Defining perturbations on submanifolds

Karim A. Malik, María Rodríguez-Martínez, David Langlois

Defining perturbations on submanifolds

Karim A. Malik, María Rodríguez-Martínez, and David Langlois
GRECO, Institut d'Astrophysique de Paris, CNRS, 98bis Boulevard Arago, 75014 Paris, France
(Received 12 May 2003; published 17 December 2003)

We study the definition of perturbations in the presence of a submanifold, such as, e.g., a brane. In the standard theory of cosmological perturbations, one compares quantities at the same *coordinate points* in the nonperturbed and the perturbed manifolds, identified via a (nonunique) mapping between the two manifolds. In the presence of a physical submanifold one needs to modify this definition in order to evaluate perturbations of quantities at the submanifold location. As an application, we compute the perturbed metric and the extrinsic curvature tensors at the brane position in a general gauge.

DOI: 10.1103/PhysRevD.68.123517 PACS number(s): 98.80.Cq, 98.80.Jk

I. INTRODUCTION

The theory of cosmological perturbations is a cornerstone of the study of the early Universe, since most of the accessible information from this epoch is believed to be contained in the cosmological fluctuations that are observed in the cosmic microwave background and, more indirectly, in the large scale structures. As a consequence, an important step in the elaboration of an early Universe model is to be able to deal with the origin and the early evolution of cosmological perturbations. A recent picture suggested (or revived the idea) to describe our Universe as that of a braneworld, i.e., a submanifold, where ordinary matter is confined or embedded in a higher dimensional spacetime [1,2]. In a cosmological context, and for one extra dimension, this picture has led to brane cosmology, which was shown to deviate from standard cosmology at high energy [3–7].

Soon after these progresses in homogeneous brane cosmology, several works were started to tackle the difficult problem of cosmological perturbations in this context. One can distinguish three main approaches in these various formalisms. One approach was to use directly a doubly gauge-invariant formalism to describe the perturbations in the bulk and in the brane [8–11]. Another approach was to use a covariant formalism [12–14]. Finally, the most common approach has been to generalize the standard metric based formalism [15–29], which has been used in standard cosmology for a long time [30–32]. For detailed reviews on braneworld perturbations see [33]. In the present work, we adopt the latter approach, which has the advantage to be of more direct access and to be more familiar to people who have already studied the standard theory of cosmological perturbations in four dimensions.

The specific purpose of the present work is to present how one can describe the brane perturbations for *any gauge* chosen in the *bulk*. Since this particular point has generated mistakes and confusion in the literature, we believe it is worthwhile to consider this question with some attention.

The usual approach to calculate tensorial quantities in the braneworld scenario is to adopt a suitable gauge in the bulk and then to transform to the Gaussian normal (GN) gauge to impose the junction conditions. One can then transform the quantities back from the GN to the original gauge. In this work, we give the end result of this procedure for a general background metric. We also allow for a metric with nontrivial background g_{44}, which is not related to the GN gauge simply by a first order coordinate transformation.

The core of the problem is that, in the standard theory of cosmological perturbations, the perturbation for any quantity is defined by comparing the perturbed and unperturbed values *at the same coordinate point*. In brane cosmology, or in fact in any model where one or several submanifolds have some specific physical role, one wishes instead to define perturbations by comparing quantities *at the same physical locus*.

In the next section we discuss the definition of perturbations on a manifold and study how this definition is affected if a submanifold is present. In Sec. III we apply these notions to a five dimensional (5D) metric tensor as an example in the braneworld scenario. In Sec. IV we then turn to the problem of calculating the extrinsic curvature tensor of the brane. We conclude in the last section.

II. THEORY

We consider the following situation: a spacetime $\mathcal{M}$ corresponding to the perturbation of a reference spacetime $\bar{\mathcal{M}}$ and a submanifold $\mathcal{S}$ in $\mathcal{M}$, which can be seen as the perturbation of an unperturbed submanifold $\bar{\mathcal{S}}$ in $\bar{\mathcal{M}}$. We then wish to define meaningful (linear) perturbations for tensorial quantities defined on the submanifold.

A. Standard linear perturbation theory

Let us ignore at this stage the submanifold $\mathcal{S}$ and let us recall some basic principles of the standard theory of linear perturbations in general relativity [30–32]. The starting point is to endow the unperturbed manifold $\bar{\mathcal{M}}$ of dimension N with a coordinate system which we call x^A (where $A=0,\ldots,N-1$), which is usually chosen according to the symmetries of $\bar{\mathcal{M}}$ so that the explicit form of the metric is as simple as possible.

