h,\mathbf{X}(x'))| \\ \leq\ & \mathrm{Var}(L(h,\mathbf{X}(x))\left(1 - T_\varepsilon\left(\mathbf{X}(x)\right)\right)^{1/2}\mathrm{Var}(L(h,\mathbf{X}(x'))^{1/2} \\ \leq\ & C\|h\|_{q,0}^2\varepsilon^{\alpha}, \end{aligned}$$

by (**LA**) using that the variance is smaller than the second order moment and the fact that T_ε has values in $[0,1]$. Moreover, we obtain similarly

$$\begin{aligned} & |\mathrm{Cov}(F_{h,\varepsilon}(\mathbf{X}(x)), L(h,\mathbf{X}(x'))\left(1 - T_\varepsilon\left(\mathbf{X}(x')\right)\right))| \\ \leq\ & \mathrm{Var}(F_{h,\varepsilon}(\mathbf{X}(x))^{1/2}\mathrm{Var}(L(h,\mathbf{X}(x'))\left(1 - T_\varepsilon\left(\mathbf{X}(x')\right)\right)^{1/2} \\ \leq\ & C\|h\|_{q,0}^2\varepsilon^{\alpha}, \end{aligned}$$

since $\mathbb{E}\left(F_{h,\varepsilon}(\mathbf{X}(x))^2\right) \leq \mathbb{E}\left(L(h,\mathbf{X}(x))^2\right)$. Hence,

$$|\mathrm{Cov}\left(L(h,\mathbf{X}(x)), L(h,\mathbf{X}(x'))\right)| \leq C^3\|h\|_q^2\left(\varepsilon^{-2}\tilde{\rho}(x-x') + 2\varepsilon^{\alpha}\right).$$

Note that if $\tilde{\rho}(x-x') = 0$ we get the upper-bound by letting ε tends to 0. If $\tilde{\rho}(x-x') \in (0,1]$ we choose $\varepsilon = \tilde{\rho}(x-x')^{\frac{1}{\alpha+2}}$ and otherwise we choose $\varepsilon = 1$ to get the result.

□

Then, under a natural decay assumption, we can define the asymptotic covariances of the sequence $(S_n(h,x))_n$. In particular this ensures the so-called finite susceptibility assumption (see (5.3) p.90).

Corollary 3. *Assuming that (25) and (**LA**) hold, we also assume that there exists $\beta > \frac{d(\alpha+2)}{\alpha}$ such that, for all $x \in \mathbb{R}^d$,*

$$\tilde{\rho}(x) \leq C(1+|x|)^{-\beta}. \tag{30}$$

Then, there exists $C_{\alpha,\beta,d} > 0$ *such that, for all* $x, y \in \mathbb{R}^d$,

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathrm{Cov}\left(L(h,\mathbf{X}(x+\mathbf{k})), L(h,\mathbf{X}(y))\right)| \leq C_{\alpha,\beta,d}\|h\|_q^2 < +\infty.$$

Hence, there exists a real-valued stationary Gaussian random field $(B(h,x))_{x\in\mathbb{R}^d}$ *with covariance function given by*

$$\mathrm{Cov}(B(h,x), B(h,y)) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathrm{Cov}\left(L(h,\mathbf{X}(x+\mathbf{k})), L(h,\mathbf{X}(y))\right). \tag{31}$$

such that

$$\mathrm{Cov}\left(S_n(h,x), S_n(h,y)\right) \underset{n\to+\infty}{\longrightarrow} \mathrm{Cov}(B(h,x), B(h,y)).$$

Proof. Note that by Proposition 4, since $\beta\frac{\alpha}{\alpha+2} > d$, one has for all $x, y \in \mathbb{R}^d$,

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathrm{Cov}\left(L(h,\mathbf{X}(\mathbf{k}+x)), L(h,\mathbf{X}(y))\right)| \leq 3C^4\|h\|_q^2 \sum_{k\in\mathbb{Z}^d} (1+|\mathbf{k}+x-y|)^{-\frac{\beta\alpha}{\alpha+2}},$$

and one can choose $C_{\alpha,\beta,d} = 3C^4\int_{\mathbb{R}^d}(1+|z|)^{-\frac{\beta\alpha}{\alpha+2}}dz < +\infty$ since $\beta > \frac{d(\alpha+2)}{\alpha}$. Moreover by stationarity of $(L(h,\mathbf{X}(x)))_{x\in\mathbb{R}^d}$ one has

$$\mathrm{Cov}\left(S_n(h,x), S_n(h,y)\right) = \sum_{\mathbf{k}\in((-n,n)\cap\mathbb{Z})^d} \prod_{j=1}^{d}\left(1-\frac{|k_i|}{n}\right)\mathrm{Cov}\left(L(h,\mathbf{X}(\mathbf{k}+x)), L(h,\mathbf{X}(y))\right).$$

Hence one can define the stationary Gaussian field $(B(h,x))_{x\in\mathbb{R}^d}$ with covariance function given by (31) since this function is of non-negative type as a limit of covariance functions. □

Note that when $\mathbf{x} \mapsto L(h,\mathbf{x})$ is Lipschitz with $\mathbb{E}(L(h,\mathbf{X}(x))^2) < +\infty$, assumption (**LA**) is satisfied for all $\alpha > 0$ since one can choose $T_\varepsilon = 1$. Then, as soon as there exists $\beta > d$ such that (30) holds true, we obtain the asymptotic normality of the $\mathbb{R}^p$-valued vector $(S_n(h,\vec{x}))_n$ for any $p \geq 1$ and $\vec{x} = (x_1,\ldots,x_p) \in ([0,1]^d)^p$, by Corollary 1.13 p.180 of [10], using the (BL,θ) dependence of Y. In the general case, noting that the ψ-dependency coefficient in (27) satisfies

$$\psi(|I|,|J|,f,g) \leq \mathrm{Lip}(f)\|g\|_\infty|I| + \mathrm{Lip}(g)\|f\|_\infty|J| + |I||J|\mathrm{Lip}(f)\mathrm{Lip}(g),$$

the field Y is also λ-dependent as defined in [14]. By (30) we obtain that $\lambda_Y(r)(=\theta_r) = \mathcal{O}_{r\to+\infty}(r^{-\lambda})$ with $\lambda = \frac{\alpha}{\alpha+2}(\beta-d)$ and θ_r given by (28). Then, using Cramer-Wold device and Theorem 2 p.219 in [14] we obtain the asymptotic normality under stronger assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ as stated in the following corollary.

