Parametric amplification of metric fluctuations through a bouncing phase

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We clarify the properties of the behavior of classical cosmological perturbations when the Universe experiences a bounce. This is done in the simplest possible case for which gravity is described by general relativity and the matter content has a single component, namely, a scalar field in a closed geometry. We show in particular that the spectrum of scalar perturbations can be affected by the bounce in a way that may depend on the wave number, even in the large scale limit. This may have important implications for string motivated models of the early Universe.

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I. INTRODUCTION

Although the idea that the Universe could have experienced a bounce in its remote past is old [1,2], it has recently come under new scrutiny [3–6] with the advent of string motivated scenarios of the pre big bang kind [7,8]. The main reason for this renewed interest is the fact that the most popular extensions of the standard model of high energy physics, such as string or M theory, when applied to cosmology, i.e., in a four dimensional time dependent background, can lead to solutions with bouncing scale factors (see, e.g. Refs. [3,6,7] and references therein).

A crucial property to decide whether these new models can be turned into realistic alternatives to the inflationary paradigm which, so far, has been so successful is the behavior of cosmological perturbations around these bouncing backgrounds. In particular, an important test is to calculate the evolution of the power spectrum of primordial fluctuations through the bounce in order to see whether it can be made close to scale invariance, i.e. if there is any possibility, given the prebounce era, to get a Harrison-Zel'dovich power spectrum.

From a technical point of view, the previous question is a nontrivial problem. Simple models, based on general relativity with flat spatial sections [8], lead to the existence of a curvature singularity at the bounce itself and therefore do not seem to represent viable physical models. In addition, it is difficult to understand how meaningful a perturbative scheme around a singular solution would be (see however Ref. [9]), so we shall assume that the question of the calculation of the cosmological perturbations cannot be addressed in this way (see Ref. [10] for more detailed discussions). In fact, it is believed that, in the vicinity of the bounce, string corrections become important [7]. Typically, these corrections add to the gravity sector terms such as R^2, $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$, where $R_{\mu\nu\rho\lambda}$ is the Riemann tensor and R the curvature scalar [11], and higher order terms in the curvature. The effect of these terms is, except in some specific instances [9], to smooth out the singularity [12] (see also Ref. [7]); this is, from a physical point of view, satisfactory and expected. Unfortunately, one can show that the process of adding up more and more high curvature corrections does not lead to convergence towards a single solution, i.e. this solution explicitly depends on the choice of the stringy corrections [13]. This means that, at each order, the bouncing scale factor looks completely different from the scale factor obtained at the previous order. Nevertheless, at a fixed order in the string corrections, one can in principle compute how the perturbations propagate through the bounce. The main disadvantage of the procedure is that it renders the computation extremely complicated and only numerical calculations are available to make the problem tractable.

An important point to be noticed is that, as already mentioned above, most models assume that the spatial sections are flat all the time whereas, at the bounce, the curvature term is expected to play a crucial role. Therefore, it seems that for a bouncing universe, one cannot just throw away the curvature term because it does not play a significant role as it is the case for an inflationary universe. In fact, in that regard, the situation is the opposite of inflation: during the final stages of inflation, one can safely assume flat spatial sections because the three-curvature is getting more and more negligible as time passes, whereas even though the curvature may be negligible either in the remote past or in the future of the bounce, it has almost certainly no reason to be so in general.

A way out of the previous difficulties, which would permit to undertake a tractable analytical calculation of the power spectrum, is the following. Far from the bounce, one usually considers the situation for which the curvature is small, even though the implementation of this particular point may not be in itself a trivial task. In this case, one can consider that, during the contracting and expanding phases, the spatial sections are essentially flat so that the well known results stemming from the theory of cosmological perturbations can be straightforwardly applied. Then, the main question becomes the effect of the bounce itself on the prebounce power spectrum. Technically, this problem can be formulated as follows [14]. Before the bounce, the perturbations are characterized by two modes, a dominant (denoted by D) and a sub-dominant (S) one. With k representing the comoving wave number of a given Fourier mode, we write the k dependence of these modes as $D_-(k)$ and $S_-(k)$. In the same manner, after the bounce, one decomposes the perturbation as $D_+(k)$ and $S_+(k)$. The effect of the bounce is

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then entirely encoded into the form of the transition matrix $T(k)$ defined by

$$\begin{pmatrix} D_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} D_- \\ S_- \end{pmatrix}. \tag{1}$$

The mode of interest is of course the dominant mode in the expanding phase, $D_+(k)=T_{11}(k)D_-(k)+T_{12}(k)S_-(k)$. This equation corresponds to a general "k-mode mixing," i.e. the dominant mode after the bounce is a general linear combination of the dominant and sub-dominant modes before the bounce.[1]

A priori, various different situations can occur: the dominant mode in the contracting phase could acquire a scale invariant spectrum which is not conveyed to the dominant mode in the expanding phase because it turns out that $T_{11}=0$ and $T_{12}\neq 0$ ("k-mode inversion," the scale invariant piece is passed to the "wrong mode" in the expanding phase); this is for instance what occurs if one applies the usual Israel junction conditions, known to apply for other cosmological transitions, at the bounce point [15].

Another possibility is that the dominant mode in the contracting phase be scale invariant but that this property is lost through the bounce due to a nontrivial k dependence of the coefficient T_{11}. Note that the opposite situation may also occur, for which the spectrum is initially not scale invariant but is turned into it because of a nontrivial k dependence of the transition matrix. In fact, the common view concerning these last possibilities is that, for scales of astrophysical interest today, the bounce, lasting a short time, is expected to have no noticeable effect on those large scales [7]. This is sometimes argued to come from general arguments such as "causality," a point which is discussed thoroughly in Ref. [16]. For instance, this is a basic assumption in the perturbation spectrum calculations in the pre big bang scenario [7]. Technically, this means that the transfer matrix is assumed not to depend on k [14]. Within this framework, the goal reduces to finding situations for which a scale invariant spectrum is produced in the contracting phase, and to ensure that this spectrum is passed to the dominant mode in the expanding phase, i.e. to insure that the matching conditions at the bounce do not imply a k-mode inversion.

The present article aims at examining whether the assumption that the transfer matrix is k-independent is generically valid or not. For this purpose, we need to specify a class of models where bouncing solutions are possible and which allows simple analytical treatment of the perturbations through the bounce. We choose general relativity, positive curvature spatial section (see the remarks above), and describe the matter content by a scalar field; a similar strategy was used in Ref. [5]. We do not assume anything relative to what happens away from the bounce, and in particular one could envisage that there the curvature is negligible; note that, as we show below, this implies the existence of a new phase. Therefore our closed geometry bounce can be viewed as an example of a transition connecting the contraction phase to the expanding phase with flat spatial sections, as already considered in the literature.

This article is organized as follows. In the following section, we set the precise model and derive the basic equations both for the background and the perturbations. We then discuss how one can model a bounce in this framework and derive an explicit form for the potential of the scalar part of the classical perturbations (Bardeen potential) whose properties we then examine in details. This leads us to the main calculation of this article, namely that of the transfer matrix of Eq. (1). We show that this matrix depends on k in a nontrivial way provided that the null energy condition (NEC) is very close to being violated at the bounce. This illustrates, by means of a specific example, that the general argument according to which the limited but nonvanishing bounce duration could not affect the spectrum of long (i.e. longer that the duration itself) wavelength modes, is incorrect. We conclude by discussing this result, also showing that in the case under consideration, the propagation of gravitational waves (tensor modes) is qualitatively different of that of scalar modes since the former are never affected by the bounce.

II. BASIC EQUATIONS

We assume that the background model is given by a Friedmann-Lemaître-Robertson-Walker (FLRW) universe, i.e.

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\right]. \tag{2}$$

In this equation, the constant parameter $\mathcal{K}$ can always be rescaled such that $\mathcal{K}=0,\pm 1$ and describes the curvature of the spatial sections. The time η is the conformal time related to the cosmic time by $\mathrm{d}t=a(\eta)\mathrm{d}\eta$. The matter is described by a homogeneous scalar field $\varphi(\eta)$ and the corresponding energy density and pressure respectively read

$$\rho = \frac{\varphi'^2}{2a^2} + V(\varphi), \quad p = \frac{\varphi'^2}{2a^2} - V(\varphi). \tag{3}$$

A prime denotes a derivative with respect to conformal time. The function $V(\varphi)$ represents the potential of the scalar field. Einstein equations relate the scale factor to the energy density and pressure of the scalar field according to

$$\frac{3}{a^2}(\mathcal{H}^2+\mathcal{K}) = \kappa\rho, \tag{4}$$

$$-\frac{1}{a^2}(2\mathcal{H}'+\mathcal{H}^2+\mathcal{K}) = \kappa p, \tag{5}$$

where we have defined $\mathcal{H}\equiv a'/a$ and $\kappa\equiv 8\pi/m_{\mathrm{Pl}}^2$, m_{Pl} being the Planck mass. The quantity $\rho+p$ is then given by

[1]We have introduced the notation "k-mode mixing" in order to emphasize that this mode mixing is valid for a fixed Fourier mode and should not be confused with the mode mixing coming from nonlinearities which involves modes with different wave numbers.

$$\kappa(\rho+p)=\frac{2}{a^2}\mathcal{H}^2\Gamma=\frac{2}{a^2}(\mathcal{H}^2-\mathcal{H}'+\mathcal{K})=\kappa\frac{\varphi'^2}{a^2}. \quad (6)$$

From the above equation, we see that the function $\Gamma(\eta)$ is defined by $\Gamma\equiv 1-\mathcal{H}'/\mathcal{H}^2+\mathcal{K}/\mathcal{H}^2$. It is directly related to the equation of state parameter $\omega\equiv p/\rho$ by the following relation $\omega=(2\Gamma/3)(1+\mathcal{K}/\mathcal{H}^2)^{-1}-1$. The function $\Gamma(\eta)$ reduces to a constant for constant equation of state and is zero in the particular case of the de Sitter manifold. At the bounce, the Hubble parameter vanishes, $\mathcal{H}=0$, while $\mathcal{H}'>0$, and therefore the only way to preserve the null energy condition $\rho+p\geqslant 0$ is to have $\mathcal{K}>0$. This is why, in this article, we restrict ourselves to the case $\mathcal{K}>0$, i.e. the spatial sections are 3-spheres. Let us also notice that, being given a bouncing scale factor $a(\eta)$, it is sufficient to check that $\Gamma\geqslant 0$ at all times in order for the scale factor to be a solution of the Einstein equations with a single real scalar field.

