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# SURVIVAL AND MAXIMUM OF SPECTRALLY NEGATIVE BRANCHING LÉVY PROCESSES WITH ABSORPTION

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ABSTRACT. We consider a spectrally negative branching Lévy process where the particles undergo dyadic branching and are killed when entering the negative half-plane. The purpose of this short note is to give conditions under which this process dies out a.s., and then study the asymptotics of its all-time maximum.

## 1. INTRODUCTION

Consider a spectrally negative branching Lévy process  $X$  where the particles undergo dyadic branching at rate  $\beta > 0$ , and are killed when entering the negative half-plane. More precisely, starting at  $t = 0$  from an initial particle located at  $a > 0$ , the reproduction and dispersion mechanisms are defined as follow :

- (1) each particle waits for an exponential random time  $\mathbf{e}$  of parameter  $\beta$  and, if still alive, splits into two independent particles whose lives start at the location of their ancestor.
- (2) between branching events, the particles move independently according to a spectrally negative Lévy process  $L$  but are killed when going below 0.

Let us denote by  $\mathbf{M}$  the all-time maximum location ever reached by a particle during the whole life of the process. The purpose of this paper is to give conditions under which the process dies out a.s. and then study the distribution of the (then finite) random variable  $\mathbf{M}$ .

As explained in [12], such model may describe for instance the evolution of a population in which the positions of the particles represent their fitness. The fitness of each individual evolves as a spectrally negative Lévy process, and when branching, the initial fitness of a child is the same as his parent. The presence of negative jumps allow to model diseases or accidents which generate an abrupt drop in the fitness, and individuals die when their fitness goes below 0.

In the following, we shall denote by  $\mathbb{P}_x$  the law of  $X$  and  $L$  when started from  $x$ . For  $\lambda \in \mathbb{C}$  such that  $\Re(\lambda) \geq 0$ , the Laplace exponent of  $L$  is given by

$$\Psi(\lambda) = \ln \mathbb{E}_0 \left[ e^{\lambda L_1} \right] = d\lambda + \frac{\eta^2}{2} \lambda^2 + \int_{-\infty}^0 \left( e^{\lambda x} - 1 - \lambda x 1_{\{|x| < 1\}} \right) \nu(dx)$$

where  $d \in \mathbb{R}$  is the drift coefficient,  $\eta \in \mathbb{R}$  the Gaussian coefficient and the Lévy measure  $\nu$  satisfies  $\int_{-\infty}^0 (x^2 \wedge 1) \nu(dx) < +\infty$ . We assume that the one-dimensional distributions of  $L$  are absolutely continuous, i.e.,

$$\mathbb{P}_0(L_t \in dx) \ll dx, \quad \text{for every } t > 0$$

and we exclude the case where  $-L$  is a subordinator (in which case  $\mathbf{M}$  is finite a.s. and equal to  $a$ ). As a consequence the function  $\Psi$  is strictly convex and tends to  $+\infty$  as  $\lambda \rightarrow +\infty$ . This implies

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that for any  $q \geq 0$ , the equation  $\Psi(\lambda) = q$  admits at most two solutions, and we denote by  $\Phi(q)$  the largest one :

$$\Phi(q) = \sup\{\lambda \geq 0, \Psi(\lambda) = q\}.$$

More precisely :

- (1) when  $\Psi'(0^+) \geq 0$ , we have  $\Phi(0) = 0$  and the function  $\Phi$  is well-defined on  $[0, +\infty)$ .
- (2) when  $\Psi'(0^+) < 0$ , we have  $\Phi(0) > 0$ . In this case,  $\Psi$  being strictly convex, it admits a unique minimum at  $\lambda_*$  which is such that  $\Psi'(\lambda_*) = 0$  and  $\Psi''(\lambda_*) = \eta^2 + \int_{-\infty}^0 x^2 e^{\lambda_* x} \nu(dx) > 0$ . As a consequence, the function  $\Phi$  is well-defined on  $[-q_*, +\infty)$ , and  $-q_* = \Psi(\lambda_*)$ .

Our first result concerns the survival of this branching process.

**Theorem 1.** *The branching process dies out a.s. if and only if  $\Psi'(0^+) < 0$  and  $\beta \leq q_*$ . In this case, the asymptotics of the extinction time  $\zeta$  satisfy :*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_a(\zeta > t) = \beta - q_*. \quad (1.1)$$

Note that the condition  $\Psi'(0^+) < 0$  implies that the Lévy process  $L$  will converge towards  $-\infty$  as might be expected for the particles to be killed a.s.. The second condition  $\beta \leq q_*$  essentially states that the decay of  $L$  towards  $-\infty$  should be fast enough to compensate the reproduction rate  $\beta$ .

When  $\Psi'(0^+) < 0$  and  $\beta \leq q_*$ , the above theorem implies that the random variable  $\mathbf{M}$  is finite, and we are now interested in studying the tail decay of its survival function. To this end, let us introduce the following scale functions  $W^{(q)}$  which are classically defined for  $q \geq 0$  by :

$$\int_0^{+\infty} e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\Psi(\lambda) - q}, \quad \lambda > \Phi(q). \quad (1.2)$$

It is known that  $W^{(q)}$  is an increasing function which is null on  $(-\infty, 0)$ . Furthermore, for every  $a > 0$ , the function  $q \rightarrow W^{(q)}(a)$  may be extended analytically on  $\mathbb{C}$ , see Kyprianou [8, Section 8.3].

