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# TKE model involving the distance to the wall.

## Part 1: the relaxed case

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### Abstract

We are considering a steady-state turbulent Reynolds-Averaged Navier-Stokes (RANS) model that couples the equation for the velocity-pressure mean field with the equation for the turbulent kinetic energy. Eddy viscosities vanish at the boundary, characterized by terms like  $d(x, \Gamma)^\eta$  and  $d(x, \Gamma)^\beta$ , where  $0 < \eta, \beta < 1$ . We determine critical values  $\eta_c$  and  $\beta_c$  for which the system has a weak solution. This solution is obtained as the limit of viscous regularizations for  $0 < \eta < \eta_c$  and  $0 < \beta < \beta_c$ .

**Key words** : Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.

**2010 MSC**: 76D05, 35Q30, 76F65, 76D03, 35Q30.

## 1 Introduction

This paper is dedicated to the analysis of a stationary coupled PDE system (see (1.4) below) that arises in incompressible turbulence modeling.

The modeling of turbulence in fluid mechanics involves decomposing the velocity and pressure fields into mean fields, denoted as  $\mathbf{u}$  and  $p$  in this work. These mean fields are assumed to be stationary, representing long-time averages. The fluctuations are denoted as  $\mathbf{u}'$  and  $p'$ . By averaging the Navier-Stokes equations, we obtain the Reynolds stress tensor  $\mathbf{R}$ , representing the long-time average of the tensor product  $\mathbf{u}' \otimes \mathbf{u}'$  [7, 22, 24]. It is crucial to model this Reynolds tensor, and it is widely accepted to be dissipative [3, 10]. According to the Boussinesq hypothesis, it is made proportional to the average strain tensor via a turbulent viscosity denoted as  $\nu_t$ . The question then arises as to whether the molecular diffusion term is negligible compared to the turbulent diffusion term.

Various approaches exist for modeling  $\nu_t$ , either within the framework of statistical averages (RANS-type models) or by filtering the high frequencies of the flow (LES-type models). In this article, we focus on a RANS model, while the case of LES models is studied in [16]. Generally speaking, in RANS models, turbulent viscosity is a function of turbulent kinetic energy,  $k$  (TKE), and turbulent dissipation  $\epsilon$ . This can be simplified by relating TKE  $k$  to  $\epsilon$  through the Prandtl mixing length, denoted by  $\varrho$ , by the dimensional law

$$\epsilon = \frac{k|k|^{1/2}}{\varrho},$$

yielding the following law for the eddy viscosity:

$$(1.1) \quad \nu_t = C\varrho\sqrt{k},$$

where  $C$  is a dimensionless constant (and  $k \geq 0$ ). After a modeling process [7, 22], one obtains the following equation for  $k$  in the steady state case, named TKE model:

$$(1.2) \quad \mathbf{u} \cdot \nabla k - \nabla \cdot (\mu_t(k)\nabla k) + \frac{k|k|^{1/2}}{\varrho} = \nu_t(k)|D\mathbf{u}|^2,$$

where  $\mathbf{u}$  is the fluid velocity,  $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$  is the strain tensor,  $\mu_t$  is an eddy diffusion similar to the eddy viscosity given by (1.1).

Assuming that  $\varrho$  is a smooth bounded function such that  $\varrho(x) \geq \varrho_0 > 0$  in the fluid domain  $\Omega$ , the coupling of the resulting steady-state Navier-Stokes equation with eddy diffusion to the TKE model (1.2) has been extensively studied [2, 7, 11, 15, 18, 19].

In most cases, we assume that  $\nu_t$  is bounded below and above. This assumption can be implemented by replacing  $C\sqrt{k}$  with a truncation function  $\tilde{\nu}$  defined on the interval  $[1/M, M]$ , where  $\tilde{\nu}$  is equal to  $C\sqrt{k}$ , and on  $[0, 1/M]$ , where it is equal to  $C/\sqrt{M}$ , and on  $[M, \infty[$ , where it is equal to  $C\sqrt{M}$ , for some  $M > 0$ .

In this paper, we remain in this framework and focus on the problem posed by the Prandtl mixing length  $\varrho$ . In many physical applications, such as meteorology or oceanography [17, 23],  $\varrho = \varrho(x)$  vanishes at the boundary, equivalent to the distance to the boundary  $\Gamma$ , denoted  $d(x, \Gamma)$ , which poses a major difficulty. Notice that it may also be of order  $\sqrt{d}$ , as in the case of shear flows [13]. Throughout the paper, we consider the case  $\varrho(x) \approx d(x, \Gamma)$  as  $x \rightarrow \Gamma$ .

A similar problem was already considered in the case of some LES models [4, 5, 25, 26], where the eddy viscosity is proportional to  $\varrho^2$  (instead of  $\varrho$  as in the TKE model), leading to challenging open problems. In these papers, the authors suggest relaxing this law and taking an eddy viscosity proportional to  $\varrho^\alpha$ , for some  $0 < \alpha < 2$ . The same situation occurs in the case of RANS problems. Therefore, in this paper, we aim to relax the law given by (1.1) by taking

$$(1.3) \quad \nu_t(k) = \varrho^\eta \tilde{\nu}(k),$$

for  $0 < \eta < 1$ , and we study the coupling of the resulting steady-state Navier-Stokes equation to (1.2), within the scope of the viscous approximations posed by the question of whether or not we can neglect the molecular viscosity in comparison with the turbulent viscosity.

To be more specific, let  $\Omega$  be a  $\mathcal{C}^{0,1}$  bounded open subset of  $\mathbb{R}^N$ , where  $N = 2, 3$ ,  $0 < \eta < 1$ ,  $0 < \beta < 1$ , and  $0 < \lambda$ . As mentioned earlier,  $\varrho : \Omega \rightarrow \mathbb{R}_+^*$  is a  $\mathcal{C}^\infty$  function equivalent to  $d(x, \Gamma)$  near  $\Gamma$  (see the precise assumption in Section 2.1). We consider the following stationary system, where  $p$  denotes the pressure<sup>12</sup>:

$$(1.4) \quad \begin{cases} \lambda\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu}(k)\nabla\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k)\nabla k) + \frac{k|k|^{1/2}}{\varrho} = \varrho^\eta \tilde{\nu}(k)|\nabla\mathbf{u}|^2 & \text{in } \Omega, \\ k \geq 0 & \text{in } \Omega, \\ u = k = 0 & \text{on } \Gamma. \end{cases}$$

<sup>1</sup>We replace  $D\mathbf{u}$  by  $\nabla\mathbf{u}$  for simplicity without loss of generality, thanks to the Korn inequality.

<sup>2</sup>Still, for simplicity, we consider homogeneous boundary conditions instead of the usual nonlinear laws at the boundary given by the turbulence modeling framework.

The term  $\lambda \mathbf{u}$  is a stabilization term, which can be seen as a discrete evolution term coming from a finite difference and plays an important role in the proofs of the main results. We next introduce the approximated system obtained by viscous regularization, where  $\varepsilon > 0$ ,

$$(1.5) \quad \begin{cases} \lambda \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}) - \varepsilon \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) - \varepsilon \Delta k + \frac{k|k|^{1/2}}{\varrho + \varepsilon} = \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 & \text{in } \Omega, \\ k \geq 0 & \text{in } \Omega, \\ u = k = 0 & \text{on } \Gamma. \end{cases}$$

For this system, we know from [18] the existence of a weak solution in an appropriate sense. We transcribe the question "Is the molecular diffusion term negligible compared to the turbulent diffusion term?" by "Do the solutions of the system (1.5) converge towards a solution -in some weak sense- of the system (1.4)" when  $\varepsilon$  goes to 0?

As we will see, the situation is seriously complicated. We only get a positive answer at the cost of severe restrictions on the exponents  $\eta$  and  $\beta$ , i.e., when  $0 < \eta < \frac{1}{2}$  and  $0 < \beta < \frac{1}{3}$  in dimension 2, and  $0 < \eta, \beta < \frac{1}{5}$  in dimension 3. A specific statement is given by Theorem 4.2 below, which is the main result of this paper.

The paper is structured into three main sections and organized as follows. In the first section, we focus on the study of the steady Navier-Stokes equations with an eddy viscosity, as given in (1.3), for a specified  $k$ . To achieve this, we meticulously define the function space in which we seek solutions. We employ the natural weighted Sobolev space of divergence-free vector fields, deduced from the energy balance. Utilizing results from Kufner [14], we embed this space into  $W^{1,p}$  for  $1 \leq p < p_\eta = \frac{2}{1+\eta}$ , and subsequently into  $H^{1-\eta/2}$  through an interpolation argument, following Amrouche-Moussaoui [1].