The next step is to introduce, in a geometrical language [34], an identification between the unperturbed and perturbed manifolds, i.e., a mapping $\phi:\bar{\mathcal{M}}\rightarrow\mathcal{M}$, which establishes a one-to-one correspondence between the points of $\bar{\mathcal{M}}$ and $\mathcal{M}$. The identification is not unique and thus chosen arbi-

trarily at first order in the perturbations. The mapping ϕ naturally induces from the coordinate system in $\bar{\mathcal{M}}$ a coordinate system in $\mathcal{M}$.

One then defines the perturbation of a tensor T as

$$\delta T(x) = T(x) - \bar{T}(x), \tag{1}$$

where x stands for a given set of coordinates (which define both a point in $\mathcal{M}$ and in $\bar{\mathcal{M}}$ via the mapping), where T is the full tensor defined in $\mathcal{M}$ and $\bar{T}$ is the corresponding "unperturbed" quantity defined in the reference spacetime $\bar{\mathcal{M}}$. In other words, one defines the perturbation by comparing the values of the tensors *at the same coordinate point*, i.e., at the pair of points identified by the mapping ϕ.

As stressed above, the identification mapping is somewhat arbitrary and one must consider slight modifications of this mapping (slight so that one stays in the domain of validity of the perturbative approach) to get the generic picture. A change of mapping can be interpreted as a change of coordinates in the perturbed spacetime $\mathcal{M}$, which we write as

$$x^A \to \tilde{x}^A = x^A + \delta x^A, \tag{2}$$

which means that the *physical point* with the old coordinates x^A has the new coordinates $x^A + \delta x^A$.

The change of any tensor under the coordinate transformation (2), evaluated *at the same old and new coordinates* (and thus at different physical points), is given by the Lie derivative with respect to the vector field δx^A [35],

$$\Delta T \equiv \tilde{T}(x) - T(x) = -\mathcal{L}_{\delta x^A} T. \tag{3}$$

Equation (3) gives for the change of a covariant two-tensor under a transformation (2)

$$\tilde{Q}_{AB} = Q_{AB} - \pounds_{\delta x^C} Q_{AB}, \tag{4}$$

or, substituting for the Lie derivative (see, e.g., [35]),

$$\tilde{Q}_{AB} = Q_{AB} - \delta x^C \partial_C Q_{AB} - (\partial_A \delta x^C) Q_{CB} - (\partial_B \delta x^C) Q_{AC}. \tag{5}$$

B. Perturbations on a submanifold

Let us now take into account the submanifold $\mathcal{S}$. We start by introducing the unperturbed submanifold $\bar{\mathcal{S}}$ in the reference manifold $\bar{\mathcal{M}}$. In general, there is no reason for the mapping ϕ to leave the submanifold invariant, i.e., we have in general $\phi(\bar{\mathcal{S}}) \neq \mathcal{S}$, as shown in Fig. 1. Of course, one can always choose a special subclass of mappings which do leave invariant this submanifold but this corresponds to a restricted choice of coordinate systems. Note that the GN coordinate system, defined below in Sec. IV B, belongs to this subclass.

If one wishes to define perturbations for quantities existing only at the geometrical locus of the submanifold, one therefore cannot use the above definition, Eq. (1). A necessary additional step is to define another mapping λ between the image of $\bar{\mathcal{S}}$ and the real $\mathcal{S}$, i.e., $\lambda : \phi(\bar{\mathcal{S}}) \to \mathcal{S}$. In the coordinate system defined by ϕ, this can be written in the form $\hat{x}^A = x^A + \epsilon^A$, i.e., λ maps a point of $\phi(\bar{\mathcal{S}})$ with coordinates x^A onto a point of $\mathcal{S}$ with coordinates $x^A + \epsilon^A$.