**Theorem 2.** *Assume that  $\Psi'(0^+) < 0$  and  $\beta \leq q_*$  so that  $\mathbf{M}$  is a.s. finite.*

- (1) *If  $\beta < q_*$ , there exists a constant  $\kappa_\beta$  independent from the starting point  $a$  such that :*

$$\mathbb{P}_a(\mathbf{M} \geq x) \underset{x \rightarrow +\infty}{\sim} \kappa_\beta W^{(-\beta)}(a) e^{-\Phi(-\beta)x}.$$

- (2) *If  $\beta = q_*$ , there exists a constant  $\kappa_{q_*}$  independent from the starting point  $a$  such that :*

$$\mathbb{P}_a(\mathbf{M} \geq x) \underset{x \rightarrow +\infty}{\sim} \kappa_{q_*} \frac{W^{(-q_*)}(a)}{x} e^{-\Phi(-q_*)x}.$$

It might be surprising to observe that there is an extra decreasing factor in the critical case, which implies that the maximum will be smaller than one might expect by simply letting  $\beta \uparrow q_*$  in the subcritical case. This phenomenon is in accordance with what is known for the extinction time in the Brownian case. Indeed, for branching drifted Brownian motion with absorption, the asymptotics of the extinction time  $\zeta$  are explicitly given by

$$\begin{cases} \mathbb{P}_a(\zeta > t) = C_\beta(a) t^{-3/2} e^{(\beta - q_*)t} & \text{if } \beta < q_*, \\ \mathbb{P}_a(\zeta > t) = C_{q_*}(a) e^{-(3\pi^2 q_* t)^{1/3}} & \text{if } \beta = q_*. \end{cases}$$

where  $C_\beta(a) > 0$ . These asymptotics were obtained through the successive works of Kesten [7], Harris & Harris [6], Berestycki, Berestycki & Schweinsberg [3] and Maillard & Schweinsberg [12]. In

particular, we observe the same phenomenon : in the critical case, the branching Brownian motion will die out faster than one might expect by simply passing to the limit in the subcritical case.

Let us finally consider the same spectrally negative branching Lévy process, but without killing. We denote by  $\mathcal{M}$  its all-time maximum. Using the invariance by translation of Lévy process, we deduce by letting the killing barrier go down to  $-\infty$  the following corollary.

**Corollary 3.** *The maximum  $\mathcal{M}$  of a (free) spectrally negative branching Lévy process is a.s. finite if  $\Psi'(0^+) < 0$  and  $\beta \leq q_*$ . In this case, there exists a constant  $\kappa_\beta > 0$  such that*

$$\mathbb{P}_0(\mathcal{M} \geq x) \underset{x \rightarrow +\infty}{\sim} \kappa_\beta e^{-\Phi(-\beta)x}.$$

Note that this result was already known in the Brownian case, see [4, Prop. 4] where the proof essentially relies on the well-known associated KPP equation.

## 2. FINITENESS OF THE EXTINCTION TIME

**2.1. Proof of the “if” part of Theorem 1.** The proof of the implication is quite straightforward. Let us denote by  $N_t$  the number of particles alive at time  $t$ , and  $\zeta = \inf\{t \geq 0, N_t = 0\}$  the extinction time. Since  $N_t$  is integer-valued, we have using Sawyer [13, Theorem 2.1],

$$\mathbb{P}_a(\zeta > t) = \mathbb{P}_a(N_t > 0) \leq \mathbb{E}_a[N_t] = \mathbb{P}_a(\tau_0^- > t)e^{\beta t}. \quad (2.1)$$

where, for  $x \in \mathbb{R}$ ,

$$\tau_x^- = \inf\{t \geq 0, L_t \leq x\}.$$

But, from Kyprianou & Palmowski [9, Corollary 4], there exists a function  $\kappa$  such that<sup>1</sup>

$$\mathbb{P}_a(\tau_0^- > t) \underset{t \rightarrow +\infty}{\sim} \kappa(a)t^{-3/2}e^{-q_*t}. \quad (2.2)$$

Passing to the limit in (2.1), we thus deduce that  $\lim_{t \rightarrow +\infty} \mathbb{P}_a(\zeta > t) = 0$  since  $\beta \leq q_*$ . As a consequence  $\mathbb{P}_a(\zeta = +\infty) = 0$ , i.e. the process dies out a.s.

**2.2. A representation formula for the extinction time.** To prove the converse part, we shall rely on the following representation formula for the survival function of  $\zeta$ .

**Proposition 4.** *Define  $v(a, t) = \mathbb{P}_a(\zeta > t)$ . Then  $v$  admits the representation :*

$$v(a, t) = e^{\beta t} \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- > t\}} e^{-\beta \int_0^t v(L_s, t-s) ds} \right].$$

In the Brownian case, this result was already used in [6] to study the asymptotic tail of  $\zeta$ . One classical way to prove such representation is to write down a pseudo-differential equation satisfied by  $u$  (involving the generator of the Lévy process  $L$ ), and then apply a Feynman-Kac theorem or a martingale argument. We shall propose here another approach, based on solving a Fredholm integral equation of the second kind.

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<sup>1</sup>Note that, with the notations of [9], a spectrally negative Lévy process such that  $\Psi'(0^+) < 0$  is of class A with  $\gamma = 2$ .

*Proof.* We start by applying the Markov property at the first branching event  $\mathbf{e}$  :

$$\begin{aligned} 1 - v(a, t) &= \mathbb{P}_a(\zeta \leq t, t < \mathbf{e}) + \mathbb{P}_a(\zeta \leq t, \tau_0^- \leq \mathbf{e}, t \geq \mathbf{e}) + \mathbb{P}_a(\zeta \leq t, \tau_0^- > \mathbf{e}, t \geq \mathbf{e}) \\ &= \mathbb{P}_a(\tau_0^- \leq t, t < \mathbf{e}) + \mathbb{P}_a(\tau_0^- \leq \mathbf{e}, t \geq \mathbf{e}) + \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} (1 - v(L_{\mathbf{e}}, t - \mathbf{e}))^2 \right]. \end{aligned}$$

Here,  $\tau_0^-$  denotes the first passage time of the initial particle  $L$ , and is independent from  $\mathbf{e}$ . Developing the square yields the integral equation :

$$v(a, t) = e^{-\beta t} \mathbb{P}_a(\tau_0^- > t) + 2\mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} v(L_{\mathbf{e}}, t - \mathbf{e}) \right] - \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} v^2(L_{\mathbf{e}}, t - \mathbf{e}) \right].$$