Once the functional framework is established, we demonstrate the existence of a weak solution (see Definition 2.1 below), obtained as a limit of the viscous regularization (SOLA<sup>3</sup>) by introducing the term  $-\varepsilon \Delta u$ , for  $0 < \varepsilon$ . However, for the coupling with the TKE equation, strong convergence of the approximations in our weighted space is necessary, and the obtained (SOLA) must satisfy the energy balance. This requirement holds only if  $0 < \eta < 1/2$  when  $N = 2$  and  $0 < \eta < 1/5$  when  $N = 3$ .

In the second section, our focus shifts to the TKE equation for a fixed vector field  $\mathbf{u}$ , especially in deriving a priori estimates in regular Sobolev spaces. This task is not straightforward due to the presence of the right-hand side in  $L^1$  and the degeneracy of the diffusive term at the boundary. To address this, we continue to use a weighted Sobolev space and adapt the ladder method initially introduced by Boccardo-Gallouët [6]. This adaptation stands as one of the principal contributions of this paper.

The third section synthesizes the findings from the previous two sections. Here, we successfully take the limit in the viscous regularizations (1.5) as  $\varepsilon$  tends to 0 and establish the main theorem, namely Theorem 4.2. This also verifies the existence of a weak solution to system (1.4), namely a (SOLA). It is important to note that the equation for TKE only holds in the sense of distributions, and achieving better results would require further reductions in the size of the  $\beta$  parameter.

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<sup>3</sup>SOLution Obtained by Approximations

## 2 Velocity equation with a fixed TKE

### 2.1 General setting and space function

Let  $\Omega$  be a  $\mathcal{C}^{0,1}$  bounded open subset of  $\mathbb{R}^N$  ( $N = 2, 3$ ),  $\Gamma$  its boundary,  $\varrho : \Omega \rightarrow \mathbb{R}_+$  a  $\mathcal{C}^\infty$  function equivalent to the distance to the boundary, which satisfies

$$(2.1) \quad \lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1,$$

$$(2.2) \quad \forall n > 0, \quad \varrho_n = \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0$$

The functions  $\tilde{\nu}, \tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, bounded by above and by below, namely

$$(2.3) \quad \forall k \in \mathbb{R}, \quad 0 < \nu_m \leq \tilde{\nu}(k), \tilde{\mu}(k) \leq \nu_M,$$

We start by studying the velocity equation for a fixed TKE function  $k \in L^1(\Omega)$ , and a source term  $\mathbf{f} \in L^2(\Omega)^N$ . The corresponding Navier-Stokes equations are as follows:

$$(2.4) \quad \begin{cases} \lambda \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div} (\varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{at } \Gamma. \end{cases}$$

To obtain an *a priori* estimate, we formally take the dot product of the momentum equation with  $\mathbf{u}$  and integrate by parts. A standard calculation, utilizing the no-slip boundary condition and incompressibility, yields  $((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{u}) = 0$  and  $(\nabla p, \mathbf{u}) = 0$ , resulting in the energy balance:

$$(2.5) \quad \lambda \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 = (\mathbf{f}, \mathbf{u}).$$

Consequently, as  $\mathbf{f} \in L^2(\Omega)^N$ , we deduce from (2.3) the *a priori* estimate:

$$(2.6) \quad \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} \varrho^\eta |\nabla \mathbf{u}|^2 \leq C = \frac{\|\mathbf{f}\|_{0,2;\Omega}^2}{2\lambda \min(\lambda/2, \nu_m)}.$$

This suggests to introduce the space function  $V_\eta$  defined as the closure of  $\mathcal{V}$  for the norm

$$(2.7) \quad \|\mathbf{u}\|_{1,2;\varrho,\eta} = \left( \|\mathbf{u}\|_{0,2,\Omega}^2 + \|\varrho^{\eta/2} \nabla \mathbf{u}\|_{0,2,\Omega}^2 \right)^{1/2},$$

where

$$(2.8) \quad \mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega)^N / \operatorname{div} \mathbf{u} = 0\}.$$

In particular,  $V_\eta$  is a Hilbert space with the scalar product

$$(2.9) \quad (\mathbf{u}, \mathbf{v})_{1,2;\varrho,\eta} = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \nabla \mathbf{u} : \nabla \mathbf{v}.$$

The norm given by (2.7) corresponds to the  $H^1$ -norm when  $\eta = 0$ . Consequently, the  $V_\eta$  functions are not likely to be in  $H^1$ . The following result from [1] gives  $H^s$ -regularity for  $1/2 < s \leq 1$ .

**Theorem 2.1.** *Let  $\eta \in ]0; 1[$ , the continuous embedding*

$$(2.10) \quad V_\eta \hookrightarrow H^{1-\frac{\eta}{2}}(\Omega)^N$$

*stands.*

*Consequently, the Sobolev embedding theorem yields the compact embedding*

$$(2.11) \quad V_\eta \hookrightarrow L^r(\Omega)^N,$$

*for all*

$$(2.12) \quad r < r_\eta^* := \frac{2N}{N + \eta - 2} = \begin{cases} \frac{4}{\eta} & \text{if } N = 2, \\ \frac{6}{1 + \eta} & \text{if } N = 3. \end{cases}$$

We present the proof of Theorem 2.1 in Appendix A for the sake of self-containment. However, this result does not guarantee that gradients of functions in  $V_\eta$  belong to certain  $L^q$  spaces, which, nevertheless, is necessary. To address this, we will rely on a result from [14, Proposition 6.5], as stated below.

**Theorem 2.2.** *Assume that  $\Omega$  satisfies the cone property, and let  $\eta \in ]0, 1[$ . For any  $p$  such that*

$$(2.13) \quad 1 \leq p < \frac{2}{1 + \eta} =: p_\eta,$$

*the following continuous embedding holds*

$$(2.14) \quad V_\eta \hookrightarrow W_0^{1,p}(\Omega)^N.$$

**Remark 2.1.** *We deduce from Theorem 2.2 that vector fields in  $V_\eta$  have a trace at  $\Gamma$  which is equal to 0, giving a sense to the no slip boundary condition for vector fields in  $V_\eta$ .*

**Remark 2.2.** *The critical exponent  $p_\eta^*$  is given by*

$$(2.15) \quad q < p_\eta^* := \frac{2N}{N(1 + \eta) - 2} = \begin{cases} \frac{2}{\eta} & \text{if } N = 2, \\ \frac{6}{1 + 3\eta} & \text{if } N = 3. \end{cases}$$

*Therefore, Theorem 2.1 establishes a superior Lebesgue regularity for vector fields in  $V_\eta$  compared to Theorem 2.2. Specifically, we have  $V_\eta \hookrightarrow L^q(\Omega)^N$  for a given  $q > 2$ , where  $N = 2, 3$ .*

**Remark 2.3.** *The following Poincaré inequality holds, for all  $p < p_\eta$  and all  $\mathbf{u} \in V_\eta$ ,*

$$(2.16) \quad \|\mathbf{u}\|_{0,p,\Omega} \leq C \|\mathbf{u}\|_{1,2;\varrho,\eta},$$

*where  $C = C(p, \Omega)$ .*

Throughout the rest of the paper, we assume that  $\Omega$  satisfies the cone property, in order to verify (2.14).

**Remark 2.4.** Let  $\eta \in ]0; 1[$ . The space  $V_\eta$  is a distributional space, and in particular we conjecture that

$$(2.17) \quad V_\eta = \left\{ \mathbf{u} \in \mathcal{D}'(\Omega)^N, \operatorname{div} \mathbf{u} = 0, \mathbf{u} \in L^2(\Omega)^N, \varrho^{\eta/2} \nabla \mathbf{u} \in L^2(\Omega)^{N \times N}, \mathbf{u}|_\Gamma = 0 \right\}.$$

Unfortunately, we were not able to prove this result, which is open.

We also denote in what follows

$$(2.18) \quad V_0 = \{ \mathbf{u} \in H_0^1(\Omega)^N; \operatorname{div} \mathbf{u} = 0 \}.$$

This space is well-defined, and  $\mathcal{V}$  is dense in it.