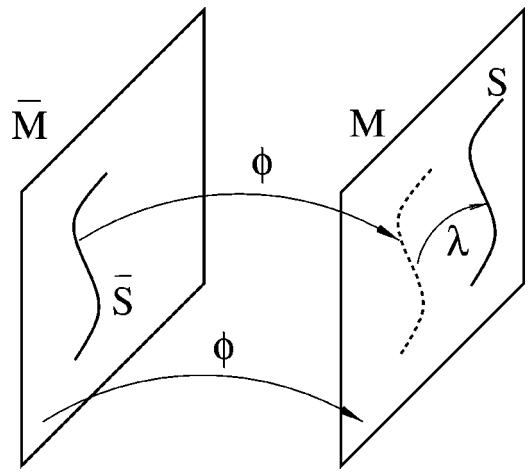

FIG. 1. The mapping ϕ, which takes the background manifold $\bar{\mathcal{M}}$ to the perturbed manifold $\mathcal{M}$, and the internal mapping λ, between the perturbed manifold and itself. To take $\bar{\mathcal{S}}$ to $\mathcal{S}$ we need ϕ and λ.

One can now define a meaningful perturbation for tensorial quantities defined at the submanifold location,

$$\delta_{\mathcal{S}} T(x) = T(\lambda[\phi(\bar{p})]) - \bar{T}(\bar{p}), \tag{6}$$

where x stands for the coordinates of $\bar{p}$, which is a point of $\bar{\mathcal{S}}$. When the quantity T is in fact defined everywhere (but one wants to use the perturbation of its value at the location of the submanifold), the defining expression above can be decomposed into

$$\delta_{\mathcal{S}} T(x) = T(\lambda[\phi(\bar{p})]) - T(\phi(\bar{p})) + T(\phi(\bar{p})) - \bar{T}(\bar{p}), \tag{7}$$

where one can recognize in the last two terms the usual definition of the linear perturbation. Reintroducing coordinates, this reads

$$\begin{aligned} \delta_{\mathcal{S}} T(x) &= T(x+\epsilon) - T(x) + \delta T(x) \\ &= \epsilon^A \partial_A T + \delta T(x), \end{aligned} \tag{8}$$

where we have used the Taylor expansion of $T(x+\epsilon)$. Equation (8) is an adequate definition of the perturbation of a tensorial quantity in the presence of a submanifold, if we impose the restriction that the perturbation has to be at the physical locus of the submanifold.

III. THE METRIC OF A BRANEWORLD

We now apply the results found in the previous section to the metric tensor of a braneworld. After briefly reviewing how the perturbed metric tensor changes under a first order coordinate transformation we show how the definition of the perturbation of the metric tensor changes in the presence of a particular submanifold, the brane.

A. Standard perturbed metric tensor and change under a gauge transformation

We study a 5D metric with maximally symmetric flat three-subspace (see, for example, [24,28]) and include only scalar perturbations, that is, perturbations that transform like scalars on spatial three-sections.

The background part of the metric tensor is given by

$$ds^2=\bar{g}_{AB}dx^A dx^B=-n^2dt^2+a^2\delta_{ij}dx^i dx^j+b^2dy^2, \quad (9)$$

where the metric factors n, a, and b are functions of coordinate time t and extra dimension y. The metric tensor, including linear perturbations, is

$$g_{AB}\equiv\bar{g}_{AB}+\delta g_{AB}=\begin{pmatrix} -n^2(1+2A) & a^2B_{,i} & nA_y \\ a^2B_{,j} & a^2[(1+2\mathcal{R})\delta_{ij}+2E_{,ij}] & a^2B_{y,i} \\ nA_y & a^2B_{y,i} & b^2(1+2A_{yy}) \end{pmatrix}, \quad (10)$$

where the scalar metric potentials A, B, $\mathcal{R}$, E, B_y, A_y, and A_{yy} are functions of the coordinates $x^A=[t,x^i,y]$.

To find the change of the metric tensor under the first order coordinate transformation defined by Eq. (2), we apply Eq. (5) to the metric tensor g_{AB}, given in Eq. (10). This gives the perturbed metric tensor in the new coordinate system:

$$\tilde{\delta}g_{AB}=\delta g_{AB}-\delta x^C\partial_C\bar{g}_{AB}-(\partial_A\delta x^C)\bar{g}_{CB}-(\partial_B\delta x^C)\bar{g}_{AC}, \quad (11)$$

where $\delta x^A=[\delta t,\delta x^i,\delta y]$. Since we are only working to linear order, the Lie derivative with respect to the first order vector δx^A only acts on the background part of the metric $\bar{g}_{AB}$.