To obtain a representation of  $v$  through this equation, we shall write down a Riemann-Liouville series. Define the linear operator  $T$  acting on positive and bounded functions  $f : \mathbb{R} \times [0, +\infty) \rightarrow [0, +\infty)$  by

$$T[f](x, t) = \mathbb{E}_x \left[ \mathbf{1}_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} (2 - v(L_{\mathbf{e}}, t - \mathbf{e})) f(L_{\mathbf{e}}, t - \mathbf{e}) \right]$$

so that  $v$  is a solution of the Fredholm integral equation :

$$v(x, t) = e^{-\beta t} \mathbb{P}_x(\tau_0^- > t) + T[v](x, t).$$

Let us set to simplify the notation  $\varphi(x, t) = e^{-\beta t} \mathbb{P}_x(\tau_0^- > t)$ . By iteration, we deduce that for  $n \geq 2$ ,

$$v(x, t) = \varphi(x, t) + \sum_{k=1}^{n-1} T^{\circ(k)}[\varphi](x, t) + T^{\circ(n)}[v](x, t)$$

where  $T^{\circ(k)}$  denotes the  $k^{\text{th}}$  composition of  $T$  with itself :

$$T^{\circ(1)} = T \quad \text{and for } k \geq 2, \quad T^{\circ(k)}[f] = T^{\circ(k-1)}[T[f]].$$

We now show that for any fixed pair  $(x, t)$ ,

$$\lim_{n \rightarrow +\infty} T^{\circ(n)}[v](x, t) = 0.$$

Indeed, fix  $A > 0$  large enough. Using the upper bound (2.1) for  $v$ , we have for any  $t \leq A$ ,

$$T[v](x, t) \leq 2\mathbb{E}_x \left[ e^{\beta(t-\mathbf{e})} \mathbb{P}_{L_{\mathbf{e}}}(\tau_0^- > t - \mathbf{e}) \mathbf{1}_{\{\tau_0^- > \mathbf{e}\}} \mathbf{1}_{\{A \geq \mathbf{e}\}} \right].$$

Applying the Markov property at the time  $\mathbf{e}$ , this is further equal to

$$T[v](x, t) \leq 2\mathbb{E}_x \left[ e^{\beta(t-\mathbf{e})} \mathbf{1}_{\{\tau_0^- > t\}} \mathbf{1}_{\{A > \mathbf{e}\}} \right] = e^{\beta t} \mathbb{P}_x(\tau_0^- > t) (1 - \delta)$$

where we have set  $e^{-2\beta A} = \delta < 1$ . By iteration, we deduce that

$$T^{\circ(n)}[v](x, t) \leq (1 - \delta)^n e^{\beta t} \mathbb{P}_x(\tau_0^- > t) \xrightarrow{n \rightarrow +\infty} 0.$$

As a consequence, we obtain the representation for any fixed  $(x, t)$ ,

$$v(x, t) = \varphi(x, t) + \sum_{k=1}^{+\infty} T^{\circ(k)}[\varphi](x, t).$$

We shall now compute the convolutions appearing in the sum.

**Lemma 5.** Let  $(\mathbf{e}_k, k \in \mathbb{N})$  be a sequence of i.i.d. exponential r.v. with parameter  $\beta$ . For any  $n \in \mathbb{N}$ , we set  $\mathbf{g}_n = \sum_{k=1}^n \mathbf{e}_k$ . Then :

$$T^{\circ(n)}[\varphi](x, t) = e^{-\beta t} \mathbb{E}_x \left[ 1_{\{\tau_0^- > t\}} 1_{\{\mathbf{g}_n \leq t\}} e^{\beta \mathbf{g}_n} \prod_{i=1}^n (2 - v(L_{\mathbf{g}_i}, t - \mathbf{g}_i)) \right].$$

*Proof.* For  $n = 1$ , we have applying the Markov property

$$\begin{aligned} T[\varphi](x, t) &= \mathbb{E}_x \left[ 1_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} (2 - v(L_{\mathbf{e}}, t - \mathbf{e})) \mathbb{P}_{L_{\mathbf{e}}}(\tau_0^- > t - \mathbf{e}) e^{-\beta(t-\mathbf{e})} \right] \\ &= e^{-\beta t} \mathbb{E}_x \left[ 1_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} (2 - v(L_{\mathbf{e}}, t - \mathbf{e})) 1_{\{\tau_0^- > t\}} e^{\beta \mathbf{e}} \right] \end{aligned}$$

which is the expected result. Assume now that the formula holds for some  $n \in \mathbb{N}$ . Applying the Markov property, we have  $\mathbb{P}_a$  a.s. :

$$T^{\circ(n)}[\varphi](L_{\mathbf{e}}, t - \mathbf{e}) = e^{-\beta t} \mathbb{E}_a \left[ 1_{\{\tau_0^- > t\}} 1_{\{\mathbf{e} + \mathbf{g}_n \leq t\}} e^{\beta(\mathbf{e} + \mathbf{g}_n)} \prod_{i=1}^n (2 - v(L_{\mathbf{e} + \mathbf{g}_i}, t - \mathbf{e} - \mathbf{g}_i)) \middle| \mathcal{F}_{\mathbf{e}} \right]$$

where  $(\mathcal{F}_t, t \geq 0)$  denotes the natural filtration of  $L$ , and  $\mathbf{e}$  is independent from  $L$  and from the sequence  $(\mathbf{g}_i, 1 \leq i \leq n)$ . Then, using the tower property of conditional expectation, we obtain :

$$\begin{aligned} T^{\circ(n+1)}[\varphi](x, t) &= \mathbb{E}_x \left[ 1_{\{\tau_0^- > \mathbf{e}, t > \mathbf{e}\}} (2 - v(L_{\mathbf{e}}, t - \mathbf{e})) T^{\circ(n)}[\varphi](L_{\mathbf{e}}, t - \mathbf{e}) \right] \\ &= e^{-\beta t} \mathbb{E}_x \left[ 1_{\{\tau_0^- > t\}} 1_{\{\mathbf{g}_{n+1} \leq t\}} e^{\beta \mathbf{g}_{n+1}} \prod_{i=1}^{n+1} (2 - v(L_{\mathbf{g}_i}, t - \mathbf{g}_i)) \right] \end{aligned}$$

which proves Lemma 5 by induction.  $\square$

Recall now that for  $n \geq 1$ , the r.v.  $\mathbf{g}_n$  is Gamma distributed with parameters  $n$  and  $\beta$  :

$$\mathbb{P}(\mathbf{g}_n \in dr) = \frac{\beta^n}{(n-1)!} r^{n-1} e^{-\beta r} dr, \quad r > 0,$$

and that conditionally on  $\{\mathbf{g}_n = r\}$ , the partial sums are distributed as ordered uniform random variables, i.e.