## 2.2 Weak solutions and existence result

We begin by presenting the weak formulation of Problem (2.4), excluding the pressure as it involves free divergence vector fields as test functions. However, it is essential to account for the transport term, which gets as  $\operatorname{div} \mathbf{u} = 0$ ,

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \operatorname{div} (\mathbf{u} \otimes \mathbf{u}),$$

and which formally acts in the sense of the distributions as

$$(2.19) \quad \forall \varphi \in \mathcal{D}'(\Omega)^N, \quad \langle \operatorname{div} (\mathbf{u} \otimes \mathbf{u}), \varphi \rangle = - \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi^T.$$

We deduce from (2.12) that for all  $0 < \eta < 1$ ,  $r_\eta^* > 2$ . Therefore,

$$\forall \mathbf{u} \in V_\eta, \forall q \in ]2; r_\eta^*[ , \quad \mathbf{u} \otimes \mathbf{u} \in L^{q/2}(\Omega)^{N \times N}.$$

Hence, for any  $\delta \in [2; r_\eta^*[$  we are led to take tests  $\mathbf{v}$  such that

$$(2.20) \quad \mathbf{v} \in \bigcup_{\delta < q < r_\eta^*} \left( V_\eta \cap W_0^{1, (q/2)'}(\Omega)^N \right) =: H_{\eta, \delta}$$

In particular, when  $\mathbf{v} \in H_{\eta, \delta}$ ,

$$\mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} \in L^1(\Omega),$$

as well as

$$\lambda \mathbf{u} \cdot \mathbf{v} + \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} \in L^1(\Omega),$$

since  $\tilde{\nu}(k) \in L^\infty(\Omega)$ , and  $\mathbf{f} \cdot \mathbf{v} \in L^1(\Omega)$ . Unfortunately, in view of the construction of weak solutions to Problem (2.4), we must reduce the space  $H_{\eta, \delta}$  a little bit for technical reasons that will be cleared in the following. Let  $r > 1$ , and let us consider  $\mathcal{V}$  given by (2.8), equipped with the norm

$$\|\mathbf{v}\|_{\eta, r} = \|\mathbf{v}\|_{1, 2; \varrho, \eta} + \|\mathbf{v}\|_{1, r, \Omega},$$

and let  $K_{\eta, r}$  denotes the closure of  $\mathcal{V}$  for the norm  $\|\cdot\|_{\eta, r}$ . Notice that we naturally have

$$(2.21) \quad K_{\eta, r} \hookrightarrow V_\eta.$$

The tests vector fields set we consider is given by

$$(2.22) \quad W_{\eta, \delta} = \bigcup_{\delta < q < r_\eta^*} K_{\eta, (q/2)'},$$

which is a subset of  $H_{\eta, \delta}$ .

**Remark 2.5.** We do not know whether, in general,  $W_{\eta,\delta} = H_{\eta,\delta}$ . However, as we shall see in the following section, we have  $W_{\eta,\delta} = V_\eta$  for  $0 < \eta < \eta_c$  and  $\delta_0 < \delta < r_\eta^*$ .

The fact that the set of vector field tests is labeled by a parameter  $\delta$  implies that the notion of a weak solution depends on this parameter. This suggests the following definition of a weak solution.

**Definition 2.1.** We say that  $\mathbf{u} \in V_\eta$  is a  $\delta$ -weak solution to Problem (2.4) if  $\forall \mathbf{v} \in W_{\eta,\delta}$ ,

$$(2.23) \quad \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} - \int_{\Omega} (\mathbf{u} \otimes \mathbf{u}) : (\nabla \mathbf{v})^T + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

**Remark 2.6.** Let  $2 \leq \delta_1 < \delta_2 < r_\eta^*$ . Then obviously  $W_{\eta,\delta_2} \subset W_{\eta,\delta_1}$ . Therefore any  $\delta_1$ -weak solution is a  $\delta_2$ -weak solution. However, it is not clear that a  $\delta_2$ -weak solution is a  $\delta_1$ -weak solution.

**Theorem 2.3.** Let  $\eta \in ]0; 1[$ ,  $k \in L^1(\Omega)$ ,  $\mathbf{f} \in L^2(\Omega)^N$ ,  $\delta \in [2; r_\eta^*[$ . Then Problem (2.4) has a  $\delta$ -weak solution obtained by approximation (SOLA).

*Proof.* We argue using viscous regularization, with the proof being divided into three steps. First, we introduce the viscous regularization we consider, and we obtain a uniform estimate in  $V_\eta$  for the resulting equations. Then we extract subsequences by weak compactness and Theorem 2.2. Finally, we pass to the limit as the viscous parameter  $\varepsilon$  goes to 0.

*Step 1. Approximations and estimate.* We argue by singular perturbations. Let  $\varepsilon > 0$ . According to standard results (see for instance in [7]), we know the existence of  $\mathbf{u}_\varepsilon \in V_0$  such that for any  $\mathbf{v} \in V_0$

$$(2.24) \quad \lambda \int_{\Omega} \mathbf{u}_\varepsilon \cdot \mathbf{v} - \int_{\Omega} (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : (\nabla \mathbf{v})^T + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} + \varepsilon \int_{\Omega} \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Taking  $\mathbf{v} = \mathbf{u}_\varepsilon$  as test, we get the following energy balance, since the transport term vanishes,

$$(2.25) \quad \lambda \int_{\Omega} |\mathbf{u}_\varepsilon|^2 + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}_\varepsilon|^2 + \varepsilon \int_{\Omega} |\nabla \mathbf{u}_\varepsilon|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_\varepsilon.$$

In particular, as  $\mathbf{f} \in L^2(\Omega)^N$ , the family  $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$  is bounded in  $V_\eta$ .

*Step 2. Extracting subsequences.* From the bound in  $V_\eta$ , we deduce the existence of a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  and  $\mathbf{u} \in V_\eta$  such that:

- 1) The sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $\mathbf{u}$  in  $V_\eta$ ,
- 2) The sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $\mathbf{u}$  in  $H^{1-\frac{\eta}{2}}(\Omega)^N$ ,
- 3) The sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  strongly converges to  $\mathbf{u}$  in  $L^r(\Omega)^N$  for all  $q < r_\eta^*$ , where  $r_\eta^*$  is given by (2.12),
- 4) The sequence  $(\nabla \mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $\nabla \mathbf{u}$  in  $W_0^{1,p}(\Omega)^{N \times N}$ , for all  $p < p_\eta$ .

Item 1) is straightforward. Items 2) 3) and 4) require additional comments. Indeed, let  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  which weakly converges to  $\mathbf{u}$  in  $V_\eta$ , as given by item 1. As the sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  is bounded in  $H^{1-\frac{\eta}{2}}(\Omega)^N$  by Theorem 2.1, we can extract a subsequence, still



denoted  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ , which weakly converges to some  $\mathbf{v} \in H^{1-\frac{\eta}{2}}(\Omega)$ , and by the way in  $\mathcal{D}'(\Omega)$ . But as  $\mathbf{u}_{\varepsilon_n} \rightarrow \mathbf{u}$  in  $V_\eta$ -weak, by (2.2) convergence in  $\mathcal{D}'(\Omega)$  also holds. Therefore by the uniqueness of the limit,  $\mathbf{u} = \mathbf{v}$ . Items 3) and 4) stand by a similar argument.

*Step 3. Passing to the limit in the equations.* Let us take  $\mathbf{v} \in \mathcal{V}$  as test in (2.24). It is easily checked that

$$(2.26) \quad \lambda \int_{\Omega} \mathbf{u}_{\varepsilon_n} \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v},$$

$$(2.27) \quad \int_{\Omega} (\mathbf{u}_{\varepsilon_n} \otimes \mathbf{u}_{\varepsilon_n}) : (\nabla \mathbf{v})^T \xrightarrow{n \rightarrow +\infty} \int_{\Omega} (\mathbf{u} \otimes \mathbf{u}) : (\nabla \mathbf{v})^T.$$

It remains to deal with the term  $\varepsilon_n \int_{\Omega} \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v}$ . As  $\nabla \mathbf{v} \in L^\infty(\Omega)$ , and in particular  $\nabla \mathbf{v} \in L^q(\Omega)^{N \times N}$  for any  $q > (p_\eta)'$ . Therefore by item 4),

$$\int_{\Omega} \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v},$$

hence

$$\varepsilon_n \int_{\Omega} \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} 0.$$

In conclusion,  $\mathbf{u}$  satisfies (2.23) for any  $\mathbf{v} \in \mathcal{V}$ . Let  $\mathbf{v} \in W_{\eta, \delta}$ . There exists  $q \in ]\delta; r_\eta^*[$  such that  $\mathbf{v} \in K_{\eta, (q/2)'}$ . It follows the existence of a sequence  $(\mathbf{v}_n)_{n \in \mathbb{N}}$  of vector fields in  $\mathcal{V}$  which converges to  $\mathbf{v}$  in both  $V_\eta$  and  $W_0^{1, (q/2)' }(\Omega)^N$ , which explains the reason of the special construction of  $W_{\eta, \delta}$ . Taking  $\mathbf{v}_n$  as test in (2.23), and passing to the limit when  $n \rightarrow +\infty$  yields  $\mathbf{u}$  satisfies (2.23) for any  $\mathbf{v} \in W_{\eta, \delta}$ .  $\square$

**Remark 2.7.** *By De Rham Theorem, we know the existence of a pressure  $p \in L_{loc}^2(\Omega)$  such that (2.4) holds in  $\mathcal{D}'(\Omega)$ .*

### 2.3 Energy balance and strong convergence

In this section, we address the question of the strong convergence in  $V_\eta$  of the approximations to Problem (2.23) constructed earlier. Our initial objective is to determine whether a  $\delta$ -weak solution satisfies the energy balance for a suitable choice of  $\delta$  and  $\eta$ . The following results hold.