A universe with closed spatial sections is characterized by two fundamental lengths. The first length is the Hubble length defined by $\ell_{\mathrm{H}}\equiv a^2/a'=a/\bar{a}\equiv H^{-1}$ (a dot denoting a derivative with respect to cosmic time t) and the second one is the curvature radius, $\ell_{\mathrm{C}}\equiv a/\sqrt{|\mathcal{K}|}$. The flat limit is recovered when $\ell_{\mathrm{C}}\gg\ell_{\mathrm{H}}$ as revealed by the equation $|1-\Omega|=\ell_{\mathrm{H}}^2/\ell_{\mathrm{C}}^2$, where Ω is the ratio of the total energy density ρ to the critical energy density. When it comes to numerical applications, let us recall that one can safely assume the preferred value [17] $H_0=100h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ with $h=0.71^{+0.04}_{-0.03}$, leading to a Hubble distance scale now of $\sim 3000h^{-1}$ Mpc $\sim 4.2\pm 0.2$ Gpc. Moreover, with $\Omega_{\mathrm{now}}=1.02\pm 0.02$, one has a curvature length, namely the scale factor as measured now [18], of order $a_0\gtrsim 15h^{-1}$ Gpc (with $\mathcal{K}=1$), the limit coming from the maximum allowed value for Ω_{now} at one σ level.

At the perturbed level, and in the presence of density perturbations only, the metric takes the following form

$$\mathrm{d}s^2=a^2(\eta)\{-(1+2\phi)\mathrm{d}\eta^2+2\partial_i B\mathrm{d}\eta\mathrm{d}x^i+[(1-2\psi)\gamma^{(3)}_{ij}+2\nabla_i\partial_j E]\mathrm{d}x^i\mathrm{d}x^j\}, \quad (7)$$

where $\gamma^{(3)}_{ij}$ is the metric of the spatial sections and the symbol ∇_i denotes the covariant derivative associated with the three-dimensional metric. The eigenfunctions $f_n(x^i)$ of the Laplace-Beltrami operator on the spatial sections satisfy the equation

$$\Delta f_n=-n(n+2)f_n, \quad (8)$$

where n is an integer. Note at this point that it is because of our normalization with a dimensionful scale factor $a(\eta)$, and hence dimensionless coordinates (η,x^i), implying a dimensionless operator itself, that the eigenvalues of Δ are dimensionless integer numbers; with a different convention, i.e. with a dimensionless, one would have $[\Delta]=L^{-2}$ and an extra factor ℓ_{C}^{-2} would appear in the right hand side of Eq. (8).

The modes $n=0$, corresponding to a homogeneous deformation, and $n=1$, being nothing but a global motion of the center of the 3-sphere, are pure gauge modes [19]: we will accordingly consider only values of n such that $n>1$. In fact, for the relevant cosmological parameters discussed above, one finds that the values of n corresponding to characteristic distance scales of cosmological interest now, namely $10^{-2}h^{-1}$ Mpc$\lesssim D_{\mathrm{cosm}}\lesssim 10^3h^{-1}$ Mpc, range between 30 and 3×10^6 for the largest possible value of the total density now. For a reasonable value of $\Omega_{\mathrm{now}}\sim 1.01$, we find that n is between 60 and 6×10^6.

The scalar perturbations are described by the four functions, ϕ, B, ψ and E and, from them, it is possible to construct two gauge-invariant quantities, called the Bardeen potentials, and defined by [20,21]

$$\Phi\equiv\phi+\frac{1}{a}[a(B-E')]', \quad \Psi\equiv\psi-\frac{a'}{a}(B-E'). \quad (9)$$

For simple form of matter with no anisotropic stress (this is the case for a scalar field), we have $\Phi=\Psi$. Notice that the form of the Bardeen potentials is the same whatever the curvature of the spatial sections is. This is related to the fact that, even if $\mathcal{K}>0$ (or $\mathcal{K}<0$), the FLRW metric remains conformally flat and the components of the perturbed Weyl tensor remain unchanged. Then, Stewart lemma guarantees that the Bardeen potentials are still defined by the same equations [22].

For the matter sector, the scalar field is written as $\varphi+\delta\varphi(\eta,x^i)$ where $\delta\varphi(\eta,x^i)$ represents the inhomogeneous fluctuations. These fluctuations can be described by the gauge invariant quantity $\delta\varphi^{(\mathrm{gi})}\equiv\delta\varphi+\varphi'(B-E')$.

The full set of Einstein equations can be written in terms of the gauge invariant quantities Φ and $\delta\varphi^{(\mathrm{gi})}$ only. Combining these equations permits to derive a master equation for the Bardeen potential (for $\varphi'\neq 0$) which reads

$$\Phi''+2\left(\mathcal{H}-\frac{\varphi''}{\varphi'}\right)\Phi'+\left[n(n+2)+2\left(\mathcal{H}'-\mathcal{H}\frac{\varphi''}{\varphi'}-2\mathcal{K}\right)\right]\Phi=0. \quad (10)$$

This equation can be cast into a more convenient form. For this purpose, one introduces a new gauge-invariant quantity, u, related to the Bardeen potential Φ by

$$\Phi\equiv\frac{\kappa}{2}(\rho+p)^{1/2}u=\frac{\sqrt{3\kappa}}{2}\frac{\mathcal{H}}{a^2\theta}u, \quad (11)$$

where the function θ is defined by

$$\theta\equiv\frac{1}{a}\left(\frac{\rho}{\rho+p}\right)^{1/2}\left(1-\frac{3\mathcal{K}}{\kappa\rho a^2}\right)^{1/2}=\frac{1}{a}\left(\frac{3}{2\Gamma}\right)^{1/2}. \quad (12)$$

Then, the equation of motion for the quantity u takes the form

$$u''+\left[n(n+2)-\frac{\theta''}{\theta}-3\mathcal{K}(1-c_{\mathrm{S}}^2)\right]u=0. \quad (13)$$

In the above equation, one has

$$c_{\rm S}^2 \equiv \frac{p'}{\rho'} = -\frac{1}{3}\left(1+2\frac{\varphi''}{\mathcal{H}\varphi'}\right), \tag{14}$$

for the scalar field, when use is made of the Klein-Gordon equation

$$\varphi'' + 2\mathcal{H}\varphi' + a^2\frac{dV(\varphi)}{d\varphi} = 0. \tag{15}$$

The quantity $c_{\rm S}$ of Eq. (14) can, in some regimes, be interpreted as the sound velocity. Let us now see how the flat case is recovered. The term θ''/θ is of order $\mathcal{H}^2$, namely $\theta''/\theta \sim a^2/\ell_{\rm H}^2$. This is a rigorous statement if the scale factor is a power law of the conformal time, which explains the usual confusion between the potential and the Hubble scale. However, this identification suffers from important exceptions, particularly relevant in the present context; see the discussion in Sec. V C. Then the above equation can be re-written as

$$u'' + a^2\left[\frac{4\pi^2}{\lambda^2} - \frac{1}{\ell_{\rm H}^2} - \frac{3}{\ell_{\rm C}^2}(1-c_{\rm S}^2)\right]u \simeq 0, \tag{16}$$

where we have used that the physical wavelength of a mode is $\lambda(\eta) = 2\pi a(\eta)/\sqrt{n(n+2)}$. In the limit $\ell_{\rm C} \gg \ell_{\rm H}$, the last term of the equation becomes negligible. Then the equation of motion for u reduces to $u'' + [n(n+2) - \theta''/\theta]u = 0$ where now θ denotes the function defined previously in Eq. (12) but without the term proportional to $\mathcal{K}$ (one can also show that, in the limit considered here, this term becomes negligible). Therefore, we have recovered the standard equation, valid for $\mathcal{K}=0$.

Note that in the flat limit for which $\mathcal{K}=0$, the mode number $n(n+2)$ that appears in Eq. (13) appears to be a large number, and even more so when $\Omega \to 1$, as is the case in the usual cosmological calculations based on a period of inflation for which the approximation $k^2 \ll 1$ is often done, permitting an expansion in powers of k^2. This is not inconsistent though, because after N e-folds of inflation have happened, in a closed situation for instance, one expects $\Omega - 1 \sim {\rm e}^{-2N}$, with $N > 55$ to set the scales, and hence a gigantic value for the scale factor normalization $a_0 = H_0^{-1}(\Omega - 1)^{-1/2}$. However, one then assumes, rightly, that the universe is almost flat, and chooses in general a different normalization for the scale factor, namely $a_0 = H_0^{-1}$, which is many orders of magnitude below what it ought to be. The relevant wave numbers, seen as eigenvalues of the Laplace-Beltrami operator, then must scale correspondingly, i.e. they are reduced by the amount

$$n(n+2) \to k^2 = n(n+2)(\Omega - 1), \tag{17}$$

which is, indeed, much smaller than unity in any inflationary scenario.

The equation of motion for the quantity u, Eq. (13), has the traditional form of a parametric oscillator equation, i.e. of a ''time-independent'' Schrödinger equation. However, in the case $\mathcal{K} \neq 0$, the effective potential cannot be written as the second time derivative of some function (θ in the flat case) over the same function because of the presence of the term $-3\mathcal{K}(1-c_{\rm S}^2)$ in the time-dependent frequency. This has for consequence that, on ''super-horizon'' scales where the term $n(n+2)$ is negligible, the solutions are not easily found contrary to the flat case where they are just $u=\theta$ and $u = \theta\int {\rm d}\tau/\theta^2$.

So far, we have discussed the quantity u which is, up to some background functions, the Bardeen potential. In the framework of cosmological perturbations, there exists another important variable, usually denoted v, that we now consider. This quantity is important because its flat case equivalent naturally appears when one studies cosmological perturbations of quantum-mechanical origin. In other words, this quantity is interesting for setting up physically well-motivated initial conditions whenever the curvature is negligible. Its definition reads

$$v = \frac{-a}{\sqrt{1-3\mathcal{K}\dfrac{1-c_{\rm S}^2}{n(n+2)}}}\left[\delta\varphi^{(\rm gi)} + \frac{\varphi'}{\mathcal{H}}\Phi - \frac{\mathcal{K}\varphi'}{\mathcal{H}^3\Gamma}\Phi\right]. \tag{18}$$

For $\mathcal{K}=0$, it reduces to the well-known definition. The presence of the factor $n(n+2)$ in the definition above suggests however that this variable, in the $\mathcal{K}\neq 0$ case, is not the canonical field that should be quantized to get initial conditions. This quantity involves $\delta\varphi^{(\rm gi)}$ and Φ. Since they are related by the perturbed Einstein equations, there is in fact only one degree of freedom as expected. The equation of motion for v reads

$$v'' + \left[n(n+2) - \frac{z''}{z} - 3\mathcal{K}(1-c_{\rm S}^2)\right]v = 0, \tag{19}$$

where the quantity z is defined by

$$z \equiv \frac{a\varphi'}{\mathcal{H}\sqrt{1-3\mathcal{K}\dfrac{1-c_{\rm S}^2}{n(n+2)}}}. \tag{20}$$

This equation was obtained previously in Ref. [23]. The same remark as for the u equation applies: in the $\mathcal{K}\neq 0$ the effective potential is not only z''/z (as in the flat case) but is corrected by the $+3\mathcal{K}(1-c_{\rm S}^2)$ term. In addition, the effective potential now depends on the wave number n through the quantity z. This has to be contrasted with the effective potential for u, θ''/θ, which is n-independent.