$$(\mathbf{g}_i, 1 \leq i \leq n-1) \stackrel{(\text{law})}{=} r \times (U_i, 1 \leq i \leq n-1).$$

where the  $(U_i)$  are uniform r.v. on  $[0, 1]$  such that  $U_1 \leq U_2 \leq \dots \leq U_{n-1}$ . As a consequence, we deduce that the expectation in Lemma 5 equals

$$e^{-\beta t} \int_0^t \mathbb{E}_x \left[ 1_{\{\tau_0^- > t\}} (2 - v(L_r, t - r)) \prod_{i=1}^{n-1} (2 - v(L_{U_i r}, t - U_i r)) \right] \frac{\beta^n r^{n-1}}{(n-1)!} dr.$$

Computing the expectations with respect to the  $(U_i)$ , we obtain :

$$\frac{\beta^n}{(n-1)!} e^{-\beta t} \int_0^t \mathbb{E}_x \left[ 1_{\{\tau_0^- > t\}} (2 - v(L_r, t - r)) \left( \int_0^r (2 - v(L_s, t - s)) ds \right)^{n-1} \right] dr.$$

Finally, plugging this expression in the representation of  $v$  and computing the remaining integral yields the result :

$$v(a, t) = \mathbb{P}_a(\tau_0^- > t) e^{-\beta t} + e^{-\beta t} \beta \int_0^t \mathbb{E}_a \left[ 1_{\{\tau_0^- > t\}} (2 - v(L_r, t - r)) e^{\beta \int_0^r (2 - v(L_s, t - s)) ds} \right] dr$$

$$= \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- > t\}} e^{\beta \int_0^t (1-v(L_s, t-s)) ds} \right].$$

□

**2.3. Proof of the “only if” part of Theorem 1.** To prove the converse part, assume that  $\Psi'(0^+) < 0$  and  $\beta > q_*$ . We set for  $x \in \mathbb{R}$ ,

$$\tau_x^+ = \inf\{t \geq 0, L_t \geq x\}.$$

Since  $x \rightarrow v(x, s)$  is increasing and  $L$  has no positive jumps, we have for  $x > a$  :

$$v(a, t) \geq e^{\beta t} \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- > t\}} e^{-\beta \int_0^t v(L_{t-s}, s) ds} \mathbf{1}_{\{\tau_x^+ > t\}} \right] \geq e^{\beta t - \beta \int_0^t v(x, s) ds} \mathbb{P}_a(\tau_0^- \wedge \tau_x^+ > t).$$

From Bertoin [1], since we have assumed that the transition density of  $L$  is absolutely continuous, the asymptotics of the exit time  $\tau_0^- \wedge \tau_x^+$  are given by

$$\mathbb{P}_a(\tau_0^- \wedge \tau_x^+ > t) \underset{t \rightarrow +\infty}{\sim} \kappa e^{-\rho(x)t} \quad (2.3)$$

where  $\kappa > 0$  and the function  $\rho$  is defined by

$$\rho(x) = \inf\{q \geq 0, W^{(-q)}(x) = 0\}.$$

Note also that from Lambert [10, Prop. 5.1],  $\rho$  is continuous and strictly decreasing, hence it converges towards

$$\lim_{x \rightarrow +\infty} \rho(x) = \inf_{x \geq 0} \rho(x) = \rho(\infty).$$

We shall now proceed by contradiction. Assume that there exists  $x > 0$  such that  $\mathbb{P}_x(\zeta = +\infty) = \lim_{t \rightarrow +\infty} v(x, t) = 0$ . Then, a standard application of the Markov property shows that we necessarily have  $\lim_{t \rightarrow +\infty} v(x, t) = 0$  for any  $x > 0$ . Let  $\varepsilon > 0$  and take  $x$  large enough such that  $\rho(x) \leq \rho(\infty) + \varepsilon$ .

By assumption, there exists  $A > 0$  such that for  $t \geq A$ , we have  $v(x, t) < \varepsilon$ . As a consequence

$$v(a, t) \geq e^{\beta(1-\varepsilon)t - \beta \int_0^A v(x, s) ds + A\beta\varepsilon} \mathbb{P}_a(\tau_0^- \wedge \tau_x^+ > t). \quad (2.4)$$

Passing to the limit as  $t \rightarrow +\infty$ , we deduce that  $\beta(1-\varepsilon) - \rho(x) < 0$  which implies that  $\beta < \rho(\infty) + (1+\beta)\varepsilon$ . Taking  $\varepsilon$  small enough leads thus to the inequality  $\beta \leq \rho(\infty)$  and the contradiction will follow from the fact that  $\rho(\infty) = q_*$  as proven in the next lemma.

**Lemma 6.** *Assume that  $\Psi'(0^+) < 0$ . Then :*

$$\rho(\infty) = q_* = \sup\{q > 0, W^{(-q)}(x) > 0 \text{ for every } x > 0\}.$$

Note that this lemma is very similar to Theorem 1.5 of K. Yamato [14], in which the author studies quasi-stationary distributions of  $-L$  and assumes implicitly that  $\Psi'(0^+) > 0$ .