**Lemma 2.1.** *Let  $\eta \in ]0; \eta_c[$ , where*

$$(2.28) \quad \eta_c = \begin{cases} 1/2 & \text{if } N = 2, \\ 1/5 & \text{if } N = 3. \end{cases}$$

*Then there exists  $\delta_0 \in ]2; r_\eta^*[$  such that for all  $\delta \in ]\delta_0; r_\eta^*[$ , any  $\delta$ -weak solution  $\mathbf{u}$  of Problem (2.23) satisfies the energy balance (2.5).*

*Proof.* The energy balance holds when we take  $\mathbf{u}$  itself as test vector field. This is the case when

$$(2.29) \quad W_{\eta, \delta} = V_\eta,$$

knowing already by that  $W_{\eta, \delta} \subset V_\eta$ . According to Theorem 2.2, a function  $\mathbf{v} \in V_\eta$  satisfies  $\nabla \mathbf{v} \in L^p(\Omega)^{N \times N}$  for  $p \in [1; p_\eta[$ .

Therefore, (2.29) is satisfied if

$$\left(\frac{r_\eta^*}{2}\right)' < p_\eta,$$

which yields the bound (2.28) after elementary calculations. Moreover, as  $[p \mapsto p']$  is non-increasing and continuous, there exists  $\delta_0 \in ]2; r_\eta^*[$ , such that

$$\left(\frac{r_\eta^*}{2}\right)' < \left(\frac{\delta_0}{2}\right)' < p_\eta.$$

Therefore, for  $\delta_0 < q < r_\eta^*$  we have  $\left(\frac{q}{2}\right)' < p_\eta$  and we deduce from Theorem 2.2,

$$V_\eta \subset W_0^{(q/2)'}(\Omega)^N.$$

Therefore, by (2.2)

$$\|\mathbf{v}\|_{\eta, (q/2)', \Omega} \leq C \|\mathbf{v}\|_{1, 2; \varrho, \eta},$$

which means  $V_\eta \hookrightarrow K_{\eta, (q/2)'}$ , hence  $V_\eta = K_{\eta, (q/2)'}$  by (2.21), leading to (2.29).  $\square$

**Remark 2.8.** *After some calculus, we find for optimal value of  $\delta_0$ :*

$$(2.30) \quad \begin{cases} \delta_0 = \frac{4}{2-\eta} & \text{when } N = 2, \\ \delta_0 = \frac{12}{5-3\eta} & \text{when } N = 3. \end{cases}$$

We are now able to prove the following convergence result.

**Theorem 2.4.** *Let  $\eta \in ]0; \eta_c[$ ,  $\delta \in ]\delta_0; r_\eta^*[$ , and let  $(k_\varepsilon)_{\varepsilon>0}$  be a family in  $L^1(\Omega)$  which converges in  $L^1(\Omega)$  to some  $k$ . Let  $\mathbf{u}_\varepsilon \in V_0$  such that for any  $\mathbf{v} \in V_0$*

$$(2.31) \quad \lambda \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{v} - \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : (\nabla \mathbf{v})^T + \int_\Omega (\varrho^\eta + \varepsilon) \tilde{\nu}(k_\varepsilon) \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

*Then there exists a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  such that the sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  strongly converges in  $V_\eta$  to a  $\delta$ -weak solution  $\mathbf{u} \in V_\eta$  of Problem (2.4) which in addition satisfies the energy balance (2.5). Moreover, the sequence  $(\varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2)_{n \in \mathbb{N}}$  converges strongly in  $L^1(\Omega)$  to  $\varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2$ .*

*Proof.* We argue in two steps. First, following the proof of Theorem 2.3, we get the existence of a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$ , such that the sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges in  $V_\eta$  to a  $\delta$ -weak solution  $\mathbf{u} \in V_\eta$  of Problem (2.4). Then we prove the strong convergence in  $V_\eta$  by the energy method.

*Step 1.* The only addition to be made, is to check how to pass to the limit in the diffusion term. We first notice that the sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  can be chosen such that the sequence  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges a.e. to  $k$ , by the inverse Lebesgue Theorem. Let  $\mathbf{v} \in \mathcal{V}$ ,  $\delta < q < r_\eta^*$ . Then

$$(2.32) \quad \varrho^{\eta/2} \nabla \mathbf{u}_{\varepsilon_n} \xrightarrow{n \rightarrow +\infty} \varrho^{\eta/2} \nabla \mathbf{u} \quad \text{weakly in } L^2(\Omega)^{N \times N},$$

while

$$(2.33) \quad \varrho^{\eta/2} \tilde{\nu}(k_{\varepsilon_n}) \nabla \mathbf{v} \xrightarrow{n \rightarrow +\infty} \varrho^{\eta/2} \tilde{\nu}(k) \nabla \mathbf{v} \quad \text{a.e. in } \Omega,$$

because  $\tilde{\nu}$  is a continuous function. Therefore, since

$$|\varrho^{\eta/2}\tilde{\nu}(k_{\varepsilon_n})\nabla\mathbf{v}| \leq \nu_M|\varrho^{\eta/2}\tilde{\nu}(k_{\varepsilon_n})\nabla\mathbf{v}| \in L^2(\Omega),$$

by (2.3), then the convergence in (2.33) also holds in  $L^2(\Omega)^{N \times N}$ , by Lebesgue Theorem. In conclusion

$$\int_{\Omega} \varrho^{\eta}\tilde{\nu}(k_{\varepsilon_n})\nabla\mathbf{u}_{\varepsilon_n} : \nabla\mathbf{v} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k)\nabla\mathbf{u} : \nabla\mathbf{v}.$$

The rest is as in the proof of Theorem 2.3, the energy balance being satisfied by Lemma (2.28).

*Step 2.* It remains to prove the strong convergence. To do so we use the so called "energy method". We deduce from (2.32) and the same argument as above that

$$(2.34) \quad \varrho^{\eta/2}\sqrt{\tilde{\nu}(k_{\varepsilon_n})}\nabla\mathbf{u}_{\varepsilon_n} \xrightarrow{n \rightarrow +\infty} \varrho^{\eta/2}\sqrt{\tilde{\nu}(k)}\nabla\mathbf{u} \quad \text{weakly in } L^2(\Omega)^{N \times N}.$$

Therefore,

$$(2.35) \quad \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k)|\nabla\mathbf{u}|^2 \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k_{\varepsilon_n})|\nabla\mathbf{u}_{\varepsilon_n}|^2.$$

Now, we write the energy balances satisfied by  $\mathbf{u}$  and  $\mathbf{u}_{\varepsilon_n}$ :

$$(2.36) \quad \begin{cases} \lambda \int_{\Omega} |\mathbf{u}_{\varepsilon_n}|^2 + \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k_{\varepsilon_n})|\nabla\mathbf{u}_{\varepsilon_n}|^2 + \varepsilon_n \int_{\Omega} |\nabla\mathbf{u}_{\varepsilon_n}|^2 = \langle \mathbf{f}, \mathbf{u}_{\varepsilon_n} \rangle \\ \lambda \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k)|\nabla\mathbf{u}|^2 = \langle \mathbf{f}, \mathbf{u} \rangle. \end{cases}$$

As  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  converges to  $\mathbf{u}$  strongly in  $L^2(\Omega)^N$ ,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\mathbf{u}_{\varepsilon_n}|^2 = \int_{\Omega} |\mathbf{u}|^2, \quad \lim_{n \rightarrow \infty} \langle \mathbf{f}, \mathbf{u}_{\varepsilon_n} \rangle = \langle \mathbf{f}, \mathbf{u} \rangle.$$

Therefore, by (2.36), we have

$$(2.37) \quad \lim_{n \rightarrow \infty} \left( \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k_{\varepsilon_n})|\nabla\mathbf{u}_{\varepsilon_n}|^2 + \varepsilon_n \int_{\Omega} |\nabla\mathbf{u}_{\varepsilon_n}|^2 \right) = \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k)|\nabla\mathbf{u}|^2,$$

which combined with (2.35) leads to

$$(2.38) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k_{\varepsilon_n})|\nabla\mathbf{u}_{\varepsilon_n}|^2 = \int_{\Omega} \varrho^{\eta}\tilde{\nu}(k)|\nabla\mathbf{u}|^2, \quad \text{and} \quad \lim_{n \rightarrow \infty} \varepsilon_n \int_{\Omega} |\nabla\mathbf{u}_{\varepsilon_n}|^2 = 0.$$