The transformation behavior of the scalar metric perturbations is therefore

$$\tilde{A}=A-\dot{\delta t}-\frac{\dot{n}}{n}\delta t-\frac{n'}{n}\delta y, \quad \tilde{B}=B+\frac{n^2}{a^2}\delta t-\dot{\delta x},$$

$$\tilde{\mathcal{R}}=\mathcal{R}-\frac{\dot{a}}{a}\delta t-\frac{a'}{a}\delta y \quad \tilde{E}=E-\delta x,$$

$$\tilde{A}y=A_y+n\,\delta t'-\frac{b^2}{n}\dot{\delta y}, \quad \tilde{B}y=B_y-\delta x'-\frac{b^2}{a^2}\delta y,$$

$$\tilde{A}yy=A_{yy}-\frac{\dot{b}}{b}\delta t-\frac{b'}{b}\delta y-\delta y'. \quad (12)$$

B. Perturbed metric tensor in the presence of a submanifold

The above definition of the perturbed metric, Eq. (10), does not depend on the presence of a submanifold and is defined everywhere in the bulk. However, our spacetime contains a brane and we are interested in computing the perturbed metric on this *physical* hypersurface, i.e., we need to evaluate it on the brane.

In our model the homogeneous brane is fixed at $x^4=$const, but after being perturbed, it is in general displaced from this location. We would like to describe the change in the metric at the brane position as a result of this displacement. This displacement is embodied in the relation

$$\hat{x}^A=x^A+\epsilon^A, \quad (13)$$

where the ϵ^A are associated with the mapping λ introduced earlier. In the general case the ϵ^A are decomposed into degrees of freedom tangential and orthogonal to the brane according to (see, e.g., [23])

$$\epsilon^A\equiv\zeta^\mu e^A_\mu+\zeta n^A, \quad (14)$$

where n^A is the normal vector to the brane, e^A_μ is a basis of vectors tangential to the brane, and $\mu=0,\ldots,3$.

From Eq. (8) it follows that the perturbed metric at the position of the brane is given by

$$\delta_{\mathcal{S}}g_{AB}=\epsilon^C\partial_C g_{AB}+\delta g_{AB}. \quad (15)$$

In our model the only relevant degree of freedom of ϵ^A is the four-component, the perturbation of the brane position in the extra dimension, which we denote by $\epsilon^4\equiv\xi(t,x^i)$, see Fig. 2. The other degrees of freedom ζ^μ, tangential to the brane, correspond to mappings that leave the brane invariant, and we are therefore allowed to choose $\zeta^\mu\equiv 0$.

The perturbed metric in the presence of the brane is thus given by

$$\delta_{\mathcal{S}}g_{AB}=\delta g_{AB}+\zeta n^A\partial_A\bar{g}_{AB}=\delta g_{AB}+\xi\partial_4\bar{g}_{AB}. \quad (16)$$

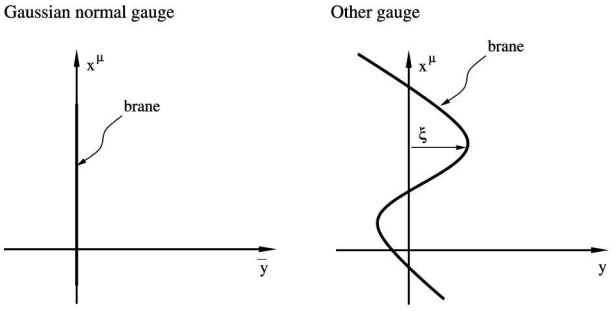

FIG. 2. The brane in Gaussian normal (left) and a different coordinate system.

Note that since we only work to linear order in the perturbations, and the perturbation ξ is first order, the additional part in the perturbed metric tensor $\xi\partial_4\bar{g}_{AB}$ only includes derivatives of the background metric.

The metric tensor at the position of the brane is therefore given by

$$g^{\mathcal{S}}_{AB}=\bar{g}_{AB}+\delta_{\mathcal{S}}g_{AB}, \tag{17}$$

where $\bar{g}_{AB}$ is the background metric.

IV. EXTRINSIC CURVATURE

In this section we calculate the extrinsic curvature tensor in the presence of a submanifold. The extrinsic curvature K_{AB} describes the local bending of the brane along the extra dimension. In the braneworld scenario this bending is determined by the local matter distribution on the brane through the junction conditions [36].

The extrinsic curvature tensor is defined as (see, e.g., [35])

$$K_{AB}=h^C_A({}^{(5)}\nabla_C n_B), \tag{18}$$

where n^B is the unit vector normal to the brane, ${}^{(5)}\nabla_A$ is the 5D covariant derivative, and h_{AB} is the projection tensor, defined as $h_{AB}\equiv g_{AB}-n_An_B$.