*Proof.* Observe first that comparing (2.2) and (2.3) immediately yields  $q_* \leq \rho(\infty)$ . Also, by definition of  $\rho(x)$  as an infimum, we have  $W^{(-\rho(\infty))}(x) > 0$  for every  $x > 0$ . As a consequence,

$$\rho(\infty) \leq \sup\{q > 0, W^{(-q)}(x) > 0 \text{ for every } x > 0\}.$$

To prove the converse inequality, recall from [8, Chapter 8] that for  $q \geq 0$  and  $a \leq x$ ,

$$\mathbb{E}_a \left[ e^{-q\tau_x^+} \right] = e^{-\Phi(q)(x-a)}$$

and that Formula (1.2) remains valid for  $q \in \mathbb{C}$  and  $\lambda > \Phi(|q|)$ . Taking  $\lambda$  large enough, since  $\Phi$  is increasing, we deduce that

$$\int_0^{+\infty} e^{-\Phi(0)y} W^{(-q)}(y) \mathbb{E}_0 \left[ e^{-\lambda\tau_y^+} \mathbf{1}_{\tau_y^+ < +\infty} \right] dy = \int_0^{+\infty} e^{-\Phi(\lambda)y} W^{(-q)}(y) dy = \frac{1}{\lambda + q}.$$

Assume now that  $q$  is such that  $W^{(-q)}(x) > 0$  for every  $x > 0$ . Applying the Fubini-Tonelli theorem, and inverting this Laplace transform gives for a.e.  $t \geq 0$  :

$$\int_0^{+\infty} e^{-\Phi(0)y} W^{(-q)}(y) \mathbb{P}_0(\tau_y^+ \in dt | \tau_y^+ < +\infty) dy = e^{-qt}$$

which implies that  $q \leq q_*$  since conditionally on  $\{\tau_y^+ < +\infty\}$ , the random variable  $\tau_y^+$  admits exponential moments of at most  $q_*$ .  $\square$

It remains to deal with the case  $\Psi'(0^+) \geq 0$ . Note that when  $\Psi'(0^+) > 0$ , as the process  $L$  goes a.s. towards  $+\infty$ , one may simply bound the probability that  $\zeta$  is infinite by the probability that one particle never go below 0 :

$$\mathbb{P}_a(\zeta = \infty) \geq \mathbb{P}_a(\tau_0^- = +\infty) = \Psi'(0^+) W^{(0)}(a) > 0.$$

Assume finally that  $\Psi'(0^+) = 0$ . The function  $\Psi'$  being continuous and strictly increasing, take  $\lambda_{\#} > 0$  small enough such that  $0 < \Psi'(\lambda_{\#})\lambda_{\#} - \Psi(\lambda_{\#}) < \beta$ . Consider now the branching process  $X^{\#}$  in which the particles evolve as spectrally negative Lévy processes  $L^{\#} = (L_t - \Psi'(\lambda_{\#})t, t \geq 0)$  whose characteristic function is given by

$$\Psi_{\#}(\lambda) = \Psi(\lambda) - \Psi'(\lambda_{\#})\lambda.$$

In particular,  $\Psi'_{\#}(0^+) < 0$  and  $\Psi'_{\#}(\lambda_{\#}) = 0$ . As a consequence, the minimum of  $\Psi_{\#}$  equals  $\Psi_{\#}(\lambda_{\#}) = \Psi(\lambda_{\#}) - \Psi'(\lambda_{\#})\lambda_{\#} > -\beta$ . From the first part of the proof, this implies that the branching process  $X^{\#}$  does not die a.s., i.e.  $\mathbb{P}_a(\zeta_{\#} = +\infty) > 0$ . Now by coupling, as we have added a negative drift, it holds  $\mathbb{P}_a(\zeta = +\infty) \geq \mathbb{P}_a(\zeta_{\#} = +\infty) > 0$  which ends the proof of the converse part of Theorem 1.  $\square$

**2.4. Proof of Formula (1.1).** Observe first that from (2.1) and (2.2), we have

$$\frac{1}{t} \ln \mathbb{P}_a(\zeta > t) \leq \beta + \frac{1}{t} \ln \mathbb{P}_a(\tau_0^- \geq t) \xrightarrow[t \rightarrow +\infty]{} \beta - q_*.$$

Conversely, from (2.4),

$$\frac{1}{t} \ln \mathbb{P}_a(\zeta > t) \geq \beta(1 - \varepsilon) + \frac{\beta}{t} \left( A\varepsilon - \int_0^A v(x, s) ds \right) + \frac{1}{t} \ln \mathbb{P}_a(\tau_0^- \wedge \tau_x^+ > t).$$

Take  $x$  large enough such that  $\rho(x) \leq q_* + \varepsilon$ . Passing to the limit as  $t \rightarrow +\infty$  and using (2.3),

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \mathbb{P}_a(\zeta > t) \geq \beta(1 - \varepsilon) - \rho(x) \geq \beta - q_* - \varepsilon(1 + \beta),$$

and the result follows by letting  $\varepsilon \downarrow 0$ .  $\square$



## 3. STUDY OF THE MAXIMUM

3.1. **An integral equation.** We set

$$u(a, x) = \mathbb{P}_a(\mathbf{M} \geq x)$$

and first prove a Feynman-Kac-like representation for  $u$  as in Proposition 4. When dealing with extreme values of branching processes, this idea already appears in Lalley & Shao [11] in their study of the maximum of a critical symmetric stable branching process.

**Proposition 7.** *The function  $u$  admits the representation :*

$$u(a, x) = \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} e^{\beta \int_0^{\tau_x^+} (1-u(L_r, x)) dr} \right].$$

*Proof.* The proof is similar to that of Proposition 4. Applying the Markov property at the first branching event, we first write :

$$\begin{aligned} 1 - u(a, x) &= \mathbb{P}_a(\mathbf{M} < x, \tau_0^- \leq \mathbf{e}) + \mathbb{P}_a(\mathbf{M} < x, \tau_0^- > \mathbf{e}) \\ &= \mathbb{P}_a(\tau_0^- < \tau_x^+, \tau_0^- \leq \mathbf{e}) + \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ \wedge \tau_0^- > \mathbf{e}\}} (1 - u(L_{\mathbf{e}}, x))^2 \right]. \end{aligned}$$

Developing the square yields the non-linear integral equation :

$$u(a, x) = \mathbb{P}_a(\tau_x^+ \leq \mathbf{e} \wedge \tau_0^-) + 2\mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- \wedge \tau_x^+ > \mathbf{e}\}} u(L_{\mathbf{e}}, x) \right] - \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_0^- \wedge \tau_x^+ > \mathbf{e}\}} u^2(L_{\mathbf{e}}, x) \right].$$

Define the linear operator :

$$T[f](a, x) = \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ \wedge \tau_0^- > \mathbf{e}\}} (2 - u(L_{\mathbf{e}}, x)) f(L_{\mathbf{e}}, x) \right]$$

so that  $u$  is a solution of the Fredholm integral equation

$$f(a, x) = \mathbb{P}_a(\tau_x^+ \leq \mathbf{e} \wedge \tau_0^-) + T[f](a, x).$$