Therefore, combining (2.34) with (2.38) yields the strong convergence in  $L^2(\Omega)^{N \times N}$  of  $(\varrho^{\eta/2}\sqrt{\tilde{\nu}(k_{\varepsilon_n})}\nabla\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  to  $\varrho^{\eta/2}\sqrt{\tilde{\nu}(k)}\nabla\mathbf{u}$ , hence the strong convergence in  $L^1(\Omega)$  of the sequence  $(\varrho^{\eta}\tilde{\nu}(k_{\varepsilon_n})|\nabla\mathbf{u}_{\varepsilon_n}|^2)_{n \in \mathbb{N}}$  to  $\varrho^{\eta}\tilde{\nu}(k)|\nabla\mathbf{u}|^2$ . Moreover, by the standard argument based on the continuity of  $\tilde{\nu}$ , (2.3) and the a.e. convergence of  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  to  $k$ , we get

$$(2.39) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \varrho^{\eta}|\nabla\mathbf{u}_{\varepsilon_n}|^2 = \int_{\Omega} \varrho^{\eta}|\nabla\mathbf{u}|^2,$$

which combined with (2.32) and the  $L^2$ -strong convergence of  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  to  $\mathbf{u}$ , yields the strong convergence in  $V_{\eta}$ .  $\square$

### 3 TKE with a fixed velocity

In this section, we tackle a substantial challenge. We aim to obtain estimates in the spirit of Boccardo-Gallouët [6] for the equation involving  $k$ . In addition to the typical difficulty arising from the source term in  $L^1$ , we must confront the degeneracy of the diffusion coefficient at the boundary, introducing additional challenges.

Let  $\beta \in ]0; 1[$ , and

$$\mathbb{D} \in L^1(\Omega), \quad 0 \leq \mathbb{D} \text{ a.e. in } \Omega.$$

We consider the following TKE equation, satisfied by  $k = k(x)$ :

$$(3.1) \quad \begin{cases} \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) + \frac{k|k|^{1/2}}{\varrho} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

**Remark 3.1.** Equation (3.1) can be written as

$$\mathbf{u} \cdot \nabla k - \varrho^\beta \tilde{\mu}(k) \Delta k - \nabla(\varrho^\beta \tilde{\mu}(k)) \cdot \nabla k + \frac{k|k|^{1/2}}{\varrho} = \mathbb{D}.$$

Therefore as  $\varrho > 0$  inside  $\Omega$ , the strong maximum principle method used in [20] can be adapted to this case, so that any  $\mathcal{C}^2$ -strong solution  $k$  to (3.1) is such that  $k > 0$  in  $\Omega$ .

Throughout this section, we assume

$$(3.2) \quad \|\mathbb{D}\|_{0,1;\Omega} \leq D = \frac{\|\mathbf{f}\|_{0,2;\Omega}^2}{2\lambda}.$$

**Remark 3.2.** In the system we are looking at,  $\mathbb{D} = \mathbb{D}(\mathbf{u}, k) = \varrho^\eta \tilde{\mu}(k) |\nabla \mathbf{u}|^2$ , which satisfies (3.2) thanks to the energy balance (2.5).

**Theorem 3.1.** Assume  $\mathbf{u} \in \mathcal{C}^0(\Omega)$ , and let  $k \in \mathcal{C}^2(\bar{\Omega})$  be any strong solution to (3.1). Then for any

$$(3.3) \quad \beta < \frac{1}{2N-1} = \beta_c, \quad q < q_\beta = N'(p_\beta - 1) = \frac{N'(1-\beta)}{1+\beta},$$

where  $p_\beta = \frac{2}{1+\beta}$ , we have

$$(3.4) \quad \|k\|_{1,q;\Omega} + \|\varrho^{-1} k^{3/2}\|_{0,1;\Omega} \leq C = C(D, \beta).$$

*Proof. Step 1.* A priori estimate over the sets  $\{n \leq k < n+1\}$ . Let  $n \in \mathbf{N}^*$ ,  $H_n : \mathbb{R} \rightarrow \mathbb{R}$  be the odd function given by

$$(3.5) \quad H_n : t \mapsto \begin{cases} 0 & \text{if } 0 \leq t < n, \\ t - n & \text{if } n \leq t < n+1, \\ 1 & \text{if } t \geq n+1. \end{cases}$$

Notice that  $H_n$  is continuous, piecewise  $\mathcal{C}^1$ , the derivative of which has a finite number of discontinuities and  $H(0) = 0$ . Therefore, Stampacchia's Theorem [27] applies, in particular, for any  $\kappa \in W_0^{1,r}(\Omega)$ ,  $H_n(\kappa) \in W_0^{1,r}(\Omega)$ ,  $\nabla H_n(\kappa) = \nabla \kappa H_n'(\kappa)$  a.e and  $\nabla \kappa = 0$  over the level sets  $\{\kappa = C\}$  that are not of null measure.

Let  $k$  be a strong solution of (3.1). We take  $H_n(k)$  as test function in (3.1) and integrate over  $\Omega$ . Due to the boundary conditions, on one hand we have,

$$(3.6) \quad \int_{\Omega} H_n(k) \mathbf{u} \cdot \nabla k = 0,$$

(see in [18] for instance). On the other hand as  $H_n$  is an odd function that satisfies  $H_n(t) \geq 0$  over  $\mathbb{R}^+$ , then  $H_n(k)k \geq 0$ , which yields

$$(3.7) \quad \int_{\Omega} \frac{H_n(k)k|k|^{1/2}}{\varrho} \geq 0.$$

According to Stampacchia Theorem, Stokes formula yields

$$(3.8) \quad \int_{\Omega} \varrho^{\beta} \tilde{\mu}(k) \nabla k \cdot \nabla H_n(k) \leq \int_{\Omega} \mathbb{D}H_n(k) \leq D,$$

given that  $|H_n| \leq 1$ .

Let  $B_n \subset \Omega$  be defined for any  $n \in \mathbb{N}^*$  by (recall that  $k > 0$  in  $\Omega$ ),

$$B_n = \{x \in \Omega / n \leq k(x) < n + 1\}.$$

We obtain by Stampacchia's Theorem

$$(3.9) \quad \nabla k \cdot \nabla H_n(k) = |\nabla k|^2 \mathbb{1}_{B_n} \quad \text{a.e. } \Omega.$$

Inequality (3.8) yields, by (3.9),

$$(3.10) \quad \int_{B_n} \varrho^{\beta} |\nabla k|^2 \leq \frac{D}{\nu_m}.$$

The left-hand side of (3.8) can be transformed another way. Indeed, since  $H'_n(k) = \mathbb{1}_{B_n}$  a.e.  $\Omega$ , we have

$$(3.11) \quad \nabla k \cdot \nabla H_n(k) = |\nabla H_n(k)|^2 \mathbb{1}_{B_n} = |\nabla H_n(k)|^2 \quad \text{a.e. } \Omega,$$

which yields by (3.9)

$$(3.12) \quad |\nabla k|^2 \mathbb{1}_{B_n} = |\nabla H_n(k)|^2 \quad \text{a.e. } \Omega.$$

Thus, (3.10) becomes

$$(3.13) \quad \int_{\Omega} \varrho^{\beta} |\nabla H_n(k)|^2 \leq \frac{D}{\nu_m}.$$

Applying Theorem 2.2, we deduce from (3.13) that for any  $p < p_{\beta} = \frac{2}{1+\beta}$ , there exists a constant  $C = C(p, \beta, \nu_m, D) > 0$  such that

$$(3.14) \quad \int_{B_n} |\nabla k|^p = \int_{\Omega} |\nabla H_n(k)|^p \leq C.$$

*Step 2. Ladder process.* We deduce now the  $W^{1,p}$  estimate (3.4) from (3.14) by adapting to this case the Boccardo-Gallouët technique [6]. To do so, Let  $p \in ]1; p_{\beta}[$ ,  $q \in ]1; p[$ . We get by Hölder's Inequality

$$(3.15) \quad \int_{B_n} |\nabla k|^q \leq \left( \int_{B_n} |\nabla k|^p \right)^{q/p} |B_n|^{1-(q/p)} \leq C^{q/p} |B_n|^{1-(q/p)}.$$