We shall study the case of a static brane, i.e., a brane that is not moving with respect to the background coordinate system. The unit vector n^A is spacelike and thus subject to the constraint

$$n_An^A=1. \tag{19}$$

In the following section we calculate the extrinsic curvature tensor in an arbitrary gauge up to first order in the perturbations. As a check, we then calculate the extrinsic curvature tensor in the GN gauge and transform the expressions found in this gauge to an arbitrary gauge.

A. Calculating K_{AB} in an arbitrary gauge

The constraint, Eq. (19), together with background metric, Eq. (9), give the unit normal vector to the brane at zeroth order as

$$\bar{n}^A=[0,\mathbf{0},b^{-1}]. \tag{20}$$

As pointed out above, we consider a brane that is nonmoving and hence the time component of $\bar{n}_A$ is zero.

The perturbed normal vector at the brane position can be computed by using the perturbed version of Eq. (19),

$$2\bar{n}_A\delta_{\mathcal{S}}n^A+\bar{n}^A\bar{n}^B\delta_{\mathcal{S}}g_{AB}=0. \tag{21}$$

It is important to stress that one has to use $\delta_{\mathcal{S}}g_{AB}$ in the above equation and not δg_{AB} defined in Eq. (10), since one is perturbing Eq. (19) defined *on the brane*. We therefore get for the normal vector to the brane at the position of the brane up to first order

$$n_A=\bar{n}_A+\delta_{\mathcal{S}}n_A=b\left[-\dot{\xi},-\xi_{,i},1+A_{yy}+\frac{b'}{b}\xi\right]. \tag{22}$$

Substituting the background metric, Eq. (9), and the background normal vector, Eq. (20), into the definition of the extrinsic curvature tensor, Eq. (18), we get for the components of the extrinsic curvature tensor at zeroth order

$$\bar{K}_{00}=-\frac{n^2}{b}\frac{n'}{n},\quad \bar{K}_{ij}=\frac{a^2}{b}\frac{a'}{a}\delta_{ij}. \tag{23}$$

To get the extrinsic curvature tensor up to first order at the position of the brane, $K_{AB}=\bar{K}_{AB}+\delta_{\mathcal{S}}K_{AB}$, we have to use perturbed quantities defined on the brane, i.e., the perturbed metric $g^{\mathcal{S}}_{AB}$, defined in Eq. (17), and the perturbed normal vector (22). Substituting into Eq. (18) then gives for the components of the perturbed part, $\delta_{\mathcal{S}}K_{AB}$,

$$\delta_{\mathcal{S}}K_{00}=-\frac{n^2}{b}\left\{A'+2\frac{n'}{n}A-\frac{n'}{n}A_{yy}+\frac{1}{n}\dot{A}_y+\frac{b^2}{n^2}\left[\ddot{\xi}+\left(2\frac{\dot{b}}{b}-\frac{\dot{n}}{n}\right)\dot{\xi}\right]+\left[\frac{n''}{n}-\frac{n'}{n}\frac{b'}{b}+\frac{n'^2}{n^2}\right]\xi\right\},$$

$$\delta_{\mathcal{S}}K_{0i}=\left\{\frac{1}{2}\frac{a^2}{b}\left[B'-\dot{B}_y+2\frac{a'}{a}B-\frac{n}{a^2}A_y\right]-\dot{\xi}+\left(\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right)\xi\right\}_{,i},$$

$$\delta_{\mathcal{S}}K_{04}=\frac{1}{b}\frac{n'}{n}(nA_y+b^2\dot{\xi}),$$

$$\delta_{\mathcal{S}}K_{ij}=\frac{a^2}{b}\left\{\mathcal{R}'-\frac{a'}{a}A_{yy}+\frac{1}{n^2}\frac{\dot{a}}{a}(nA_y+b^2\dot{\xi})+\left(\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'b'}{ab}\right)\xi\right\}\delta^i_j+\frac{a^2}{b}\left\{E'-B_y+2\frac{a'}{a}E-\frac{b^2}{a^2}\xi\right\}_{,ij},$$

$$\delta_{\mathcal{S}}K_{i4}=\frac{1}{b}\frac{a'}{a}(a^2B_y+b^2\xi)_{,i},\quad \delta_{\mathcal{S}}K_{44}=0. \tag{24}$$

B. Starting from Gaussian normal gauge

In the following section we outline the calculation of the extrinsic curvature tensor in the GN gauge. To get the components of the extrinsic curvature tensor in an arbitrary gauge, we then have to transform these results to an arbitrary gauge which we shall do in the following section.