Setting to simplify the notations

$$\varphi(a, x) = \mathbb{P}_a(\tau_x^+ \leq \mathbf{e} \wedge \tau_0^-) = \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} e^{-\beta \tau_x^+} \right] = \frac{W^{(\beta)}(a)}{W^{(\beta)}(x)},$$

we obtain by iteration that for  $n \geq 2$ ,

$$u(a, x) = \varphi(a, x) + \sum_{k=1}^{n-1} T^{\circ(k)}[\varphi](a, x) + T^{\circ(n)}[u](a, x), \quad (3.1)$$

and it remains to evaluation the convolutions. Following the same proof as for Proposition 4, we deduce that for any  $k \geq 1$  :

$$T^{\circ(k)}[f](a, x) = \frac{\beta^{k-1}}{(k-1)!} \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ \wedge \tau_0^- > \mathbf{e}\}} f(L_{\mathbf{e}}, x) (2 - u(L_{\mathbf{e}}, x)) \left( \int_0^{\mathbf{e}} (2 - u(L_r, x)) dr \right)^{k-1} \right]. \quad (3.2)$$

In particular, taking  $f = u$ , we obtain since  $u$  is positive and bounded by 1 :

$$T^{\circ(n)}[u](a, x) \leq 2 \frac{(2\beta)^{n-1}}{(n-1)!} \mathbb{E}_a \left[ \mathbf{e}^{n-1} \mathbf{1}_{\{\tau_x^+ \wedge \tau_0^- > \mathbf{e}\}} \right] \leq \frac{(2\beta)^n}{(n-1)!} \mathbb{E}_a \left[ \int_0^{\tau_x^+ \wedge \tau_0^-} z^{n-1} e^{-\beta z} dz \right].$$

From the asymptotics (2.3), we deduce that there exists a constant  $C > 0$  such that :

$$\begin{aligned} T^{\circ(n)}[u](a, x) &\leq C \frac{(2\beta)^{n-1}}{(n-1)!} \left( 1 + \int_0^{+\infty} \int_0^s z^{n-1} e^{-\beta z} dz e^{-\rho(x)s} ds \right) \\ &= C \frac{(2\beta)^{n-1}}{(n-1)!} \left( 1 + \frac{(n-1)!}{\rho(x)(\beta + \rho(x))^n} \right) \xrightarrow{n \rightarrow +\infty} 0 \end{aligned}$$

since  $\beta \leq q_* < \rho(x)$  for any fixed  $x > 0$ . As a consequence, we obtain the series formula

$$u(a, x) = \varphi(a, x) + \sum_{k=1}^{+\infty} T^{\circ(k)}[\varphi](a, x).$$

Plugging (3.2) and computing the sum yields, after another application of the Markov property,

$$\begin{aligned} u(a, x) &= \mathbb{P}_a(\tau_x^+ \leq \mathbf{e} \wedge \tau_0^-) + \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} \mathbf{1}_{\{\tau_x^+ \geq \mathbf{e}\}} e^{-\beta(\tau_x^+ - \mathbf{e})} (2 - u(L_{\mathbf{e}}, x)) e^{\beta \int_0^{\mathbf{e}} (2 - u(L_r, x)) dr} \right] \\ &= \mathbb{P}_a(\tau_x^+ \leq \mathbf{e} \wedge \tau_0^-) + \int_0^{+\infty} \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} \mathbf{1}_{\{\tau_x^+ \geq s\}} e^{-\beta \tau_x^+} (2 - u(L_s, x)) e^{\beta \int_0^s (2 - u(L_r, x)) dr} \right] \beta ds \\ &= \mathbb{E}_a \left[ \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} e^{\beta \int_0^{\tau_x^+} (1 - u(L_r, x)) dr} \right], \end{aligned}$$

which ends the proof of Proposition 7.  $\square$

**Remark 8.** If we denote by  $T_x$  the first time when a particle of the branching process  $X$  hits the level  $x$ , we have  $\mathbb{P}_a(\mathbf{M} \geq x) = \mathbb{P}_a(\zeta > T_x)$ . As a consequence, it is not surprising that the representation result of Proposition 7 is similar to that of Proposition 4 but with  $t$  replaced by  $\tau_x^+$ .

**3.2. Study of the limit.** We now study the limit of  $u(a, x)$  as  $x \rightarrow +\infty$ . To this end, let us recall the classical Esscher transform. For  $c \geq -q_*$ , we denote by  $\mathbb{P}_a^{(c)}$  the probability defined by

$$\left. \frac{d\mathbb{P}_a^{(c)}}{d\mathbb{P}_a} \right|_{\mathcal{F}_t} = e^{\Phi(c)(Lt - a) - ct}. \quad (3.3)$$

Under  $\mathbb{P}_a^{(c)}$ , the process  $L$  is still a spectrally negative Lévy process starting from  $a$  but with characteristic function  $\Psi_c(\lambda) = \Psi(\lambda + \Phi(c)) - c$ . In particular, since  $\Psi'_c(0) = \Psi'(\Phi(c)) \geq 0$ , the process  $L$  no longer drifts a.s. towards  $-\infty$ . Denoting by  $W_c^{(q)}$  its scale function, we have

$$\int_0^{+\infty} e^{-\lambda x} W_c^{(q)}(x) dx = \frac{1}{\Psi(\lambda + \Phi(c)) + c - q}, \quad \lambda > \Phi(q - c) - \Phi(c),$$

which implies that for  $q \in \mathbb{C}$  and  $c \geq -q_*$ ,

$$W_c^{(q)}(x) = e^{\Phi(c)x} W^{(q-c)}(x), \quad x \geq 0.$$

Applying (3.3) to Proposition 4 and using the absence of positive jumps yields

$$e^{\Phi(-\beta)a} \mathbb{E}_a^{(-\beta)} \left[ e^{-\int_0^{\tau_x^+} u(L_r, x) dr} \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} \right] = \mathbb{E}_a \left[ e^{\Phi(-\beta)x + \beta \tau_x^+} e^{-\int_0^{\tau_x^+} u(L_r, x) dr} \mathbf{1}_{\{\tau_x^+ < \tau_0^-\}} \right].$$