Let  $n_0 \in \mathbb{N}^*$ . On one hand we have

$$(3.16) \quad \sum_{n=0}^{n_0-1} \int_{B_n} |\nabla k|^q \leq n_0 C^{q/p} |\Omega|^{1-(q/p)} \leq n_0 \max(C; |\Omega|).$$

On the other hand, let  $r \geq 1$ , the definition of  $B_n$  yields

$$(3.17) \quad \int_{B_n} |k|^r \geq n^r |B_n|.$$

We get by (3.15)

$$(3.18) \quad \sum_{n=n_0}^{+\infty} \int_{B_n} |\nabla k|^q \leq C^{q/p} \sum_{n=n_0}^{+\infty} |B_n|^{1-(q/p)} \leq C^{q/p} \sum_{n=n_0}^{+\infty} \frac{1}{n^{\frac{r(p-q)}{p}}} \left( \int_{B_n} |k|^r \right)^{\frac{p-q}{p}}.$$

Then the discrete Hölder Inequality yields

$$(3.19) \quad \sum_{n=n_0}^{+\infty} \int_{B_n} |\nabla k|^q \leq C^{q/p} \left[ \sum_{n=n_0}^{+\infty} \int_{B_n} |k|^r \right]^{\frac{p-q}{p}} \left[ \sum_{n=n_0}^{+\infty} \frac{1}{n^{\frac{r(p-q)}{q}}} \right]^{q/p}.$$

Inequality (3.19) brings a condition on  $r \geq 1$ . Indeed, the right-hand side of (3.19) is finite if and only if the Riemann series converges, which means

$$(3.20) \quad r > \frac{q}{p-q}.$$

Let the two constants  $\lambda_1(n_0)$  and  $\lambda_2(n_0)$  be defined by

$$\begin{cases} \lambda_1(n_0) = n_0 \max(C; |\Omega|) \\ \lambda_2(n_0) = C^{q/p} \left[ \sum_{n=n_0}^{+\infty} \frac{1}{n^{\frac{r(p-q)}{q}}} \right]^{q/p}, \end{cases}$$

with  $r$  satisfying (3.20). Consequently, (3.16) and (3.19) ensure the existence of two constants  $\lambda_1(n_0)$  and  $\lambda_2(n_0)$  such that  $\lambda_2(n_0) \xrightarrow{n_0 \rightarrow +\infty} 0$  and

$$(3.21) \quad \int_{\Omega} |\nabla k|^q \leq \lambda_1(n_0) + \lambda_2(n_0) \left( \int_{B_n} |k|^r \right)^{\frac{p-q}{p}} = \lambda_1(n_0) + \lambda_2(n_0) \|k\|_{0,r;\Omega}^{\frac{r(p-q)}{p}}.$$

In order to obtain an inequality involving the norm  $\|k\|_{1,q;\Omega}$ , the Sobolev embedding  $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$  must stand, which brings another condition on  $r$ :

$$(3.22) \quad 1 \leq r \leq q^* = \frac{Nq}{N-q}.$$

For both conditions (3.20) and (3.22) to be satisfied, one needs to have

$$(3.23) \quad \frac{q}{p-q} < \frac{Nq}{N-q} \quad \text{hence} \quad q < \frac{N(p-1)}{N-1}.$$

As this must be satisfied for any  $p < p_\beta$ , we get the following bound for  $q$ ,

$$(3.24) \quad q < N'(p_\beta - 1) = q_\beta.$$

Condition (3.24) makes sense if and only if  $\beta$  is such that

$$(3.25) \quad 1 < q_\beta < p_\beta \quad \text{hence} \quad 2 - \frac{1}{N} < p_\beta < N.$$

Since  $0 \leq \beta < 1$ , the condition  $p_\beta < N$  is always satisfied. However,  $p_\beta > 2 - \frac{1}{N}$  is equivalent to

$$(3.26) \quad \beta < \frac{1}{2N-1} =: \beta_c.$$

Conditions (3.26) and (3.24) then yield the bounds

$$(3.27) \quad \begin{aligned} \beta_c &= \frac{1}{3} \text{ and } q_\beta = 2 \left( \frac{2}{1+\beta} - 1 \right) & \text{if } N = 2 \\ \beta_c &= \frac{1}{5} \text{ and } q_\beta = \frac{3}{2} \left( \frac{2}{1+\beta} - 1 \right) & \text{if } N = 3. \end{aligned}$$

Therefore, if  $\beta < \beta_c$  and  $q < q_\beta$ , considering  $r$  satisfying (3.20) and (3.22) for all  $p < p_\beta$ ,  $q < q_\beta$ ,  $\beta < \beta_c$ , Inequality (3.21) becomes

$$(3.28) \quad \|k\|_{1,q;\Omega}^q \leq \lambda_1(n_0) + c\lambda_2(n_0) \|k\|_{1,q;\Omega}^{\frac{r(p-q)}{p}}.$$

As when  $0 < \beta < \beta_c$ , we verify  $\gamma = \frac{q_\beta^*(p_\beta - q_\beta)}{p_\beta q_\beta} < 1$ , then we deduce from (3.28) that for  $q < q_\beta$  and  $0 < \beta < \beta_c$ ,

$$(3.29) \quad \|k\|_{1,q;\Omega} \leq C = C(\beta, \nu_m, D),$$

a constant that blows up when  $q \rightarrow q_\beta$  or  $\beta \rightarrow \beta_c$ .

*Step 3.  $L^{3/2}(\Omega, \rho^{-1})$  estimate.* Let  $G_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  be the odd function given by

$$(3.30) \quad G_\varepsilon : \begin{cases} G_\varepsilon(x) = \frac{x}{\varepsilon} & \text{if } 0 \leq x < \varepsilon, \\ G_\varepsilon(x) = 1 & \text{if } x \geq \varepsilon. \end{cases}$$

We take  $G_\varepsilon(k)$  as test function in (3.1). By the same argument as above,

$$\int_{\Omega} \mathbf{u} \cdot \nabla k G_\varepsilon(k) = 0, \quad \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \cdot \nabla G_\varepsilon(k) = \int_{\Omega} G'_\varepsilon(k) \varrho^\beta \tilde{\mu}(k) |\nabla k|^2 \geq 0,$$

hence, since  $0 \leq G_\varepsilon(k) \leq 1$ ,

$$\int_{\Omega} \frac{G_\varepsilon(k) k^{\frac{3}{2}}}{\varrho} \leq \|\mathbb{D}\|_{0,1;\Omega}.$$

As  $k > 0$  and  $\varrho > 0$  in  $\Omega$ ,

$$\frac{G_\varepsilon(k) k^{\frac{3}{2}}}{\varrho} \rightarrow \frac{k^{\frac{3}{2}}}{\varrho}$$

a.e. in  $\Omega$ . Then by Fatou's Lemma,

$$(3.31) \quad 0 \leq \int_{\Omega} \frac{k^{\frac{3}{2}}}{\varrho} \leq \|\mathbb{D}\|_{0,1;\Omega}.$$

Notice that this last estimate does not require any restriction about  $\beta$ . □

**Remark 3.3.** Let  $T_n$  be the truncation at height  $n$ . Then by taking  $T_n(k)$  as test in (3.1), we see that  $T_n(k) \in W_0^{1,p}(\Omega)$ , uniformly in  $n$ , namely for any  $p < p_\beta$ ,

$$(3.32) \quad \|T_n(k)\|_{1,p;\Omega} \leq C(D, \nu_m, \beta).$$

**Remark 3.4.** We have assumed in Theorem 3.1  $\mathbf{u} \in \mathcal{C}^0(\Omega)$  and  $k \in \mathcal{C}^2(\bar{\Omega})$  for convenience. However such regularity is not crucial. Indeed, we obtain the same estimate if, instead of (3.1), we consider its following regularization, where  $\varepsilon > 0$ ,

$$(3.33) \quad \begin{cases} \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) - \Delta k + \frac{k|k|^{1/2}}{\varrho} = \mathbb{D} & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

for a given  $\mathbf{u} \in V_0$ . We know the existence of a distributional solution  $k$  to this equation such that:

$$k \in \bigcap_{r < N'} W_0^{1,r}(\Omega),$$

and such that

$$\forall n \in \mathbb{N}, \quad T_n(k) \in H_0^1(\Omega)$$

(see [18]). For this solution, tests can be taken in  $W_0^{1,\infty}(\Omega)$ , as well as functions of the form  $h(k)\varphi$ , where  $\varphi \in W_0^{1,\infty}(\Omega)$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, bounded,  $\mathcal{C}^1$ -piecewise, such that  $h'$  is in  $L^\infty(\mathbb{R})$  with a finite number of discontinuities. This justifies that the entire procedure above, to obtain estimates (3.4), also applies to this regularized equation considering only this distributional solution.