1. Calculating the extrinsic curvature tensor in a Gaussian normal gauge

We begin by computing the extrinsic curvature tensor in a slightly more general gauge than the GN gauge, so that we can still work directly with the background metric defined in Eq. (9), denoting quantities in this gauge by a "hat."

In our nearly GN gauge the metric has a particularly simple form

$$\hat{g}_{44}=b^2, \quad \hat{g}_{4\mu}=0, \tag{25}$$

which is in terms of the scalar metric perturbations

$$\hat{A}_y=0, \quad \hat{A}_{yy}=0, \quad \hat{B}_y=0. \tag{26}$$

The brane is located at $\hat{y}=0$, i.e., there is no perturbation in the brane position

$$\hat{\xi}=0. \tag{27}$$

Note that to go from this gauge to the GN, one only needs to put $b=1$.

The normal vector to the brane in this gauge is

$$\hat{n}^A=[0,\mathbf{0},b^{-1}], \tag{28}$$

and the definition of the extrinsic curvature tensor, Eq. (18), simplifies to

$$\hat{K}_{AB}=\frac{1}{2b}\frac{\partial}{\partial\hat{y}}\hat{g}_{AB} \quad \text{for} \quad A,B\neq 4,$$

$$\hat{K}_{A4}=\hat{K}_{4B}=0. \tag{29}$$

The extrinsic curvature is therefore in components

$$\hat{K}_{00}=-\frac{n^2}{b}\left[\frac{n'}{n}+2\frac{n'}{n}\hat{A}+\hat{A}'\right],$$

$$\hat{K}_{0i}=\frac{a^2}{b}\left[\frac{a'}{a}\hat{B}+\frac{1}{2}\hat{B}'\right]_{,i},$$

$$\hat{K}_{ij}=\frac{a^2}{b}\left[\left(\frac{a'}{a}+2\frac{a'}{a}\hat{\mathcal{R}}+\hat{\mathcal{R}}'\right)\delta_{ij}+\hat{E}'_{,ij}+2\frac{a'}{a}\hat{E}_{,ij}\right]. \tag{30}$$

Again the components in the GN gauge are obtained by putting $b=1$.

2. Transforming the extrinsic curvature to an arbitrary gauge

We can now transform the extrinsic curvature tensor in the hat gauge, Eq. (30), to an arbitrary gauge by using the transformation rules of the tensor components themselves, Eq. (5), and by the transformation rules of the metric potentials, Eq. (12).

Under a first order coordinate transformation the extrinsic curvature tensor changes according to Eq. (5):

$$\tilde{K}_{AB}=\hat{K}_{AB}-\delta x^C\partial_C\overline{K}_{AB}-(\partial_A\delta x^C)\overline{K}_{CB}-(\partial_B\delta x^C)\overline{K}_{AC}. \tag{31}$$

The components therefore change as

$$\tilde{K}_{00}=\hat{K}_{00}-\partial_t\overline{K}_{00}\delta t-\overline{K}'_{00}\delta y-2\dot{\delta}t\overline{K}_{00},$$

$$\tilde{K}_{0i}=\hat{K}_{0i}-\delta t_{,i}\overline{K}_{00}-\dot{\delta}x_{,}{}^k\overline{K}_{ki},$$

$$\tilde{K}_{ij}=\hat{K}_{ij}-\partial_t\overline{K}_{ij}\delta t-\overline{K}'_{00}\delta y-2\,\delta x^k_{,i}\overline{K}_{kj}. \tag{32}$$

From the definition of the hat gauge, Eq. (26), and the transformation behavior of the metric perturbations, Eq. (12), it follows that the hat gauge only restricts the y derivatives of the coordinate transformation δx^A, i.e., the degrees of freedom orthogonal to the brane, but leaves the gauge transformations on the brane, that is, tangential to it, arbitrary [24]. We can solve Eqs. (26) and (12) for $\delta t'$, $\delta x'$, and $\delta y'$ and get