Having defined the various quantities needed to study the evolution of cosmological perturbations through a bouncing phase, we now turn to the description of the bounce itself.

III. MODELING THE BOUNCE

In this section, we define precisely the behavior of the scale factor during the bouncing epoch, then discuss its relation with the following eras of standard cosmology and de-

rive the relevant perturbation potentials.

A. The de Sitter–like bounce

Once the background is fixed, the effective potentials for the quantities u and v are completely specified. In this section, our aim is therefore to discuss how one can model the scale factor of a bouncing universe. At this point, one should notice the differences (and similarities) with inflation. In an inflationary universe, the behavior of the scale factor is known: essentially, this is $a \propto |\eta|^{-1}$, i.e. the de Sitter phase. However, one can also treat slightly more complicated backgrounds by means of an expansion around this de Sitter solution. This expansion is characterized by the so-called slow-roll parameters [24], which are constrained to be small. The de Sitter solution also exists in the bounce case [25] and, as we shall see, it can be used in much the same way. However, contrary to the inflation case, there is no fundamental reason why the background equation of state should be close to vacuum. Despite this fact, one can nevertheless expand around the $\mathcal{K}=1$ de Sitter spacetime and similarly define parameters which control the departure from it. Obviously, those parameters are not subject to tight constraints, and in particular are not required to be small.

For $\mathcal{K}>0$, the de Sitter solution [25] corresponds to the scale factor $a(t)=a_0\cosh(\omega t)$, which is expressed as a function of the cosmic time t, with $\omega=1/a_0$. More general solutions are obtained by relaxing this last constraint and considering a general value for ω. These de Sitter–like solutions are the ones we shall be concerned with in what follows: our expansion will be based on these solutions. In terms of conformal time, one can integrate the relation $a\,\mathrm{d}\eta=\mathrm{d}t$ to get

$$a(\eta)=a_0\sqrt{1+\tan^2\left(\frac{\eta}{\eta_0}\right)}, \tag{21}$$

where the conformal time is bounded within the range $-\pi/2<\eta/\eta_0<\pi/2$ and the conformal time duration η_0 is related to the de Sitter coefficient ω through $\eta_0=(a_0\omega)^{-1}$ [the solution (21) is shown in Fig. 1].

In order to understand the dynamics of this solution, one needs to obtain the evolution of the scalar field. It can be integrated straightforwardly with the scale factor (21): from Eqs. (4) and (5), one obtains

$$\varphi=\varphi_0+\sqrt{\frac{2\Upsilon}{\kappa}}\left(\eta+\frac{\pi}{2}\eta_0\right), \tag{22}$$

where we have set $\varphi\to\varphi_0$ as the cosmic time $t\to-\infty$, i.e. as $\eta/\eta_0\to-\pi/2$, and we also have defined a parameter

$$\Upsilon\equiv 1-\frac{1}{\eta_0^2} \tag{23}$$

for further convenience. We shall keep this definition later on for more general bounces than the quasi–de Sitter ones.

It should be noted that the parameter Υ, in the case of de Sitter like expansion (21) is, according to the definition (6), $\Upsilon_{\mathrm{dS}}=\mathcal{H}^2\Gamma$, which is proportional to $\rho+p$. As a result, the null energy condition at the bounce can only be satisfied provided $\Upsilon>0$, i.e. if $|\eta_0|\geqslant 1$: indeed, one has

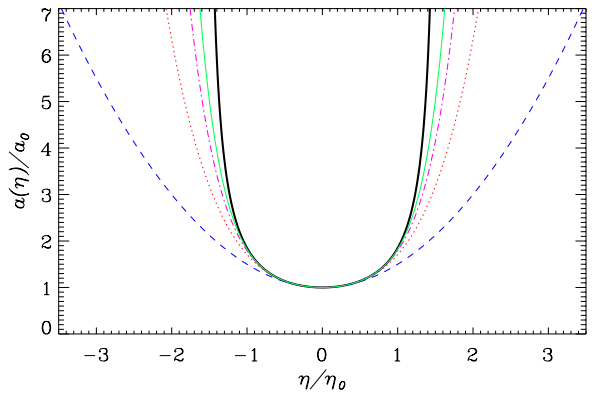

FIG. 1. Scale factors as functions of the conformal time η corresponding to the de Sitter–like solution [Eq. (21), full line] and its various levels of approximations stemming from Eq. (30), namely up to quadratic (dashed), quartic (dotted), sixth (dot-dashed) and eighth power (dot-dot-dashed). The last two approximations, although clearly better from the point of view of the scale factor, do not lead to any new qualitative information as far as the evolution of the perturbations is concerned.

$$\lim_{\eta\to 0}(\rho+p)=2\frac{\Upsilon}{a_0^2}, \tag{24}$$

a relation which we shall use in the rest of the paper to define Υ in a solution-independent way. As emphasized before, the case $\eta_0=1$ corresponds to a constant scalar field potential and to an equation of state $\rho=-p$ and is thus the exact counterpart of the inflationary de Sitter solution. The scalar field time derivative is now simply obtained as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{a\,\mathrm{d}\eta}=\frac{1}{a_0}\left\{\frac{2\Upsilon}{\kappa\left[1+\tan^2\left(\frac{\eta}{\eta_0}\right)\right]}\right\}^{1/2}. \tag{25}$$

Both the field and its time derivative are displayed in Fig. 2

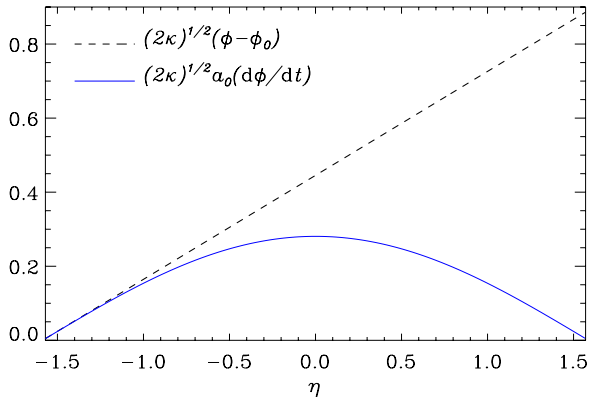

FIG. 2. Behavior of the scalar field and its coordinate time derivative as functions of the conformal time η (varying between $-\pi/2$ and $\pi/2$ for the overall evolution of the Universe) for the de Sitter–like solution with $\eta_0=1.01$.

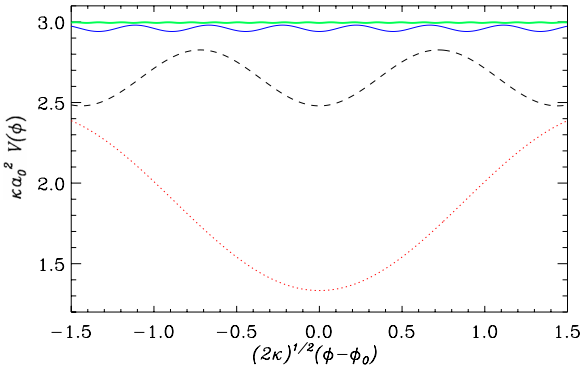

FIG. 3. The shape (27) of the potential for the scalar field φ (in units of the Planck mass $\kappa^{-1/2}=m_{\rm Pl}/\sqrt{8\pi}$) for different values of the bounce characteristic conformal time η_0. The full lines are respectively for $\eta_0=1.001$ (above) and $\eta_0=1.01$ (below), the dashed line corresponds to $\eta_0=1.1$, and the dotted line is for $\eta_0=1.5$. In the strict de Sitter limit $\eta_0\to 1$, the potential goes to the constant value $V(\varphi)=3/(\kappa a_0^2)$, which explains why the $\eta_0=1.001$ seems almost constant as it oscillates with a very small amplitude around its central value $[(3-1/\eta_0^2)$ in these units].

for a case having $\eta_0\neq 1$ as functions of the conformal time.

It is now a simple matter to derive the corresponding potential for the scalar field which solves Einstein equations (4) and (5). It reads

$$\kappa a_0^2 V(\varphi)=\frac{\mathcal{H}'+2(\mathcal{H}^2+\mathcal{K})}{(a/a_0)^2}, \tag{26}$$

i.e., with the solution (21) above,

$$\kappa a_0^2 V(\varphi)=\frac{3}{\eta_0^2}+2\Upsilon\,\sin^2\left[\frac{\sqrt{2\kappa}}{\eta_0}\Upsilon^{-1/2}(\varphi-\varphi_0)\right], \tag{27}$$

and it is displayed in Fig. 3 with a specific choice of initial conditions for the field. From Figs. 2 and 3, one sees that the universe starts at either a maximum or a minimum of the potential, in both cases with a nonvanishing amount of kinetic energy in the scalar field $\varphi'(-\pi\eta_0/2)=\sqrt{2\Upsilon/\kappa}$.

One remarkable property of the above model is that the effective potential for density perturbations remains very simple even if $\eta_0\neq 1$. Indeed, assuming the de Sitter like solution (21) and plugging it into the form (12) yields

$$\theta_{\rm dS}=\frac{1}{a_0}\sqrt{\frac{3}{2(\eta_0^2-1)}}\sin\left(\frac{\eta}{\eta_0}\right), \tag{28}$$

which, together with $c_{\rm s}^2=-1/3$ [this stems from Eq. (14) with the solution (21) and the scalar field (22)] leads to

$$V_u^{(\rm dS)}=4-\frac{1}{\eta_0^2}, \tag{29}$$

or, in other words, the potential for the variable u does not depend on time for the de Sitter–like solution. Besides, the maximum value achievable by this potential is given for the limiting case $\eta_0\to 1$, and it is $V_u^{(\rm dS,max)}=3$: this potential can only interact with the modes $n=0$ and $n=1$, which we already mentioned are gauge modes. This was to be expected since the exact de Sitter solution, in this bouncing situation as in the more usual inflationary scenario, does not amplify scalar perturbations by any amount. We believe that this is what happens in Fig. 4 of Ref. [5] in which the Bardeen potential mode $n=10$ is seen to oscillate while passing through a de Sitter–like bounce, reflecting the nondomination of the potential at this point.

