



**HAL**  
open science

# A new perspective on the tensor product of semi-lattices

Eric Buffenoir

► **To cite this version:**

| Eric Buffenoir. A new perspective on the tensor product of semi-lattices. 2023. hal-04111800v1

**HAL Id: hal-04111800**

**<https://hal.science/hal-04111800v1>**

Preprint submitted on 31 May 2023 (v1), last revised 4 Mar 2024 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# A new perspective on the tensor product of semi-lattices

Eric Buffenoir\*

Université de la Côte d'Azur, CNRS, InPhyNi, FRANCE

May 31, 2023

## Abstract

We adopt a new perspective on the tensor product of arbitrary semi-lattices. Our basic construction exploits a description of semi-lattices in terms of bi-extensional Chu spaces associated to a target space defined to be the boolean domain. The comparison between our tensor product and the canonical tensor product, introduced by G.A. Fraser, is made in the distributive case and in the general case. Some properties of our tensor products are also given.

---

2020 *Mathematics subject classification*. 18C50, 18B35

*Keywords*: Categorical semantics of formal languages / Preorders, orders, domains and lattices (viewed as categories).

\*Email: eric.buffenoir@cnrs.fr

# 1 Preliminaries

The set  $\mathfrak{B} := \{\mathbf{Y}, \mathbf{N}, \perp\}$  will be equipped with the following poset structure :

$$\forall u, v \in \mathfrak{B}, \quad (u \leq v) :\Leftrightarrow (u = \perp \text{ or } u = v). \quad (1)$$

$(\mathfrak{B}, \leq)$  is also an Inf semi-lattice which infima will be denoted  $\bigwedge$ . We have

$$\forall x, y \in \mathfrak{B}, \quad x \wedge y = \begin{cases} x & \text{if } x = y \\ \perp & \text{if } x \neq y \end{cases} \quad (2)$$

We will also introduce a commutative monoid law denoted  $\bullet$  and defined by

$$\forall x \in \mathfrak{B}, \quad x \bullet \mathbf{Y} = x, \quad x \bullet \mathbf{N} = \mathbf{N}, \quad \perp \bullet \perp = \perp. \quad (3)$$

This product law verifies the following properties

$$\forall x \in \mathfrak{B}, \forall B \subseteq \mathfrak{B} \quad x \bullet \bigwedge B = \bigwedge_{b \in B} (x \bullet b), \quad (4)$$

$$\forall x \in \mathfrak{B}, \forall C \subseteq_{Chain} \mathfrak{B} \quad x \bullet \bigvee C = \bigvee_{b \in C} (x \bullet b). \quad (5)$$

$(\mathfrak{B}, \leq)$  will be also equipped with the following involution map :

$$\overline{\perp} := \perp \quad \overline{\mathbf{Y}} := \mathbf{N} \quad \overline{\mathbf{N}} := \mathbf{Y}. \quad (6)$$

$(\mathfrak{B}, \leq)$  will be called *boolean domain*.

## 2 Inf semi-lattices and States/Effects Chu spaces

### 2.1 States/Effects Chu spaces

We will say that the triple  $(\mathfrak{S}, \mathfrak{E}, \varepsilon^{\mathfrak{S}})$  is a *States/Effects Chu space* iff

- the set  $\mathfrak{S}$ , called *space of states*, is a down-complete Inf semi-lattice (i.e.  $\forall S \subseteq \mathfrak{S}$  the infimum  $(\bigcap^{\mathfrak{S}} S)$  exists in  $\mathfrak{S}$ ), which admits a bottom element denoted  $\perp_{\mathfrak{S}}$ ;
- the set  $\mathfrak{E}$ , called *space of effects*, is a down-complete Inf semi-lattice (i.e.  $\forall E \subseteq \mathfrak{E}$  the infimum  $(\bigcap^{\mathfrak{E}} E)$  exists in  $\mathfrak{E}$ );
- $\varepsilon^{\mathfrak{S}}$  is a map from  $\mathfrak{E}$  to  $\mathfrak{B}^{\mathfrak{S}}$ , called *evaluation map*, and satisfying

$$\forall l \in \mathfrak{E}, \forall S \subseteq \mathfrak{S}, \quad \varepsilon_l^{\mathfrak{S}}(\bigcap^{\mathfrak{S}} S) = \bigwedge_{\sigma \in S} \varepsilon_l^{\mathfrak{S}}(\sigma), \quad (7)$$

$$\forall \sigma \in \mathfrak{S}, \forall E \subseteq \mathfrak{E}, \quad \varepsilon_{\bigcap^{\mathfrak{E}} E}^{\mathfrak{S}}(\sigma) = \bigwedge_{l \in E} \varepsilon_l^{\mathfrak{S}}(\sigma), \quad (8)$$

and

$$\forall l, l' \in \mathfrak{E}, \quad (\forall \sigma \in \mathfrak{S}, \varepsilon_l^{\mathfrak{S}}(\sigma) = \varepsilon_{l'}^{\mathfrak{S}}(\sigma)) \Leftrightarrow (l = l'), \quad (9)$$

$$\forall \sigma, \sigma' \in \mathfrak{S}, \quad (\forall l \in \mathfrak{E}, \varepsilon_l^{\mathfrak{S}}(\sigma) = \varepsilon_l^{\mathfrak{S}}(\sigma')) \Leftrightarrow (\sigma = \sigma'). \quad (10)$$

We will say that *the space of states admits a description in terms of pure states* iff we have moreover

- the set of completely meet-irreducible elements of  $\mathfrak{S}$ , denoted  $\mathfrak{S}^{pure}$  and called *set of pure states*, is equal to the set of maximal elements  $Max(\mathfrak{S})$  and it is a generating set for  $\mathfrak{S}$ , i.e.

$$\forall \sigma \in \mathfrak{S}, \sigma = \prod_{\sigma \in \mathfrak{S}}^{\sigma} \underline{\sigma}_{\sigma}, \text{ where } \underline{\sigma}_{\sigma} := (\mathfrak{S}^{pure} \cap (\uparrow^{\mathfrak{S}} \sigma)) \text{ and } \mathfrak{S}^{pure} = Max(\mathfrak{S}). \quad (11)$$

We will introduce the following notations :

$$\forall l \in \mathfrak{E}, \exists \bar{l} \in \mathfrak{E} \mid \forall \sigma \in \mathfrak{S}, \varepsilon_l^{\mathfrak{S}}(\sigma) = \overline{\varepsilon_{\bar{l}}^{\mathfrak{S}}(\sigma)}, \quad (12)$$

$$\exists \mathfrak{Y}_{\mathfrak{E}} \in \mathfrak{E} \mid \forall \sigma \in \mathfrak{S}, \varepsilon_{\mathfrak{Y}_{\mathfrak{E}}}^{\mathfrak{S}}(\sigma) = \mathbf{Y}, \quad (13)$$

$$\exists \perp_{\mathfrak{E}} \in \mathfrak{E} \mid \forall \sigma \in \mathfrak{S}, \varepsilon_{\perp_{\mathfrak{E}}}^{\mathfrak{S}}(\sigma) = \perp. \quad (14)$$

We note that, for any  $l \in \mathfrak{E}$ , if  $(\varepsilon_l^{\mathfrak{S}})^{-1}(\mathbf{Y}) \neq \emptyset$  (resp.  $(\varepsilon_l^{\mathfrak{S}})^{-1}(\mathbf{N}) \neq \emptyset$ ) then  $(\varepsilon_l^{\mathfrak{S}})^{-1}(\mathbf{Y})$  (resp.  $(\varepsilon_l^{\mathfrak{S}})^{-1}(\mathbf{N})$ ) is a principal filter. Hence, we will introduce the following notations for effects. Let us consider  $\Sigma, \Sigma' \in \mathfrak{S}$  such that  $\neg \widehat{\Sigma \Sigma'}^{\mathfrak{S}}$ .<sup>[1]</sup> We denote  $l_{(\Sigma, \Sigma')}$  the effect defined by<sup>[2]</sup>

$$(\varepsilon_{l_{(\Sigma, \Sigma')}}^{\mathfrak{S}})^{-1}(\mathbf{Y}) := \uparrow^{\mathfrak{S}} \Sigma \text{ and } (\varepsilon_{l_{(\Sigma, \Sigma')}}^{\mathfrak{S}})^{-1}(\mathbf{N}) := \uparrow^{\mathfrak{S}} \Sigma'. \quad (15)$$

By extension, we denote  $l_{(\Sigma, \cdot)}$  the effect defined by

$$(\varepsilon_{l_{(\Sigma, \cdot)}}^{\mathfrak{S}})^{-1}(\mathbf{Y}) := \uparrow^{\mathfrak{S}} \Sigma \text{ and } (\varepsilon_{l_{(\Sigma, \cdot)}}^{\mathfrak{S}})^{-1}(\mathbf{N}) := \emptyset \quad (16)$$

and by  $l_{(\cdot, \Sigma')}$  the effect defined by

$$l_{(\cdot, \Sigma')} := \overline{l_{(\Sigma', \cdot)}} \quad (17)$$

We note that the order on these effects is "inversed" with respect to the order on states. More precisely,

$$\left\{ \begin{array}{l} (l_{(\Sigma_1, \Sigma'_1)} \sqsubseteq_{\mathfrak{E}} l_{(\Sigma_2, \Sigma'_2)}) \Leftrightarrow (\Sigma_1 \sqsupseteq_{\mathfrak{S}} \Sigma_2 \text{ and } \Sigma'_1 \sqsupseteq_{\mathfrak{S}} \Sigma'_2) \\ (l_{(\Sigma_1, \cdot)} \sqsubseteq_{\mathfrak{E}} l_{(\Sigma_2, \Sigma'_2)}) \Leftrightarrow (\Sigma_1 \sqsupseteq_{\mathfrak{S}} \Sigma_2) \\ (l_{(\cdot, \Sigma'_1)} \sqsubseteq_{\mathfrak{E}} l_{(\Sigma_2, \Sigma'_2)}) \Leftrightarrow (\Sigma'_1 \sqsupseteq_{\mathfrak{S}} \Sigma'_2) \end{array} \right. \quad (18)$$

**Theorem 1.** Let us consider a map  $(A : \mathfrak{S} \rightarrow \mathfrak{B}, \sigma \mapsto a_{\sigma})$  satisfying

$$\forall \sigma, \sigma' \in \mathfrak{S}, (\sigma \sqsubseteq_{\mathfrak{S}} \sigma') \Rightarrow (a_{\sigma} \leq a_{\sigma'}), \quad (19)$$

$$\forall \{\sigma_i \mid i \in I\} \subseteq \mathfrak{S}, a_{\prod_{i \in I}^{\mathfrak{S}} \sigma_i} = \bigwedge_{i \in I} a_{\sigma_i}, \quad (20)$$

Then, we have

$$\exists ! l \in \mathfrak{E} \mid \forall \sigma \in \mathfrak{S}, \varepsilon_l^{\mathfrak{S}}(\sigma) = a_{\sigma}. \quad (21)$$

■

<sup>1</sup> $\widehat{\Sigma \Sigma'}^{\mathfrak{S}}$  means that  $\Sigma$  and  $\Sigma'$  have a common upper-bound in  $\mathfrak{S}$ , and  $\neg \widehat{\Sigma \Sigma'}^{\mathfrak{S}}$  means they have none.  
<sup>2</sup> $\uparrow^{\mathfrak{S}} \Sigma$  denotes the upper subset  $\{\sigma \in \mathfrak{S} \mid \sigma \sqsupseteq_{\mathfrak{S}} \Sigma\}$ .

*Proof.* Straightforward. If  $\{\sigma \mid \alpha_\sigma = \mathbf{Y}\}$  and  $\{\sigma \mid \alpha_\sigma = \mathbf{N}\}$  are not empty, it suffices to define  $\Sigma_A := \prod^{\mathfrak{E}} \{\sigma \mid \alpha_\sigma = \mathbf{Y}\}$ ,  $\Sigma'_A := \prod^{\mathfrak{E}} \{\sigma \mid \alpha_\sigma = \mathbf{N}\}$  and  $l := l_{(\Sigma_A, \Sigma'_A)}$  (the case where some or all of these subsets are empty is treated immediately).  $\square$

**Theorem 2.** Let us consider a map  $(B : \mathfrak{E} \longrightarrow \mathfrak{B}, l \mapsto b_l)$  satisfying

$$\forall l, l' \in \mathfrak{E}, \quad (l \sqsubseteq_{\mathfrak{E}} l') \Rightarrow (b_l \leq b_{l'}), \quad (22)$$

$$\forall \{l_i \mid i \in I\} \subseteq \mathfrak{E}, \quad b_{\prod_{i \in I} l_i} = \bigwedge_{i \in I} b_{l_i}, \quad (23)$$

$$\forall l \in \mathfrak{E}, \quad b_{\top} = \overline{b_l}, \quad (24)$$

$$b_{\eta_{\mathfrak{E}}} = \mathbf{Y}. \quad (25)$$

Then, we have

$$\exists! \sigma \in \mathfrak{S} \mid \forall l \in \mathfrak{E}, \varepsilon_l^{\mathfrak{S}}(\sigma) = b_l. \quad (26)$$