$$\delta x'=B_y-\frac{b^2}{a^2}\delta y,$$

$$\delta t'=-\frac{1}{n}A_y+\frac{b^2}{n^2}\dot{\delta}y,$$

$$\delta y'=A_{yy}-\frac{\dot{b}}{b}\delta t-\frac{b'}{b}\delta y. \tag{33}$$

We now substitute the expression for the extrinsic curvature tensor in the nearly GN gauge, Eq. (30), into Eq. (32), use the expressions for the change of the scalar metric potentials under a coordinate transformation, Eq. (12), together with Eq. (33) and the expression for the background part of the extrinsic curvature tensor, Eq. (23), and get after a straightforward but tiresome calculation the extrinsic curvature in an arbitrary gauge. But we are not completely there, yet. As pointed out in Sec. II A, a gauge transformation leaves the coordinate point unchanged. Hence the extrinsic curvature tensor is still at the coordinate position of the brane in the nearly GN gauge at $y=0$ and we therefore have to shift it to the physical position of the perturbed brane at $y=\xi$. We can do this by using a Taylor expansion of K_{AB} around $y=0$, and get

$$K_{AB}|_{y=\xi}=K_{AB}|_{y=0}+\epsilon^C K_{AB,C}, \tag{34}$$

where, see Eq. (14) above, $\epsilon^A=[0,\mathbf{0},\xi]$ is the shift between the coordinate systems in which the brane is at $y=0$ and $y=\xi$.

The final piece of information that we need for the complete extrinsic curvature tensor in an arbitrary gauge is the relation between δy and ξ, since the expression we have derived so far still contains δy and its time derivatives. The relation is readily found by calculating the change of the normal vector defined above in Eq. (22) under an infinitesimal coordinate transformation δx^A, which is given by $\tilde{n}_A=n_A-\pounds_{\delta x^B}n_A$ [35]. We therefore find for the transformation behavior of ξ under an infinitesimal coordinate transformation

$$\tilde{\xi} = \xi + \delta y, \tag{35}$$

and since in the nearly GN gauge $\hat{\xi} = 0$, we get

$$\delta y = -\xi. \tag{36}$$

Hence we get the same expression for the components of the perturbed curvature tensor in an arbitrary gauge, that is, Eq. (24), by transforming from the nearly GN gauge as by direct calculation, as expected.

V. CONCLUSIONS

In this paper we have studied the definition of first order perturbations in the presence of a submanifold. Perturbations in standard cosmological perturbation theory can be defined via a mapping ϕ of tensorial quantities between a background manifold $\bar{\mathcal{M}}$ and a perturbed manifold $\mathcal{M}$. The introduction of a submanifold, or hypersurface, $\mathcal{S}$ into the spacetime does not present a problem in itself, since it does not affect the definition of the perturbations. A problem might arise, if we require the perturbations to be restricted to the submanifold, since in general $\phi(\bar{\mathcal{S}}) \neq \mathcal{S}$. This problem can be alleviated either by choosing a particular coordinate system, e.g., a GN one, which enforces the correct mapping between the submanifold in the background and its perturbed image, or by leaving the coordinate system unrestricted, but then using a second mapping λ which takes the image of the submanifold, $\phi(\bar{\mathcal{S}})$, to the correct position of the submanifold, i.e., $\lambda[\phi(\bar{\mathcal{S}})] = \mathcal{S}$.

The usage of a non-GN gauge requires care in defining perturbations at the physical position of the submanifold, as we have demonstrated. Although the problem of defining perturbations on a submanifold does not arise if we work in a GN coordinate system or gauge from the outset, as already pointed out, one looses in this case the freedom to adopt a particularly suitable gauge for the problem or a gauge that simplifies the calculations, by having ''used up'' some of the gauge freedom. Of course it is always possible to start in the GN coordinate system and then transform to a different gauge, but this in itself is nontrivial, as we have shown as well.

As an example, we have calculated the perturbed metric tensor for a braneworld scenario at the position of the brane. Using this result we then calculated the perturbed extrinsic curvature tensor at the brane position. We have also shown how to calculate the extrinsic curvature tensor in the GN gauge and its transformation into an arbitrary gauge.

In Ref. [10] Mukohyama uses a doubly gauge-invariant formalism to investigate the perturbed junction conditions in braneworld cosmology. In his formalism the gauge transformations on the submanifold, or brane, are allowed to differ from the gauge transformations in the bulk spacetime. This additional freedom is not required for most applications and makes the formalism difficult to use. If we limit the gauge transformations in the bulk and on the brane to be identical, the doubly gauge-invariant formalism and our approach give the same results.

