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TKE model involving the distance to the wall.

Part 1: the relaxed case

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Abstract

We consider a steady-state turbulent RANS model that couples the equation for the velocity-pressure field to the equation for the turbulent kinetic energy, with eddy viscosities vanishing at the boundary like $d(x, \Gamma)^\eta$ and $d(x, \Gamma)^\beta$, where $0 < \eta, \beta < 1$. We find critical values η_c and β_c for which the system has a weak solution obtained as the limit of viscous regularizations for $0 < \eta < \eta_c$, $0 < \beta < \beta_c$.

Key words : Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.

2010 MSC: 76D05, 35Q30, 76F65, 76D03, 35Q30.

1 Introduction

The modeling of turbulence in fluid mechanics is based on the decomposition of the velocity and pressure fields into the sum of mean fields, which we denote \mathbf{u}, p in this work and which we will assume stationary for simplicity, and fluctuations denoted \mathbf{u}' and p' . By averaging the Navier-Stokes equations, we get the Reynolds stress $\boldsymbol{\sigma}^{(R)}$, which is the average of the tensor product $\mathbf{u}' \otimes \mathbf{u}'$ [6, 18, 20]. We must model this Reynolds tensor. It is quite admitted that the latter is dissipative [2, 8], and according to the boussinesq hypothesis, it is made proportional to the average strain tensor via a turbulent viscosity, denoted ν_{turb} . The question then arises as to whether the molecular diffusion term is negligible compared to the turbulent diffusion term.

There are different ways to model ν_{turb} , whether in the framework of statistical averages (RANS type models), or whether the high frequencies of the flow are filtered (LES type models). In this article we consider the case of a RANS model, the case of LES models being studied in [13].

Generally speaking, in RANS models the turbulent viscosity is a function of the turbulent kinetic energy, k (TKE) and the turbulent dissipation \mathcal{E} . This can be simplified by relating the TKE k to \mathcal{E} by the Prandtl mixing length, denoted by ϱ , which yields the following law:

$$(1.1) \quad \nu_{\text{turb}} = C\varrho\sqrt{k},$$

for some dimensionless constant C (and $k \geq 0$). After a modeling process [6, 18], one gets the following equation for k , named TKE model:

$$(1.2) \quad \mathbf{u} \cdot \nabla k - \operatorname{div}(\mu_{\text{turb}}(k) \nabla k) + \frac{k|k|^{1/2}}{\varrho} = \nu_{\text{turb}}(k) |D\mathbf{u}|^2,$$

where \mathbf{u} is the fluid velocity, $D\mathbf{u} = 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain tensor, μ_{turb} an eddy diffusion similar to the eddy viscosity given by (1.1), and $\mathcal{E} = \frac{k|k|^{1/2}}{\varrho}$. When we assume that ϱ is a smooth bounded function such that $\varrho(x) \geq \varrho_0 > 0$ in the fluid domain Ω , the coupling of the resulting steady-state Navier-Stokes equation with an eddy diffusion to the TKE model (1.2), has been studied extensively [1, 6, 9, 12, 15, 16].

In most cases, we assume that ν_{turb} is bounded below and above, which amounts to replacing $C\sqrt{k}$ by a truncation function $\tilde{\nu}$ equal to $C\sqrt{k}$ on the interval $[1/M, M]$, and equal to C/\sqrt{M} over $[0, 1/M]$ and $C\sqrt{M}$ on $[M, \infty[$, for some $M > 0$.

In this paper, we remain in this framework and focus on the problem posed by the Prandtl mixing length ϱ . Indeed, in most physical applications, such as meteorology or oceanography [14, 19], $\varrho = \varrho(x)$ vanishes at the boundary equivalently to the distance to the boundary Γ , denoted $d(x, \Gamma)$, where Γ is the boundary of Ω , which poses a major difficulty. Notice that it also may be of order \sqrt{d} , as for instance for shear flows [10]. Throughout the paper, we consider the case $\varrho(x) \approx d(x, \Gamma)$ when $x \rightarrow \Gamma$.

A similar problem was already considered in the case of some LES models [3, 4, 21, 22], for which the eddy viscosity is proportional to ϱ^2 (instead of ϱ as in the TKE model), which yields difficult open problems. In these papers, the authors suggest to relax this law and to take an eddy viscosity proportional to ϱ^α , for some $0 < \alpha < 2$. The same situation occurs in the case of RANS problem. Therefore, in this paper we aim to relax the law given by (1.1) by taking

$$(1.3) \quad \nu_{\text{turb}}(k) = \varrho^\eta \tilde{\nu}(k),$$

for $0 < \eta < 1$, and we study the coupling of the resulting steady-state Navier-Stokes equation to (1.2), in the scope of the viscous approximations posed by the question of whether or not we can neglect the molecular viscosity in comparison with the turbulent viscosity.