In the following we set

$$(3.34) \quad \mathbb{K}_\beta = \bigcap_{1 \leq q < q_\beta} W_0^{1,q}(\Omega).$$

## 4 Main result

Recall that  $\eta_c$  is given in (2.28),  $\beta_c$  in (3.3). According to the previous sections, the assumptions about the data are the following:

$$(4.1) \quad \begin{cases} \tilde{\nu}, \tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R} \text{ are continuous and } \forall k \in \mathbb{R}, \quad 0 < \nu_m \leq \tilde{\nu}(k), \tilde{\mu}(k) \leq \nu_M, \\ \mathbf{f} \in L^2(\Omega)^N, \quad \lambda > 0 \\ 0 < \eta < \eta_c, \quad \beta < \beta_c. \end{cases}$$

**Definition 4.1.** When (4.1) holds, we say that  $(\mathbf{u}, k) \in V_\eta \times \mathbb{K}_\beta$  is a weak solution to the initial problem (1.4) if  $k \geq 0$  a.e in  $\Omega$ , and if for all  $(\mathbf{v}, \varphi) \in V_\eta \times \mathcal{D}(\Omega)$ ,

$$(4.2) \quad \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} - \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : (\nabla \mathbf{v})^T + \int_\Omega \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

$$(4.3) \quad - \int_\Omega k \mathbf{u} \cdot \nabla \varphi + \int_\Omega \varrho^\beta \tilde{\mu}(k) \nabla k \nabla \varphi + \int_\Omega \frac{k^{3/2} \varphi}{\varrho} = \int_\Omega \mathbb{D}(\mathbf{u}, k) \varphi,$$

where  $\mathbb{D}(\mathbf{u}, k) = \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2$ .



**Remark 4.1.** As shown in section 2.3, the assumption about  $\eta$  makes sure that all the integrals in (4.2) are well defined. Moreover, by (2.5) and (3.2),  $\mathbb{D}(\mathbf{u}, k) \in L^1(\Omega)$ . Therefore, by (3.4), it is guaranteed that all the integrals in (4.3) are well defined, except the transport term about which we must take care. However,  $k \in L^{3/2}(\Omega)$  and as  $V_\eta \subset H_{loc}^1(\Omega)^N$ , then for a given  $\varphi \in \mathcal{D}(\Omega)$ , we have at least  $\mathbf{u} \in L^3(\text{supp}(\varphi))$ . Therefore,  $k \mathbf{u} \in L^1(\text{supp}(\varphi))$ , which makes well defined the integral  $\int_\Omega k \mathbf{u} \cdot \nabla \varphi$  since  $\nabla \varphi \in L^\infty(\Omega)$ .

**Definition 4.2.** We say that  $(\mathbf{u}, k) \in V_0 \times (\cap_{q < N'} W_0^{1,q}(\Omega))$  is a weak solution to the perturbed Problem (1.5) if  $k \geq 0$  a.e in  $\Omega$ , and if for all  $(\mathbf{v}, \varphi) \in V_0 \times \mathcal{D}(\Omega)$ ,

$$(4.4) \quad \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} - \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : (\nabla \mathbf{v})^T + \int_\Omega \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} + \varepsilon \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

$$(4.5) \quad - \int_\Omega k \mathbf{u} \cdot \nabla \varphi + \int_\Omega \varrho^\beta \tilde{\mu}(k) \nabla k \cdot \nabla \varphi + \varepsilon \int_\Omega \nabla k \cdot \nabla \varphi + \int_\Omega \frac{k^{3/2} \varphi}{\varrho + \varepsilon} = \int_\Omega \mathbb{D}(\mathbf{u}, k) \varphi.$$

By [18], we already know:

**Theorem 4.1.** Let  $\varepsilon > 0$ . Assume that  $\tilde{\nu}, \tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, bounded and nonnegative,  $\mathbf{f} \in V_0'$ ,  $\lambda \geq 0$ . Let  $0 \leq \eta, \beta \leq 1$ . Then Problem (1.5) has a weak solution  $(\mathbf{u}_\varepsilon, k_\varepsilon) \in V_0 \times (\cap_{q < N'} W_0^{1,q}(\Omega))$ .

We prove the following result:

**Theorem 4.2.** Assume that (4.1) holds. Then the initial Problem (1.4) has a weak solution obtained by approximation (SOLA). More precisely, let  $(\mathbf{u}_\varepsilon, k_\varepsilon)$  be a weak solution to the perturbed problem (1.5). Then there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0 and  $(\mathbf{u}, k) \in V_\eta \times \mathbb{K}_\beta$  a weak solution to the initial problem such that the sequence  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  strongly converges to  $\mathbf{u}$  in  $V_\eta$  and the sequence  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  weakly converges to  $k$  in  $W_0^{1,q}(\Omega)$  for all  $1 \leq q < q_\beta$ .

*Proof.* The proof is the synthesis of the sections 2 and 3. Passing to the limit in the fluid equation was already done in the proof of Theorem 2.3. Theorem 2.4 ensures the strong convergence of the source term in  $L^1(\Omega)$ . We already know estimates for the TKE from Theorem 3.1 and Remark 3.4. Therefore, we summarize in a first step the sequence extraction process, and we proceed in a second step to pass to the limit in the TKE equation.

*Step 1. Extracting subsequences* - Due to (4.1), particularly the choices of  $\eta$  and  $\beta$ , we know from step 1 of the proof of Theorem 2.3 and Theorem 3.1, that there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  that converges to 0, such that the family  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  is bounded in  $V_\eta$ , and the family  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  is bounded in  $W_0^{1,q}(\Omega)$  for any  $1 < q < q_\beta$  (recall that  $q_\beta$  is given by (3.3)). Arguing as Step 2 of the proof of Theorem 2.3, based on a standard compactness argument combined to an uniqueness argument and Lebesgue inverse Theorem, we deduce that there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$ ,  $\mathbf{u} \in V_\eta$  and  $k \in \mathbb{K}_\beta$  such that

1.  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly to  $\mathbf{u}$  in  $V_\eta$ ,
2.  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly to  $\mathbf{u}$  in  $W_0^p(\Omega)^N$  for all  $1 < p < p_\eta$ ,
3.  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  converges strongly to  $\mathbf{u}$  in  $L^q(\Omega)^N$  for all  $q < r_\eta^*$ ,
4.  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly to  $k$  in  $W_0^q(\Omega)$  for all  $1 < q < q_\beta$ ,
5.  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges strongly to  $k$  in  $L^r(\Omega)$  for all  $1 < r < q_\beta^*$ ,

6.  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges a.e. to  $k$ .

Moreover, by Theorem 2.4 and from the proof of Theorem 2.3, we can add to the list that  $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$  converges strongly to  $\mathbf{u}$  in  $V_\eta$ , and that  $(\mathbf{u}, k)$  satisfy (4.2). We also know that the sequence  $(\varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2)_{n \in \mathbb{N}}$  converges strongly in  $L^1(\Omega)$  to  $\varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2$ . Finally, as  $V_\eta \subset H_{loc}^1(\Omega)$ , by the same argument as in the proof of 2.3 based on the uniqueness of the limit,  $(\varepsilon_n)_{n \in \mathbb{N}}$  can be chosen so that for any  $\omega \subset\subset \Omega$ ,  $(\mathbf{u}_{\varepsilon_n}|_\omega)_{n \in \mathbb{N}}$  converges strongly to  $\mathbf{u}|_\omega$  in  $L^p(\Omega)^N$ , for all  $p < \infty$  when  $N = 2$ , and  $p < 6$  when  $N = 3$ .