Corollary 4. *Assuming that* (25), (**LA**) *and* (30) *hold, assuming moreover that*

$(\mathbf{A_1})$ *there exists* $\delta > 0$ *such that* $\mathbb{E}(|L(h,\mathbf{X}(0))|^{2+\delta}) < +\infty$,
$(\mathbf{A_2})$ $\frac{\alpha}{\alpha+2}(\beta-d) > \max\left(2d, d(1+\frac{1}{\delta})\right)$.

Then

$$(S_n(h,x))_{x\in[0,1]^d} \underset{n\to+\infty}{\overset{fdd}{\longrightarrow}} (B(h,x))_{x\in[0,1]^d}.$$

This result will allow us to get a central limit theorem for the sequence of generalized processes given by (26), by approximating integrals by Riemann sums, under an appropriate continuity assumption.

Proposition 5. *Assuming that* (25) *and* (**LA**) *hold, if there exists* $\delta > 0$ *such that for all* $x \in \mathbb{R}^d$

$$\|x\| \leq 1 \Rightarrow \omega_X(x) := \mathbb{E}\left[\|\mathbf{X}(x)-\mathbf{X}(0)\|_\infty^2\right]^{1/2} \leq C\|x\|^\delta. \tag{32}$$

Then, there exists $C_\alpha > 0$ such that, for all $x \in \mathbb{R}^d$ with $\|x\| \leq 1$,

$$\mathbb{E}\left[|L(h,\mathbf{X}(x)) - L(h,\mathbf{X}(0))|^2\right]^{1/2} \leq C_\alpha \|h\|_q \omega_X(x)^{\frac{\alpha}{1+\alpha}}. \tag{33}$$

Assuming moreover (30) for $\beta > \frac{d(\alpha+2)}{\alpha}$, one can choose a version of $(B(h,x))_{x\in\mathbb{R}^d}$, given in Corollary 3, that is γ-Hölder continuous on any compact sets of $\mathbb{R}^d$ for all $\gamma < \frac{\delta}{2}\frac{\alpha}{1+\alpha}\left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)$.

Proof. Let $x \in \mathbb{R}^d$ with $\|x\| \leq 1$. First note that we may assume that $\omega_X(x) \in (0, C]$. Let us remark that, as previously, using the fact that for $\varepsilon \in (0,1]$

$$L(h,\mathbf{X}(x)) = F_{h,\varepsilon}(X(x)) + L(h,\mathbf{X}(x))(1 - T_\varepsilon(\mathbf{X}(x))),$$

we have

$$\begin{aligned}\mathbb{E}\left[|L(h,\mathbf{X}(x)) - L(h,\mathbf{X}(0))|^2\right]^{1/2} &\leq \mathrm{Lip}(F_{h,\varepsilon})\mathbb{E}\left[\|\mathbf{X}(x) - \mathbf{X}(0)\|_\infty\right] + 2C^{1/2}\|h\|_{q,0}\varepsilon^\alpha \\ &\leq C\|h\|_q\left[\varepsilon^{-1}\omega_X(x) + 2\varepsilon^\alpha\right].\end{aligned}$$

Taking $\varepsilon = \left(\frac{\omega_X(x)}{C}\right)^{\frac{1}{1+\alpha}}$ we obtain (33).
Assuming moreover (30) for $\beta > \frac{d(\alpha+2)}{\alpha}$, for $K \geq 1$,

$$\begin{aligned}\mathbb{E}\left[|B(h,x) - B(h,0)|^2\right] &= 2\sum_{\mathbf{k}\in\mathbb{Z}^d}\mathrm{Cov}(L(h,\mathbf{X}(\mathbf{k})), L(h,\mathbf{X}(0)) - L(h,\mathbf{X}(x)), \\ &\leq 2C^{1/2}\|h\|_{q,0}\mathbb{E}\left[|L(h,\mathbf{X}(x)) - L(h,\mathbf{X}(0))|^2\right]^{1/2}(2K+3)^d \\ &+ 6C^3\|h\|_q^2\sum_{|\mathbf{k}|>K+1}\left[\max\left(\tilde\rho(\mathbf{k}+x)^{\frac{\alpha}{\alpha+2}}, \tilde\rho(\mathbf{k}+x)\right) + \max\left(\tilde\rho(\mathbf{k})^{\frac{\alpha}{\alpha+2}}, \tilde\rho(\mathbf{k})\right)\right],\end{aligned}$$

using Cauchy-Schwarz inequality and Proposition 4. Hence, by (30) and (33), we can find $\tilde C_{\alpha,\beta,d} > 0$ such that

$$\mathbb{E}\left[|B(h,x) - B(h,0)|^2\right] \leq \tilde C_{\alpha,\beta,d}\|h\|_q^2\left(K^d\omega_X(x)^{\frac{\alpha}{1+\alpha}} + K^{-\left(\frac{\beta\alpha}{\alpha+2} - d\right)}\right).$$

Then, choosing $K = \omega_X(x)^{-\frac{\alpha+2}{\alpha\beta}\frac{\alpha}{1+\alpha}}$, there exists $C_{\alpha,\beta,d} > 0$ such that

$$\mathbb{E}\left[|B(h,x) - B(h,0)|^2\right] \leq C_{\alpha,\beta,d}\|h\|_q^2\omega_X(x)^{\frac{\alpha}{1+\alpha}\left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)}.$$