B. General bouncing scale factor

We now assume that the universe experiences a regular bounce at the time $\eta=0$. This means there exists a particular function $a(\eta)$ which can always be Taylor expanded in the vicinity of $\eta=0$. Since we are interested in understanding the behavior of the perturbations through the bounce, and because the effective potentials for density perturbations involve derivatives of the scale factor only up to the fourth order, a description of the scale factor up to η^4 only is sufficient. We therefore set

$$a(\eta)=a_0\left[1+\frac{1}{2}\left(\frac{\eta}{\eta_0}\right)^2+\delta\left(\frac{\eta}{\eta_0}\right)^3+\frac{5}{24}(1+\xi)\left(\frac{\eta}{\eta_0}\right)^4\right], \tag{30}$$

which defines the parameters a_0, the radius of the universe at the bounce, η_0, the typical conformal time scale of the bounce, δ and ξ. They control the magnitude of each term of the expansion.

For the scale factor (30) to be a solution of Einstein equations with a scalar field as matter content, the function $a(\eta)$ must be chosen such that η_0 is greater than unity, which is only a necessary condition as discussed below. The parameters a_0 and η_0 also provide the tangent de Sitter–like solution (21), whereas δ and ξ measure the deviation with respect to this de Sitter–like solution. Equation (30) represents a double expansion, both around $\eta=0$ and around the de Sitter–like solution discussed in the previous section since $\delta=\xi=0$ exactly corresponds to the small η expansion of the scale factor (21); this explains, among others, the factor 5/24 in this equation. The parameters δ and ξ are in a certain sense similar to the traditional slow-roll parameters. Both the de Sitter form and its approximations are shown in Fig. 1 as functions of the conformal time.

We now discuss how the general expansion (30) can be related to an underlying particle physics model. The scale factor is entirely specified once the scalar field potential $V(\varphi)$ and the initial conditions, for instance the values of the scalar field and its first derivative at the bounce: φ_0, φ_0', have been chosen. This means that there exists a relation between these last quantities and the parameters a_0, η_0, δ and ξ characterizing the expansion (30). We now establish what this relation is. This can be done easily by solving the Einstein equations (4) and (5) in the vicinity of the bounce. In practice, we insert the Taylor expansion of the scalar field,

$$\varphi(\eta)=\varphi_0+\varphi_0'\eta+\frac{1}{2}\varphi_0''\eta^2+\mathcal{O}(\eta^3), \tag{31}$$

and of the scale factor, Eq. (30), in Eqs. (4) and (5) and identify, order by order, the various terms appearing in the resulting expressions. To zeroth order, this gives

$$a_0^2=\frac{6-\kappa\varphi_0'^2}{2\kappa V(\varphi_0)}, \quad \eta_0^2=\left(1-\frac{\kappa}{2}\varphi_0'^2\right)^{-1}. \tag{32}$$

The last relation can also be re-written as

$$\Upsilon=\frac{\kappa}{2}\varphi_0'^2, \tag{33}$$

as expected from Eqs. (6) and (24). We see that the magnitude of the scalar field conformal time gradient at the bounce determines the value of the parameter Υ. Typically, one expects $\Upsilon\ll 1$ since the order of magnitude of the scalar field and its derivatives should be such that $\varphi_0'\ll m_{\rm Pl}$ in order for the field theory to make sense. If the velocity of the field vanishes at the bounce, then $\Upsilon=0$.

To first order, the Einstein equations yield

$$\kappa\varphi_0'\left(\varphi_0''+a_0^2\left.\frac{\mathrm{d}V}{\mathrm{d}\varphi}\right|_{\varphi_0}\right)=0, \tag{34}$$

$$\kappa\varphi_0'\left(\varphi_0''-a_0^2\left.\frac{\mathrm{d}V}{\mathrm{d}\varphi}\right|_{\varphi_0}\right)+\frac{12\delta}{\eta_0^3}=0. \tag{35}$$

In the following, we will be mainly interested in the situation where the bounce is symmetric, that is, we shall demand that $\delta=0$. Then, there are two ways of satisfying the Einstein equations. Either the kinetic energy vanishes at the bounce or $\varphi_0'\neq 0$ but then $\varphi_0''=0$ and $\mathrm{d}V/\mathrm{d}\varphi|_{\varphi_0}=0$. This means that the bounce occurs at the minimum of the scalar field potential. This also implies that, in this case, the minimum of the potential cannot vanish, $V(\varphi_0)\neq 0$, see Eq. (32). If, for instance, the potential is given by $V(\varphi)\propto\varphi^n$, as is the case for instance of the model studied in Ref. [5], then the only way to have a symmetric bounce is to satisfy the condition $\varphi_0'=0$ at the bounce and, as a consequence, one necessarily has $\Upsilon=0$. In the following, we will be mainly interested in the second situation, i.e. $\varphi_0'\neq 0$, since we will show that amplitude of the spectrum is controlled by the parameter Υ. In this case, going to the next order allows us to determine what the parameter ξ is. The result reads

$$\xi=\frac{1}{5(2-\kappa\varphi_0'^2)^2\kappa V^2(\varphi_0)}\left\{(2-\kappa\varphi_0'^2)V_0[(6-\kappa\varphi_0'^2)^2+6(-2+\kappa\varphi_0'^2)\kappa V(\varphi_0)]+(6-\kappa\varphi_0'^2)^2\varphi_0'^2\left.\frac{\mathrm{d}^2V}{\mathrm{d}\varphi^2}\right|_{\varphi_0}\right\}. \tag{36}$$

Let us notice that this parameter depends on the second order derivative of the potential at the bounce.

Assuming a symmetric bounce for now on, i.e. setting $\delta=0$, some restrictions can be put on the numerical value of ξ. They stem from the fact that we demand $a(\eta)$ to be positive in the range $-\eta_0\leqslant\eta\leqslant\eta_0$ and to describe a bounce, i.e. $a'>0$ for $0<\eta<\eta_0$. This latter condition turns out to be more stringent and implies $\xi>-11/5$. Moreover, if we further require that the scale factor (30) be solution of Einstein equation sourced by a single scalar field, we see from Eq. (6) that $\mathcal{H}^2\Gamma$ must be positive. Around the bounce, this is

$$\mathcal{H}^2\Gamma\simeq\Upsilon-\frac{5}{2}\xi(1-2\Upsilon)\left(\frac{\eta}{\eta_0}\right)^2+\mathcal{O}\left[\left(\frac{\eta}{\eta_0}\right)^4\right], \tag{37}$$

which will be positive definite in a small but finite neighborhood of $\eta=0$ provided $\xi<0$ in the limit $\Upsilon\to 0$ we will be concerned with. Combining both constraints, we arrive at

$$-\frac{11}{5}<\xi<0. \tag{38}$$

The approximation method we discuss later does not allow to consider very small values for ξ, so that in practice, we shall use $-11/5<\xi\lesssim -0.1$.

We now assume the fiducial expansion (30) for the bounce through which we want to propagate the perturbations. Let us however first examine the connection of this bounce to the standard cosmological epochs of radiation and matter domination.

C. Connecting the bounce to standard cosmology

In this section, we study how the bounce that we described previously can be connected to an epoch of the standard hot big bang model. In particular, we study the connection with a radiation dominated era. In this case, the scale factor can be written as

$$a(\eta)=a_{\rm r}\sin(\eta-\eta_{\rm r}), \tag{39}$$

where $a_{\rm r}$ and $\eta_{\rm r}$ are two parameters to be fixed with the help of the matching conditions. We match this scale factor to the bouncing scale factor given by Eq. (30), using the junction conditions, known to be valid even in the curved spatial section case, as derived in Ref. [26], namely $[a]=[a']=0$. The matching is performed at $\eta=\eta_{\rm j}$ such that $\eta_{\rm j}\ll\eta_0$ in order for our quartic approximation of the scale factor to still be meaningful. The matching conditions imply that the Hubble parameter at the matching time is given by

$$\mathcal{H}(\eta_{\rm j})=\frac{x}{\eta_0}\frac{1+\frac{5}{6}(1+\xi)x^2}{1+\frac{1}{2}x^2+\frac{5}{24}(1+\xi)x^4}, \tag{40}$$

where $x\equiv\eta_{\rm j}/\eta_0\ll 1$. From the above formula, one sees that it is not possible to connect the bounce to an epoch where the curvature is negligible, provided the null energy condition, which demands $\eta_0\geqslant 1$ [see discussion around Eq. (24)], is

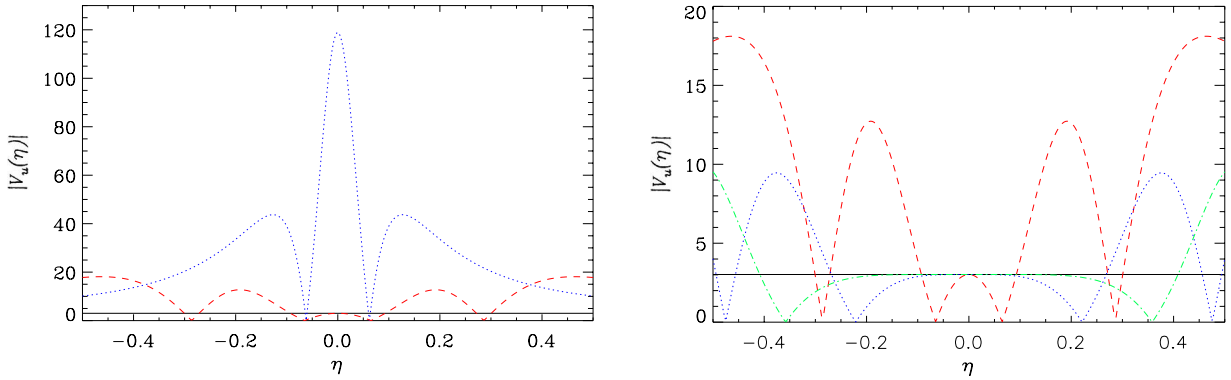

FIG. 4. Absolute value of the effective potential $V_u(\eta)$ for the perturbation variable $u(\eta)$ for the de Sitter–like case (full line on both panels), for which it is constant and for the various approximation levels (from quadratic to eighth power of the scale factor). The left panel shows the potential as obtained by using the quadratic (dotted line) and quartic (dashed) expansions of the scale factor only, whereas the right panel presents the situation when quartic (dashed), sixth (dotted) and eighth (dot-dashed) terms are used. It is clear that the quadratic approximation is qualitatively wrong and cannot be used to describe a de Sitter bounce. The value $\eta_0=1.01$ has been used to derive these plots.

still satisfied. As a consequence, this implies that $\mathcal{H}(\eta_j)$ cannot be large in comparison to $\mathcal{K}=1$; in fact, since $\eta_0\simeq 1$ and $x\ll 1$, $\mathcal{H}^2$ is expected to be negligibly small compared to unity right after the bounce. This means that one necessarily connects the bounce to a regime where the curvature is important or, in other words, in a region where the sine function appearing in the scale factor (39) cannot be approximated by the first term of the Taylor expansion, $a(\eta)\simeq a_r(\eta-\eta_r)$. The only way to avoid this conclusion would be to violate the null energy condition, as already noticed in Ref. [4] and to have a small η_0 but then it would have been useless to consider the case $\mathcal{K}=1$ for modeling the bounce since this was done precisely in order to satisfy this condition. Therefore, we conclude that between the bounce and the standard hot big bang, another phase must necessary occur whose main effect will be to drive $\mathcal{H}$ to sufficiently large values. This is usually the role played by a phase of inflation.