■

*Proof.* Let us consider  $l_B := \prod^{\mathfrak{E}} \{l \in \mathfrak{E} \mid b_l = \mathbf{Y}\}$ . Note that  $l_B$  exists because  $\mathfrak{E}$  is a down-complete Inf semi-lattice. Moreover,  $b_{l_B} = \mathbf{Y}$  because of the relation (23). Note also that  $l \sqsubseteq_{\mathfrak{E}} l_B$  implies  $b_l = \mathbf{Y}$  because of the relation (22), and conversely  $b_l = \mathbf{Y}$  implies  $l \sqsubseteq_{\mathfrak{E}} l_B$  due to the definition of  $l_B$ . Let us now introduce  $\Sigma_{l_B} = \prod^{\mathfrak{E}} (\varepsilon_{l_B}^{\mathfrak{S}})^{-1}(\mathbf{Y})$ . For any  $l$  such that  $l \sqsubseteq_{\mathfrak{E}} l_B$ , we have  $\varepsilon_l^{\mathfrak{S}}(\Sigma_{l_B}) \geq \varepsilon_{l_B}^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{Y}$ , i.e.  $\varepsilon_l^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{Y}$ . We could suppose that  $l_B = l_{(\Sigma_{l_B}, \Sigma'_{l_B})}$  for a certain  $\Sigma'_{l_B} \in \mathfrak{S}$ . However, we note that, because of (23) and (25), we have  $l_{(\Sigma_{l_B}, \cdot)} \sqsubseteq_{\mathfrak{E}} l_{(\Sigma_{l_B}, \Sigma'_{l_B})}$  and  $b_{l_{(\Sigma_{l_B}, \cdot)}} = b_{l_{(\Sigma_{l_B}, \Sigma'_{l_B})}} \sqcap_{\mathfrak{E}} \eta_{\mathfrak{E}} = b_{l_{(\Sigma_{l_B}, \Sigma'_{l_B})}} \wedge b_{\eta_{\mathfrak{E}}} = \mathbf{Y}$  which would contradict the definition of  $l_B$ . Hence, we have to accept that  $l_B = l_{(\Sigma_{l_B}, \cdot)}$ . Thus, we note that, for any  $l_{(\Sigma, \Sigma')}$ , the property  $l_{(\Sigma, \Sigma')} \not\sqsubseteq_{\mathfrak{E}} l_B$  is then equivalent to the property  $\Sigma \not\sqsubseteq_{\mathfrak{S}} \Sigma_{l_B}$ . Then, if  $l_{(\Sigma, \Sigma')} \not\sqsubseteq_{\mathfrak{E}} l_B$  we cannot have  $\varepsilon_{l_{(\Sigma, \Sigma')}}^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{Y}$ . We then conclude that the property  $\varepsilon_l^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{Y}$  is equivalent to the property  $l \sqsubseteq_{\mathfrak{E}} l_B$ , or in other words  $\varepsilon_l^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{Y}$  is equivalent to  $b_l = \mathbf{Y}$ . Using (24) and (12), we deduce that  $(\varepsilon_l^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{N}) \Leftrightarrow (\varepsilon_l^{\mathfrak{S}}(\Sigma_{l_B}) = \mathbf{Y}) \Leftrightarrow (b_l = \mathbf{Y}) \Leftrightarrow (b_l = \mathbf{N})$ . As a final conclusion, we have for any  $l \in \mathfrak{E}$  the equality  $\varepsilon_l^{\mathfrak{S}}(\Sigma_B) = b_l$ . This concludes the proof.  $\square$

**Theorem 3.**

$$\forall \{\sigma_i \mid i \in I\} \subseteq_{Chain} \mathfrak{S}, \exists \sigma \in \mathfrak{S} \mid \forall l \in \mathfrak{E}, \varepsilon_l^{\mathfrak{S}}(\sigma) = \bigvee_{i \in I} \varepsilon_l^{\mathfrak{S}}(\sigma_i), \quad (27)$$

$$\sigma = \bigsqcup_{i \in I}^{\mathfrak{E}} \sigma_i. \quad (28)$$

As a consequence, using Zorn's Lemma, we deduce that

$$\forall \sigma \in \mathfrak{S}, \quad \exists \sigma' \in Max(\mathfrak{S}) \mid \sigma \sqsubseteq_{\mathfrak{S}} \sigma'. \quad (29)$$

■

*Proof.* First of all, we note that  $\{\sigma_i \mid i \in I\} \subseteq_{Chain} \mathfrak{S}$  and the monotonicity property of  $\varepsilon^{\mathfrak{S}}$  implies that  $\{\varepsilon_l^{\mathfrak{S}}(\sigma_i) \mid i \in I\} \subseteq_{Chain} \mathfrak{B}$  for any  $l \in \mathfrak{E}$  and then  $\bigvee_{i \in I} \varepsilon_l^{\mathfrak{S}}(\sigma_i)$  exists for any  $l \in \mathfrak{E}$

due to the chain-completeness of  $\mathfrak{B}$ .

Using the properties (8)(12)(13) of the map  $\varepsilon$  and the complete-distributivity properties satisfied by  $\mathfrak{B}$ , we can check easily that the map  $\iota \mapsto \bigvee_{i \in I} \varepsilon_i^{\mathfrak{G}}(\sigma_i)$  satisfies properties (22) (23) (24) (25). As a consequence, the property (27) is a direct consequence of Theorem 2.

By definition of the poset structure on  $\mathfrak{G}$ , we deduce, from the property  $(\forall \iota \in \mathfrak{E}, \varepsilon_i^{\mathfrak{G}}(\sigma) = \bigvee_{i \in I} \varepsilon_i^{\mathfrak{G}}(\sigma_i))$ , that  $\sigma \sqsupseteq_{\mathfrak{G}} \sigma_i, \forall i \in I$  and  $(\sigma' \sqsupseteq_{\mathfrak{G}} \sigma_i, \forall i \in I) \Rightarrow (\sigma \sqsupseteq_{\mathfrak{G}} \sigma')$ . In other words,  $\sigma = \bigsqcup_{i \in I}^{\mathfrak{G}} \sigma_i$ .  $\square$

*Remark 1.* We now observe that, if  $\mathfrak{G}$  has a description in terms of pure states, then  $\mathfrak{E}$  inherits a description in terms of pure states as well.

To check this point, we first note that the space of effects has a bottom element denoted  $\perp_{\mathfrak{E}}$  and defined by

$$\forall \sigma \in \mathfrak{G}, \quad \varepsilon_{\perp_{\mathfrak{E}}}(\sigma) = \perp. \quad (30)$$

Secondly,  $\mathfrak{E}$  appears to be an algebraic domain.

To prove this point, we firstly observe that  $\mathfrak{E}$  satisfies the following chain-completeness property

$$\forall \{\iota_i \mid i \in I\} \subseteq_{Chain} \mathfrak{E}, \exists \iota \in \mathfrak{E} \quad | \quad \forall \sigma \in \mathfrak{G}, \varepsilon_{\iota}(\sigma) = \bigvee_{i \in I} \varepsilon_{\iota_i}(\sigma), \quad (31)$$

$$\iota = \bigsqcup_{i \in I}^{\mathfrak{E}} \iota_i. \quad (32)$$

This is an immediate consequence of the down-completeness of  $\mathfrak{G}$ , using the general expression of effects introduced in (15) and subsequents, because of the properties (18).

We secondly observe that  $\mathfrak{E}$  is atomistic, i.e. <sup>[3]</sup>

$$\exists \mathcal{A}_{\mathfrak{E}} \subseteq \mathfrak{E} \quad | \quad \begin{cases} \forall \iota \in \mathcal{A}_{\mathfrak{E}}, \perp_{\mathfrak{E}} \bar{\sqsubseteq}_{\mathfrak{E}} \iota \\ \forall \iota \in \mathfrak{E} \setminus \{\perp_{\mathfrak{E}}\}, \exists \iota' \in \mathcal{A}_{\mathfrak{E}} \mid \iota' \sqsubseteq_{\mathfrak{E}} \iota \end{cases} \quad (33)$$

$$\mathcal{A}_{\mathfrak{E}} = \{ \iota_{(\Sigma, \cdot)} \mid \Sigma \in \mathfrak{G}^{pure} \} \cup \{ \iota_{(\cdot, \Sigma)} \mid \Sigma \in \mathfrak{G}^{pure} \} \quad (34)$$

and that

$$\forall \iota \in \mathfrak{E}, \quad \iota = \bigsqcup^{\mathfrak{E}} \{ \iota' \in \mathcal{A}_{\mathfrak{E}} \mid \iota' \sqsubseteq_{\mathfrak{E}} \iota \}. \quad (35)$$

The property (29) implies directly the second condition of (33). The first condition of (33) is easy to check using the expression of the order (18). The property (35) is a direct consequence of property (11).

Endly, the compacity of atoms is trivial.

The algebraicity of  $\mathfrak{E}$  follows.

As a conclusion of this analysis, if we define the "pure effects" as the completely meet-irreducible elements in  $\mathfrak{E}$ , we can check the following property analog to (11) :

$$\forall \iota \in \mathfrak{E}, \quad \iota = \bigsqcap^{\mathfrak{E}} \underline{\iota}_{\mathfrak{E}}, \quad \text{where } \underline{\iota}_{\mathfrak{E}} = (\mathfrak{E}^{pure} \cap (\uparrow^{\mathfrak{E}} \iota)). \quad (36)$$

Indeed, from previous results  $\mathfrak{E}$  is a bounded-complete algebraic domain. The property (36) is then a direct consequence of [4, Theorem I-4.26 p.126].

We can conclude our analysis by characterizing explicitly the elements of  $Max(\mathfrak{E})$ .<sup>[4]</sup> We have explicitly

$$\mathfrak{E}^{pure} = Max(\mathfrak{E}) = \{ \iota_{(\Sigma, \Sigma')} \mid \Sigma \check{\boxtimes}_{\mathfrak{E}} \Sigma' \} \cup \{ \mathfrak{Y}_{\mathfrak{E}} \} \cup \{ \overline{\mathfrak{Y}}_{\mathfrak{E}} \}. \quad (38)$$

<sup>3</sup>We adopt the notation  $\forall \iota \in \mathfrak{E}, \perp_{\mathfrak{E}} \bar{\sqsubseteq}_{\mathfrak{E}} \iota \Leftrightarrow (\perp_{\mathfrak{E}} \sqsubseteq_{\mathfrak{E}} \iota \text{ and } \forall \iota' \in \mathfrak{E}, \perp_{\mathfrak{E}} \sqsubseteq_{\mathfrak{E}} \iota' \sqsubseteq_{\mathfrak{E}} \iota \Rightarrow (\perp_{\mathfrak{E}} = \iota' \text{ or } \iota' = \iota))$ .

<sup>4</sup>We introduce the following binary relation, denoted  $\check{\boxtimes}_{\mathfrak{E}}$  and defined on  $\mathfrak{G}$  by

$$\forall (\sigma, \sigma') \in \mathfrak{G}^{\times 2}, \quad \sigma \check{\boxtimes}_{\mathfrak{E}} \sigma' \quad :\Leftrightarrow \quad (\forall \sigma'' \sqsubseteq_{\mathfrak{E}} \sigma', \widehat{\sigma \sigma''}^{\mathfrak{E}} \text{ and } \forall \sigma'' \sqsubseteq_{\mathfrak{E}} \sigma, \widehat{\sigma' \sigma''}^{\mathfrak{E}} \text{ and not } \widehat{\sigma \sigma'}^{\mathfrak{E}}). \quad (37)$$

## 2.2 Morphisms

We turn the collection of States/Effects Chu spaces into a category by defining the following morphisms.

**Definition 1.** We will consider the morphisms from a States/Effects Chu space  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to another States/Effects Chu space  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ , i.e. pairs of maps  $f : \mathfrak{S}_A \rightarrow \mathfrak{S}_B$  and  $f^* : \mathfrak{E}_B \rightarrow \mathfrak{E}_A$  satisfying the following properties (see [6])

$$\forall \sigma_A \in \mathfrak{S}_A, \forall l_B \in \mathfrak{E}_B \quad \varepsilon_{l_B}^{\mathfrak{S}_B}(f(\sigma_A)) = \varepsilon_{f^*(l_B)}^{\mathfrak{S}_A}(\sigma_A). \quad (39)$$

*Remark 2.* Note that, the eventual surjectivity of  $f^*$  implies the injectivity of  $f$ . This point uses the property (10). Explicitly,

$$\begin{aligned} \forall \sigma_A, \sigma'_A \in \mathfrak{S}_A, f(\sigma_A) = f(\sigma'_A) &\Leftrightarrow (\forall l_B \in \mathfrak{E}_B, \varepsilon_{l_B}^{\mathfrak{S}_B}(f(\sigma_A)) = \varepsilon_{l_B}^{\mathfrak{S}_B}(f(\sigma'_A))) \\ &\Leftrightarrow (\forall l_B \in \mathfrak{E}_B, \varepsilon_{f^*(l_B)}^{\mathfrak{S}_B}(\sigma_A) = \varepsilon_{f^*(l_B)}^{\mathfrak{S}_B}(\sigma'_A)) \\ &\Leftrightarrow (\forall l'_B \in \mathfrak{E}_B, \varepsilon_{l'_B}^{\mathfrak{S}_B}(\sigma_A) = \varepsilon_{l'_B}^{\mathfrak{S}_B}(\sigma'_A)) \\ &\Leftrightarrow (\sigma_A = \sigma'_A). \end{aligned} \quad (40)$$

In the same way, using the properties (9) and the surjectivity of  $f$ , we can deduce the injectivity of  $f^*$ .

The duality property (39) suffices to deduce the following properties.

**Theorem 4.** The left-component  $f$  of a Chu morphism from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  satisfies

$$\forall S \subseteq \mathfrak{S}_A, \quad f(\prod_{\sigma \in S}^{\mathfrak{S}_A} \sigma) = \prod_{\sigma \in S}^{\mathfrak{S}_B} f(\sigma) \quad (41)$$

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}_A, \quad f(\bigsqcup_{\sigma \in \mathfrak{C}}^{\mathfrak{S}_A} \sigma) = \bigsqcup_{\sigma \in \mathfrak{C}}^{\mathfrak{S}_B} f(\sigma). \quad (42)$$

As a consequence of (42),  $f$  is in particular monotonic.