*Step 2. Passing to the limit in the TKE equation* - Let  $\varphi \in \mathcal{D}(\Omega)$ , and let  $\omega \subset\subset \Omega$  denotes the support of  $\varphi$ . From the above, we already know that

$$\int_{\Omega} \mathbb{D}(\mathbf{u}_{\varepsilon_n}, k_{\varepsilon_n}) \varphi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \mathbb{D}(\mathbf{u}, k) \varphi.$$

By a usual argument [7, 18], we also have

$$\int_{\Omega} \varrho^\beta \tilde{\mu}(k_{\varepsilon_n}) \nabla k_{\varepsilon_n} \cdot \nabla \varphi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \cdot \nabla \varphi.$$

Moreover, let  $q \in ]1; q_\beta[$  be fixed. By the weak convergence of  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  to  $k$  in  $W_0^{1,q}(\Omega)$  to  $k$  and  $\mathcal{D}(\Omega) \subset W_0^{1,q'}(\Omega)$ ,

$$\int_{\Omega} \nabla k_{\varepsilon_n} \cdot \nabla \varphi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \nabla k \cdot \nabla \varphi.$$

hence

$$\varepsilon_n \int_{\Omega} \nabla k_{\varepsilon_n} \cdot \nabla \varphi \xrightarrow{n \rightarrow +\infty} 0.$$

Whether we are in dimension 2 or in dimension 3, we easily check that  $3/2 < q_\beta^*$ . Therefore, as  $\varrho|_\omega \geq C_\omega > 0$ ,

$$\frac{k_{\varepsilon_n}^{3/2}}{\varrho + \varepsilon_n} \xrightarrow{n \rightarrow +\infty} \frac{k^{3/2}}{\varrho} \quad \text{in } L^1(\omega),$$

which leads to

$$\int_{\Omega} \frac{k_{\varepsilon_n}^{3/2} \varphi}{\varrho + \varepsilon_n} \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \frac{k^{3/2} \varphi}{\varrho}.$$

Finally, by the same argument using  $\mathbf{u}_{\varepsilon_n} \rightarrow \mathbf{u}$  in  $L^3(\omega)$ ,

$$\int_{\Omega} k_{\varepsilon_n} \mathbf{u}_{\varepsilon_n} \cdot \nabla \varphi \xrightarrow{n \rightarrow +\infty} \int_{\Omega} k \mathbf{u} \cdot \nabla \varphi.$$

As we were able to pass to the limit in the TKE equations, we have shown that  $(\mathbf{u}, k)$  also satisfies (4.3), which concludes the proof.  $\square$

## A Appendix : Proof of Theorem 2.1

We start by giving an interpolation lemma.

**Lemma A.1.** *Let  $\Omega$  be a bounded Lipschitz domain,  $0 < s < 1$ , and  $u \in \mathcal{D}'(\Omega)$  such that  $\nabla u \in H^{-s}(\Omega)^N$  and  $u \in H^{-s}(\Omega)$ . Then  $u \in H^{1-s}(\Omega)$  and there exists a constant  $C > 0$ , which does not depend on  $u$ , be such that*

$$(A.1) \quad \|u\|_{1-s,2,\Omega} \leq C (\|u\|_{-s,2,\Omega} + \|\nabla u\|_{-s,2,\Omega}).$$

*Proof.* We start by recalling two inequalities. First, the definition of the  $H^1$  norm reads for any  $v \in H^1(\Omega)$

$$(A.2) \quad \|v\|_{1,2,\Omega} = \|v\|_{0,2,\Omega} + \|\nabla v\|_{0,2,\Omega}.$$

Then a result due to Nečas and proved in [9] ensures that for any  $v \in \mathcal{D}'(\Omega)$  such that  $v \in H^{-1}(\Omega)$  and  $\nabla v \in H^{-1}(\Omega)^N$ , we have

$$(A.3) \quad \|v\|_{0,2,\Omega} \leq \|v\|_{-1,2,\Omega} + \|\nabla v\|_{-1,2,\Omega}.$$

In order to obtain an inequality for spaces with a noninteger exponent, we use the interpolation Theorem from [21]. To do so, let the spaces  $W^0$  and  $W^{-1}$  be defined as the closure of  $\mathcal{D}(\Omega)$  respectively for the norms

$$(A.4) \quad \|u\|_{W^0} := \|u\|_{0,2,\Omega} + \|\nabla u\|_{0,2,\Omega} \quad \text{and} \quad \|u\|_{W^{-1}} := \|u\|_{-1,2,\Omega} + \|\nabla u\|_{-1,2,\Omega}.$$

Notice that  $W^0 = H_0^1(\Omega)$ .

**Remark A.1.** *The embedding  $\Lambda : W^0 \hookrightarrow W^{-1}$  is continuous and dense. Moreover, because both  $W^0$  and  $W^{-1}$  are Hilbert spaces, for all  $u, v \in W^0$ ,*<sup>4</sup>

$$(u, v)_2 + (\nabla u, \nabla v)_2 = (\Lambda u, \Lambda v)_{-1} + (\nabla \Lambda u, \nabla \Lambda v)_{-1}.$$

As the interpolation space  $[W^0, W^{-1}]_s = \mathcal{D}(\Lambda^s)$  and  $H^{-s}(\Omega) = [L^2(\Omega), H^{-1}(\Omega)]_s$ , we have

$$(u, v)_2 + (\nabla u, \nabla v)_2 = (\Lambda^s u, \Lambda^s v)_{-s} + (\nabla \Lambda^s u, \nabla \Lambda^s v)_{-s}.$$

so that the interpolation norm is defined by

$$(A.5) \quad \begin{aligned} \|u\|_{[W^0, W^{-1}]_s} &= \|u\|_{[L^2(\Omega), H^{-1}(\Omega)]_s} + \|\nabla u\|_{[L^2(\Omega)^N, H^{-1}(\Omega)^N]_s} \\ &= \|u\|_{-s,2,\Omega} + \|\nabla u\|_{-s,2,\Omega}. \end{aligned}$$

Inequality (A.2) shows that the identity map,  $\text{Id} : W^0 \rightarrow H^1(\Omega)$  is continuous, and similarly (A.3) yields that  $\text{Id} : W^{-1} \rightarrow L^2(\Omega)$  is continuous. Then, the interpolation theorem yields

$$(A.6) \quad \text{Id} : [W^0, W^{-1}]_s \rightarrow [H^1(\Omega), L^2(\Omega)]_s = H^{1-s}(\Omega)$$

is continuous.

This means there exists a constant  $C > 0$  such that for any  $u \in [W^0, W^{-1}]_s$

$$(A.7) \quad \|u\|_{1-s,2,\Omega} \leq C \|u\|_{[W^0, W^{-1}]_s},$$

hence (A.1) by using (A.5). □

As  $L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$ , we deduce from (A.1) the following.

**Corollary A.1.** *Let  $u \in L^2(\Omega)$  be such that  $\nabla u \in H^{-s}(\Omega)$ . Then  $u \in H^{1-s}(\Omega)$  and*

$$(A.8) \quad \|u\|_{1-s,2,\Omega} \leq C (\|u\|_{0,2,\Omega} + \|\nabla u\|_{-s,2,\Omega}).$$

<sup>4</sup>This can be easily seen by recalling that  $F \in H^{-1}(\Omega)$  if and only if there exists  $f_0, \dots, f_N \in L^2(\Omega)$  be such that  $\forall \varphi \in H_0^1(\Omega)$ ,  $\langle F, \varphi \rangle = \sum_i (f_i, \partial_i \varphi)_2$ . Therefore we have,  $(F, G)_{-1} = \sum_i (f_i, g_i)_2$ .

### Proof of Theorem 2.1.

Let  $\eta \in ]0; 1[$  and  $\Omega$  a lipschitz domain. Let  $\mathbf{u} \in V_\eta$  and  $\varphi \in \mathcal{D}(\Omega)^{N \times N}$ .

$$(A.9) \quad \left| \int_{\Omega} \nabla \mathbf{u} : \varphi \right| = \left| \int_{\Omega} \varrho^{\eta/2} \nabla \mathbf{u} : \varrho^{-\eta/2} \varphi \right| \leq \|\varrho^{\eta/2} \nabla \mathbf{u}\|_{0,2,\Omega} \|\varrho^{-\eta/2} \varphi\|_{0,2,\Omega}.$$

Given that  $\eta < 1$ , Hardy inequality (see [12]) yields

$$(A.10) \quad \|\varrho^{-\eta/2} \varphi\|_{0,2,\Omega} \leq C \|\varphi\|_{\eta/2,2,\Omega},$$

where  $C > 0$  depends neither on  $\varphi$  nor on  $\mathbf{u}$ .

Therefore, we obtain

$$(A.11) \quad |\langle \nabla \mathbf{u}, \varphi \rangle| \leq C \|\mathbf{u}\|_{1,2,\varrho,\eta} \|\varphi\|_{\eta/2,2,\Omega}.$$

Thus  $\nabla \mathbf{u}$  is in  $(H^{\eta/2}(\Omega)^{N \times N})' = H^{-\eta/2}(\Omega)^{N \times N}$ , by [21, Theorem 6.2], and we deduce from (A.11),

$$\|\nabla \mathbf{u}\|_{-\eta/2,2,\Omega} \leq C \|u\|_{1,2,\varrho,\eta}$$

Finally, as  $\|u\|_{0,2,\Omega} \leq \|u\|_{1,2,\varrho,\eta}$ , we deduce from (A.8)

$$\|\mathbf{u}\|_{1-\eta/2,2,\Omega} \leq C \|u\|_{1,2,\varrho,\eta}.$$

Therefore, we have indeed  $\mathbf{u} \in H^{1-\frac{\eta}{2}}(\Omega)^N$ , which concludes the proof.

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