In view of (32) and by stationarity of $(B(h,x))_{x\in\mathbb{R}^d}$, we can choose a version such that $(B(h,x))_{x\in\mathbb{R}^d}$ is γ-Hölder continuous on any compact set of $\mathbb{R}^d$ for $\gamma < \frac{\delta}{2}\frac{\alpha}{1+\alpha}\left(1 - \frac{d(\alpha+2)}{\alpha\beta}\right)$ (see Proposition 9 in [6] for instance). □

Theorem A.2. *Under the assumptions of Corollary 4, assuming moreover (32), one has for $h \in \mathcal{S}$,*

$$\langle \mathcal{T}_n, h\rangle = \sqrt{\mathcal{L}^d(U_n)}\left(\frac{1}{\mathcal{L}^d(U_n)}\int_{U_n} L(h,\mathbf{X}(x))dx - \mathbb{E}(L(h,\mathbf{X}(0)))\right) \xrightarrow[n\to+\infty]{d} \int_{[0,1]^d} B(h,x)dx,$$

where $(B(h,x))_{x\in[0,1]^d}$ is the continuous Gaussian random field with covariance function given by (31). The random variable $\int_{[0,1]^d} B(h,x)dx$ is centered Gaussian with variance given by

$$\int_{\mathbb{R}^d}\mathrm{Cov}\left(L(h,\mathbf{X}(z)), L(h,\mathbf{X}(0))\right)dz. \tag{34}$$

Hence, there exists B a generalized Gaussian random process on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$, with characteristic functional given by

$$\mathcal{L}_B(h) := \mathbb{E}\left(e^{i\langle B,h\rangle}\right) = \mathbb{E}\left(e^{i\int_{[0,1]^d} B(h,x)dx}\right) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Cov}\left(L(h,\mathbf{X}(z)), L(h,\mathbf{X}(0))\right) dz\right)$$

such that $\mathcal{T}_n \overset{distrib.}{\underset{n\to+\infty}{\longrightarrow}} B$, where the convergence holds in distribution with respect to the strong topology of $\mathcal{S}'(\mathbb{R})$.

Proof. In view of Proposition 5, by continuity of $(B(h,x))_{x\in[0,1]^d}$ we have the convergence of the Riemann sums

$$\int_{[0,1]^d} B(h,x)dx = \lim_{m\to+\infty} \frac{1}{m^d} \sum_{\mathbf{l}\in(\mathbb{Z}\cap[0,m))^d} B\left(h, \frac{\mathbf{l}}{m}\right).$$

Hence the random variable $\int_{[0,1]^d} B(h,x)dx$ is centered Gaussian with variance given by

$$\begin{aligned}\int_{[0,1]^d}\int_{[0,1]^d} \mathrm{Cov}\left(B(h,x), B(h,y)\right) dxdy &= \int_{[0,1]^d}\int_{[0,1]^d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathrm{Cov}\left(L(h,\mathbf{X}(x+\mathbf{k})), L(h,\mathbf{X}(y))\right) dxdy \\ &= \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(h,\mathbf{X}(z)), L(h,\mathbf{X}(0))\right) dz,\end{aligned}$$

by change of variables and Fubini's theorem.
Note also that for $x\in[0,1]^d$, $m>0$ and $\mathbf{l}\in(\mathbb{Z}\cap[0,m))^d$ we have

$$S_n\left(h, x+\frac{\mathbf{l}}{m}\right) - S_n\left(h,\frac{\mathbf{l}}{m}\right) = \frac{1}{n^{d/2}} \sum_{\mathbf{k}'\in nB\cap\mathbb{Z}^d} \left(L\left(h,\mathbf{X}\left(x+\frac{\mathbf{l}}{m}+\mathbf{k}'\right)\right) - L\left(h,\mathbf{X}\left(\frac{\mathbf{l}}{m}+\mathbf{k}'\right)\right)\right).$$

Then, by stationarity,

$$\mathrm{Var}\left(S_n\left(h,x+\frac{\mathbf{l}}{m}\right) - S_n\left(h,\frac{\mathbf{l}}{m}\right)\right) \le \sum_{\|\mathbf{k}\|_\infty\le n} |r(h,\mathbf{k},x)|,$$

where

$$r(h,\mathbf{k},x) = \mathrm{Cov}\left(L(h,\mathbf{X}(x)) - L(h,\mathbf{X}(0)), L(h,\mathbf{X}(x+\mathbf{k})) - L(h,\mathbf{X}(\mathbf{k}))\right)$$

satisfies, by Proposition 5 and Proposition 4 together with (30),

$$|r(h,\mathbf{k},x)| \le C_\alpha^2 \|h\|_q^2 \omega_X(x)^{\frac{2\alpha}{1+\alpha}} \quad \text{and} \quad |r(h,\mathbf{k},x)| \le 12C^3\|h\|_q^2 |k|^{-\frac{\beta\alpha}{\alpha+2}}.$$

It follows that we can find $C_{\alpha,\beta,d}>0$ such that

$$\begin{aligned}\mathbb{E}\left(\left(S_n\left(h,x+\frac{\mathbf{l}}{m}\right)\right) - S_n\left(h,\frac{\mathbf{l}}{m}\right)\right)^2 &= \mathrm{Var}\left(S_n\left(h,x+\frac{\mathbf{l}}{m}\right) - S_n\left(h,\frac{\mathbf{l}}{m}\right)\right) \\ &\le C_{\alpha,\beta,d}\|h\|_q^2 \omega_X(x)^{\frac{2\alpha}{1+\alpha}\left(1-\frac{d(\alpha+2)}{\alpha\beta}\right)}.\end{aligned}$$

Then,

$$\begin{aligned}&\left\| \int_{[0,1]^d} S_n(h,x)dx - \frac{1}{m^d} \sum_{\mathbf{l}\in(\mathbb{Z}\cap[0,m])^d} S_n\left(h,\frac{\mathbf{l}}{m}\right)\right\|_{L^2(\Omega)} \\ \le &\sum_{\mathbf{l}\in(\mathbb{Z}\cap[0,m])^d} \left\| \int_{[0,1/m]^d} \left(S_n(h,x+\frac{\mathbf{l}}{m}) - S_n\left(h,\frac{\mathbf{l}}{m}\right)\right) dx\right\|_{L^2(\Omega)}\end{aligned}$$