With the general framework thus clarified, let us turn to the evolution of the scalar gravitational perturbations through the bounce by means of evaluating the effective potential for the variable u related with the Bardeen potential through Eq. (11). We discuss the potential for the variable v in the discussion Sec. V A below.

D. The potential $V_u(\eta)$

The effective potential for the variable u in the de Sitter–like solution is, according to Eq. (29), constant in time. This is however very specific to this particular solution, as any displacement away from it immediately leads to a different form of the potential. This is illustrated in Fig. 4 which shows the relative accuracy of the expansion (30) around the de Sitter–like solution (21). It is also clear from the figure that the expansion (30), if pushed to sufficiently high orders in η, gives back the correct constant value over a large range of conformal times. Let us now turn to the more general bounce case of Eq. (30).

Arbitrary values for the parameter ξ restricted to the range of interest discussed above lead to the generic shape illustrated in Fig. 5. The calculation of the effective potential is extremely complicated even with the quartic approximation of the scale factor. Even if it can be done in full generality since, for a scale factor given by Eq. (30), the potential $V_u(\eta)$ reads

$$V_u(\eta)\equiv\frac{\theta''}{\theta}+3\mathcal{K}(1-c_S^2)=\frac{P_{24}(\eta)}{Q_{24}(\eta)}, \tag{41}$$

where $P_{24}(\eta)$ and $Q_{24}(\eta)$ are two polynomials of order 24, in practice the calculation is not tractable. However, since in practice we always have $\eta/\eta_0\ll 1$, only the first monomials are important. One can check that the following approximation

$$V_u^{(\mathrm{app})}(\eta)=3\frac{c_0+c_2\eta^2}{d_0+d_2\eta^2+d_4\eta^4}, \tag{42}$$

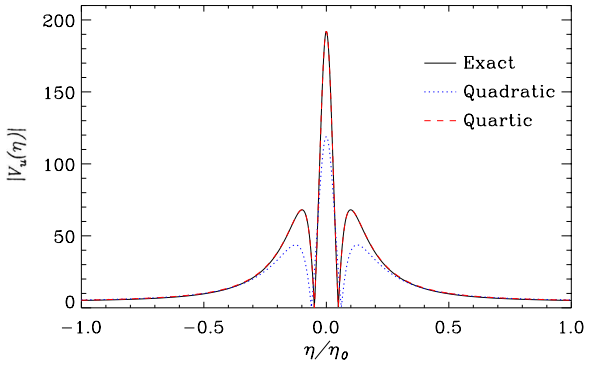

FIG. 5. Absolute value of the potential $V_u(\eta)$ as a function of rescaled conformal time η/η_0 for $\eta_0=1.01$ as derived using either the assumption that the scale factor behaves as a square root, i.e., $a=a_0\sqrt{1+(\eta/\eta_0)^2}$, (full line) or Eq. (30) up to quadratic (dotted line) and quartic order with $\delta=0$ and $\xi=-2/5$ (dashed line). The quartic approximation is extremely close to the exact solution, exemplifying its accuracy, while the quadratic approximation appears to be at best qualitatively correct.

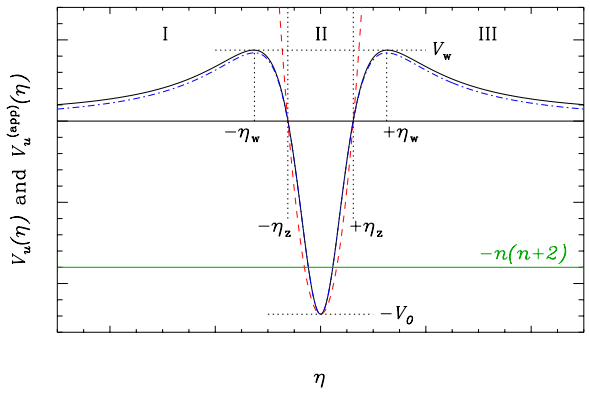

FIG. 6. The potential $V_u(\eta)$ [full line, Eq. (41)] and its approximation stemming from Eq. (42) $V_u^{(app)}(\eta)$ (dot-dashed line) in full details with the same parameters as in Fig 5. Also shown is the parabolic approximation that will be used in Sec. IV A (dashed line), the various conformal times involved in the calculations in the text, as well as a pictorial definition of the regions I, II and III used for the matching of the perturbations also in Sec. IV A. The mode $n(n+2)$ interacts, in this example, only with the central part of the potential.

is extremely good, see Fig. 6. In this expression, we have only kept the first two monomials at the numerator and the first three ones at the denominator. The coefficients c_i's and d_i's can be written as

$$c_0 = 2\,\eta_0^2(\,\eta_0^2-1)(2-10\,\eta_0^2+8\,\eta_0^4+5\,\xi), \tag{43}$$

$$c_2 = 49+48\,\eta_0^6+55\xi+50\xi^2-18\,\eta_0^4(6+5\,\xi) + \eta_0^2(11+35\xi), \tag{44}$$

$$d_0 = 12\,\eta_0^4(\,\eta_0^2-1)^2 \tag{45}$$

$$d_2 = 12\,\eta_0^2(\,\eta_0^2-1)(-3+3\,\eta_0^2-5\,\xi) \tag{46}$$

$$d_4 = -1+15\,\eta_0^4(4+\xi)+5\,\xi(16+15\xi)-\eta_0^2(59+95\xi). \tag{47}$$

Equipped with Eq. (42), we can now compute the height and the position of the central peak and of the wings. Let us start with the central peak. The absolute value of $V_u(\eta)$ at $\eta=0$ is given by

$$V_0 = \frac{3c_0}{d_0} = \frac{2-10\,\eta_0^2+8\,\eta_0^4+5\,\xi}{2\,\eta_0^2(\,\eta_0^2-1)}. \tag{48}$$

The most important property of the above formula is that it diverges as $\eta_0 \to 1$. It is shown in Ref. [16] that this property also holds in the case $\delta \neq 0$, and is therefore generic, i.e. not restricted to symmetric bounces. We have seen previously that the values of n of astrophysical interest are such that $n \gg 1$. Therefore, a necessary condition for the bounce to affect the spectrum of the fluctuations is that η_0 be close to one. As already discussed, the physical interpretation is that one must be very close to a violation of the null energy condition. In this case, it is more convenient to work with the variable Υ introduced in Eq. (23). In practice, Υ must be a tiny number in order to get a modification of the spectrum. We have seen that this is to be expected since Υ is the square of the ratio of the scalar field conformal time gradient at the bounce to the Planck mass. The amplitude of Υ controls the maximum value of n below which the perturbation modes will be affected by the bounce. The crucial point is that for $\Upsilon \ll 1$, large scales, having cosmological and astrophysical relevance, can be modified as they evolve through the bounce.

Assuming for now on that $\Upsilon \ll 1$, it is sufficient to Taylor expand everything in terms of this parameter to get an accurate approximation. For V_0, one gets

$$V_0 = -\frac{5\,\xi}{2\,\Upsilon} - (3-5\,\xi) + \mathcal{O}(\Upsilon). \tag{49}$$

Another interesting quantity is the time η_z for which $V_u = 0$, see Fig. 6. This time is given by $\eta_z^2 = -c_0/c_2$ which leads to

$$\eta_z = \sqrt{-\frac{\Upsilon}{5\,\xi}} + \mathcal{O}(\Upsilon^{3/2}). \tag{50}$$

The above equations means that, in the limit $\Upsilon \to 0$, the width of the potential goes to zero while its height increases unboundedly. Finally, let us describe the wings of the potential. The position of the wings can be derived from the condition $V_u' = 0$ $(\eta \neq 0)$. This gives

$$\eta_w^2 = -\frac{1}{c_2 d_4}[c_0 d_4 \pm \sqrt{c_2 d_4 (c_2 d_0 - c_0 d_2) + c_0^2 d_4^2}], \tag{51}$$

and the Taylor expansion in Υ reads

$$\eta_w = \sqrt{-\frac{4\,\Upsilon}{5\,\xi}} + \mathcal{O}(\Upsilon^{3/2}). \tag{52}$$

One sees that $\eta_w \simeq 2\,\eta_z$ at first order in Υ. Therefore, η_w also goes to zero when Υ tends to zero. The height of the wings is just given by $V_u(\eta_w)$ and can be expressed as

$$V_w = -\frac{5\,\xi}{6\,\Upsilon} + \left(3+\frac{5}{3}\,\xi\right) + \mathcal{O}(\Upsilon). \tag{53}$$

The height of the wing also diverges as one approaches the violation of the null energy condition and, at first order in Υ, one has $V_0/V_w \simeq 3$. This concludes the description of the perturbation potential with which we now examine the fate of the perturbations themselves.

IV. CALCULATION OF THE TRANSFER MATRIX

The purpose of this section, which is also our main result, is to show that the transfer matrix **T** of Eq. (1) may depend on the wave number n in a way which we derive. We found that two completely different and independent methods, one based on a piecewise expansion of the potential and the other assuming the potential to behave mathematically as a distribution rather than a simple function in the limit $\Upsilon \to 0$, lead

to comparable results, both for the final spectrum itself and its magnitude. We examine these methods in turn.