The right-component  $f^*$  of a Chu morphism from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  satisfies

$$\forall E \subseteq \mathfrak{E}_B, \quad f^*(\prod_{l \in E}^{\mathfrak{E}_B} l) = \prod_{l \in E}^{\mathfrak{E}_A} f^*(l) \quad (43)$$

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{E}_B, \quad f^*(\bigsqcup_{l \in \mathfrak{C}}^{\mathfrak{E}_B} l) = \bigsqcup_{l \in \mathfrak{C}}^{\mathfrak{E}_A} f^*(l) \quad (44)$$

$$\forall l \in \mathfrak{E}_B, \quad f^*(\overline{l}) = \overline{f^*(l)} \quad (45)$$

$$f^*(\mathfrak{I}_{\mathfrak{E}_B}) = \mathfrak{I}_{\mathfrak{E}_A}. \quad (46)$$

In particular,  $f^*$  is monotonic. ■

*Proof.* All proofs follow the same trick based on the duality relation (39) and the separation property (10). For example, for any  $S \subseteq \mathfrak{S}_A$  and any  $l \in \mathfrak{E}_B$ , we have, using (39) and (8) :

$$\begin{aligned} \varepsilon_l^{\mathfrak{S}_B}(f(\prod_{\sigma \in S}^{\mathfrak{S}_A} \sigma)) &= \varepsilon_{f^*(l)}^{\mathfrak{S}_A}(\prod_{\sigma \in S}^{\mathfrak{S}_A} \sigma) \\ &= \bigwedge_{\sigma \in S} \varepsilon_{f^*(l)}^{\mathfrak{S}_A}(\sigma) \\ &= \bigwedge_{\sigma \in S} \varepsilon_l^{\mathfrak{S}_B}(f(\sigma)) \\ &= \varepsilon_l^{\mathfrak{S}_B}(\prod_{\sigma \in S}^{\mathfrak{S}_B} f(\sigma)) \end{aligned} \quad (47)$$

We now use the property (10) to deduce (41).  $\square$

**Theorem 5.** For any monotonic map  $f : \mathfrak{S} \longrightarrow \mathfrak{S}'$  satisfying  $\forall \{\sigma_i \mid i \in I\} \subseteq \mathfrak{S}, f(\prod_{i \in I}^{\mathfrak{S}} \sigma_i) = \prod_{i \in I}^{\mathfrak{S}'} f(\sigma_i)$ , there exists a unique map  $f^* : \mathfrak{E}' \longrightarrow \mathfrak{E}$  such that

$$\forall l \in \mathfrak{E}, \varepsilon_{f^*(l)}^{\mathfrak{S}}(\sigma) = \varepsilon_l^{\mathfrak{S}'}(f(\sigma)). \quad (48)$$

■

*Proof.* Direct consequence of Theorem 1.  $\square$

As a consequence of this theorem, the couple of maps  $(f, f^*)$  defining a morphism from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  can then be reduced to the single data  $f$ . We will then speak shortly of "the morphism  $f$  from the space of states  $\mathfrak{S}_A$  to the space of states  $\mathfrak{S}_B$ " rather than "the morphism from the states/effects Chu space  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to the states/effects Chu space  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ ".

**Definition 2.** The space of morphisms from the space of states  $\mathfrak{S}_A$  to the space of states  $\mathfrak{S}_B$  will be denoted  $\mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ . It is the space of maps from  $\mathfrak{S}_A$  to  $\mathfrak{S}_B$  that is order-preserving and satisfies the homomorphic property (41).

**Theorem 6.** The composition of a morphism  $(f, f^*)$  from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  by another morphism  $(g, g^*)$  defined from  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  to  $(\mathfrak{S}_C, \mathfrak{E}_C, \varepsilon^{\mathfrak{S}_C})$  is given by  $(g \circ f, f^* \circ g^*)$  defining a valid morphism from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_C, \mathfrak{E}_C, \varepsilon^{\mathfrak{S}_C})$ . ■

*Proof.* Using two times the duality property, we obtain

$$\varepsilon_{l_C}^{\mathfrak{S}_C}(g \circ f(\sigma_A)) = \varepsilon_{g^*(l_C)}^{\mathfrak{S}_B}(f(\sigma_A)) = \varepsilon_{f^* \circ g^*(l_C)}^{\mathfrak{S}_A}(\sigma_A). \quad (49)$$

$\square$

**Definition 3.** We define the infimum of two maps  $f$  and  $g$  satisfying (41) (resp. two maps  $f^*$  and  $g^*$  satisfying (43)) by  $\forall \sigma \in \mathfrak{S}_A, (f \sqcap g)(\sigma) := f(\sigma) \sqcap_{\mathfrak{S}_B} g(\sigma)$  (resp.  $\forall l \in \mathfrak{E}_B, (f^* \sqcap g^*)(l) := f^*(l) \sqcap_{\mathfrak{E}_A} g^*(l)$ ).

**Theorem 7.** The infimum of a morphism  $(f, f^*)$  from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  with another morphism  $(g, g^*)$  defined from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$  is given by  $(g \sqcap f, f^* \sqcap g^*)$  defining a valid morphism from  $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$  to  $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ . ■

*Proof.* Using two times the duality property and the homomorphic property of  $\varepsilon$ , we obtain

$$\varepsilon_{l_B}^{\mathfrak{S}_B}((f \sqcap g)(\sigma_A)) = \varepsilon_{l_B}^{\mathfrak{S}_B}(f(\sigma_A)) \wedge \varepsilon_{l_B}^{\mathfrak{S}_B}(g(\sigma_A)) = \varepsilon_{f^*(l_B)}^{\mathfrak{S}_A}(\sigma_A) \wedge \varepsilon_{g^*(l_B)}^{\mathfrak{S}_A}(\sigma_A) = \varepsilon_{(f^* \sqcap g^*)(l_B)}^{\mathfrak{S}_A}(\sigma_A). \quad (50)$$

$\square$



### 3 A new perspective on the construction of the tensor product of semi-lattices

#### 3.1 The canonical tensor product construction

Let us first introduce the classical construction of G.A. Fraser for the tensor product of semi-lattices [2, 3]. As it will be clarified in the next subsection a new proposal can be made for the tensor product of semi-lattices.

**Definition 4.** Let  $A, B$  and  $C$  be semi-lattices. A function  $f : A \times B \rightarrow C$  is a bi-homomorphism if the functions  $g_a : B \rightarrow C$  defined by  $g_a(b) = f(a, b)$  and  $h_b : A \rightarrow C$  defined by  $h_b(a) = f(a, b)$  are homomorphisms for all  $a \in A$  and  $b \in B$ .

**Theorem 8. [2, Definition 2.2 and Theorem 2.3]**

The tensor product  $S_{AB} := \mathfrak{S}_A \otimes \mathfrak{S}_B$  of the two Inf semi-lattices  $\mathfrak{S}_A$  and  $\mathfrak{S}_B$  is obtained as a solution of the following universal problem : there exists a bi-homomorphism, denoted  $\iota$  from  $\mathfrak{S}_A \times \mathfrak{S}_B$  to  $S_{AB}$ , such that, for any Inf semi-lattice  $\mathfrak{S}$  and any bi-homomorphism  $f$  from  $\mathfrak{S}_A \times \mathfrak{S}_B$  to  $\mathfrak{S}$ , there is a unique homomorphism  $g$  from  $S_{AB}$  to  $\mathfrak{S}$  with  $f = g \circ \iota$ . We denote  $\iota(\sigma, \sigma') = \sigma \otimes \sigma'$  for any  $\sigma \in \mathfrak{S}_A$  and  $\sigma' \in \mathfrak{S}_B$ .

The tensor product  $S_{AB}$  exists and is unique up to isomorphism, it is built as the homomorphic image of the free  $\sqcap$  semi-lattice generated by the set  $\mathfrak{S}_A \times \mathfrak{S}_B$  under the congruence relation determined by identifying  $(\sigma_1 \sqcap_{\mathfrak{S}_A} \sigma_2, \sigma')$  with  $(\sigma_1, \sigma') \sqcap (\sigma_2, \sigma')$  for all  $\sigma_1, \sigma_2 \in \mathfrak{S}_A, \sigma' \in \mathfrak{S}_B$  and identifying  $(\sigma, \sigma'_1 \sqcap_{\mathfrak{S}_B} \sigma'_2)$  with  $(\sigma, \sigma'_1) \sqcap (\sigma, \sigma'_2)$  for all  $\sigma \in \mathfrak{S}_A, \sigma'_1, \sigma'_2 \in \mathfrak{S}_B$ .

In other words,  $S_{AB}$  is the Inf semi-lattice (the infimum of  $S \subseteq S_{AB}$  will be denoted  $\sqcap^{S_{AB}} S$ ) generated by the elements  $\sigma_A \otimes \sigma_B$  with  $\sigma_A \in \mathfrak{S}_A, \sigma_B \in \mathfrak{S}_B$  and subject to the conditions

$$(\sigma_A \sqcap_{\mathfrak{S}_A} \sigma'_A) \otimes \sigma_B = (\sigma_A \otimes \sigma_B) \sqcap_{S_{AB}} (\sigma'_A \otimes \sigma_B), \quad \sigma_A \otimes (\sigma_B \sqcap_{\mathfrak{S}_B} \sigma'_B) = (\sigma_A \otimes \sigma_B) \sqcap_{S_{AB}} (\sigma_A \otimes \sigma'_B). \quad (51)$$

■

**Definition 5.** The space  $S_{AB} = \mathfrak{S}_A \otimes \mathfrak{S}_B$  is turned into a partially ordered set with the following binary relation

$$\forall \sigma_{AB}, \sigma'_{AB} \in S_{AB}, \quad (\sigma_{AB} \sqsubseteq_{S_{AB}} \sigma'_{AB}) \Leftrightarrow (\sigma_{AB} \sqcap_{S_{AB}} \sigma'_{AB} = \sigma_{AB}). \quad (52)$$

**Definition 6.** A non-empty subset  $\mathfrak{R}$  of  $\mathfrak{S}_A \times \mathfrak{S}_B$  is called a bi-filter of  $\mathfrak{S}_A \times \mathfrak{S}_B$  iff

$$\forall \sigma_A, \sigma_{1,A}, \sigma_{2,A} \in \mathfrak{S}_A, \forall \sigma_B, \sigma_{1,B}, \sigma_{2,B} \in \mathfrak{S}_B, \\ ((\sigma_{1,A}, \sigma_{1,B}) \leq (\sigma_{2,A}, \sigma_{2,B}) \text{ and } (\sigma_{1,A}, \sigma_{1,B}) \in \mathfrak{R}) \Rightarrow (\sigma_{2,A}, \sigma_{2,B}) \in \mathfrak{R}, \quad (53)$$

$$(\sigma_{1,A}, \sigma_B), (\sigma_{2,A}, \sigma_B) \in \mathfrak{R} \Rightarrow (\sigma_{1,A} \sqcap_{\mathfrak{S}_A} \sigma_{2,A}, \sigma_B) \in \mathfrak{R}, \quad (54)$$

$$(\sigma_A, \sigma_{1,B}), (\sigma_A, \sigma_{2,B}) \in \mathfrak{R} \Rightarrow (\sigma_A, \sigma_{1,B} \sqcap_{\mathfrak{S}_B} \sigma_{2,B}) \in \mathfrak{R}. \quad (55)$$

**Definition 7.** If  $\{(\sigma_{1,A}, \sigma_{1,B}), \dots, (\sigma_{n,A}, \sigma_{n,B})\}$  is a non-empty finite subset of  $\mathfrak{S}_A \times \mathfrak{S}_B$ , then the intersection of all bi-filters of  $\mathfrak{S}_A \times \mathfrak{S}_B$  which contain  $(\sigma_{1,A}, \sigma_{1,B}), \dots, (\sigma_{n,A}, \sigma_{n,B})$  is a bi-filter, which we denote by  $\mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \dots, (\sigma_{n,A}, \sigma_{n,B})\}$ .

**Lemma 1.** If  $F$  is a filter of  $S_{AB} = \mathfrak{S}_A \otimes \mathfrak{S}_B$  then the set  $\alpha(F) := \{(\sigma_A, \sigma_B) \in \mathfrak{S}_A \times \mathfrak{S}_B \mid \sigma_A \otimes \sigma_B \in F\}$  is a bi-filter of  $\mathfrak{S}_A \times \mathfrak{S}_B$ . ■

**Lemma 2. [3, Lemma 1]** Let us choose  $\sigma_A, \sigma_{1,A}, \dots, \sigma_{n,A} \in \mathfrak{S}_A$  and  $\sigma_B, \sigma_{1,B}, \dots, \sigma_{n,B} \in \mathfrak{S}_B$ . Then,

$$(\sigma_A, \sigma_B) \in \mathfrak{F}\{(\sigma_{1,A}, \sigma_{1,B}), \dots, (\sigma_{n,A}, \sigma_{n,B})\} \Leftrightarrow \left(\prod_{1 \leq i \leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right) \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B. \quad (56)$$

■

**Lemma 3. [3, Theorem 1]**

Let us choose  $\sigma_A, \sigma_{1,A}, \dots, \sigma_{n,A} \in \mathfrak{S}_A$  and  $\sigma_B, \sigma_{1,B}, \dots, \sigma_{n,B} \in \mathfrak{S}_B$ . Then,

$$\left(\prod_{1 \leq i \leq n}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}\right) \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B \Leftrightarrow \begin{array}{l} \text{there exists a } n\text{-ary lattice polynomial } p \mid \sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A}) \\ \text{and } \sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma_{1,B}, \dots, \sigma_{n,B}). \end{array} \quad (57)$$

where  $p^*$  denotes the lattice polynomial obtained from  $p$  by dualizing the lattice operations. ■

## 3.2 The maximal tensor product

Let us now consider a radically different approach of tensor product, exploiting the notion of States/Effects Chu spaces.

**Definition 8.** We will denote by  $\check{S}_{AB}$  (or equivalently by  $\mathfrak{S}_A \check{\otimes} \mathfrak{S}_B$ ) the set of maps  $\Phi$  from  $\mathfrak{E}_{\mathfrak{S}_A} \times \mathfrak{E}_{\mathfrak{S}_B}$  to  $\mathfrak{E}_{\perp} \cong \mathfrak{B}$  satisfying

$$\forall \{l_{i,A} \mid i \in I\} \subseteq \mathfrak{E}_{\mathfrak{S}_A}, \forall l_B \in \mathfrak{E}_{\mathfrak{S}_B}, \quad \Phi\left(\prod_{i \in I}^{\mathfrak{E}_{\mathfrak{S}_A}} l_{i,A}, l_B\right) = \bigwedge_{i \in I} \Phi(l_{i,A}, l_B) \quad (58)$$

$$\forall \{l_{j,B} \mid j \in J\} \subseteq \mathfrak{E}_{\mathfrak{S}_B}, \forall l_A \in \mathfrak{E}_{\mathfrak{S}_A}, \quad \Phi(l_A, \prod_{j \in J}^{\mathfrak{E}_{\mathfrak{S}_B}} l_{j,B}) = \bigwedge_{j \in J} \Phi(l_A, l_{j,B}), \quad (59)$$

$$\forall l_A \in \mathfrak{E}_{\mathfrak{S}_A} \quad \Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \overline{\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}})}, \quad (60)$$

$$\forall l_B \in \mathfrak{E}_{\mathfrak{S}_B}, \quad \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{l_B}) = \overline{\Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, l_B)}, \quad (61)$$

$$\Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}, \quad (62)$$

$$\Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) = \mathbf{N}. \quad (63)$$

$\check{S}_{AB}$  is called *the maximal tensor product* of  $\mathfrak{S}_A$  and  $\mathfrak{S}_B$ .