Using Cauchy-Schwarz inequality,

$$\mathbb{E}\left(\int_{[0,1/m]^d}\left(S_n(h,x+\frac{\mathbf{1}}{m})-S_n\left(h,\frac{\mathbf{1}}{m}\right)\right)dx\right)^2 \le \frac{1}{m^d}\int_{[0,1/m]^d}\mathbb{E}\left(S_n(h,x+\frac{\mathbf{1}}{m})-S_n\left(h,\frac{\mathbf{1}}{m}\right)\right)^2 dx.$$

Then,

$$\begin{aligned}&\left\|\int_{[0,1]^d} S_n(h,x)dx-\frac{1}{m^d}\sum_{\mathbf{l}\in(\mathbb{Z}\cap[0,m])^d} S_n\left(h,\frac{\mathbf{l}}{m}\right)\right\|_{L^2(\Omega)}\\ \le\ & C^{1/2}_{\alpha,\beta,d}\|h\|_q m^{d/2}\left(\int_{[0,1/m]^d}\omega_X(x)^{\frac{2\alpha}{1+\alpha}\left(1-\frac{d(\alpha+2)}{\alpha\beta}\right)}dx\right)^{1/2}.\end{aligned}$$

In view of (32), this implies that

$$\int_{[0,1]^d} S_n(h,x)dx=\lim_{m\to+\infty}\frac{1}{m^d}\sum_{\mathbf{l}\in\mathbb{Z}\cap[0,m]^d} S_n\left(h,\frac{\mathbf{l}}{m}\right),$$

where the convergence holds in $L^2(\Omega)$, uniformly in n. Since

$$\sqrt{\mathcal{L}^d(U_n)}\left(\frac{1}{\mathcal{L}^d(U_n)}\int_{U_n} L(h,\mathbf{X}(x))dx-\mathbb{E}(L(h,\mathbf{X}(0)))\right)=\int_{[0,1]^d} S_n(h,x)dx,$$

the result will simply follows from the fact that for $m>0$, by Corollary 4, one has

$$\frac{1}{m^d}\sum_{\mathbf{l}\in\mathbb{Z}\cap[0,m]^d} S_n\left(h,\frac{\mathbf{l}}{m}\right)\xrightarrow[n\to+\infty]{d}\frac{1}{m^d}\sum_{\mathbf{l}\in\mathbb{Z}\cap[0,m]^d} B\left(h,\frac{\mathbf{l}}{m}\right).$$

Note that the variance of the Gaussian variable $\int_{[0,1]^d} B(h,x)dx$ is explicitly given by

$$\sigma^2(h):=\int_{[0,1]^2}\int_{[0,1]^2}\mathrm{Cov}(B(h,x),B(h,y))dxdy,$$

with

$$\mathrm{Cov}(B(h,x),B(h,y))=\sum_{\mathbf{k}\in\mathbb{Z}^d}\mathrm{Cov}\left(L(h,\mathbf{X}(x+\mathbf{k})),L(h,\mathbf{X}(y))\right).$$

It follows by Proposition 4 that, on the one hand we can use Fubini's theorem to obtain (34), and on the other hand $\sigma^2(h)\le C\|h\|_q^2$ and therefore the characteristic functional $\mathcal{L}_B:\mathcal{S}(\mathbb{R})\to\mathbb{C}$ given by

$$\mathcal{L}_B(h)=e^{-\frac{1}{2}\sigma^2(h)},$$

is continuous at 0 and the conclusion comes from Theorem 2.3 of [9].

□

A.2. Gaussian case and Proof of Theorem 2.1

Assuming that X is a Gaussian field satisfying the assumptions of Theorem 2.1, by the Cramer-Wold device, it is enough to prove that for $(a_j)_{0\le j\le d}\in\mathbb{R}^{d+1}$ one has

$$\sqrt{\mathcal{L}^d(U_n)}\sum_{j=0}^d a_j\left(\frac{C_j(E_X(\cdot),U_n)}{\mathcal{L}^d(U_n)}-\overline{C}_j^X,\right)\xrightarrow[n\to+\infty]{distrib.}\sum_{j=0}^d a_j B_j,$$

where $(B_j)_{0\le j\le d}$ are the centered Gaussian generalized processes introduced in Theorem 2.1 and the convergence holds in distribution with respect to the strong topology of $\mathcal{S}'(\mathbb{R})$. Therefore we let

$$\mathcal{T}_n = \sqrt{\mathcal{L}^d(U_n)}\sum_{j=0}^{d} a_j\left(\frac{C_j(E_X(\cdot),U_n)}{\mathcal{L}^d(U_n)} - \overline{C}_j^X\right) \quad \text{and} \quad B=\sum_{j=0}^{d} a_j B_j.$$

Hence we need to check that the assumptions of Theorem A.2 are satisfied for $h\in\mathcal{S}$ and

$$L(h,\mathbf{X}(x)) := \sum_{j=0}^{d} a_j L_j(h,\mathbf{X}(x)),$$

since

$$\langle \mathcal{T}_n, h\rangle = \sqrt{\mathcal{L}^d(U_n)}\left(\frac{1}{\mathcal{L}^d(U_n)}\int_{U_n} L(h,\mathbf{X}(x))dx - \mathbb{E}\left(L(h,\mathbf{X}(0))\right)\right).$$