A. Method I: Piecewise solution approach

When trying to evaluate the transfer matrix of Eq. (1) with the potential $V_u(\eta)$ derived in the previous section, one immediately faces a difficulty, namely that unfortunately, even with the simple form given by Eq. (42), the equation of motion of the variable u is not integrable analytically. However, one can find piecewise solutions. For a mode which interacts only with the central peak of the barrier, i.e. which is above the wings, the potential is essentially zero for $|\eta| > \eta_z$. This corresponds to regions I and III in Fig. 6. In the central region, region II in Fig. 6 corresponding to $|\eta| < \eta_z$, we model the bounce by a parabola with a minimum at $-V_0$ and which vanishes at $\eta = \pm\eta_z$. To summarize, our piecewise potential is given by

$$V_u(\eta) = \begin{cases} 0, & \eta < -\eta_z, \\ -V_0\left[1-\left(\dfrac{\eta}{\eta_z}\right)^2\right], & -\eta_z < \eta < \eta_z, \\ 0, & \eta > \eta_z. \end{cases} \quad (54)$$

In each region, the function u is the sum of two modes and can be expressed as

$$u_i(n,\eta) = A_i(n) f_i(n,\eta) + B_i(n) g_i(n,\eta), \quad i = \mathrm{I,II,III}. \quad (55)$$

Before and after the interaction with the barrier, the solution are plane waves,

$$f_{\mathrm{I,III}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}, \quad g_{\mathrm{I,III}}(\eta) = \frac{1}{\sqrt{2k}} e^{ik\eta}, \quad (56)$$

where we have introduced the quantity $k \equiv \sqrt{n(n+2)}$ and the normalization is chosen such as to simplify further calculations (unit Wronskian). In region II, one has an even and an odd mode, i.e.

$$f_{\mathrm{II}}(-\eta) = f_{\mathrm{II}}(\eta), \quad g_{\mathrm{II}}(-\eta) = -g_{\mathrm{II}}(\eta). \quad (57)$$

For the moment, we do not specify what $f_{\mathrm{II}}(\eta)$ and $g_{\mathrm{II}}(\eta)$ are since we are trying to keep the calculation as general as possible (for example, we could imagine other parametrization of the potential in the central region for which f_{II} and g_{II} would be different). Our goal is to predict what A_{III} and B_{III} are. For this purpose, we match u and its derivative u' at $\eta = \pm\eta_z$. Straightforward calculations lead to

$$\begin{bmatrix} A_{\mathrm{III}} \\ B_{\mathrm{III}} \end{bmatrix} = \frac{1}{2ikW(n)} e^{2ik\eta_z} \begin{bmatrix} -f_{\mathrm{II}}' + ikf_{\mathrm{II}} & -g_{\mathrm{II}}' + ikg_{\mathrm{II}} \\ e^{-2ik\eta_z}(f_{\mathrm{II}}' + ikf_{\mathrm{II}}) & e^{-2ik\eta_z}(g_{\mathrm{II}}' + ikg_{\mathrm{II}}) \end{bmatrix} \cdot \begin{bmatrix} g_{\mathrm{II}}' - ikg_{\mathrm{II}} & e^{-2ik\eta_z}(g_{\mathrm{II}}' + ikg_{\mathrm{II}}) \\ f_{\mathrm{II}}' - ikf_{\mathrm{II}} & e^{-2ik\eta_z}(f_{\mathrm{II}}' + ikf_{\mathrm{II}}) \end{bmatrix} \begin{bmatrix} A_{\mathrm{I}} \\ B_{\mathrm{I}} \end{bmatrix}, \quad (58)$$

where $W(n)$ is the Wronskian of the function f_{II} and g_{II}, namely $W(n) = f_{\mathrm{II}} g_{\mathrm{II}}' - f_{\mathrm{II}}' g_{\mathrm{II}}$. In the previous expressions, all the functions are expressed at the point $\eta = \eta_z$ (we have used the parity properties of the function f_{II} and g_{II}). The above matrix is general and is parametrized by only four numbers: f_{II}, g_{II}, f_{II}' and g_{II}'. Any model permitting to calculate what these numbers are allows us to estimate the transfer matrix on the bounce given above.

We now use the parabolic model introduced before. If we perform the following change of variable, $\eta \equiv \sqrt{\eta_z/(2\sqrt{V_0})}\,x$, then the equation of motion for u in region II takes the form

$$\frac{d^2u}{dx^2} - \left(\frac{x^2}{4} + \alpha\right)u = 0, \quad (59)$$

where the parameter α is given by

$$\alpha = -\frac{1}{2}\eta_z\sqrt{V_0}\left[1 + \frac{n(n+2)}{V_0}\right]. \quad (60)$$

Equation (59) can be solved exactly in terms of cylinder parabolic functions [27]. Since the potential is symmetric, the solutions can always be chosen to be even and odd. The explicit expression of the even and odd solutions are respectively

$$f_{\mathrm{II}}(\eta) = e^{-\sqrt{V_0}\eta^2/(2\eta_z)}\,{}_1F_1\left(\frac{\alpha}{2} + \frac{1}{4}; \frac{1}{2}; \frac{\sqrt{V_0}}{\eta_z}\eta^2\right), \quad (61)$$

$$g_{\mathrm{II}}(\eta) = \eta e^{-\sqrt{V_0}\eta^2/(2\eta_z)}\,{}_1F_1\left(\frac{\alpha}{2} + \frac{3}{4}; \frac{3}{2}; \frac{\sqrt{V_0}}{\eta_z}\eta^2\right), \quad (62)$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function. As already mentioned previously, these functions and their derivatives must be evaluated at $\eta = \eta_z$ and then expanded in the parameter Υ. The first step is to calculate the parameter α. This gives

$$\alpha = -\frac{1}{2\sqrt{2}} + \frac{\{-61 + 5[-53 + 8n(n+2)]\xi\}}{200\sqrt{2}\,\xi^2}\Upsilon + \mathcal{O}(\Upsilon^2). \quad (63)$$

Using this expansion and that of V_0 and η_z, one obtains at first order in Υ

$$f_{\rm II}(\eta_z)=\mathrm{e}^{-1/(2\sqrt{2})}{}_1F_1\left(\frac{2-\sqrt{2}}{8},\frac{1}{2},\frac{1}{\sqrt{2}}\right)+\mathcal{O}(\Upsilon^{1/2})$$
$$\simeq 0.798+\mathcal{O}(\Upsilon^{1/2}), \tag{64}$$

$$g_{\rm II}(\eta_z)=\frac{1}{\sqrt{-5\xi}}\mathrm{e}^{-1/(2\sqrt{2})}{}_1F_1\left(\frac{6-\sqrt{2}}{8},\frac{3}{2},\frac{1}{\sqrt{2}}\right)\Upsilon^{1/2}+\mathcal{O}(\Upsilon^{3/2})$$
$$\simeq \frac{0.422}{\sqrt{-\xi}}\Upsilon^{1/2}+\mathcal{O}(\Upsilon^{3/2}), \tag{65}$$

$$f'_{\rm II}(\eta_z)=\frac{\sqrt{-5\xi}}{2}\mathrm{e}^{-1/(2\sqrt{2})}\left[-\sqrt{2}\,{}_1F_1\left(\frac{2-\sqrt{2}}{8},\frac{1}{2},\frac{1}{\sqrt{2}}\right)+(\sqrt{2}-1)\,{}_1F_1\left(\frac{10-\sqrt{2}}{8},\frac{3}{2},\frac{1}{\sqrt{2}}\right)\right]\frac{1}{\Upsilon^{1/2}}+\mathcal{O}(\Upsilon^{1/2}), \tag{66}$$
$$\simeq -\frac{0.711\sqrt{-\xi}}{\Upsilon^{1/2}}+\mathcal{O}(\Upsilon^{1/2}), \tag{67}$$

$$g'_{\rm II}(\eta_z)=\frac{1}{6}\mathrm{e}^{-1/(2\sqrt{2})}\left[(6-3\sqrt{2})\,{}_1F_1\left(\frac{6-\sqrt{2}}{8},\frac{3}{2},\frac{1}{\sqrt{2}}\right)+(3\sqrt{2}-1)\,{}_1F_1\left(\frac{14-\sqrt{2}}{8},\frac{5}{2},\frac{1}{\sqrt{2}}\right)\right]+\mathcal{O}(\Upsilon^{1/2}), \tag{68}$$
$$\simeq 0.878+\mathcal{O}(\Upsilon^{1/2}). \tag{69}$$

The expression for the derivatives can be easily recovered if one uses the following expression giving the derivative of a Kummer hypergeometric function, ${}_1F'_1(\alpha,\beta,z)=(\alpha/\beta)\,{}_1F_1(\alpha+1,\beta+1,z)$, where a prime in this context means a derivative with respect to the argument z of the hypergeometric function.

The next step consists in inserting these relations into the general form of the transfer matrix and then in expanding the resulting expression in the parameter Υ. The result reads

$$T_u\simeq -0.624i\sqrt{\frac{-\xi}{n(n+2)}}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\frac{1}{\Upsilon^{1/2}}. \tag{70}$$

Several remarks are in order at this point. First, the formula above applies only for the modes actually interacting with the potential, namely those having $n(n+2)\leqslant V_0$, otherwise, T_u is obviously the identity. Note also that, in the former case, the transfer matrix is n-dependent. This means that the bounce affects the spectrum and therefore disproves *a priori* any general argument stating that the spectrum should propagate through the bounce without being modified [7,14]. Besides, we see that the amplitude is divergent as Υ goes to zero. However, one should remember that we are not interested in the spectrum of u itself but rather in the spectrum of the Bardeen potential Φ. The relation between Φ and u, Eq. (11), together with Eq. (24), leads to the remarkable result that the terms in Υ cancel out exactly and that the resulting spectrum is Υ-independent, and thus perfectly finite even in the $\Upsilon\to 0$ limit. Finally, the ξ dependence of the overall amplitude is also predicted by this calculation. As expected, there is no net effect in the limit $\xi\to 0$ at which the bounce is effectively de Sitter and thus can amplify no amount of perturbation. In this last case, the calculation leading to Eq. (70) is not accurate enough and should be done at a higher order in Υ since the leading order vanishes; one should then find that the transition matrix is essentially the identity (de Sitter) plus some correction vanishing in the limit $\Upsilon\to 0$.

B. Method II: Distributional approach

We show in this section that the previous result can be understood in very simple terms and that the result of the previous section can be reproduced by a back-of-the-envelope calculation. The crucial observation is that the height of the potential V_u diverges as Υ goes to zero while its width shrinks to zero. This suggests that there is something like a Dirac δ-function at play.[2] To study this point we calculate the integral of the potential. One gets

$$\int_{-\infty}^{+\infty}[V_u(\tau)-4]\mathrm{d}\tau\simeq\int_{-\eta_0}^{+\eta_0}V_u^{(\rm app)}(\tau)\mathrm{d}\tau \tag{71}$$
$$=\left(\frac{-5\pi^2\xi}{8\Upsilon}\right)^{1/2}+\mathcal{O}(\Upsilon^0). \tag{72}$$

Thus the potential can be re-written as

$$V_u(\eta)=-C_\Upsilon\Delta_\Upsilon(\eta), \tag{73}$$

where the constant C_Υ is given by $C_\Upsilon\equiv[-5\pi^2\xi/(8\Upsilon)]^{1/2}$ and where the function $\Delta_\Upsilon(\eta)$ is a representation of the Dirac δ-function,i.e.

$$\lim_{\Upsilon\to 0}\Delta_\Upsilon(\eta)=\delta(\eta). \tag{74}$$

In a certain sense, the potential $V_u(\eta)$ possesses divergences ''worst'' than a Dirac δ-function. The equation of motion of the quantity u can now be written as

$$u''+[n(n+2)+C_\Upsilon\delta(\eta)]u=0, \tag{75}$$

[2]Note that even though we send $\Upsilon\to 0$ in this section, this is merely a computational artifact allowing an easy calculation of the effect. The true value of Υ must be nonvanishing, although tiny, so the calculation of this section is accurate only for those modes interacting with the potential. Therefore, the presence of the Dirac distribution in no way implies the existence of a singular behavior either of the potential $V_u(\eta)$ or of the modes $u(\eta)$ themselves as long as $\Upsilon\neq 0$.