Although the examples given in this paper are concerned with the definitions of perturbations on the brane in a five-dimensional bulk, it has some connections with the question of the hypersurface matching in standard 4D [37] or the question of the matching of perturbations across a cosmological bounce [38]. In these cases, one finds a physical hypersurface and although this hypersurface is timelike, instead of spacelike as the brane, our approach could be certainly useful.

ACKNOWLEDGMENTS

The authors would like to thank Helen Bridgman, David Matravers, Alain Riazuelo, Filippo Vernizzi, and David Wands for useful discussions. K.M. is supported by a Marie Curie Fellowship under the contract No. HPMF-CT-2000-00981. Algebraic computations of tensor components were performed using the GRTENSORII package for Maple.

[1] N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Lett. B **429**, 263 (1998).
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[3] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000).
[4] C. Csaki, M. Graesser, C. Kolda, and J. Terning, Phys. Lett. B **462**, 34 (1999).
[5] J.M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999).
[6] P. Binétruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B **477**, 285 (2000).
[7] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[8] S. Mukohyama, Phys. Rev. D **62**, 084015 (2000).
[9] H. Kodama, A. Ishibashi, and O. Seto, Phys. Rev. D **62**, 064022 (2000).
[10] S. Mukohyama, Class. Quantum Grav. **17**, 4777 (2000).
[11] S. Mukohyama, Phys. Rev. D **64**, 064006 (2001).
[12] R. Maartens, Phys. Rev. D **62**, 084023 (2000).
[13] C. Gordon and R. Maartens, Phys. Rev. D **63**, 044022 (2001).
[14] B. Leong, P. Dunsby, A. Challinor, and A. Lasenby, Phys. Rev. D **65**, 104012 (2002).
[15] D. Langlois, Phys. Rev. D **62**, 126012 (2000).
[16] C. van de Bruck, M. Dorca, R.H. Brandenberger, and A. Lukas, Phys. Rev. D **62**, 123515 (2000).
[17] K. Koyama and J. Soda, Phys. Rev. D **62**, 123502 (2000).
[18] D. Langlois, R. Maartens, and D. Wands, Phys. Lett. B **489**, 259 (2000).
[19] D. Langlois, Phys. Rev. Lett. **86**, 2212 (2001).
[20] H.A. Bridgman, K.A. Malik, and D. Wands, Phys. Rev. D **63**, 084012 (2001).
[21] A. Neronov and I. Sachs, Phys. Lett. B **513**, 173 (2001).
[22] D. Langlois, R. Maartens, M. Sasaki, and D. Wands, Phys. Rev. D **63**, 084009 (2001).

[23] N. Deruelle, T. Dolezel, and J. Katz, Phys. Rev. D **63**, 083513 (2001).
[24] H.A. Bridgman, K.A. Malik, and D. Wands, Phys. Rev. D **65**, 043502 (2002).
[25] K. Koyama and J. Soda, Phys. Rev. D **65**, 023514 (2002).
[26] D.S. Gorbunov, V.A. Rubakov, and S.M. Sibiryakov, J. High Energy Phys. **10**, 015 (2001).
[27] C. Deffayet, Phys. Rev. D **66**, 103504 (2002).
[28] A. Riazuelo, F. Vernizzi, D. Steer, and R. Durrer, hep-th/0205220.
[29] G.F. Giudice, E.W. Kolb, J. Lesgourgues, and A. Riotto, Phys. Rev. D **66**, 083512 (2002).
[30] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[31] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[32] V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[33] R. Maartens, in *Reference Frames and Gravitomagnetism*, edited by J. Pascual-Sanchez *et al.* (World Scientific, Singapore, 2001), pp. 93–119, gr-qc/0101059; D. Langlois, Prog. Theor. Phys. Suppl. **148**, 181 (2003); N. Deruelle, contributed in Spanish Relativity Meeting on Gravitation and Cosmology, ERE, 2002, Mao, Menorca, Spain, 2002, gr-qc/0301036; P. Brax and C. van de Bruck, Class. Quantum Grav. **20**, R201 (2003).
[34] J.M. Stewart, Class. Quantum Grav. **7**, 1169 (1990).
[35] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[36] W. Israel, Nuovo Cimento B **44S10**, 1 (1966); **48**, 463(E) (1966).
[37] N. Deruelle and V.F. Mukhanov, Phys. Rev. D **52**, 5549 (1995).
[38] C. Gordon and N. Turok, Phys. Rev. D **67**, 123508 (2003).