**Theorem 9.**  $\check{S}_{AB}$  is equipped with the pointwise partial order. It is a down-complete Inf semi-lattice with

$$\forall \{\Phi_i \mid i \in I\} \subseteq \check{S}_{AB}, \forall (l_A, l_B) \in \mathfrak{E}_{\mathfrak{S}_A} \times \mathfrak{E}_{\mathfrak{S}_B}, \quad \left(\prod_{i \in I}^{\check{S}_{AB}} \Phi_i\right)(l_A, l_B) := \bigwedge_{i \in I} \Phi_i(l_A, l_B). \quad (64)$$

■

**Theorem 10.** The following maps are homomorphisms

$$\begin{aligned} \eta : \check{S}_{AB} &\longrightarrow \mathfrak{E}_{\mathfrak{S}_A}^* & \lambda : \check{S}_{AB} &\longrightarrow \mathfrak{E}_{\mathfrak{S}_B}^* \\ \Phi &\mapsto \Phi(\cdot, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) & \Phi &\mapsto \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \cdot) \end{aligned} \quad (65)$$

with

$$\mathfrak{E}_{\mathfrak{S}}^* := \{ \psi \in \mathfrak{C}(\mathfrak{E}_{\mathfrak{S}}, \mathfrak{B}) \mid \forall l \in \mathfrak{E}_{\mathfrak{S}}, \psi(\bar{l}) = \overline{\psi(l)} \text{ and } \psi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}}) = \mathbf{Y} \}. \quad (66)$$

Moreover, we have, for any space of states  $\mathfrak{S}$ , the following isomorphism :

$$\mathfrak{E}_{\mathfrak{S}}^* \cong \mathfrak{S} \quad (67)$$

■

*Proof.* Let  $\Phi$  be an element of  $\check{S}_{AB}$ .

The map  $\psi$  from  $\mathfrak{E}_{\mathfrak{S}_A}$  to  $\mathfrak{B}$  defined by  $\psi(l_A) := \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}})$  is an element of  $\mathfrak{C}(\mathfrak{E}_{\mathfrak{S}_A}, \mathfrak{B})$  because of relation (58). We have  $\forall l \in \mathfrak{E}_{\mathfrak{S}}, \psi(\bar{l}) = \overline{\psi(l)}$  because of relation (60) and  $\psi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}}) = \mathbf{Y}$  because of relation (62). As a result,  $\psi$  is an element of  $\mathfrak{E}_{\mathfrak{S}_A}^*$ .

In the same way, the map  $\psi'$  from  $\mathfrak{E}_{\mathfrak{S}_B}$  to  $\mathfrak{B}$  defined by  $\psi'(l_B) := \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, l_B)$  is an element of  $\mathfrak{E}_{\mathfrak{S}_B}^*$  because of relations (59)(61) and (62).

Secondly, we note the following isomorphism of Inf semi-lattices :

$$\begin{aligned} \rho : \mathfrak{S} &\xrightarrow{\cong} \mathfrak{E}_{\mathfrak{S}}^* \\ \sigma &\mapsto \rho(\sigma) \mid \rho(\sigma)(l) := \varepsilon_l^{\mathfrak{S}}(\sigma), \forall l \in \mathfrak{E}_{\mathfrak{S}}. \end{aligned} \quad (68)$$

Indeed, for any  $\sigma \in \mathfrak{S}$ , we can define a map  $\varphi$  from  $\mathfrak{E}_{\mathfrak{S}}$  to  $\mathfrak{B}$  by  $\varphi(l) := \varepsilon_l^{\mathfrak{S}}(\sigma)$ . Using the properties (12)(8)(13) of  $\varepsilon^{\mathfrak{S}}$ , we deduce that  $\varphi \in \mathfrak{E}_{\mathfrak{S}}^*$ .

Reciprocally, using Theorem 2, we know that

$$\forall \varphi \in \mathfrak{E}_{\mathfrak{S}}^*, \quad \exists! \sigma \in \mathfrak{S} \mid \forall l \in \mathfrak{E}, \varepsilon_l^{\mathfrak{S}}(\sigma) = \varphi(l). \quad (69)$$

The bijective character of the map  $\rho$  is then established. We have also trivially, for any  $\{\sigma_i \mid i \in I\} \subseteq \mathfrak{S}$  the homomorphic property  $\rho(\prod_{i \in I}^{\mathfrak{S}} \sigma_i) = \prod_{i \in I} \rho(\sigma_i)$  due to the property (7).  $\square$

**Theorem 11.** The inclusion of pure tensors in  $\check{S}_{AB}$  is realized as follows :

$$\begin{aligned} \iota^{\check{S}_{AB}} : \mathfrak{S}_A \times \mathfrak{S}_B &\hookrightarrow \check{S}_{AB} \\ (\sigma_A, \sigma_B) &\mapsto \iota^{\check{S}_{AB}}(\sigma_A, \sigma_B) \mid \forall (l_A, l_B) \in \mathfrak{E}_{\mathfrak{S}_A} \times \mathfrak{E}_{\mathfrak{S}_B}, \iota^{\check{S}_{AB}}(\sigma_A, \sigma_B)(l_A, l_B) := \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_A) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_B) \in \mathfrak{B}. \end{aligned} \quad (70)$$

■

*Proof.* The properties (58) and (59) are direct consequences of the properties (8) and (4). The properties (60) (61) (62) and (63) are direct consequences of the properties (12) (13) (3). As a conclusion,  $\iota^{\check{S}_{AB}}(\sigma_A, \sigma_B) \in \check{S}_{AB}$  for any  $(\sigma_A, \sigma_B)$  in  $\mathfrak{S}_A \times \mathfrak{S}_B$ .

Let us now consider  $(\sigma_A, \sigma_B)$  and  $(\sigma'_A, \sigma'_B)$  in  $\mathfrak{S}_A \times \mathfrak{S}_B$  such that  $\iota^{\check{S}_{AB}}(\sigma_A, \sigma_B) = \iota^{\check{S}_{AB}}(\sigma'_A, \sigma'_B)$ . We choose first of all  $l_A := l_{(\sigma_A, \cdot)}$  and  $l_B := l_{(\sigma_B, \cdot)}$ . We have  $\iota^{\check{S}_{AB}}(\sigma_A, \sigma_B)(l_A, l_B) = \mathbf{Y} \bullet \mathbf{Y} = \mathbf{Y}$  and then must have  $\varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma'_A) = \mathbf{Y}$  and  $\varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma'_B) = \mathbf{Y}$ , i.e.  $\sigma_A \sqsubseteq_{\mathfrak{S}_A} \sigma'_A$  and  $\sigma_B \sqsubseteq_{\mathfrak{S}_B} \sigma'_B$ . Choosing now  $l_A := l_{(\sigma'_A, \cdot)}$  and  $l_B := l_{(\sigma'_B, \cdot)}$ , we justify also  $\sigma_A \supseteq_{\mathfrak{S}_A} \sigma'_A$  and  $\sigma_B \supseteq_{\mathfrak{S}_B} \sigma'_B$ . The map  $\iota^{\check{S}_{AB}}$  is then injective.  $\square$

**Theorem 12.** We have the following relations

$$\forall \{\sigma_{i,A} \mid i \in I\} \subseteq \mathfrak{S}_A, \forall \sigma_B \in \mathfrak{S}_B, \quad \iota^{\check{S}_{AB}}(\prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}, \sigma_B) = \prod_{i \in I}^{\check{S}_{AB}} \iota^{\check{S}_{AB}}(\sigma_{i,A}, \sigma_B), \quad (71)$$

$$\forall \{\sigma_{i,B} \mid i \in I\} \subseteq \mathfrak{S}_B, \forall \sigma_A \in \mathfrak{S}_A, \quad \iota^{\check{S}_{AB}}(\sigma_A, \prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,B}) = \prod_{i \in I}^{\check{S}_{AB}} \iota^{\check{S}_{AB}}(\sigma_A, \sigma_{i,B}). \quad (72)$$

■

*Proof.* This is a direct consequence of properties (7) and (4). More explicitly,

$$\begin{aligned} \forall (l_A, l_B) \in \mathfrak{E}_{\mathfrak{S}_A} \times \mathfrak{E}_{\mathfrak{S}_B}, \quad \iota^{\check{S}_{AB}}(\prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}, \sigma_B)(l_A, l_B) &= \varepsilon_{l_A}^{\mathfrak{S}_A}(\prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_B) \\ &= (\bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A})) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_B) \\ &= \bigwedge_{i \in I} (\varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_B)) \\ &= \bigwedge_{i \in I} \iota^{\check{S}_{AB}}(\sigma_{i,A}, \sigma_B)(l_A, l_B) \\ &= (\prod_{i \in I}^{\check{S}_{AB}} \iota^{\check{S}_{AB}}(\sigma_{i,A}, \sigma_B))(l_A, l_B). \quad (73) \end{aligned}$$

In other words,  $\iota^{\check{S}_{AB}}(\prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}, \sigma_B) = \prod_{i \in I}^{\check{S}_{AB}} \iota^{\check{S}_{AB}}(\sigma_{i,A}, \sigma_B)$ .  $\square$

**Definition 9.** We define  $\check{S}_{AB}$  to be the sub Inf semi-lattice of  $\check{S}_{AB}$  generated by the elements  $\iota^{\check{S}_{AB}}(\sigma_A, \sigma_B)$  for any  $(\sigma_A, \sigma_B) \in \mathfrak{S}_A \times \mathfrak{S}_B$ .

$\check{S}_{AB}$  will be equivalently denoted  $\mathfrak{S}_A \check{\otimes} \mathfrak{S}_B$  and called *the minimal tensor product* of  $\mathfrak{S}_A$  and  $\mathfrak{S}_B$

### 3.3 The minimal tensor product

In the following, the set  $\mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$  will be equipped with the Inf semi-lattice structure  $\cup$ .

**Definition 10.**  $\mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$  is equipped with a congruence relation denoted  $\approx$  and defined between any two elements  $\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  and  $\{(\sigma'_{j,A}, \sigma'_{j,B}) \mid j \in J\}$  of  $\mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$  by

$$\begin{aligned} (\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} \approx \{(\sigma'_{j,A}, \sigma'_{j,B}) \mid j \in J\}) &:\Leftrightarrow \\ (\forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) &= \bigwedge_{j \in J} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma'_{j,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma'_{j,B})). \quad (74) \end{aligned}$$

The congruence class associated with  $U \subseteq \mathfrak{S}_A \times \mathfrak{S}_B$  will be denoted  $U_{\approx}$ .

**Definition 11.** We introduce the following injective Inf semi-lattice homomorphism

$$\begin{aligned} \Omega : \quad & \mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B) / \approx \hookrightarrow \check{S}_{AB} \\ & \{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}_{\approx} \mapsto \Omega(\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}_{\approx}) \mid \forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \\ & \quad \Omega(\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}_{\approx})(l_A, l_B) := \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}). \end{aligned} \quad (75)$$

We note that  $\check{S}_{AB}$  is the image of  $\Omega$  in  $\check{S}_{AB}$ .

If we adopt the following notation

$$\begin{aligned} \forall \tilde{\sigma} \in \check{S}_{AB}, \quad \langle \tilde{\sigma} \rangle &:= \text{Max}\{U \in \mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B) \mid \Omega(U_{\approx}) \sqsubseteq_{\check{S}_{AB}} \tilde{\sigma}\} \\ &= \{(\sigma_A, \sigma_B) \mid \Omega(\{(\sigma_A, \sigma_B)\}_{\approx}) \sqsubseteq_{\check{S}_{AB}} \tilde{\sigma}\}, \end{aligned} \quad (76)$$

we note the following Galois relation

$$\forall \tilde{\sigma} \in \check{S}_{AB}, \forall U \in \mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B), \quad \langle \tilde{\sigma} \rangle \supseteq U \Leftrightarrow \tilde{\sigma} \sqsubseteq_{\check{S}_{AB}} \Omega(U_{\approx}). \quad (77)$$

Indeed, let us fix  $U := \{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  and  $\tilde{\sigma} \in \check{S}_{AB}$ . We have

$$\begin{aligned} \langle \tilde{\sigma} \rangle \supseteq U &\Leftrightarrow \forall i \in I, \Omega(\{(\sigma_{i,A}, \sigma_{i,B})\}_{\approx}) \sqsubseteq_{\check{S}_{AB}} \tilde{\sigma} \\ &\Leftrightarrow \forall i \in I, \forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \tilde{\sigma}(l_A, l_B) \leq \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) \\ &\Leftrightarrow \forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \tilde{\sigma}(l_A, l_B) \leq \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) \\ &\Leftrightarrow \tilde{\sigma} \sqsubseteq_{\check{S}_{AB}} \Omega(U_{\approx}). \end{aligned} \quad (78)$$

As a consequence of this Galois relation, we obtain that

**Theorem 13.**  $\check{S}_{AB}$  is a down-complete Inf semi-lattice with

$$\forall \{U_i \mid i \in I\} \subseteq \mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B), \quad \prod_{i \in I}^{\check{S}_{AB}} \Omega((U_i)_{\approx}) = \Omega(\left(\bigcup_{i \in I} U_i\right)_{\approx}). \quad (79)$$

We will adopt the notation  $\prod_{i \in I}^{\check{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} := \Omega(\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}_{\approx})$  for any  $\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  in  $\mathcal{P}(\mathfrak{S}_A \times \mathfrak{S}_B)$ . ■

**Theorem 14.** We have the following relations

$$\forall \{\sigma_{i,A} \mid i \in I\} \subseteq \mathfrak{S}_A, \forall \sigma_B \in \mathfrak{S}_B, \quad \left(\prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}\right) \tilde{\otimes} \sigma_B = \prod_{i \in I}^{\check{S}_{AB}} (\sigma_{i,A} \tilde{\otimes} \sigma_B), \quad (80)$$

$$\forall \{\sigma_{i,B} \mid i \in I\} \subseteq \mathfrak{S}_B, \forall \sigma_A \in \mathfrak{S}_A, \quad \sigma_A \tilde{\otimes} \left(\prod_{i \in I}^{\mathfrak{S}_B} \sigma_{i,B}\right) = \prod_{i \in I}^{\check{S}_{AB}} (\sigma_A \tilde{\otimes} \sigma_{i,B}). \quad (81)$$

■

*Proof.* Rewriting of Theorem 12. □

**Theorem 15.** The following maps are homomorphisms

$$\eta : \begin{array}{ccc} \tilde{S}_{AB} & \longrightarrow & \mathfrak{E}_A \\ \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} & \mapsto & \prod_{i \in I}^{\mathfrak{E}_A} \sigma_{i,A} \end{array} \quad \lambda : \begin{array}{ccc} \tilde{S}_{AB} & \longrightarrow & \mathfrak{E}_B \\ \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} & \mapsto & \prod_{i \in I}^{\mathfrak{E}_B} \sigma_{i,B} \end{array} \quad (82)$$

■

We can obviously clarify the poset structure on  $\tilde{S}_{AB}$ .