i.e. a well-known equation in the context of quantum mechanics. The matching conditions are $[u]=0$ and $[u']=-C_{\Upsilon}u(0)$, the last one coming from an integration of the equation of motion across a thin shell around $\eta=0$. This reduces to

$$A_{\rm III}+B_{\rm III}=A_{\rm I}+B_{\rm I}, \tag{76}$$

$$A_{\rm III}-B_{\rm III}=A_{\rm I}-B_{\rm I}-\frac{C_{\Upsilon}}{i\sqrt{n(n+2)}}(A_{\rm I}+B_{\rm I}). \tag{77}$$

Straightforward algebraic manipulations lead to the following transfer matrix, under the assumption that the second term of the last equation dominates over the first since $C_{\Upsilon}\to\infty$ as $\Upsilon\to 0$,

$$T_u=-i\sqrt{\frac{-5\pi^2\xi}{32n(n+2)}}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\frac{1}{\Upsilon^{1/2}}. \tag{78}$$

It is interesting to compare Eq. (70) with Eq. (78). The numerical coefficient in the above equation is $\pi\sqrt{5}/(4\sqrt{2})\simeq 1.242$, to be compared with the coefficient 0.624 found in Eq. (70). The difference is approximatively a factor 1/2 in the amplitude. This difference can be interpreted in the following way. When the matrix transfer is computed using the matching procedure, one uses the parabola formula for the potential and one neglects the wings of the potential. The area of the central part of the potential is given by

$$\int_{-\eta_z}^{+\eta_z}V_u(\tau)\mathrm{d}\tau=\sqrt{-\frac{100\xi}{45\Upsilon}}=\frac{4\sqrt{2}}{3\pi}\int_{-\eta_0}^{+\eta_0}V_u^{(\rm app)}(\tau)\mathrm{d}\tau. \tag{79}$$

We see that there is factor $4\sqrt{2}/(3\pi)$ between the area below the central part and the area below the whole potential including the wings. Since the matching procedure is sensitive to the central part only whereas the calculation of the Dirac δ-function is sensitive to the whole potential, we therefore expect a factor $4\sqrt{2}/(3\pi)$ between the corresponding two amplitudes. We have $4\sqrt{2}/(3\pi)\simeq 0.600$ and hence we recover approximatively the factor 1/2 mentioned above. The correct amplitude is the one given by the Dirac δ-function calculation and is $\simeq 1.25$.

V. DISCUSSION

We now complete the description of the propagation of perturbations through a general relativistic bounce by some considerations regarding the variables u and v, the spectrum of tensor modes and a comparison with other known transitions in cosmology.

A. *u* versus *v*

An interesting issue, debated at length in the literature [3–6], is how the variables u and v behave as they go through the bounce. As a first step towards understanding what is the variable that is the most useful, let us construct the potentials for both, as in Fig. 7. It is clear from this figure

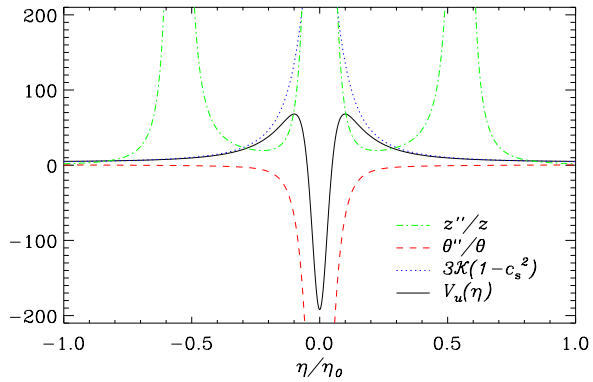

FIG. 7. Construction of the potentials V_u and V_v for the perturbation variables u and v in the special case of the square root form for the scale factor as in Fig. 5 [or Eq. (30) with $\delta=0$ and $\xi=-2/5$, the corresponding curves being visually undistinguishable] with $\eta_0=1.01$. According to Eqs. (13) and (19), the potentials depend on three possible terms, namely θ''/θ, $3\mathcal{K}(1-c_{\rm S}^2)$ and z''/z, respectively plotted as the dashed, dotted, and dot-dashed curves. The potential for u is also shown as the full line. The pole at $\eta=0$ in either θ''/θ and $3\mathcal{K}(1-c_{\rm S}^2)$ appears with opposite sign but is otherwise the same, so there is an exact compensation, so that the full potential is everywhere well-behaved. This is clearly not possible for the potential V_v, since there are more poles in z''/z than there are in $3\mathcal{K}(1-c_{\rm S}^2)$, so no compensation can occur at these points, but also the pole at $\eta=0$ appear with the same sign; the potential for the function v follows z''/z, up to small corrections and was therefore not plotted here.

that the terms appearing in the potential for the variable v, namely

$$V_v(\eta)\equiv\frac{z''}{z}+3\mathcal{K}(1-c_{\rm S}^2), \tag{80}$$

never compensate each others, as was found to be the case for V_u of Eq. (41), so the resulting potential is divergent at some points which, furthermore, depend on the wavelength index n. This provides a first hint that v is not the correct variable to work with, and indicates that as one approaches the bounce, or as the curvature becomes non-negligible, v ceases to be the good quantum variable (see e.g. Ref. [21]).

In the present context, it is easy to show that u and v are related by the following relation:

$$v=-\frac{1}{\sqrt{1-3\mathcal{K}\dfrac{1-c_{\rm S}^2}{n(n+2)}}}\left[u'+\frac{(a\sqrt{\Gamma})'}{a\sqrt{\Gamma}}u\right]. \tag{81}$$

We know from the previous considerations that u is continuous and that u' may have a finite jump at $\eta=0$ provided Υ is small enough. From the above equation, we conclude that the variable v possesses divergences during the bounce. These divergences are given by the zeros of the argument of the square root at the denominator of the previous equation. In other words, v diverges when

$$c_s^2 = 1 - \frac{n(n+2)}{3}, \tag{82}$$

as found in Fig. 7.

Some remarks are in order at that point. First, an interesting feature is that $v=0$ at the bounce (hence is regular) and that the divergences occur before and after the bounce but not at the bounce itself even though its potential actually diverges at this point. This is because the effective velocity of sound diverges at the bounce. Secondly, the time at which v divergences is n-dependent as can clearly be seen from Eq. (82). Thirdly, the physical interpretation of this divergence is subtle. If v had the usual interpretation (i.e. the variable that is canonically quantized), the divergence would clearly be a problem. Roughly speaking, this would mean explosive particle creations and, as a consequence that there is a back-reaction problem. More seriously, this divergence would be at odd with the fact that the Bardeen potential remains finite and small. As already discussed below Eq. (18), there are reasons to believe that, in the case $\mathcal{K}=1$, the variable v introduced before is not the variable that appears in the action for cosmological perturbations. This last variable should remain finite during the bounce.

B. Density perturbations versus gravitational waves

The evolution of the tensorial modes of perturbations $\mu \equiv ah$, where h is, roughly speaking, the amplitude of the gravitational wave, stems from the relation [26,28]

$$\mu'' + \left[n(n+2) - \mathcal{K} - \frac{a''}{a} \right] \mu = 0, \tag{83}$$

i.e. an equation similar to that valid for the scalar modes but with a potential simply given by $V_h = \mathcal{K} + a''/a$. Within the framework of our bouncing solution, this is

$$V_h = 1 + \frac{1}{\eta_0^2} \frac{1 + \frac{5}{2}(1+\xi)\left(\frac{\eta}{\eta_0}\right)^2}{1 + \frac{1}{2}\left(\frac{\eta}{\eta_0}\right)^2 + \frac{5}{24}(1+\xi)\left(\frac{\eta}{\eta_0}\right)^4}, \tag{84}$$

which can be simply analyzed as follows.

As direct calculation reveals, the potential V_h of Eq. (84) has either a single maximum located at $\eta=0$ if the expansion parameter $\xi \leqslant -4/5$, or a minimum at $\eta=0$ and two maxima at the points $\eta_{\max}$ given by

$$\eta_{\max}^2 = \frac{2}{5}\eta_0^2 \frac{\sqrt{5(5+6\xi)} - 1}{1+\xi}, \tag{85}$$

provided $-6/5 \leqslant \xi \leqslant -4/5$. In the latter case, the maximum value attained by the gravitational wave potential is

$$V_h^{\max} = 1 + \frac{1}{\eta_0^2} \begin{cases} \dfrac{15(1+\xi)}{2+\sqrt{5(5+6\xi)}}, & \text{if} -\dfrac{6}{5} \leqslant \xi \leqslant -\dfrac{4}{5}, \\ 1, & \text{otherwise.} \end{cases} \tag{86}$$

Since $\xi<0$ and $\eta_0>1$, this means that the maximum value for the potential is less than $22/7 \simeq 3.14$ in all cases of physical interest. In other words, and since in these units the cosmologically relevant modes are those having $n \gg 1$, the potential is dominated at all times during the bounce itself, and therefore cannot lead to tensor mode production. There is therefore a qualitative difference between the tensor and the scalar modes since the latter can be affected by the bounce provided the NEC is almost violated, while the former are never affected, regardless of the underlying parameter values.

C. Comparison with other transitions

In order to make a comparison of our bouncing era to other known transitions, we first consider below the radiation to matter transition under the hypothesis that this occurs, as observation demands, at some time $\eta_{\rm eq}$ such that the three-space curvature is negligible. In other words, we study this transition with $\mathcal{K}=0$. The scale factor can be given the form

$$a(\eta) = a_{\rm eq}\left[b_2^2 \left(\frac{\eta}{\eta_{\rm eq}}\right)^2 + 2 b_1 \left(\frac{\eta}{\eta_{\rm eq}}\right) \right], \tag{87}$$

where $b_1 = b_2 = \sqrt{2} - 1$ is chosen such that $a(\eta = \eta_{\rm eq}) = a_{\rm eq}$. We have emphasized the two different normalization factors b_1 and b_2 because they play a different role in the potentials for either u or v. Indeed, the potential for v in this case is $(a\sqrt{\Gamma})''/(a\sqrt{\Gamma})$, which, for a purely radiation dominated universe (i.e. with $b_2=0$ and $b_1 \neq 0$), is identically vanishing, whereas the potential for u is $[(a\sqrt{\Gamma})^{-1}]''/[(a\sqrt{\Gamma})^{-1}]$, which in the same situation would be $\sim 2/\eta^2$. During the transition however, the presence of an amount of matter, however tiny, leads to a nonvanishing b_2, and hence a nonzero diverging term for small conformal times $\sim b_2^2/(2 b_1 \eta_{\rm eq} \eta)$: the radiation dominated universe represents a singular limiting case. This means that both potentials are large already at small times, deep into the radiation era, and the approximation $k^2 \ll V_{\rm R\text{-}M}$ is accurate both before and after the transition and for both variables. This accounts for the fact that the Bardeen potential changes during the transition, but only insofar as the amplitude is concerned, leaving its spectrum unaltered. This is because, in this situation, there is no potential crossing: the modes are always below the barrier. Figure 8 illustrates this fact and summarizes the situation by showing a sketch of the perturbation potential $V_{\rm R\text{-}M}$ together with the evolution of the gravitational potential.