To be more specific, let Ω be a $\mathcal{C}^{0,1}$ bounded open subset of \mathbb{R}^N , $N = 2, 3$, $0 < \eta < 1$, $0 < \beta < 1$, $0 < \lambda$. As said before, $\varrho : \Omega \rightarrow \mathbb{R}_*^+$ is a C^∞ function equivalent to $d(x, \Gamma)$ near Γ (see the precise assumption in the beginning of section 2.1). We consider the following system, where p denotes the pressure¹²:

$$(1.4) \quad \begin{cases} \lambda \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) + \frac{k|k|^{1/2}}{\varrho} = \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 & \text{in } \Omega, \\ k \geq 0 & \text{in } \Omega, \\ u = k = 0 & \text{at } \Gamma. \end{cases}$$

The term $\lambda \mathbf{u}$ is a stabilization term, which can be seen as a discrete evolution term coming from a finite difference, and which plays an important role in the proofs of the main results.

¹we replace $D\mathbf{u}$ by $\nabla\mathbf{u}$ for the simplicity without loss of generality, thanks to the Korn inequality.

²Still for the simplicity, we consider homogeneous boundary conditions instead of usual non linear laws at the boundary given by the turbulence modeling framework.

We next introduce the regularized system, where $\varepsilon > 0$,

$$(1.5) \quad \begin{cases} \lambda \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div}(\varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}) - \varepsilon \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) - \varepsilon \Delta k + \frac{k|k|^{1/2}}{\varrho + \varepsilon} = \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 & \text{in } \Omega, \\ k \geq 0 & \text{in } \Omega, \\ u = k = 0 & \text{at } \Gamma, \end{cases}$$

for which we know from [15] the existence of a weak solution in an appropriate sense. We transcribe the question "Is the molecular diffusion term negligible compared to the turbulent diffusion term" by "do the solutions of the system (1.5) converge towards a solution -in some weak sense- of the system (1.4) when ε goes to 0" ?

As we will see, the situation is seriously complicated. We only get a positive answer at the cost of severe restrictions on the exponents η and β , i.e. when $0 < \eta, \beta < 1/3$ in dimension 2, and $0 < \eta < 1/9, \beta < 1/5$ in dimension 3. A specific statement is given by Theorem 4.2 below, which is the main result of this paper.

The paper is divided in 3 main sections and is organized as follows. In the first section we study the steady Navier-Stokes equations with an eddy viscosity such as (1.3), with a given k . By using a weighted Sobolev spaces of divergence free vector fields and results in Kufner [11], we are able to prove the existence of a weak solution in a particular weak sense (see definition 2.1 below), obtained as a limit of the viscous regularization by adding the term $-\varepsilon \Delta \mathbf{u}$, when $0 < \eta < 1$ for $N = 2$, $0 < \eta < 2/3$ for $N = 3$. However, in view of the coupling with the TKE equation, we need the strong convergence of the approximations in our weighted space, to be able to pass to the limit in the production term in the r.h.s of the equation for the TKE. Indeed, this term is only in $L^1(\Omega)$, which is one of the main characteristics of this range of problems. We find that the strong convergence holds when $0 < \eta < 1/3$ for $N = 2$, and $0 < \eta < 1/9$ for $N = 3$, and we can't do better at the moment. In the second session, we focus on the TKE equation for a fixed vector field \mathbf{u} , especially the derivation of a priori estimates in regular Sobolev spaces, which is not straightforward due to the r.h.s in L^1 and the degeneracy of the diffusive term at the boundary. To do so, we use a weighted Sobolev space, and then we adapt to this case the ladder method initially due to Boccardo-Gallouët [5], which is one of the main contributions of this paper. In the third section we synthesize the two previous sections. We are then able to pass to the limit in the viscous regularizations (1.5) when ε tends to 0 and to prove the main theorem, namely Theorem 4.2. This also proves the existence of a weak solution to system (1.4) in a certain way, solutions that we call "SOLA" (solutions obtained by approximations). Note that the equation for TKE only holds in the sense of the distributions, and we are not able to do better without further decreasing the size of the β parameter.