**Lemma 4.**

$$\begin{aligned} & \left( \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \sqsubseteq_{\tilde{S}_{AB}} \prod_{j \in J}^{\tilde{S}_{AB}} \sigma'_{j,A} \tilde{\otimes} \sigma'_{j,B} \right) \Leftrightarrow \\ & \left( \forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{E}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{E}_B}(\sigma_{i,B}) \leq \bigwedge_{j \in J} \varepsilon_{l_A}^{\mathfrak{E}_A}(\sigma'_{j,A}) \bullet \varepsilon_{l_B}^{\mathfrak{E}_B}(\sigma'_{j,B}) \right). \end{aligned} \quad (83)$$

■

This poset structure can be "explicitated" according to following lemma addressing the word problem in  $\tilde{S}_{AB}$ .

**Lemma 5.**

$$\begin{aligned} & \left( \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \sqsubseteq_{\tilde{S}_{AB}} \sigma_A \tilde{\otimes} \sigma_B \right) \\ & \Leftrightarrow \left( \left( \prod_{k \in I}^{\mathfrak{E}_A} \sigma_{k,A} \right) \sqsubseteq_{\mathfrak{E}_A} \sigma_A \text{ and } \left( \prod_{m \in I}^{\mathfrak{E}_B} \sigma_{m,B} \right) \sqsubseteq_{\mathfrak{E}_B} \sigma_B \text{ and} \right. \\ & \quad \left. \left( \forall \emptyset \subsetneq K \subsetneq I, \left( \prod_{k \in K}^{\mathfrak{E}_A} \sigma_{k,A} \right) \sqsubseteq_{\mathfrak{E}_A} \sigma_A \text{ or } \left( \prod_{m \in I-K}^{\mathfrak{E}_B} \sigma_{m,B} \right) \sqsubseteq_{\mathfrak{E}_B} \sigma_B \right) \right) \quad (84) \\ & \Leftrightarrow \left( \exists \mathcal{K}, \mathcal{K}' \subseteq 2^I \text{ with } \mathcal{K} \cup \mathcal{K}' = 2^I, \mathcal{K} \cap \mathcal{K}' = \emptyset, \{\emptyset\} \in \mathcal{K}', I \in \mathcal{K} \mid \right. \\ & \quad \left. \left( \bigsqcup_{K \in \mathcal{K}} \prod_{k \in K}^{\mathfrak{E}_A} \sigma_{k,A} \right) \sqsubseteq_{\mathfrak{E}_A} \sigma_A \text{ and } \left( \bigsqcup_{K' \in \mathcal{K}'} \prod_{m \in I-K'}^{\mathfrak{E}_B} \sigma_{m,B} \right) \sqsubseteq_{\mathfrak{E}_B} \sigma_B \right). \end{aligned} \quad (85)$$

■

*Proof.* The inequality  $\prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \sqsubseteq_{\tilde{S}_{AB}} \sigma_A \tilde{\otimes} \sigma_B$  is equivalent to

$$\forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \quad \left( \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{E}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{E}_B}(\sigma_{i,B}) \right) \leq \varepsilon_{l_A}^{\mathfrak{E}_A}(\sigma_A) \bullet \varepsilon_{l_B}^{\mathfrak{E}_B}(\sigma_B). \quad (86)$$

We intent to choose a pertinent set of effects  $l_A \in \mathfrak{E}_A$  and  $l_B \in \mathfrak{E}_B$  to reformulate this inequality. Let us firstly choose  $l_B = \mathfrak{J}_{\mathfrak{E}_B}$ . Using (3), we obtain

$$\varepsilon_{l_A}^{\mathfrak{E}_A} \left( \prod_{i \in I}^{\mathfrak{E}_A} \sigma_{i,A} \right) \leq \varepsilon_{l_A}^{\mathfrak{E}_A}(\sigma_A), \forall l_A \in \mathfrak{E}_A, \quad (87)$$

which leads immediately (using (10))

$$\prod_{i \in I}^{\mathfrak{E}_A} \sigma_{i,A} \sqsubseteq_{\mathfrak{E}_A} \sigma_A. \quad (88)$$

Choosing  $l_A = \mathfrak{J}_{\mathfrak{E}_A}$ , we obtain along the same line

$$\prod_{i \in I}^{\mathfrak{E}_B} \sigma_{i,B} \sqsubseteq_{\mathfrak{E}_B} \sigma_B. \quad (89)$$

Let us now consider  $\emptyset \subsetneq K \subsetneq I$  and let us choose  $l_A$  and  $l_B$  according to

$$\varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma) := \mathbf{N}, \forall \sigma \sqsupseteq_{\mathfrak{S}_A} \prod_{k \in K}^{\mathfrak{S}_A} \sigma_{k,A} \quad \text{and} \quad \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma) := \perp, \text{ elsewhere,} \quad (90)$$

$$\varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma) := \mathbf{N}, \forall \sigma \sqsupseteq_{\mathfrak{S}_B} \prod_{m \in I-K}^{\mathfrak{S}_B} \sigma_{m,B} \quad \text{and} \quad \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma) := \perp, \text{ elsewhere.} \quad (91)$$

We deduce, from the assumption (86), that, for this  $\emptyset \subsetneq K \subsetneq I$ , we have

$$\left( \prod_{k \in K}^{\mathfrak{S}_A} \sigma_{k,A} \sqsubseteq_{\mathfrak{S}_A} \sigma_A \right) \text{ or } \left( \prod_{m \in I-K}^{\mathfrak{S}_B} \sigma_{m,B} \sqsubseteq_{\mathfrak{S}_B} \sigma_B \right). \quad (92)$$

We let the reader check that we have obtained the whole set of independent inequalities reformulating the property (86).  $\square$

**Theorem 16.** If  $\mathfrak{S}_A$  and  $\mathfrak{S}_B$  admit  $\perp_{\mathfrak{S}_A}$  and  $\perp_{\mathfrak{S}_B}$  respectively as bottom elements, then  $\tilde{S}_{AB}$  admits a bottom element explicitly given by  $\perp_{\tilde{S}_{AB}} = \perp_{\mathfrak{S}_A} \tilde{\otimes} \perp_{\mathfrak{S}_B}$ .  $\blacksquare$

*Proof.* Trivial using the expansion (84).  $\square$

### 3.4 Canonical vs. minimal tensor product

**Definition 12.** We denote  $\tilde{S}_{AB}^{fin}$  the sub-poset of  $\tilde{S}_{AB}$  defined as follows :

$$\tilde{S}_{AB}^{fin} := \{ \Omega(U_{\approx}) \mid U \subseteq_{fin} \mathfrak{S}_A \times \mathfrak{S}_B \}. \quad (93)$$

It is also a sub- Inf semi-lattice of  $\tilde{S}_{AB}$ .

**Theorem 17.** We have the following obvious property relating the partial orders of  $\tilde{S}_{AB}^{fin}$  and  $S_{AB}$ . For any  $\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} \subseteq_{fin} \mathfrak{S}_A \times \mathfrak{S}_B$ ,

$$\left( \prod_{i \in I}^{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B} \right) \sqsubseteq_{S_{AB}} \sigma'_A \otimes \sigma'_B \quad \Rightarrow \quad \left( \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \right) \sqsubseteq_{\tilde{S}_{AB}} \sigma'_A \tilde{\otimes} \sigma'_B. \quad (94)$$

We denote

$$\tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} := \{(\sigma'_A, \sigma'_B) \mid \left( \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \right) \sqsubseteq_{\tilde{S}_{AB}} \sigma'_A \tilde{\otimes} \sigma'_B\} = \langle \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \rangle. \quad (95)$$

*Proof.* First of all, it is clear that  $\tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  is a bi-filter.

Secondly, it is easy to check that  $(\sigma_{k,A}, \sigma_{k,B}) \in \tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  for any  $k \in I$ . Indeed, for any  $K \subseteq I$ , if  $k \in K$  we have  $\left( \prod_{l \in K}^{\mathfrak{S}_A} \sigma_{l,A} \right) \sqsubseteq_{\mathfrak{S}_A} \sigma_{k,A}$  and if  $k \notin K$  we have  $\left( \prod_{m \in I-K}^{\mathfrak{S}_B} \sigma_{m,B} \right) \sqsubseteq_{\mathfrak{S}_B} \sigma_{k,B}$ .

As a conclusion, and by definition of  $\tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  as the intersection of all bi-filters containing  $(\sigma_{i,A}, \sigma_{i,B})$  for any  $i \in I$ , we have then  $\tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} \supseteq \tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$ .

We now use Lemma 2 to obtain the announced result.  $\square$

**Definition 13.** [5, definition p.117 and Section 11 Lemma 1 p.118] A space of states  $\mathfrak{S}$  is said to be *distributive* iff

$$\forall \sigma, \sigma_1, \sigma_2 \in \mathfrak{S} \mid \sigma \neq \sigma_1, \sigma_2, \quad (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqsubseteq_{\mathfrak{S}} \sigma \Rightarrow \\ \exists \sigma'_1, \sigma'_2 \in \mathfrak{S} \mid (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma'_1, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'_2 \text{ and } \sigma = \sigma'_1 \sqcap_{\mathfrak{S}} \sigma'_2). \quad (96)$$

When  $\mathfrak{S}$  is distributive, we have the following standard properties satisfied, as soon as the implied suprema are well defined

$$\sigma_1 \sqcap_{\mathfrak{S}} (\sigma_2 \sqcup_{\mathfrak{S}} \sigma_3) = (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqcup_{\mathfrak{S}} (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_3) \quad (97)$$

$$\sigma_1 \sqcup_{\mathfrak{S}} (\sigma_2 \sqcap_{\mathfrak{S}} \sigma_3) = (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2) \sqcap_{\mathfrak{S}} (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_3). \quad (98)$$

**Theorem 18.** If  $\mathfrak{S}_A$  or  $\mathfrak{S}_B$  are distributive, then  $\tilde{S}_{AB}^{fin}$  and  $S_{AB}$  are isomorphic posets.  $\blacksquare$

*Proof.* We now suppose that  $\mathfrak{S}_A$  or  $\mathfrak{S}_B$  is distributive and we intent to prove that  $\tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} = \tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  for any  $\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} \subseteq_{fin} \mathfrak{S}_A \times \mathfrak{S}_B$ .

Let us prove the following fact : every bi-filter  $F$  which contains  $(\sigma_{k,A}, \sigma_{k,B})$  for any  $k \in I$  contains also  $\tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$ . In fact, we can show that, for any bi-filter  $F$  we have

$$(\forall k \in I, (\sigma_{k,A}, \sigma_{k,B}) \in F) \Rightarrow \left( \bigsqcup_{K \in \mathcal{K}}^{\mathfrak{S}_A} \prod_{k \in K}^{\mathfrak{S}_A} \sigma_{k,A}, \bigsqcup_{K' \in \mathcal{K}'}^{\mathfrak{S}_A} \prod_{m \in I-K'}^{\mathfrak{S}_B} \sigma_{m,B} \right) \in F, \\ \forall \mathcal{K}, \mathcal{K}' \subseteq 2^I, \mathcal{K} \cup \mathcal{K}' = 2^I, \mathcal{K} \cap \mathcal{K}' = \emptyset, \{\emptyset\} \in \mathcal{K}', I \in \mathcal{K}. \quad (99)$$

The first step towards (99) is obtained by checking that  $\forall \mathcal{K}, \mathcal{K}' \subseteq 2^I, \mathcal{K} \cup \mathcal{K}' = 2^I, \mathcal{K} \cap \mathcal{K}' = \emptyset, \{\emptyset\} \in \mathcal{K}', I \in \mathcal{K}$ ,

$$\left( \bigsqcup_{K' \in \mathcal{K}'}^{\mathfrak{S}} \prod_{m \in I-K'}^{\mathfrak{S}} \sigma_m \right) \supseteq_{\mathfrak{S}} \left( \prod_{K \in \mathcal{K}}^{\mathfrak{S}} \bigsqcup_{k \in K}^{\mathfrak{S}} \sigma_k \right) \quad (100)$$

for any distributive  $\mathfrak{S}$  and any collection of elements of  $\mathfrak{S}$  denoted  $\sigma_k$  for  $k \in I$  for which these two sides of inequality exist. To check this fact, we have to note that, using [1, Lemma 8 p. 50], we have first of all

$$\left( \prod_{K \in \mathcal{K}}^{\mathfrak{S}} \bigsqcup_{k \in K}^{\mathfrak{S}} \sigma_k \right) = \bigsqcup^{\mathfrak{S}} \left\{ \prod_{K \in \mathcal{K}}^{\mathfrak{S}} \pi_K(A) \mid A \in \prod_{K \in \mathcal{K}} K \right\}, \quad (101)$$

where  $\pi_K$  denotes the projection of the component indexed by  $K$  in the cardinal product  $\prod_{K \in \mathcal{K}} K$ . Moreover, for any  $A \in \prod_{K \in \mathcal{K}} K$ , there exists  $L \in \mathcal{K}'$  such that  $\bigcup \{ \pi_K(A) \mid K \in \mathcal{K} \} \supseteq (I \setminus L)$  and then  $\left( \prod_{K \in \mathcal{K}}^{\mathfrak{S}} \pi_K(A) \right) \sqsubseteq_{\mathfrak{S}} \left( \prod_{m \in I-L}^{\mathfrak{S}} \sigma_m \right) \sqsubseteq_{\mathfrak{S}} \left( \bigsqcup_{K' \in \mathcal{K}'}^{\mathfrak{S}} \prod_{m \in I-K'}^{\mathfrak{S}} \sigma_m \right)$ . As a result, we obtain the property (100).