Another situation of cosmological interest to compare the bounce with is a phase of quasi-exponential inflation followed by preheating and the subsequent epoch of radiation domination. When only one field is present, the potentials for either u or v are essentially undistinguishable and both coincide numerically with the inverse Hubble size $\mathcal{H}^2$, as shown schematically in Fig. 9. For more than one field, the situation is qualitatively different and cannot be understood by means of a simple potential [29]. For a given wave number k, the spectrum is frozen when the wavelength hits the potential, which is often phrased, because of the similarity with the

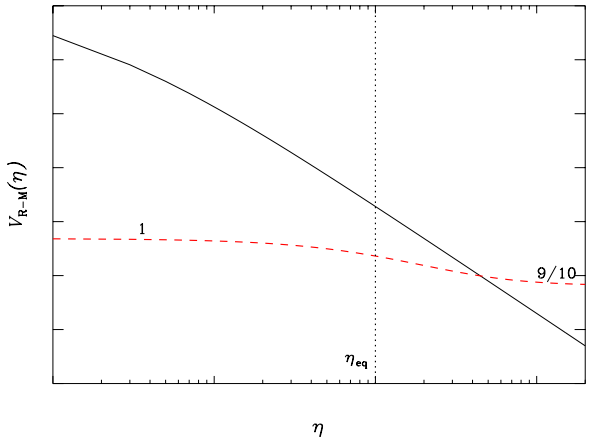

FIG. 8. The effective potential $V_{\mathrm{R\text{-}M}}(\eta)$ for the perturbation variables $u(\eta)$ or $v(\eta)$ for the radiation to matter transition, derived from the scale factor given by Eq. (87). This log-log sketch shows the potential (full line) for either of the variables (they differ by numerical factors) as well as the exact solution (for $k=0$) for the Bardeen potential (dashed line) as a function of conformal time.

Hubble scale, as ''horizon exit'' (see Ref. [15] for a more detailed discussion of this point, and Ref. [16] in the bounce context). As illustrated in Fig. 9, the crucial difference between the two situations, namely preheating and bounce transitions, is that in the latter case the potential and the Hubble scale behave in completely different ways whereas they correspond in the former, at least in the region of potential crossing. Far from the bounce itself, however, the potential tends to $\mathcal{H}^2$ again, in a fashion similar to what happens during inflation. We conclude that in the bounce case, the potential is the quantity that matters and the Hubble scale is irrelevant for the calculation of the amplification of perturbations. As a consequence, for practical calculatory purposes, the phrase ''Hubble crossing'' appears misleading in this context [16] and the phrase ''potential crossing'' should be used instead.

VI. CONCLUSIONS

In this section, we summarize the main results obtained above and discuss them in a more general framework.

Assuming general relativity as the theory describing gravitation during a bouncing stage happening in the early universe, letting the matter content be in the form of a scalar field, and restricting attention to the closed spatial section case in order to satisfy the null energy condition, we were able to develop a general formalism by expanding any bouncing scale factor around the $\mathcal{K}=1$ de Sitter–like bouncing solution. This expansion is characterized by two parameters δ and ξ which, in some sense, are the counterparts of the slow-roll parameters in the usual inflationary models [24]. Because this expansion permits a general calculation of the potential for the primordial scalar gravitational perturbations, this allows to fully determine the structure of their evolution as they propagate across the bounce.

The potential V_u obtained is radically different from the Hubble scale at the relevant times. This has to be contrasted with the inflationary paradigm for which $\mathcal{H}^2$ and V_u are almost identical.

An important conclusion of this work is that a bounce phase, even a short one, can affect large scales of perturbations. General arguments aiming at showing the contrary therefore suffer from our counter-example. The bounce itself is part of the mechanism described in the Introduction, so that the transfer matrix we obtained participates to the one of Eq. (1) through

$$\lim_{\eta_0 \to 1} \mathbf{T} \propto \mathbf{T}_?^{<} \cdot k^{-1} \cdot \mathbf{T}_?^{>}, \tag{88}$$

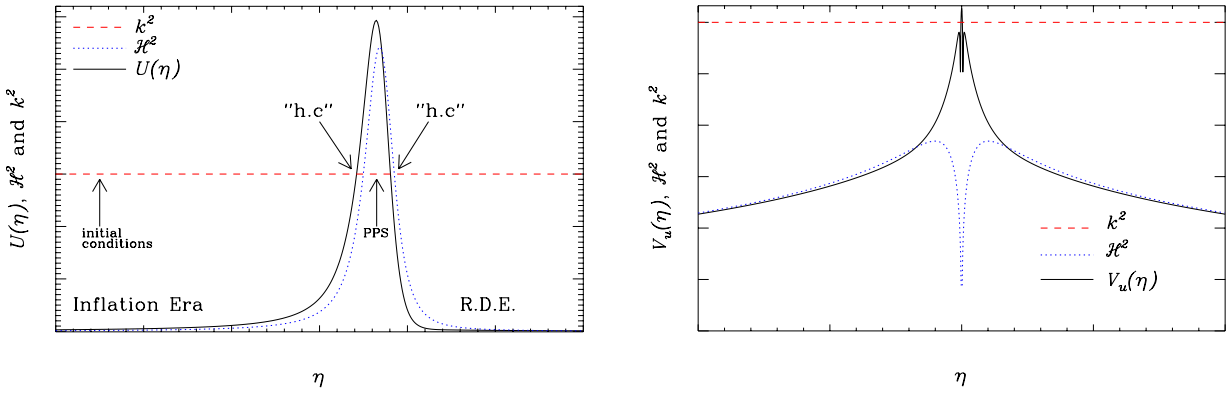

FIG. 9. Left panel: Effective potential U and inverse horizon size $\mathcal{H}^2$ relative to the scale k^2 of the perturbations in inflation models as functions of the conformal time η. The inflation phase, in this sketch, is smoothly linked with the radiation dominated epoch (RDE). The times at which the effect of the potential is comparable with the scale, i.e. $k^2 \sim U$ are seen to be essentially the times at which the scale enters and exits the horizon, i.e. $k \sim \mathcal{H}$, and are hence labeled ''h.c.,'' standing for horizon crossing. The primordial power spectrum (PPS) is understood to be the spectrum that is obtained in the phase for which the modes are frozen and indicated by an arrow. The actual power spectrum, in such a model, also needs to pass the radiation to matter domination transition later on. Right panel: Effective potential V_u and inverse horizon size $\mathcal{H}^2$ relative to the scale k^2 of the perturbations in the bounce model as functions of the conformal time η. The difference with the inflation case is striking.

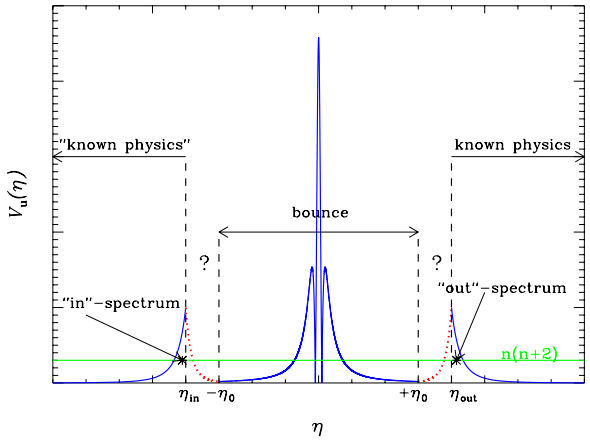

FIG. 10. The effective potential $V_u(\eta)$ for the perturbation variables $u(\eta)$ for our bounce model when one connects this bounce transition to both a previous contracting phase on one side and to the usual radiation dominated phase later on the other side.

where the k dependence stems from the solution (78) and the unknown matrices $\mathbf{T}_?^<$ and $\mathbf{T}_?^>$ refer to the unknown parts sketched in Fig. 10. The coefficients one is interested in, namely T_{11} and T_{12}, giving the amplitude of the growing mode in the expanding phase as functions of the modes in the contracting phase, accordingly can depend on k. In addition, it is important to notice that, as shown in Ref. [16], this mechanism does not violate causality; a similar statement was also emphasized in Ref. [30].

Paradoxically, obtaining a spectral modification at the bounce is possible provided the bounce lasts the minimal amount of conformal time compatible with the NEC preservation. Nevertheless, the assumption of no effect can be justified provided the constraint $\eta_0 - 1 \not\ll 1$ is satisfied, or in the pure de Sitter case having $\eta_0 = 1$ strictly. This last situation is what happens in models in which the bounce takes place for a vanishing value of the scalar field kinetic energy [5], whereas the former case implies a kinetic energy density (not the scalar field itself) for the scalar field comparable to the Planck scale, which may render the semi-classical field theory dubious.

This can be particularly important in view of the string motivated potential alternatives to inflation of the pre big bang kind if it turns out that these models might lead to such spectral corrections as discussed above. This condition needs be verified in each particular situation. For instance, in the pre big bang case, one would need to model the bounce occurring in the Einstein frame, in which our formalism is well suited, to see what the behavior of V_u is in this context. Therefore, and unfortunately, one consequence of the failure of any general argument preventing any alteration of the spectrum is that one needs to explicitly model a regime in which higher order string corrections are dominant. Avoiding this was the main interest of the general argument in question.

We also obtained that the relevant propagation variable is not v, whose flat space equivalent is commonly used for quantization, i.e. for setting up the initial conditions, but rather the intermediate variable u, directly related to the Bardeen potential. This is to be compared with what was recently obtained in Ref. [6], based on a completely different theory of gravity, in which neither variable happens to be bounded at the bounce.

The spectrum of gravitational wave cannot be affected by propagating through these bounces. This exemplifies the fact that there is no fundamental reason according to which scalar and tensor modes should propagate similarly through a bounce.

The picture that emerges for the construction of a complete model of the universe is shown in Fig. 10 and consists in a regime in which quantum field theory in a time-dependent background is well suited, as is the case for instance in many string motivated scenarios [7,8]; this first phase allows an easy calculation of a spectrum of perturbation that would be sort of pre-primordial. Then, unless the curvature was always important in this first period, it is followed by an unknown epoch which connects to the bounce itself, which should also be followed by yet another unknown epoch in order for the curvature to be negligible [16]. This reveals the most important difference between bouncing scenarios and inflation, namely the need for a high curvature phase, which we have seen may drastically modify the physical predictions.

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