Using the invariance by translation of Lévy processes, this is further equal to

$$e^{\Phi(-\beta)(a-x)} u(a, x) = \mathbb{E}_0^{(-\beta)} \left[ e^{-\beta \int_0^{\tau_x^+ - a} u(a + L_r, x) dr} \mathbf{1}_{\{\tau_{x-a}^+ < \tau_{-a}^-\}} \right]. \quad (3.4)$$

Note that under  $\mathbb{P}_a^{(-\beta)}$  the random variable  $\tau_{x-a}^+$  is a.s. finite. As a consequence, one may apply the time reversal result of Bertoin [2, Chapter VII, Th. 8] to obtain

$$\mathbb{E}_0^{(-\beta)} \left[ e^{-\beta \int_0^{\tau_{x-a}^+} u(a+L_r, x) dr} 1_{\{\tau_{x-a}^+ < \tau_{-a}^-\}} \right] = \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{g_x^-} u(x-L_r, x) dr} 1_{\{g_x^- < \tau_x^+\}} \right] \quad (3.5)$$

where  $g_x^-$  denotes the last passage time of  $L$  below the level  $x$ , i.e.  $g_x^- = \sup\{t \geq 0, L_t \leq x\}$ , and  $\mathbb{P}_a^{(-\beta)\uparrow}$  denotes the law of  $L$  (under  $\mathbb{P}_a^{(-\beta)}$ ) conditioned to stay positive, which is defined by :

$$\forall \Lambda_t \in \mathcal{F}_t, \quad \mathbb{P}_a^{(-\beta)\uparrow}(\Lambda_t) = \frac{1}{W_{-\beta}^{(0)}(a)} \mathbb{E}_a^{(-\beta)} \left[ W_{-\beta}^{(0)}(L_t) 1_{\{\tau_0^- > t\}} 1_{\Lambda_t} \right]. \quad (3.6)$$

Observe next that, by translation,  $u(x-a, x)$  corresponds to the probability that a branching process starting at 0, and where the particles are killed at the level  $a-x$ , reaches the level  $a$  before dying. Letting  $x \rightarrow +\infty$ , we deduce that  $u(x-a, x) \xrightarrow{x \rightarrow +\infty} \mathbb{P}_0(\mathcal{M} \geq a)$  where  $\mathcal{M}$  is the all-time maximum of a free spectrally negative branching Lévy process. Also, passing to the limit in the inequality

$$u(x-a, x) \leq \mathbb{E}_{x-a} \left[ 1_{\{\tau_x^+ < \tau_0^-\}} e^{\beta \tau_x^+} \right] = e^{-\Phi(-\beta)a} \frac{W_{-\beta}^{(0)}(x-a)}{W_{-\beta}^{(0)}(x)} \leq e^{-\Phi(-\beta)a}$$

we further obtain the bound

$$\mathbb{P}_0(\mathcal{M} \geq a) \leq e^{-\Phi(-\beta)a} < 1 \quad (3.7)$$

which shows that  $\mathcal{M}$  is a.s. finite when  $\beta \leq q_*$ . Applying the Markov property, we obtain

$$\begin{aligned} \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{g_x^-} u(x-L_r, x) dr} 1_{\left\{ \inf_{s \geq \tau_x^+} L_s > x-a \right\}} \right] \\ \geq \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{\tau_x^+} u(x-L_r, x) dr} \right] \mathbb{P}_x^{(-\beta)\uparrow} \left( \inf_{s \geq 0} L_s > x-a \right). \end{aligned} \quad (3.8)$$

To compute the last term, we shall come back to the absolute continuity formula (3.6) :

$$\mathbb{P}_x^{(-\beta)\uparrow} \left( \inf_{s \leq t} L_s > x-a \right) = \mathbb{P}_x^{(-\beta)\uparrow} (\tau_{x-a}^- > t) = \mathbb{E}_x^{(-\beta)} \left[ \frac{W_{-\beta}^{(0)}(L_t)}{W_{-\beta}^{(0)}(x)} 1_{\{t < \tau_0^-\}} 1_{\{t < \tau_{x-a}^-\}} \right].$$

By translation, this is further equal to

$$\mathbb{E}_a^{(-\beta)} \left[ \frac{W_{-\beta}^{(0)}(x-a+L_t)}{W_{-\beta}^{(0)}(x)} 1_{\{t < \tau_0^-\}} \right] = \frac{W_{-\beta}^{(0)}(a)}{W_{-\beta}^{(0)}(x)} \mathbb{E}_a^{(-\beta)\uparrow} \left[ \frac{W_{-\beta}^{(0)}(x-a+L_t)}{W_{-\beta}^{(0)}(L_t)} \right].$$

We shall now let  $t \rightarrow +\infty$  and use the fact that  $L$  goes to  $+\infty$  a.s. under  $\mathbb{P}_a^{(-\beta)\uparrow}$ . Recall from Hubalek & Kyprianou [5, Section 3] that the asymptotics of  $W_{-\beta}^{(0)}$  are given by

$$\begin{cases} \lim_{z \rightarrow +\infty} W_{-\beta}^{(0)}(z) = \frac{1}{\Psi'_{-\beta}(0^+)} = \frac{1}{\Psi'(\Phi(-\beta))} < +\infty & \text{if } \beta < q_*, \\ W_{-q_*}^{(0)}(z) \underset{z \rightarrow +\infty}{\sim} \frac{2}{\Psi''(\lambda_*)} z & \text{if } \beta = q_*. \end{cases} \quad (3.9)$$

As a consequence, we obtain in both cases

$$\mathbb{P}_x^{(-\beta)\uparrow} \left( \inf_{s \geq 0} L_s > x - a \right) = \frac{W_{-\beta}^{(0)}(a)}{W_{-\beta}^{(0)}(x)}.$$