The second step towards (99) consists in showing that

$$(\forall k \in I, (\sigma_{k,A}, \sigma_{k,B}) \in F) \Rightarrow \left( \bigsqcup_{K \in \mathcal{K}}^{\mathfrak{S}_A} \prod_{k \in K}^{\mathfrak{S}_A} \sigma_{k,A}, \prod_{K \in \mathcal{K}}^{\mathfrak{S}_B} \bigsqcup_{k \in K}^{\mathfrak{S}_B} \sigma_{k,B} \right) \in F \quad (102)$$

for any  $\mathcal{K} \subseteq 2^I$ . This intermediary result is obtained by induction on the complexity of the polynomial  $\left( \bigsqcup_{K \in \mathcal{K}}^{\mathfrak{S}_A} \prod_{k \in K}^{\mathfrak{S}_A} \sigma_{k,A} \right)$  by using the following elementary result

$$\forall \sigma_A, \sigma'_A \in \mathfrak{S}_A, \sigma_B, \sigma'_B \in \mathfrak{S}_B, \quad ((\sigma_A, \sigma_B), (\sigma'_A, \sigma'_B)) \in F \Rightarrow \begin{cases} (\sigma_A \sqcup_{\mathfrak{S}_A} \sigma'_A, \sigma_B \sqcap_{\mathfrak{S}_B} \sigma'_B) \in F \\ (\sigma_A \sqcap_{\mathfrak{S}_A} \sigma'_A, \sigma_B \sqcup_{\mathfrak{S}_B} \sigma'_B) \in F \end{cases}$$

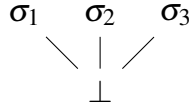


trivially deduced using the bi-filter character of  $F$ , i.e. properties (53)(54)(55).

As a final conclusion, using the explicit definition of  $\mathfrak{F}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$  as the intersection of all bi-ideals containing  $(\sigma_{k,A}, \sigma_{k,B})$  for any  $k \in I$ , we obtain  $\mathfrak{F}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\} = \tilde{\mathfrak{F}}\{(\sigma_{i,A}, \sigma_{i,B}) \mid i \in I\}$ .

$\tilde{\mathcal{S}}_{AB}^{fin}$  and  $\mathcal{S}_{AB}$  are then isomorphic posets.  $\square$

*Remark 3.* We note that the distributivity property is a key condition to obtain previous isomorphism between  $\tilde{\mathcal{S}}_{AB}^{fin}$  and  $\mathcal{S}_{AB}$ . Indeed, let us consider that  $\mathfrak{S}_A$  and  $\mathfrak{S}_B$  are both defined as the lattice associated to the following Hasse diagram:



According to (84), we have  $(\perp_{\mathfrak{S}_A}, \perp_{\mathfrak{S}_B}) \in \tilde{\mathfrak{F}}\{(\sigma_1, \sigma_1), (\sigma_2, \sigma_2), (\sigma_3, \sigma_3)\}$ . However, we have obviously  $(\perp_{\mathfrak{S}_A}, \perp_{\mathfrak{S}_B}) \notin \mathfrak{F}\{(\sigma_1, \sigma_1), (\sigma_2, \sigma_2), (\sigma_3, \sigma_3)\}$ .

### 3.5 Properties of the minimal tensor product

Let us now consider that  $\mathfrak{S}_A$  and  $\mathfrak{S}_B$  have a description in terms of pure states. We intent to prove that  $\tilde{\mathcal{S}}_{AB}$  inherits a description in terms of pure states.

**Theorem 19.**

$$\tilde{\mathcal{S}}_{AB}^{pure} = \{ \sigma_A \tilde{\otimes} \sigma_B \mid \sigma_A \in \mathfrak{S}_A^{pure}, \sigma_B \in \mathfrak{S}_B^{pure} \} = \text{Max}(\tilde{\mathcal{S}}_{AB}) \quad (103)$$

■

*Proof.* First of all, it is a trivial fact that the completely meet-irreducible elements of  $\tilde{\mathcal{S}}_{AB}$  are necessarily pure tensors of  $\tilde{\mathcal{S}}_{AB}$ , i.e. elements of the form  $\sigma_A \tilde{\otimes} \sigma_B$ .

Let us then consider  $\sigma_A \tilde{\otimes} \sigma_B$  a completely meet-irreducible element of  $\tilde{\mathcal{S}}_{AB}$  and let us assume that  $\sigma_A = \prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}$  for  $\sigma_{i,A} \in \mathfrak{S}_A$  for any  $i \in I$ . We have then  $(\sigma_A \tilde{\otimes} \sigma_B) = ((\prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A}) \tilde{\otimes} \sigma_B) = \prod_{i \in I}^{\tilde{\mathcal{S}}_{AB}} (\sigma_{i,A} \tilde{\otimes} \sigma_B)$ . On another part,  $\sigma_A \tilde{\otimes} \sigma_B$  being completely meet-irreducible in  $\tilde{\mathcal{S}}_{AB}$ , there exists  $k \in I$  such that  $\sigma_A \tilde{\otimes} \sigma_B = \sigma_{k,A} \tilde{\otimes} \sigma_B$ , i.e.  $\sigma_A = \sigma_{k,A}$ . As a conclusion,  $\sigma_A$  is completely meet-irreducible. In the same way,  $\sigma_B$  is completely meet-irreducible. As a first result, pure states of  $\tilde{\mathcal{S}}_{AB}$  are necessarily of the form  $\sigma_A \tilde{\otimes} \sigma_B$  with  $\sigma_A \in \mathfrak{S}_A^{pure}, \sigma_B \in \mathfrak{S}_B^{pure}$ .

Conversely, let us consider  $\sigma_A$  a pure state of  $\mathfrak{S}_A$  and  $\sigma_B$  a pure state of  $\mathfrak{S}_B$ , and let us suppose that  $(\prod_{i \in I}^{\tilde{\mathcal{S}}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B}) = (\sigma_A \tilde{\otimes} \sigma_B)$  with  $\sigma_{i,A} \in \mathfrak{S}_A$  and  $\sigma_{i,B} \in \mathfrak{S}_B$  for any  $i \in I$ . We now exploit the two conditions  $(\prod_{k \in I}^{\mathfrak{S}_A} \sigma_{k,A}) = \sigma_A$  and  $(\prod_{m \in I}^{\mathfrak{S}_B} \sigma_{m,B}) = \sigma_B$  derived from the expansion (84). From  $\sigma_A \in \text{Max}(\mathfrak{S}_A)$  and  $\sigma_B \in \text{Max}(\mathfrak{S}_B)$ , we deduce that  $\sigma_{i,A} = \sigma_A$  and  $\sigma_{j,B} = \sigma_B$  for any  $i, j \in I$ . As a second result, we have then obtained that the state  $(\sigma_A \tilde{\otimes} \sigma_B)$ , with  $\sigma_A$  a pure state of  $\mathfrak{S}_A$  and  $\sigma_B$  a pure state of  $\mathfrak{S}_B$ , is completely meet-irreducible.

From the expansion (84), we deduce also immediately that  $(\sigma_A \tilde{\otimes} \sigma_B) \in \text{Max}(\tilde{\mathcal{S}}_{AB})$  as long as  $\sigma_A \in \text{Max}(\mathfrak{S}_A)$  and  $\sigma_B \in \text{Max}(\mathfrak{S}_B)$ .  $\square$

**Theorem 20.**

$$\forall \sigma \in \tilde{S}_{AB}, \sigma = \prod_{i \in I}^{\tilde{S}_{AB}} \underline{\sigma}_{\tilde{S}_{AB}}, \text{ where } \underline{\sigma}_{\tilde{S}_{AB}} = (\tilde{S}_{AB}^{pure} \cap (\uparrow^{\tilde{S}_{AB}} \sigma)). \quad (104)$$

■

*Proof.* Let us fix  $\sigma \in \tilde{S}_{AB}$ .

We note that  $\sigma \sqsubseteq_{\tilde{S}_{AB}} \sigma'$  for any  $\sigma' \in (\tilde{S}_{AB}^{pure} \cap (\uparrow^{\tilde{S}_{AB}} \sigma))$  and then  $\sigma \sqsubseteq_{\tilde{S}_{AB}} \prod_{i \in I}^{\tilde{S}_{AB}} \underline{\sigma}_{\tilde{S}_{AB}}$ .

Secondly, denoting  $\sigma := (\prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B})$ , we note immediately that, for any  $\sigma_A \in \mathfrak{G}_A^{pure}$  and  $\sigma_B \in \mathfrak{G}_B^{pure}$ , if  $\sigma_A \sqsupseteq_{\mathfrak{G}_A} \sigma_{i,A}$  and  $\sigma_B \sqsupseteq_{\mathfrak{G}_B} \sigma_{i,B}$ , then  $(\sigma_A \tilde{\otimes} \sigma_B) \sqsupseteq_{\mathfrak{G}_{AB}} \sigma$ , i.e.  $(\sigma_A \tilde{\otimes} \sigma_B) \in \underline{\sigma}_{\tilde{S}_{AB}}$ . As a consequence, we have

$$\left( \prod_{i \in I}^{\tilde{S}_{AB}} \prod_{\sigma_A \in \mathfrak{G}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{G}_A} \sigma_{i,A}}^{\tilde{S}_{AB}} \prod_{\sigma_B \in \mathfrak{G}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{G}_B} \sigma_{i,B}}^{\tilde{S}_{AB}} \sigma_A \tilde{\otimes} \sigma_B \right) \sqsupseteq_{\tilde{S}_{AB}} \prod_{i \in I}^{\tilde{S}_{AB}} \underline{\sigma}_{\tilde{S}_{AB}}. \quad (105)$$

Endly, using Theorem 14, we have

$$\begin{aligned} \sigma &= \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} = \prod_{i \in I}^{\tilde{S}_{AB}} \left( \prod_{\sigma_A \in \mathfrak{G}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{G}_A} \sigma_{i,A}}^{\tilde{S}_{AB}} \sigma_A \right) \tilde{\otimes} \left( \prod_{\sigma_B \in \mathfrak{G}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{G}_B} \sigma_{i,B}}^{\tilde{S}_{AB}} \sigma_B \right) \\ &= \prod_{i \in I}^{\tilde{S}_{AB}} \prod_{\sigma_A \in \mathfrak{G}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{G}_A} \sigma_{i,A}}^{\tilde{S}_{AB}} \prod_{\sigma_B \in \mathfrak{G}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{G}_B} \sigma_{i,B}}^{\tilde{S}_{AB}} \sigma_A \tilde{\otimes} \sigma_B. \end{aligned} \quad (106)$$

As a final conclusion, we obtain

$$\sigma = \left( \prod_{i \in I}^{\tilde{S}_{AB}} \prod_{\sigma_A \in \mathfrak{G}_A^{pure} \mid \sigma_A \sqsupseteq_{\mathfrak{G}_A} \sigma_{i,A}}^{\tilde{S}_{AB}} \prod_{\sigma_B \in \mathfrak{G}_B^{pure} \mid \sigma_B \sqsupseteq_{\mathfrak{G}_B} \sigma_{i,B}}^{\tilde{S}_{AB}} \sigma_A \tilde{\otimes} \sigma_B \right) = \prod_{i \in I}^{\tilde{S}_{AB}} \underline{\sigma}_{\tilde{S}_{AB}}. \quad (107)$$

□

**Theorem 21.** Let  $\tilde{\sigma}_{AB}$  and  $\tilde{\sigma}'_{AB}$  be two elements of  $\tilde{S}_{AB}$  having a common upper-bound. Then the supremum of  $\{\tilde{\sigma}_{AB}, \tilde{\sigma}'_{AB}\}$  exists in  $\tilde{S}_{AB}$  and its expression is given by

$$\tilde{\sigma}_{AB} \sqcup_{\tilde{S}_{AB}} \tilde{\sigma}'_{AB} = \prod_{\tilde{\sigma} \in (\underline{\tilde{\sigma}_{AB}} \cap \underline{\tilde{\sigma}'_{AB}})}^{\tilde{S}_{AB}} \tilde{\sigma} \quad (108)$$

■

*Proof.* As long as  $\tilde{\sigma}_{AB}$  and  $\tilde{\sigma}'_{AB}$  have a common upper-bound,  $\underline{\tilde{\sigma}_{AB}} \cap \underline{\tilde{\sigma}'_{AB}}$  is not empty.

Secondly, it is clear that  $\tilde{\sigma}_{AB} = (\prod_{\tilde{\sigma} \in \underline{\tilde{\sigma}_{AB}}}^{\tilde{S}_{AB}} \tilde{\sigma}) \sqsubseteq_{\tilde{S}_{AB}} \prod_{\tilde{\sigma} \in \underline{\tilde{\sigma}_{AB}} \cap \underline{\tilde{\sigma}'_{AB}}}^{\tilde{S}_{AB}} \tilde{\sigma}$  and  $\tilde{\sigma}'_{AB} = (\prod_{\tilde{\sigma} \in \underline{\tilde{\sigma}'_{AB}}}^{\tilde{S}_{AB}} \tilde{\sigma}) \sqsubseteq_{\tilde{S}_{AB}} \prod_{\tilde{\sigma} \in \underline{\tilde{\sigma}_{AB}} \cap \underline{\tilde{\sigma}'_{AB}}}^{\tilde{S}_{AB}} \tilde{\sigma}$ . Then, if we suppose there exists  $\tilde{\sigma}''_{AB}$  such that  $\tilde{\sigma}_{AB}, \tilde{\sigma}'_{AB} \sqsubseteq_{\tilde{S}_{AB}} \tilde{\sigma}''_{AB}$  we can use Theorem 20 to obtain the decomposition  $\tilde{\sigma}''_{AB} = (\prod_{\tilde{\sigma} \in \underline{\tilde{\sigma}''_{AB}}}^{\tilde{S}_{AB}} \tilde{\sigma})$  with necessarily  $\forall \tilde{\sigma} \in \underline{\tilde{\sigma}''_{AB}}, \tilde{\sigma}_{AB} \sqsubseteq_{\tilde{S}_{AB}} \tilde{\sigma}$  and  $\tilde{\sigma}'_{AB} \sqsubseteq_{\tilde{S}_{AB}} \tilde{\sigma}$ , i.e.  $\tilde{\sigma} \in \underline{\tilde{\sigma}_{AB}} \cap \underline{\tilde{\sigma}'_{AB}}$ , and then  $(\prod_{\tilde{\sigma} \in \underline{\tilde{\sigma}_{AB}} \cap \underline{\tilde{\sigma}'_{AB}}}^{\tilde{S}_{AB}} \tilde{\sigma}) \sqsubseteq_{\tilde{S}_{AB}} \tilde{\sigma}''_{AB}$ . □

**Theorem 22.** If  $\mathfrak{G}_A$  and  $\mathfrak{G}_B$  are distributive, then  $\tilde{S}_{AB}$  is also distributive.