Then we will identify the Gaussian limit. First, since X is a Gaussian field the quasi-association assumption (25) is satisfied (see [32]). Second, for $\mathbf{x}=(u,v,w)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d(d+1)/2}$ with a slight abuse of notation we write

$$F_0(\mathbf{x}) = \|v\|, \quad F_1(\mathbf{x}) = -\mathrm{Tr}(w) + \frac{v^t w v}{\|v\|^2} \quad \text{and} \quad F_2(\mathbf{x}) = -\frac{v^t \mathrm{adj}(w) v}{\|v\|^3},$$

and remark that $\mathbf{x}\mapsto L_d(h,\mathbf{x})$ is bounded Lipschitz and clearly satisfies (**LA**). However for $0\le j\le d-1$ we need to conveniently approximate $\mathbf{x}\mapsto L_j(h,\mathbf{x}) = h(u)F_{d-1-j}(\mathbf{x})$. For $\delta>0$ we can find a C^1 function $\mathsf{T}_\delta:\mathbb{R}^d\to[0,1]$ such that $\mathsf{T}_\delta(v)=1$ for $\|v\|>\delta$ and $\|\nabla\mathsf{T}_\delta\|\le\delta^{-1}$. Moreover for $M>0$ we can find a C^1 function $\mathsf{T}_M:\mathbb{R}^d\times\mathbb{R}^{d(d+1)/2}\to[0,1]$ such that $\mathsf{T}_M(v,w)=1$ for $\|(v,w)\|<M$ and $\mathsf{T}_M(v,w)=0$ for $\|(v,w)\|>2M$ with $\|\nabla\mathsf{T}_M\|_\infty\le 1$. Then $\mathbf{x}\mapsto L(h,\mathbf{x})\mathsf{T}_M(v,w)\mathsf{T}_\delta(v)$ is C^1 with gradient bounded by $C\|h\|_q\left(\frac{M}{\delta}\right)^2$ with C that does not depend on M,δ, $q=2$, and $\|h\|_q=\|h\|_{q,0}+\|h\|_{q,1}$. Now $\mathbb{P}(\|\nabla X(x)\|\le\delta)\le C\delta^d$ and one can find $c_d>1$ such that for M large enough $\mathbb{P}(\|(\nabla X(x),D^2X(x))\|\ge M)\le e^{-c_d M}$. For $\varepsilon\in(0,1]$, we set $\delta=\frac{d}{2c_d}\varepsilon^{1/2}|\log(\varepsilon/2)|$ and $M=\frac{d}{2c_d}|\log(\varepsilon/2)|$ such that $\frac{\delta}{M}=\varepsilon$ and for all $p'\ge 1$, we have

$$\begin{aligned}\mathbb{E}((1-\mathsf{T}_M(\nabla X(x),D^2X(x))\mathsf{T}_\delta(\nabla X(x)))^{p'}) &\le \mathbb{P}(\|\nabla X(x)\|\le\delta)+\mathbb{P}(\|(\nabla X(x),D^2X(x))\|\ge M)\\ &\le C\varepsilon^{d/2}(1+|\log(\varepsilon/2)|^d).\end{aligned}$$

Note also that for any $p>1$, choosing $p<d$ for $d=3$, one has $\mathbb{E}(L(h,\mathbf{X}(x))^{2p})\le C_p\|h\|_{q,0}^{2p}$. Hence, by Hölder inequality, inequalities (**LA**) are satisfied for $\alpha\in(0,\frac{d}{2}(1-1/p))$. Then, under Assumption (12) with $\beta-d>12$, when $d=2$, since $(\mathbf{A_1})$ is satisfied for all $\delta>0$, and $2d(1+4/d)=12$, one can find $p>1$ large enough such that $2d(1+\frac{4}{d(1-1/p)})<\beta-d$ and take $p'>p$ with $\alpha\in(0,\frac{d}{2}(1-1/p'))$ and then $\frac{\alpha}{\alpha+2}(\beta-d)>2d$ so that $(\mathbf{A_2})$ and together with (30) are also satisfied. When $d=3$, since $2d\frac{4}{d}\frac{1}{1-1/d}=12$, one can find $\delta<1$ and $p'<d$ such that $d(1+\frac{1}{\delta})\frac{4}{d}\frac{1}{1-1/p}<\beta-d$. Then taking $\alpha=\frac{d}{2}(1-1/p)$ we can check both $(\mathbf{A_1})$, $(\mathbf{A_2})$ and (30). Finally, note that for $\|x\|\le 1$ one has

$$\mathbb{E}\left([X_{ij}(x)-X_{ij}(0)]^2\right) = 2(-1)^{\varepsilon(i)+\varepsilon(j)}\left(\partial_{ijij}^{2\varepsilon(i)+2\varepsilon(j)}\rho(0)-\partial_{ijij}^{2\varepsilon(i)+2\varepsilon(j)}\rho(x)\right),$$

such that (32) is satisfied by (**H**).

It follows that all assumptions and therefore conclusions of Theorem 2.1 hold true.

To conclude it only remains to identify the asymptotic covariance. By linearity of B and L for

$h, \tilde{h} \in \mathcal{S}$ we obtain that

$$\begin{aligned}
\mathrm{Var}(\langle B, h+\tilde{h}\rangle) &= \mathrm{Var}(\langle B, h\rangle) + \mathrm{Var}(\langle B, \tilde{h}\rangle) + 2\mathrm{Cov}(\langle B, h\rangle, \langle B, \tilde{h}\rangle) \\
&= \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(h+\tilde{h}, \mathbf{X}(z)), L(h+\tilde{h}, \mathbf{X}(0))\right) dz \\
&= \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(h, \mathbf{X}(z)), L(h, \mathbf{X}(0))\right) dz + \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(\tilde{h}, \mathbf{X}(z)), L(\tilde{h}, \mathbf{X}(0))\right) dz \\
&+ 2\int_{\mathbb{R}^d} \mathrm{Cov}\left(L(h, \mathbf{X}(z)), L(\tilde{h}, \mathbf{X}(0))\right) dz,
\end{aligned}$$