Coming back to (3.8), we have thus obtained that

$$\liminf_{x \rightarrow +\infty} W_{-\beta}^{(0)}(x) e^{\Phi(-\beta)(a-x)} u(a, x) \geq W_{-\beta}^{(0)}(a) \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{+\infty} \mathbb{P}_0(\mathcal{M} \geq L_r) dr} \right] \quad (3.10)$$

which gives a lower bound, provided the expectation on the right-hand side is non null. To get an upper bound, we start back from (3.5) and take  $A > 0$ . On the one hand, we write :

$$\mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{g_{x-a}^-} u(x-L_r, x) dr} \mathbf{1}_{\left\{ \inf_{s \geq \tau_x} L_s > x-a \right\}} \mathbf{1}_{\{g_{x-a}^- \leq A\}} \right] \leq \mathbb{P}_0^{(-\beta)\uparrow} (g_{x-a}^- \leq A) \xrightarrow{x \rightarrow +\infty} 0$$

since  $L$  goes to  $+\infty$  a.s. under  $\mathbb{P}_a^{(-\beta)\uparrow}$ . On the other hand, we have

$$\mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{g_{x-a}^-} u(x-L_r, x) dr} \mathbf{1}_{\left\{ \inf_{s \geq \tau_x^+} L_s > x-a \right\}} \mathbf{1}_{\{g_{x-a}^- > A\}} \right] \leq \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^A u(x-L_r, x) dr} \mathbf{1}_{\left\{ \inf_{s \geq \tau_x^+} L_s > x-a \right\}} \right].$$

Decomposing further according as whether  $\tau_x^+ \leq A$  or  $\tau_x^+ > A$ , we deduce that

$$\begin{aligned} & \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{g_{x-a}^-} u(x-L_r, x) dr} \mathbf{1}_{\left\{ \inf_{s \geq \tau_x^+} L_s > x-a \right\}} \right] \\ & \leq \mathbb{P}_0^{(-\beta)\uparrow} (\tau_x^+ \leq A) + \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^A u(x-L_r, x) dr} \mathbf{1}_{\{A < \tau_x^+\}} \right] \mathbb{P}_x^{(-\beta)\uparrow} \left( \inf_{s \geq 0} L_s > x-a \right). \end{aligned}$$

The first term being smaller than

$$\mathbb{P}_0^{(-\beta)\uparrow} (\tau_x^+ \leq A) = \mathbb{P}_0^{(-\beta)} (g_{x-a}^- \leq A) \leq \mathbb{P}_0^{(-\beta)} (\tau_x^+ \leq A) \leq e^{\beta A} \mathbb{E}_0^{(-\beta)} \left[ e^{-\beta \tau_x^+} \right] = e^{\beta A} e^{(\Phi(-\beta) - \Phi(0))x}$$

we conclude by letting  $x \rightarrow +\infty$  that

$$\limsup_{x \rightarrow +\infty} W_{-\beta}^{(0)}(x) e^{\Phi(-\beta)(a-x)} u(a, x) \leq W_{-\beta}^{(0)}(a) \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^A \mathbb{P}_0(\mathcal{M} \geq L_r) dr} \right]. \quad (3.11)$$

Letting finally  $A \uparrow +\infty$  and gathering (3.10) and (3.11), we have thus proven that

$$\lim_{x \rightarrow +\infty} W_{-\beta}^{(0)}(x) e^{\Phi(-\beta)(a-x)} u(a, x) = W_{-\beta}^{(0)}(a) \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^{+\infty} \mathbb{P}_0(\mathcal{M} \geq L_r) dr} \right]$$

and it remains to check that the expectation on the right-hand side is not null. Using the bound (3.7), it is sufficient to prove that  $\int_0^{+\infty} e^{-\Phi(-\beta)L_r} dr < +\infty$  a.s. under  $\mathbb{P}_0^{(-\beta)\uparrow}$ . Applying the Fubini-Tonelli theorem and Bertoin [2, Chapter VII.3, Cor.16], we have

$$\mathbb{E}_0^{(-\beta)\uparrow} \left[ \int_0^{+\infty} e^{-\Phi(-\beta)L_r} dr \right] = \int_0^{+\infty} \int_0^{+\infty} e^{-\Phi(-\beta)x} \frac{x W_{-\beta}^{(0)}(x)}{t} \mathbb{P}_0^{(-\beta)}(L_r \in dx) dr.$$

Using the Kendall identity  $t\mathbb{P}_0^{(-\beta)}(\tau_x^+ \in dt)dx = x\mathbb{P}_0^{(-\beta)}(L_t \in dx)dt$ , this is further equal to

$$\int_0^{+\infty} \int_0^{+\infty} e^{-\Phi(-\beta)x} W_{-\beta}^{(0)}(x) \mathbb{P}_0^{(-\beta)}(\tau_x^+ \in dt) dx = \int_0^{+\infty} e^{-\Phi(-\beta)x} W_{-\beta}^{(0)}(x) dx < +\infty$$

since  $\tau_x^+$  is a.s. finite under  $\mathbb{P}_0^{(-\beta)}$  and the asymptotics of  $W_{-\beta}^{(0)}$  are given by (3.9).  $\square$

**3.3. Proof of Corollary 3.** Let us set  $m(a) = \mathbb{P}_0(\mathcal{M} > a)$ . Passing to the limit in Proposition 7 and applying the Esscher transform, we deduce that

$$m(a) = \mathbb{E}_0 \left[ 1_{\{\tau_a^+ < +\infty\}} e^{\beta \int_0^{\tau_a^+} (1-m(a-L_s)) ds} \right] = e^{-\Phi(-\beta)a} \mathbb{E}_0^{(-\beta)} \left[ 1_{\{\tau_a^+ < +\infty\}} e^{-\beta \int_0^{\tau_a^+} m(a-L_s) ds} \right].$$

Since  $\mathbb{P}_0^{(-\beta)}(\tau_a^+ < +\infty) = 1$ , the time reversal result [2, Chapter VII, Th. 8] yields

$$m(a) = e^{-\Phi(-\beta)a} \mathbb{E}_0^{(-\beta)\uparrow} \left[ e^{-\beta \int_0^a m(L_s) ds} \right].$$

The result then follows by letting  $a \rightarrow +\infty$ , since as before, the expectation on the right-hand side will converge towards a strictly positive constant.  $\square$

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