Note, using Theorem 18, that, in this situation, we have also  $\tilde{S}_{AB}^{fin} = S_{AB}$ .

In that case, the explicit expression for the supremum of two elements in  $\tilde{S}_{AB}^{fin}$  is given by

$$\left(\prod_{i \in I} \tilde{\sigma}_{i,A} \otimes \tilde{\sigma}_{i,B}\right) \sqcup_{\tilde{S}_{AB}} \left(\prod_{j \in J} \tilde{\sigma}'_{j,A} \otimes \tilde{\sigma}'_{j,B}\right) = \prod_{i \in I, j \in J} \tilde{\sigma}_{i,A} \sqcup_{\mathfrak{S}_A} \tilde{\sigma}'_{j,A} \otimes (\tilde{\sigma}_{i,B} \sqcup_{\mathfrak{S}_B} \tilde{\sigma}'_{j,B}). \quad (109)$$

■

*Proof.* First of all, using Theorem 18, we note that, as soon as  $\mathfrak{S}_A$  or  $\mathfrak{S}_B$  is distributive, we have  $\tilde{S}_{AB} = S_{AB}$  as Inf semi-lattices. We are then reduced to prove the distributivity of  $S_{AB}$ .

In reference to the definition of distributivity of an Inf semi-lattice given in Definition 13, we have then to prove that if  $\prod_{1 \leq i \leq n} \sigma_{i,A} \otimes \sigma_{i,B} \sqsubseteq_{S_{AB}} \sigma_A \otimes \sigma_B$ , then there exists  $\sigma'_{i,A} \otimes \sigma'_{i,B} \sqsupseteq_{S_{AB}} \sigma_{i,A} \otimes \sigma_{i,B}$  for any  $1 \leq i \leq n$  such that  $\prod_{1 \leq i \leq n} \sigma'_{i,A} \otimes \sigma'_{i,B} = \sigma_A \otimes \sigma_B$ . From Lemma 3, we conclude that it is sufficient to prove that, for any  $n$ -ary polynomial  $p$ , if  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A})$  and  $\sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma_{1,B}, \dots, \sigma_{n,B})$ , then there exist  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  and  $\sigma'_{i,B} \sqsupseteq_{\mathfrak{S}_B} \sigma_{i,B}$  for  $1 \leq i \leq n$  such that  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \dots, \sigma'_{n,A})$  and  $\sigma_B \sqsupseteq_{\mathfrak{S}_B} p^*(\sigma'_{1,B}, \dots, \sigma'_{n,B})$ , and  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  and  $\sigma'_{i,B} \sqsupseteq_{\mathfrak{S}_B} \sigma_B$  for  $1 \leq i \leq n$ .

The proof of this fact is sketched in [3, Theorem 3], and we give here a developed version of it.

Let us prove the following statement for any  $n$ -ary polynomial  $p$  :

$$\begin{aligned} &\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A}) \Rightarrow \\ &\exists \sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}, \forall 1 \leq i \leq n \mid \left( \sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \dots, \sigma'_{n,A}) \text{ and } \sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A, \forall 1 \leq i \leq n \right). \end{aligned} \quad (110)$$

This statement is obviously true for  $p(\sigma_{1,A}, \dots, \sigma_{n,A}) := \sigma_{k,A}$ , it suffices to chose  $\sigma_{k,A} = \sigma_A$ .

Let us assume that the induction statement is true for two  $n$ -ary polynomials  $p$  and  $q$ , and let us prove the statement is also true for  $(p \sqcap q)$ .

We will assume  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A}) \sqcap_{\mathfrak{S}_A} q(\sigma_{1,A}, \dots, \sigma_{n,A})$ . Then, there exist  $\gamma_A, \delta_A \in \mathfrak{S}_A$  such that  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} (\gamma_A \sqcap_{\mathfrak{S}_A} \delta_A)$  and  $\gamma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A})$  and  $\delta_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma_{1,A}, \dots, \sigma_{n,A})$ .

From distributivity of  $\mathfrak{S}_A$ , we deduce that there exist  $\gamma'_A$  and  $\delta'_A$  such that  $\sigma_A = (\gamma'_A \sqcap_{\mathfrak{S}_A} \delta'_A)$  and  $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} \gamma_A$  and  $\delta'_A \sqsupseteq_{\mathfrak{S}_A} \delta_A$ . As a result, we have  $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A})$  and  $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma_{1,A}, \dots, \sigma_{n,A})$ .

By assumption, there exist  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  and  $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  for  $1 \leq i \leq n$  with  $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \dots, \sigma'_{n,A})$  and  $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \dots, \sigma''_{n,A})$ , and with  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \gamma'_A$  and  $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \delta'_A$  for  $1 \leq i \leq n$ .

Let us denote  $\bar{\sigma}_{i,A} := \sigma'_{i,A} \sqcap_{\mathfrak{S}_A} \sigma''_{i,A}$ .

We first note that  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  for  $1 \leq i \leq n$ .

From  $\bar{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma'_{i,A}$  and  $\bar{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma''_{i,A}$  for any  $1 \leq i \leq n$ , and  $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \dots, \sigma'_{n,A})$  and  $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \dots, \sigma''_{n,A})$ , we deduce  $\gamma'_A \sqsupseteq_{\mathfrak{S}_A} p(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$  and  $\delta'_A \sqsupseteq_{\mathfrak{S}_A} q(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$ .

As a consequence,  $\sigma_A = (\gamma'_A \sqcap_{\mathfrak{S}_A} \delta'_A) \sqsupseteq_{\mathfrak{S}_A} p(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A}) \sqcap_{\mathfrak{S}_A} q(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$ .

From  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \gamma'_A$  and  $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \delta'_A$  for  $1 \leq i \leq n$ , we deduce also  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \gamma'_A \sqcap_{\mathfrak{S}_A} \delta'_A = \sigma_A$  for  $1 \leq i \leq n$ .

As a summary, there exist  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  for  $1 \leq i \leq n$ , such that  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A}) \sqcap_{\mathfrak{S}_A} q(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$ , and  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  for  $1 \leq i \leq n$ . In other words, the  $n$ -ary polynomial  $(p \sqcap q)$  satisfies also the induction assumption.

Let us assume that the induction statement is true for two  $n$ -ary polynomials  $p$  and  $q$ , and let us now prove the statement is also true for  $(p \sqcup q)$ .

We will assume  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A}) \sqcup_{\mathfrak{S}_A} q(\sigma_{1,A}, \dots, \sigma_{n,A})$ . Then, we have  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma_{1,A}, \dots, \sigma_{n,A})$

and  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma_{1,A}, \dots, \sigma_{n,A})$ .

By assumption, there exist  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  and  $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  for  $1 \leq i \leq n$  with  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \dots, \sigma'_{n,A})$  and  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \dots, \sigma''_{n,A})$ , and with  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  and  $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  for  $1 \leq i \leq n$ .

Let us denote  $\bar{\sigma}_{i,A} := \sigma'_{i,A} \sqcap_{\mathfrak{S}_A} \sigma''_{i,A}$ .

We first note that  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  for  $1 \leq i \leq n$ .

From  $\bar{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma'_{i,A}$  and  $\bar{\sigma}_{i,A} \sqsubseteq_{\mathfrak{S}_A} \sigma''_{i,A}$  for any  $1 \leq i \leq n$ , and  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\sigma'_{1,A}, \dots, \sigma'_{n,A})$  and  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\sigma''_{1,A}, \dots, \sigma''_{n,A})$ , we deduce  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$  and  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} q(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$ .

As a consequence,  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A}) \sqcup_{\mathfrak{S}_A} q(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$ .

From  $\sigma'_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  and  $\sigma''_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  for  $1 \leq i \leq n$ , we deduce also  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  for  $1 \leq i \leq n$ .

As a summary, there exist  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_{i,A}$  for  $1 \leq i \leq n$ , such that  $\sigma_A \sqsupseteq_{\mathfrak{S}_A} p(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A}) \sqcup_{\mathfrak{S}_A} q(\bar{\sigma}_{1,A}, \dots, \bar{\sigma}_{n,A})$ , and  $\bar{\sigma}_{i,A} \sqsupseteq_{\mathfrak{S}_A} \sigma_A$  for  $1 \leq i \leq n$ . In other words, the  $n$ -ary polynomial  $(p \sqcup q)$  satisfies also the induction assumption.

By induction on the complexity of the  $n$ -ary polynomial  $p$  we have then proved the statement. As a final consequence,  $S_{AB}$  and then also  $\tilde{S}_{AB}$  is a distributive Inf semi-lattice.

As a consequence of this distributivity property, we obtain the following simplification

$$\left( \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \right) \sqcup_{\tilde{S}_{AB}} \left( \prod_{j \in J}^{\tilde{S}_{AB}} \sigma'_{j,A} \tilde{\otimes} \sigma'_{j,B} \right) = \prod_{i \in I}^{\tilde{S}_{AB}} \prod_{j \in J}^{\tilde{S}_{AB}} \left( (\sigma_{i,A} \tilde{\otimes} \sigma_{i,B}) \sqcup_{\tilde{S}_{AB}} (\sigma'_{j,A} \tilde{\otimes} \sigma'_{j,B}) \right). \quad (111)$$

Using the expansion (84), we know that

$$(\sigma_{i,A} \tilde{\otimes} \sigma_{i,B}) \sqcup_{\tilde{S}_{AB}} (\sigma'_{j,A} \tilde{\otimes} \sigma'_{j,B}) = (\sigma_{i,A} \sqcup_{\mathfrak{S}_A} \sigma'_{j,A}) \tilde{\otimes} (\sigma_{i,B} \sqcup_{\mathfrak{S}_B} \sigma'_{j,B}) \quad (112)$$

This concludes the proof of the formula (109).  $\square$

**Theorem 23.** Let us consider  $\sigma_{1,A}$  and  $\sigma_{2,A}$  two distinct elements of  $\mathfrak{S}_A$ , and  $\sigma_{1,B}$  and  $\sigma_{2,B}$  two distinct elements of  $\mathfrak{S}_B$ . We have then

$$(\Phi \sqcap_{\tilde{S}_{AB}} (\sigma_{1,A} \tilde{\otimes} \sigma_{1,B} \sqcap_{\tilde{S}_{AB}} \sigma_{2,A} \tilde{\otimes} \sigma_{2,B}) \text{ and } \Phi \in \text{Max}(\tilde{S}_{AB})) \Rightarrow \Phi \in \{ \sigma_{1,A} \tilde{\otimes} \sigma_{1,B}, \sigma_{2,A} \tilde{\otimes} \sigma_{2,B} \}. \quad (113)$$

■

*Proof.* Direct consequence of the expansion (84) with Theorem 19.  $\square$

### 3.6 Maximal vs. minimal tensor product

**Lemma 6.** For any  $\Phi \in \tilde{S}_{AB}$ , we have

$$\forall l_B \in \mathfrak{E}_{\mathfrak{S}_B}, \quad \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, l_B}) = \mathbf{N} \quad (114)$$

$$\forall l_A \in \mathfrak{E}_{\mathfrak{S}_A}, \quad \Phi(l_A, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) = \mathbf{N}. \quad (115)$$

■

*Proof.* Using (60) (61) and (62), we deduce for any  $\Phi \in \tilde{S}_{AB}$  the following equations

$$\Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) = \mathbf{N} \quad (116)$$

$$\Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) = \mathbf{N} \quad (117)$$

Using (116)(117) (62)(63) and (58)(59), we deduce for any  $\Phi \in \check{S}_{AB}$  the following equations

$$\Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N} \quad (118)$$

$$\Phi(\perp_{\mathfrak{E}_{\mathfrak{S}_B}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}) = \mathbf{N} \quad (119)$$

$$\Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp \quad (120)$$

$$\Phi(\perp_{\mathfrak{E}_{\mathfrak{S}_B}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}) = \perp. \quad (121)$$

We have also, using (118) (120) and (58), for any  $\Phi \in \check{S}_{AB}$  the following equation

$$\Phi(\perp_{\mathfrak{E}_{\mathfrak{S}_A}}, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp. \quad (122)$$

From (118) and (59), we deduce that, for any  $l_B \in \mathfrak{E}_{\mathfrak{S}_B}$  we have  $\Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, l_B) \wedge \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{l_B}) = \mathbf{N}$  and then  $\Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, l_B) = \mathbf{N}$ . Here we have used the obvious property  $l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} \overline{l_B} = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}$  satisfied by any  $l_B \in \mathfrak{E}_{\mathfrak{S}_B}$ . In the same way, using (119) and (58), we obtain the symmetric property. As a result of our investigations of the consequences of (118) and (119), we have obtained for any  $\Phi \in \check{S}_{AB}$  the equations (114) and (115).  $\square$

Let us now investigate the consequences of (114) and (115).

**Lemma 7.** Let us consider any  $\Phi$  in  $\check{S}_{AB}$  and any  $l_A \in \mathfrak{E}_{\mathfrak{S}_A}$ . We are necessarily in one of the following three cases

1.

$$\left\{ \begin{array}{l} \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N}, \quad \Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}, \\ \forall l_B \in \mathfrak{E}_{\mathfrak{S}_B}, \Phi(l_A, l_B) = \mathbf{N}, \\ \forall l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B} \mid l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}, (\Phi(\overline{l_A}, l_B), \Phi(\overline{l_A}, l'_B)) \notin \{(\mathbf{N}, \mathbf{N}), (\mathbf{Y}, \mathbf{Y})\}. \end{array} \right. \quad (123)$$

2.

$$\left\{ \begin{array}{l} \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}, \quad \Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N}, \\ \forall l_B \in \mathfrak{E}_{\mathfrak{S}_B}, \Phi(\overline{l_A}, l_B) = \mathbf{N}, \\ \forall l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B} \mid l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}, (\Phi(l_A, l_B), \Phi(l_A, l'_B)) \notin \{(\mathbf{N}, \mathbf{N}), (\mathbf{Y}, \mathbf{Y})\}. \end{array} \right. \quad (124)$$

3.

$$\left\{ \begin{array}{l} \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp, \quad \Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp, \\ \forall l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B} \mid l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}, (\Phi(l_A, l_B), \Phi(l_A, l'_B)) \notin \{(\mathbf{N}, \mathbf{N}), (\mathbf{Y}, \mathbf{Y})\}, \\ \forall l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B} \mid l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}, (\Phi(\overline{l_A}, l_B), \Phi(\overline{l_A}, l'_B)) \notin \{(\mathbf{N}, \mathbf{N}), (\mathbf{Y}, \mathbf{Y})\}. \end{array} \right. \quad (125)$$

■

*Proof.* The distinction between the three cases is directly inherited from (60).