where we use the fact that, by stationarity and change of variables, one has

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{Cov}\left(L(\tilde{h}, \mathbf{X}(z)), L(h, \mathbf{X}(0))\right) dz &= \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(\tilde{h}, \mathbf{X}(0)), L(h, \mathbf{X}(-z))\right) dz \\
&= \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(\tilde{h}, \mathbf{X}(0)), L(h, \mathbf{X}(z))\right) dz.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Cov}(\langle B, h\rangle, \langle B, \tilde{h}\rangle) &= \int_{\mathbb{R}^d} \mathrm{Cov}\left(L(h, \mathbf{X}(z)), L(\tilde{h}, \mathbf{X}(0))\right) dz \\
&= \sum_{j=0}^{d}\sum_{k=0}^{d} a_j a_k \int_{\mathbb{R}^d} \mathrm{Cov}\left(L_j(h, \mathbf{X}(z)), L_k(\tilde{h}, \mathbf{X}(0))\right) dz,
\end{aligned}$$

and $B = \sum_{j=0}^{d} a_j B_j$ where $(B_j)_{0\le j\le d}$ are centered Gaussian generalized processes with covariance given by

$$\forall j,k \in \{0,\dots,d\}, \quad \mathrm{Cov}(\langle B_j, h\rangle, \langle B_k, \tilde{h}\rangle) = \int_{\mathbb{R}^d} \mathrm{Cov}\left(L_j(h, \mathbf{X}(z)), L_k(\tilde{h}, \mathbf{X}(0))\right) dz.$$

Appendix B: Additional numerical experiments

We provide in this section several additional figures to illustrate, through numerical experiments, the statistical estimation of the anisotropy from the Lipschitz-Killing curvatures of an excursion set.

B.1. In the 2D case

On Figures 8 and 9 we show error bars on the estimation of the anisotropy from one excursion set of a 2D stationary Gaussian field, and we illustrate the central limit behavior by plotting the standard deviation of the estimation of the anisotropy ratio R as a function of the observation window size. See the caption of the figures for detailed comments on these numerical experiments.

B.2. In the 3D case

On Figures 10 and 11, we show some additional 3D experiments in the isotropic case. As in the anisotropic case, the estimation of $R = R_G = 1$ here is inaccurate when the effective level is close to 0 (for R) or close to ± 1 for R_G. Now, these two ratios could be combined to accurately estimate the isotropy whatever the effective level.

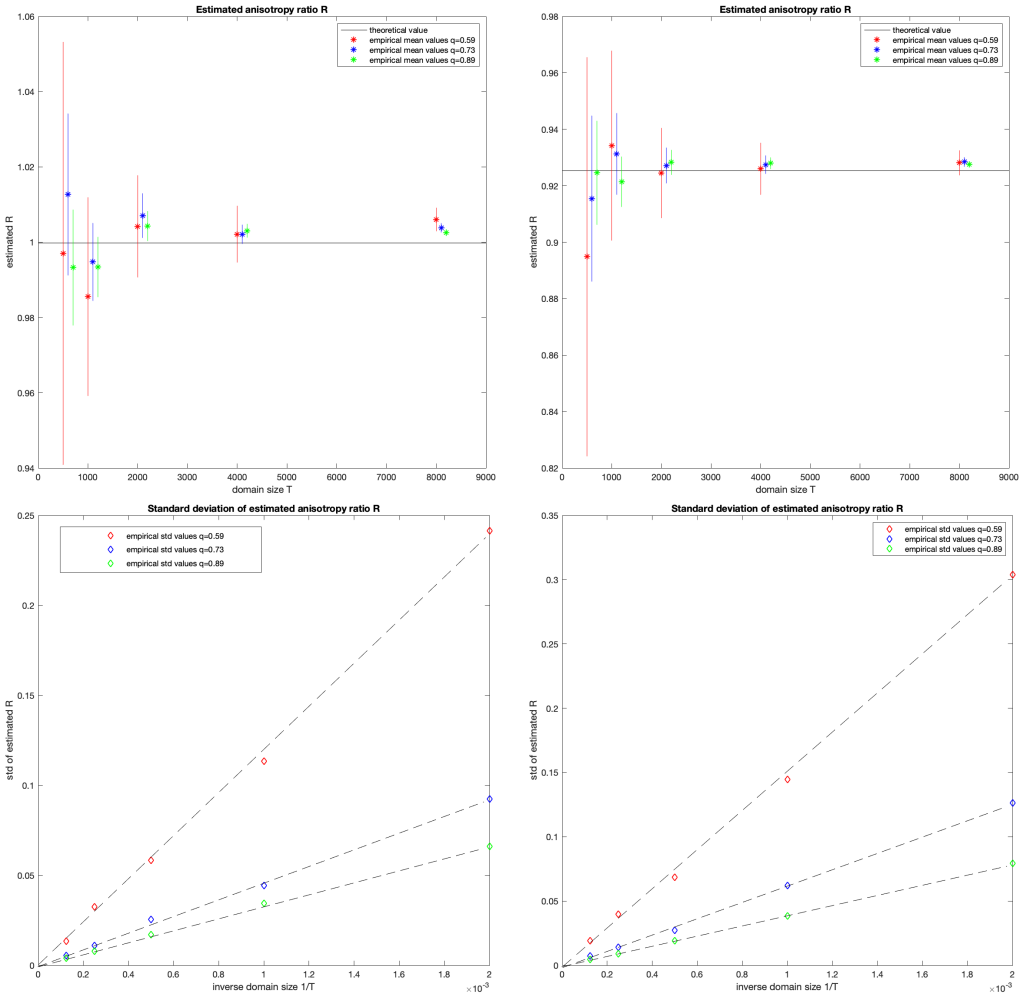

FIG 8. *Checking the central limit behavior. First line: estimation of R in the isotropic case (left) and in the anisotropic one (right, here $R \simeq 0.92$). The estimation is done for three different quantiles (corresponding to the three colors) and on different domain size T (x-axis of the plot). The confidence intervals have been obtained with* 100 *samples. Second line: standard deviation of the estimated value of R in the isotropic case (left) and in the anisotropic one (right), for three different quantiles as a function of the inverse domain size $1/T$. This plot shows experimentally that the empirical standard deviation behaves like $1/T$, with a slope that depends on the quantile.*