2 Velocity equation with a fixed TKE

2.1 General setting and space function

Let Ω is be a $\mathcal{C}^{0,1}$ bounded open subset of \mathbb{R}^N ($N = 2, 3$), $\varrho : \Omega \rightarrow \mathbb{R}^+$ is of \mathcal{C}^∞ class function equivalent to the distance to the boundary., which satisfies

$$(2.1) \quad \lim_{\substack{d(x, \Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x, \Gamma)} = 1,$$

$$(2.2) \quad \forall n > 0, \quad \varrho_n = \inf_{\substack{d(x, \Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0$$

The functions $\tilde{\nu}, \tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, bounded by above and by below, namely

$$(2.3) \quad \forall k \in \mathbb{R}, \quad 0 < \nu_m \leq \tilde{\nu}(k), \tilde{\mu}(k) \leq \nu_M,$$

We start by studying the velocity equation for a fixed TKE function k at least in $L^1(\Omega)$, and a source term \mathbf{f} , the regularity of which will be discussed later. The corresponding Navier-Stokes system is as follows:

$$(2.4) \quad \begin{cases} \lambda \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \operatorname{div}(\varrho^\eta(x) \tilde{\nu}(k) \nabla \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{at } \Gamma. \end{cases}$$

To get an *à priori* estimate, we formally take the dot product of the transport equation with \mathbf{u} and we integrate by parts, using a standard calculus that uses the no slip boundary condition, giving in particular $((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{u}) = 0$ and $(\nabla p, \mathbf{u}) = 0$. Therefore, the following energy balance holds:

$$(2.5) \quad \lambda \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 = \langle \mathbf{f}, \mathbf{u} \rangle.$$

Consequently, for a large class of source terms \mathbf{f} , for instance $\mathbf{f} \in L^2(\Omega)^N$, and using (2.3), we get the *à priori* estimate:

$$(2.6) \quad \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} \varrho^\eta |\nabla \mathbf{u}|^2 \leq C = \frac{\|\mathbf{f}\|_{0,2;\Omega}^2}{2\lambda \min(\lambda/2, \nu_m)}.$$

This suggests to introduce the space function V_η defined as the closure of \mathcal{V} for the norm

$$(2.7) \quad \|u\|_{1,2;\varrho,\eta} = \left(\|u\|_{0,2,\Omega}^2 + \|\varrho^{\eta/2} \nabla u\|_{0,2,\Omega}^2 \right)^{1/2}.$$

where

$$(2.8) \quad \mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega)^N / \operatorname{div} \mathbf{u} = 0\},$$

In particular, V_η is a Hilbert space with the scalar product

$$(2.9) \quad (\mathbf{u}, \mathbf{v})_{1,2;\varrho,\eta} = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \nabla \mathbf{u} : \nabla \mathbf{v}.$$

According to [11, Proposition 6.5], the following holds.

Theorem 2.1. *Assume that Ω satisfies the cone property, and $\eta \in]0, 1[$. For any p such that*

$$(2.10) \quad 1 \leq p < \frac{2}{1+\eta} = p_\eta,$$

the following continuous embedding holds

$$(2.11) \quad V_\eta \hookrightarrow W_0^{1,p}(\Omega)^N.$$

The Sobolev embedding theorem yields the following.

Corollary 1. *We have*

$$(2.12) \quad V_\eta \hookrightarrow L^q(\Omega)^N,$$

for all

$$(2.13) \quad q < p_\eta^* = \frac{2N}{N(1+\eta) - 2} = \begin{cases} \frac{6}{1+3\eta} & \text{when } N = 3, \\ \frac{2}{\eta} & \text{when } N = 2, \end{cases}$$

with compact embedding.

Remark 2.1. *We deduce from Theorem 2.1 that vector fields in V_η have a trace at Γ which is equal to 0, giving a sense to the no slip boundary condition for vector fields in V_η .*

Remark 2.2. *Vector fields in V_η are naturally in $L^2(\Omega)^N$ by construction. In the case $N = 2$, $p^* > 2$, then (2.12) brings additional regularity. However, in the case $N = 3$, more regularity is reached only for $0 < \eta < 2/3$, where $p^* > 2$.*

Remark 2.3. *The following Poincaré inequality holds, for all $p < p_\eta$ and all $\mathbf{u} \in V_\eta$,*

$$(2.14) \quad \|\mathbf{u}\|_{0,p,\Omega} \leq C \|\mathbf{u}\|_{1,2;\varrho,\eta},$$

where $C = C(p, \Omega)$.