Let us consider the first case :  $\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N}$  and  $\Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}$ . Using  $\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N}$  and (115) and (59), we obtain  $\Phi(l_A, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N}$  and then,  $\forall l_B \in \mathfrak{E}_{\mathfrak{S}_B}, \Phi(l_A, l_B) = \mathbf{N}$ . Secondly, using  $\Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}$  and (115) and (59), we obtain  $\Phi(\overline{l_A}, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$ , which means that for

any  $l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B}$  such that  $l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}$  we have  $\Phi(\overline{l_A}, l_B) \wedge \Phi(\overline{l_A}, l'_B) = \perp$ .

The second case ( $\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}$  and  $\Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{N}$ ) is treated exactly in the same way as the first case.

Let us conclude with the third case :  $\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$  and  $\Phi(\overline{l_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$ . Using (115) and (59), we obtain  $\Phi(l_A, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$  and  $\Phi(\overline{l_A}, \perp_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$ , which means that, for any  $l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B}$  such that  $l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}$  we have  $\Phi(\overline{l_A}, l_B) \wedge \Phi(\overline{l_A}, l'_B) = \perp$  and  $\Phi(l_A, l_B) \wedge \Phi(l_A, l'_B) = \perp$ . This concludes the proof.  $\square$

If we restrict ourselves to the elements of  $\widetilde{\mathfrak{S}}_{AB}$ , the conditions are in fact more severe.

**Lemma 8.** Let us now fix  $\Phi \in \widetilde{\mathfrak{S}}_{AB}$  and  $l_A \in \mathfrak{E}_{\mathfrak{S}_A}$ , and let us suppose that  $\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \mathbf{Y}$ . Then, for any  $l_B \in \mathfrak{E}_{\mathfrak{S}_B}$ , we have  $(\Phi(l_A, l_B), \Phi(l_A, \overline{l_B})) \in \{(\mathbf{Y}, \mathbf{N}), (\mathbf{N}, \mathbf{Y}), (\perp, \perp)\}$ .  $\blacksquare$

*Proof.* Let us consider that  $\Phi = \prod_{i \in I} \iota^{\widetilde{\mathfrak{S}}_{AB}}(\sigma_{i,A}, \sigma_{i,B}) = \prod_{i \in I} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}$ . We have then  $\mathbf{Y} = \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}^{\mathfrak{S}_B}(\sigma_{i,B}) = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \mathbf{Y} = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) = \varepsilon_{l_A}^{\mathfrak{S}_A}(\prod_{i \in I} \sigma_{i,A})$ . As a consequence, we obtain  $\varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) = \mathbf{Y}$  for any  $i \in I$ . As a result, we obtain  $\Phi(l_A, l_B) = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) = \bigwedge_{i \in I} \mathbf{Y} \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) = \bigwedge_{i \in I} \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) = \varepsilon_{l_B}^{\mathfrak{S}_B}(\prod_{i \in I} \sigma_{i,B})$ . We now observe that  $\Phi(l_A, \overline{l_B}) = \varepsilon_{l_B}^{\mathfrak{S}_B}(\prod_{i \in I} \sigma_{i,B}) = \varepsilon_{\overline{l_B}}^{\mathfrak{S}_B}(\prod_{i \in I} \sigma_{i,B}) = \overline{\Phi(l_A, l_B)}$ . This concludes the proof.  $\square$

**Lemma 9.** Let us now fix  $\Phi \in \widetilde{\mathfrak{S}}_{AB}$  and  $l_A \in \mathfrak{E}_{\mathfrak{S}_A}$ , and let us suppose that  $\Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$ . Then, for any  $l_B, l'_B \in \mathfrak{E}_{\mathfrak{S}_B}$  such that  $l_B \sqcap_{\mathfrak{E}_{\mathfrak{S}_B}} l'_B = \perp_{\mathfrak{E}_{\mathfrak{S}_B}}$ , we have  $(\Phi(l_A, l_B), \Phi(l_A, l'_B)) \in \{(\perp, \mathbf{N}), (\mathbf{N}, \perp), (\perp, \perp)\}$ .  $\blacksquare$

*Proof.* Let us consider that  $\Phi = \prod_{i \in I} \iota^{\widetilde{\mathfrak{S}}_{AB}}(\sigma_{i,A}, \sigma_{i,B}) = \prod_{i \in I} \sigma_{i,A} \widetilde{\otimes} \sigma_{i,B}$ . As it has been clarified in the third case of Lemma 7, we have then necessarily  $(\Phi(l_A, l_B), \Phi(l_A, \overline{l_B})) \notin \{(\mathbf{N}, \mathbf{N}), (\mathbf{Y}, \mathbf{Y})\}$ . Let us suppose that  $\Phi(l_A, l_B) = \mathbf{Y}$ . Due to the expression (3), we have then necessarily  $\varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) = \mathbf{Y}$  and  $\varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_{i,B}) = \mathbf{Y}$  for any  $i \in I$  and then, in particular,  $\mathbf{Y} = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \mathbf{Y} = \bigwedge_{i \in I} \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_{i,A}) \bullet \varepsilon_{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}^{\mathfrak{S}_B}(\sigma_{i,B}) = \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) = \perp$ , which is contradictory. As a conclusion, we cannot have  $\Phi(l_A, l_B) = \mathbf{Y}$ . In the same way, we cannot have  $\Phi(l_A, \overline{l_B}) = \mathbf{Y}$ . This concludes the proof.  $\square$

**Theorem 24.** Let us fix for example  $\mathfrak{S}_A := \mathfrak{B}$  and  $\mathfrak{S}_B := \mathfrak{B}$ . We have

$$\widetilde{\mathfrak{S}}_{AB} \subsetneq \widetilde{\mathfrak{S}}_{AB} \quad (126)$$

$\blacksquare$

*Proof.* Let us denote by  $u$  the pure effect  $l_{(\mathbf{Y}, \mathbf{N})}$ . The description of pure effects given in (38) is here explicitly given by

$$\mathfrak{E}_{\mathfrak{B}}^{\text{pure}} = \{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, u, \bar{u}\}. \quad (127)$$

According to Lemma 7 and Lemma 8 and Lemma 9, we can identify an element  $\Phi$  in  $\tilde{S}_{AB}$  which is NOT in  $\tilde{S}_{AB}$ . For example, the map defined from  $\mathfrak{E}_{\mathfrak{S}_A} \times \mathfrak{E}_{\mathfrak{S}_B}$  to  $\mathfrak{B}$  by

$$\Phi(l_A, l_B) := \bigwedge_{l'_A \in \underline{l}_A \mathfrak{E}_{\mathfrak{S}_A}} \bigwedge_{l'_B \in \underline{l}_B \mathfrak{E}_{\mathfrak{S}_B}} \Phi(l'_A, l'_B) \quad (128)$$

and

$$\begin{aligned} \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, & \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, u_B) &= \mathbf{N}, & \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \mathbf{N}, & \Phi(u_A, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi(\overline{u_A}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, u_B) &= \perp, & \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \perp, & \Phi(u_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \perp, & \Phi(\overline{u_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \perp, \\ \Phi(u_A, u_B) &= \mathbf{Y}, & \Phi(u_A, \overline{u_B}) &= \mathbf{N}, & \Phi(\overline{u_A}, u_B) &= \mathbf{N}, & \Phi(\overline{u_A}, \overline{u_B}) &= \mathbf{Y}, \end{aligned} \quad (129)$$

is a solution of this problem.  $\square$

### 3.7 Remarks on the maximal tensor product

Let us fix once again for example  $\mathfrak{S}_A := \mathfrak{B}$  and  $\mathfrak{S}_B := \mathfrak{B}$ .

*Remark 4.* We can exhibit an interesting property of the element  $\Phi$  of  $\tilde{S}_{AB}$  defined by (128) and (129).

Let us denote  $\Phi_1 := \mathbf{Y} \tilde{\otimes} \mathbf{Y}$ ,  $\Phi_2 := \mathbf{N} \tilde{\otimes} \mathbf{N}$  and  $\Phi_{12} := \mathbf{Y} \tilde{\otimes} \mathbf{Y} \sqcap_{\tilde{S}_{AB}} \mathbf{N} \tilde{\otimes} \mathbf{N}$ . We have

$$\begin{aligned} \Phi_1(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi_1(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi_1(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, & \Phi_1(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi_1(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, u_B) &= \mathbf{N}, & \Phi_1(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \mathbf{N}, & \Phi_1(u_A, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi_1(\overline{u_A}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi_1(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, u_B) &= \mathbf{Y}, & \Phi_1(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \mathbf{N}, & \Phi_1(u_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi_1(\overline{u_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, \\ \Phi_1(u_A, u_B) &= \mathbf{Y}, & \Phi_1(u_A, \overline{u_B}) &= \mathbf{N}, & \Phi_1(\overline{u_A}, u_B) &= \mathbf{N}, & \Phi_1(\overline{u_A}, \overline{u_B}) &= \mathbf{N}, \end{aligned} \quad (130)$$

$$\begin{aligned} \Phi_2(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi_2(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi_2(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, & \Phi_2(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi_2(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, u_B) &= \mathbf{N}, & \Phi_2(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \mathbf{N}, & \Phi_2(u_A, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi_2(\overline{u_A}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi_2(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, u_B) &= \mathbf{N}, & \Phi_2(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \mathbf{Y}, & \Phi_2(u_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, & \Phi_2(\overline{u_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, \\ \Phi_2(u_A, u_B) &= \mathbf{N}, & \Phi_2(u_A, \overline{u_B}) &= \mathbf{N}, & \Phi_2(\overline{u_A}, u_B) &= \mathbf{N}, & \Phi_2(\overline{u_A}, \overline{u_B}) &= \mathbf{Y}, \end{aligned} \quad (131)$$

$$\begin{aligned} \Phi_{12}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi_{12}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi_{12}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, & \Phi_{12}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi_{12}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, u_B) &= \mathbf{N}, & \Phi_{12}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \mathbf{N}, & \Phi_{12}(u_A, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi_{12}(\overline{u_A}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\ \Phi_{12}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, u_B) &= \perp, & \Phi_{12}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{u_B}) &= \perp, & \Phi_{12}(u_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \perp, & \Phi_{12}(\overline{u_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \perp, \\ \Phi_{12}(u_A, u_B) &= \perp, & \Phi_{12}(u_A, \overline{u_B}) &= \mathbf{N}, & \Phi_{12}(\overline{u_A}, u_B) &= \mathbf{N}, & \Phi_{12}(\overline{u_A}, \overline{u_B}) &= \perp, \end{aligned} \quad (132)$$

We have then

$$\begin{cases} \Phi \sqcap_{\tilde{S}_{AB}} (\mathbf{Y} \tilde{\otimes} \mathbf{Y} \sqcap_{\tilde{S}_{AB}} \mathbf{N} \tilde{\otimes} \mathbf{N}) \\ \Phi \in \text{Max}(\tilde{S}_{AB}) \\ \Phi \notin \{\mathbf{Y} \tilde{\otimes} \mathbf{Y}, \mathbf{N} \tilde{\otimes} \mathbf{N}\} \end{cases} \quad (133)$$

This result has to be confronted with the result of Theorem 23.

*Remark 5.* Let us consider the map  $\Phi'$  defined from  $\mathfrak{E}_{\mathfrak{S}_A} \times \mathfrak{E}_{\mathfrak{S}_B}$  to  $\mathfrak{B}$  by

$$\Phi'(l_A, l_B) := \bigwedge_{l'_A \in \underline{l}_A \mathfrak{E}_{\mathfrak{S}_A}} \bigwedge_{l'_B \in \underline{l}_B \mathfrak{E}_{\mathfrak{S}_B}} \Phi'(l'_A, l'_B) \quad (134)$$

and

$$\begin{aligned}
\Phi'(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi'(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi'(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, & \Phi'(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\
\Phi'(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \mathfrak{u}_B) &= \mathbf{N}, & \Phi'(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}}, \overline{\mathfrak{u}_B}) &= \mathbf{N}, & \Phi'(\mathfrak{u}_A, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, & \Phi'(\overline{\mathfrak{u}_A}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}}) &= \mathbf{N}, \\
\Phi'(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \mathfrak{u}_B) &= \mathbf{Y}, & \Phi'(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_A}}, \overline{\mathfrak{u}_B}) &= \mathbf{N}, & \Phi'(\mathfrak{u}_A, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{Y}, & \Phi'(\overline{\mathfrak{u}_A}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}_B}}) &= \mathbf{N}, \\
\Phi'(\mathfrak{u}_A, \mathfrak{u}_B) &= \perp, & \Phi'(\mathfrak{u}_A, \overline{\mathfrak{u}_B}) &= \mathbf{N}, & \Phi'(\overline{\mathfrak{u}_A}, \mathfrak{u}_B) &= \mathbf{N}, & \Phi'(\overline{\mathfrak{u}_A}, \overline{\mathfrak{u}_B}) &= \mathbf{N}.
\end{aligned} \tag{135}$$

The map  $\Phi'$  is such that  $((\uparrow^{\check{S}_{AB}} \Phi') \setminus \{\Phi'\})$  admits a minimum element. More precisely,

$$\text{Min} \left( (\uparrow^{\check{S}_{AB}} \Phi') \setminus \{\Phi'\} \right) = \{\mathbf{Y} \tilde{\otimes} \mathbf{Y}\}. \tag{136}$$

As a result,  $\Phi'$  is a completely meet-irreducible element of  $\check{S}_{AB}$  but is NOT a maximal element of  $\check{S}_{AB}$ .

## Statements and Declarations

The author did not receive any support from any organization for the present work. The author declares that there is no conflict of interest. There is no associated data to this article.

## References

- [1] Balbes, R., Dwinger, P. *Distributive Lattices*. University of Missouri Press, Columbia, Missouri, 1975.
- [2] Fraser, G.A. The Semilattice Tensor Product of Distributive Lattices. *Transactions of the American Mathematical Society*, 217:183–194, 1976.
- [3] Fraser, G.A. The tensor product of semilattices. *algebra universalis*, 8:1–3, 1978.
- [4] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., Scott, D. S. *Continuous Lattices and Domains*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2003.
- [5] Grätzer, G. *Lattice Theory: First Concepts and Distributive Lattices*. 1971.
- [6] Pratt, V.R. *Chu Spaces*, 1999.