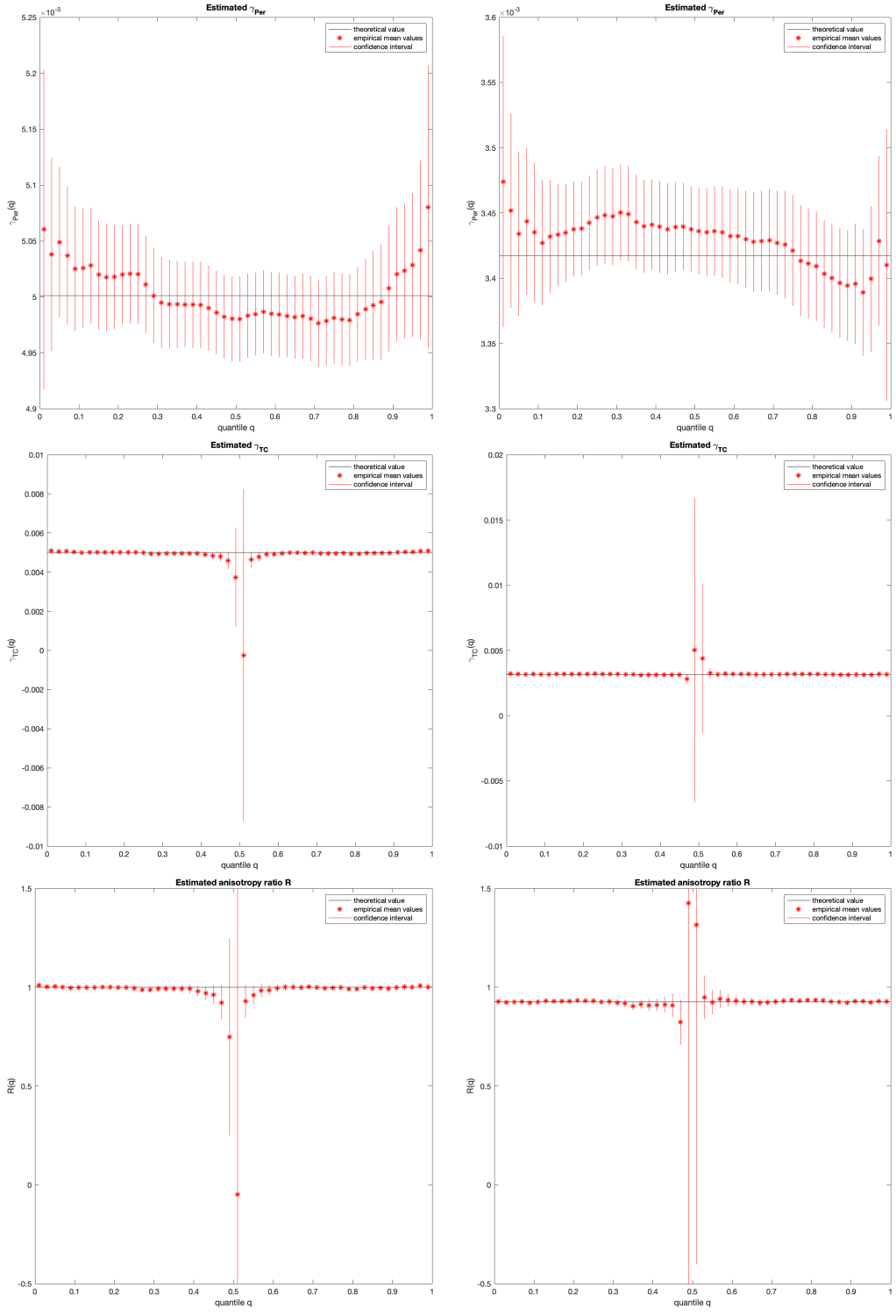

FIG 9. *Estimating the anisotropy from one level. We consider here the same two Gaussian random fields as in Figure 3. First line: estimation of* $\overline{\gamma}_{\mathrm{Per}}$ *from one level set of one sample (of size* 1000×1000*), as a function of the quantile, in the isotropic case (left figure) and in the anisotropic one (right figure). Second line: same experiment but for the estimation of* $\overline{\gamma}_{\mathrm{TC}}$*. Third line : same experiment but for the estimation of the anisotropy ratio* R*, that is* 1 *on the left and* 0.92 *on the right. The confidence intervals have been obtained with* 100 *samples. Notice how the estimations of* $\overline{\gamma}_{\mathrm{TC}}$ *and of* R *degenerate around the quantile* $q = 0.5$ *since it corresponds to the effective level* $\widehat{t} = 0$*, and the division by* $\widehat{t}$ *is therefore unstable.*

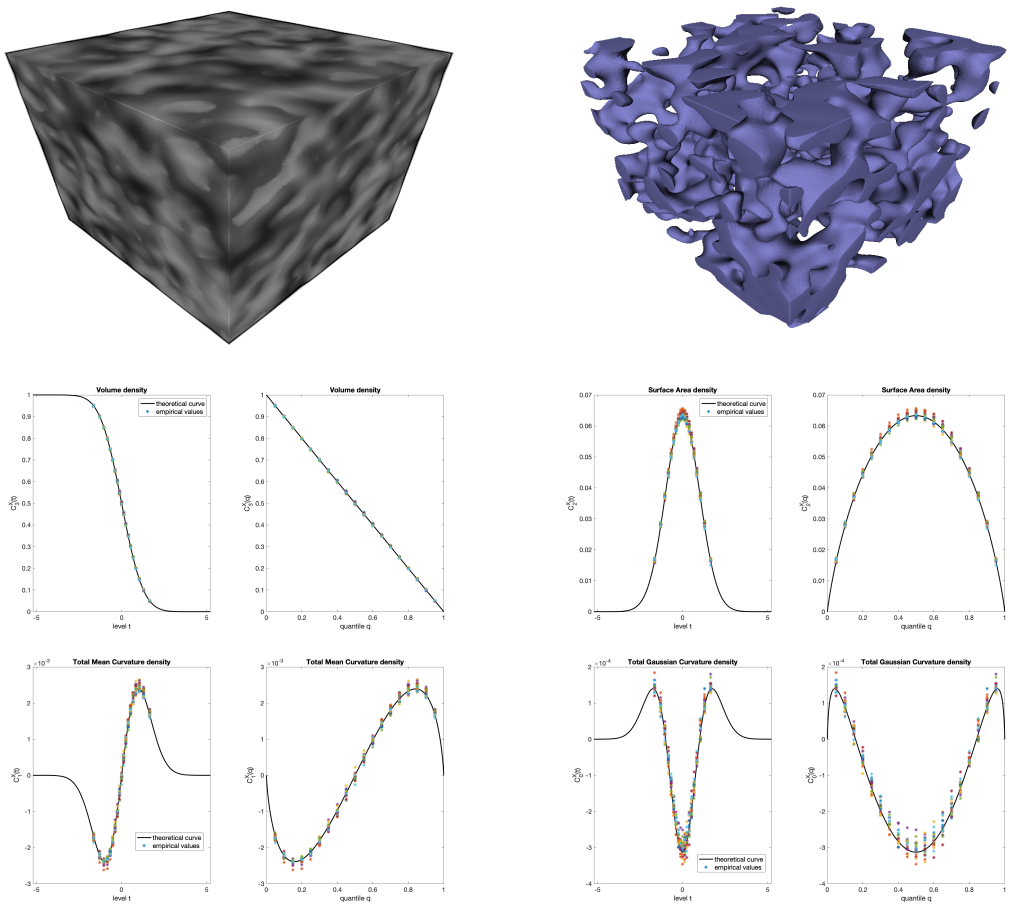

FIG 10. *Lipschitz-Killing curvature densities of the excursion sets of an isotropic Gaussian volume. Here we have taken* 20 *samples of a 3D volume of size* 200^3 *voxels, of a Gaussian field with mean* $m = 0$*, variance* $\sigma^2 = 1$ *and covariance of the form* $\exp(-\gamma_1 x_1^2 - \gamma_2 x_2^2 - \gamma_3 x_3^2)$ *with here* $\gamma_1 = \gamma_2 = \gamma_3 = 0.01$*. An example of such a Gaussian volume is shown on the left of the first line, with one excursion set on the right. Using the Matlab toolbox ImMinkowski [27], we have estimated the volume densities (second line, left), the surface area densities (second line, right), the total mean curvature densities (third line, left) and the total Gaussian curvature densities (third line, right). The densities are plot else as functions of the level* t *or as functions of the quantile* q*. The stars are the values for the* 20 *different samples.*

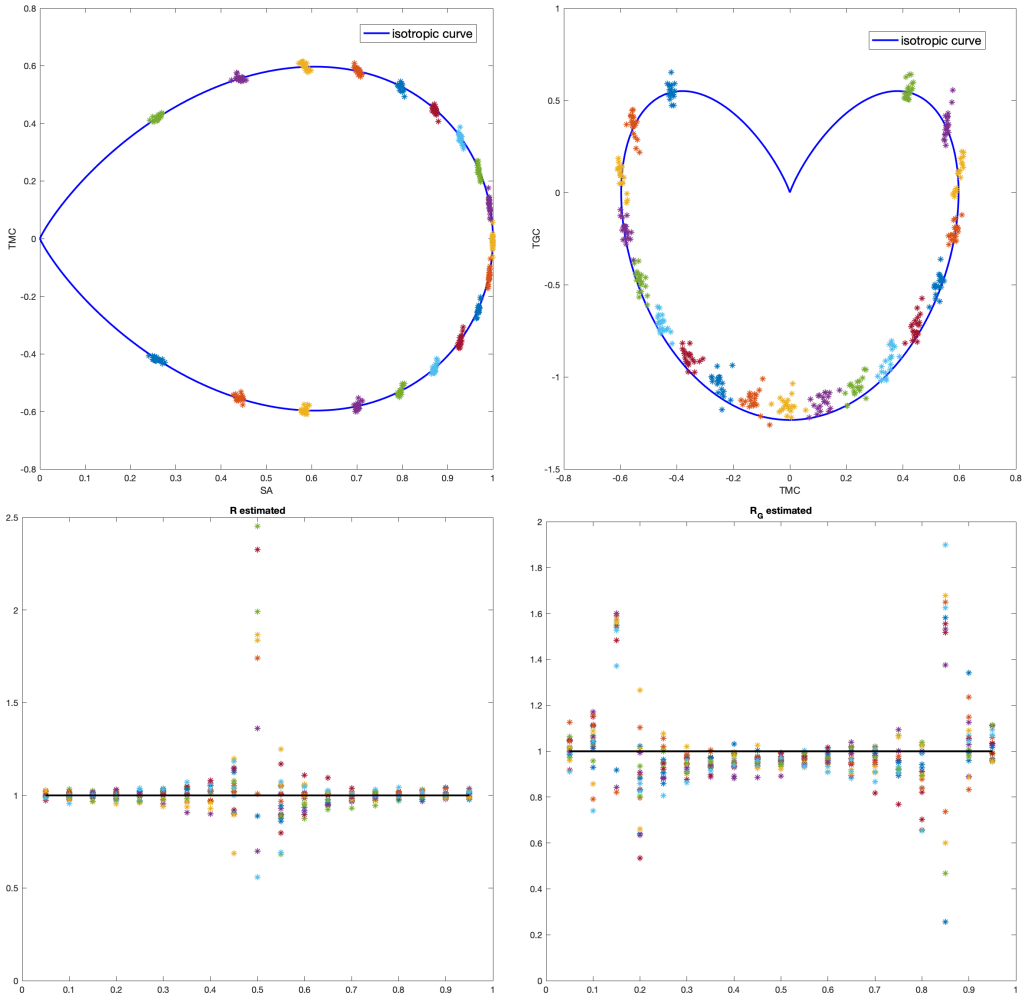

FIG 11. *First line: the samples of the previous figure are plotted as points on the almond and heart curve of isotropy. Second line: estimation of R and R_G from the Lipschitz-Killing curvature densities of the excursion sets of the samples.*