Throughout the rest of the paper, we assume that Ω satisfies the cone property, in order to verify (2.11).

Remark 2.4. *Let $0 < \eta < 1$. The space V_η is a distributional space, and in particular we conjecture that*

$$(2.15) \quad V_\eta = \{\mathbf{u} \in \mathcal{D}'(\Omega)^N, \operatorname{div} \mathbf{u} = 0, \mathbf{u} \in L^2(\Omega)^N, \sqrt{\varrho^\eta} \nabla \mathbf{u} \in L^2(\Omega)^{N \times N}, \mathbf{u}|_\Gamma = 0\}.$$

We also denote in what follows

$$(2.16) \quad V_0 = \{\mathbf{u} \in H_0^1(\Omega)^N; \operatorname{div} \mathbf{u} = 0\}.$$

2.2 Weak solutions and existence result

We start by giving the weak formulation of Problem (2.4) in which the pressure is not involved, since we take free divergence vector fields as tests. However, we must take in consideration the transport term $(\mathbf{u} \cdot \nabla) \mathbf{u} = \operatorname{div}(\mathbf{u} \otimes \mathbf{u})$. We deduce from (2.13) that for any $\mathbf{u} \in V_\eta$, $\mathbf{u} \otimes \mathbf{u} \in L^{q/2}(\Omega)^{N \times N}$ for all $2 < q < p_\eta^*$. The constrain $p_\eta^* > 2$ is equivalent to $0 < \eta < 2/3 = \eta_c$ when $N = 3$, and no additional condition about η when $N = 2$, therefore $\eta_c = 1$ in this case. Throughout the rest of the paper, we assume this condition fulfilled

Hence, for any $2 \leq \delta < p_\eta^*$ we could consider

$$(2.17) \quad \mathbf{v} \in \bigcup_{\delta < q < p_\eta^*} \left(V_\eta \cap W_0^{1,(q/2)'}(\Omega)^N \right) = H_{\eta,\delta}$$

as test vector fields set. In particular, when $\mathbf{v} \in H_{\eta,\delta}$, $\mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} \in L^1(\Omega)$, as well as $\lambda \mathbf{u} \cdot \mathbf{v} + \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} \in L^1(\Omega)$, since $\tilde{\nu}(k) \in L^\infty(\Omega)$, and $\mathbf{f} \cdot \mathbf{v} \in L^1(\Omega)$. Unfortunately, in view of the construction of weak solutions to Problem (2.4), we must reduce the space

$H_{\eta,\delta}$ a little bit for reasons that will be cleared in the following. Let $r > 1$, and let us consider \mathcal{V} (see (2.8)) equipped with the norm

$$\|\mathbf{v}\|_{\eta,r} = \|\mathbf{v}\|_{1,2;\varrho,\eta} + \|\mathbf{v}\|_{1,r,\Omega},$$

and let $K_{\eta,r}$ denotes the closure of \mathcal{V} for the norm $\|\cdot\|_{\eta,r}$. Notice that we naturally have

$$(2.18) \quad K_{\eta,r} \hookrightarrow V_\eta.$$

The tests vector fields set we consider is given by:

$$(2.19) \quad W_{\eta,\delta} = \bigcup_{\delta < q < p_\eta^*} K_{\eta,(q/2)'},$$

which is a subset of $H_{\eta,\delta}$.

Remark 2.5. *We do not know if generally $W_{\eta,\delta} = H_{\eta,\delta}$. However, as we shall see it in the remainder, when η is small enough and δ close enough from p^* , then $W_{\eta,\delta} = V_\eta$.*

Definition 2.1. *We say that $\mathbf{u} \in V_\eta$ is a δ -weak solution to Problem (2.4) if $\forall \mathbf{v} \in W_{\eta,\delta}$,*

$$(2.20) \quad \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}^T + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Remark 2.6. *Let $2 \leq \delta_1 < \delta_2 < p_\eta^*$. Then obviously $W_{\eta,\delta_2} \subset W_{\eta,\delta_1}$. Therefore any δ_1 -weak solution is a δ_2 -weak solution. However, it is not clear that a δ_2 -weak solution is a δ_1 -weak solution as well.*

Theorem 2.2. *Let $0 < \eta < \eta_c$, where $\eta_c = 1$ when $N = 2$, $\eta_c = 2/3$ when $N = 3$, $k \in L^1(\Omega)$, $\mathbf{f} \in L^2(\Omega)^N$, $2 \leq \delta < p_\eta$. Then Problem (2.4) a δ -weak solution obtained by approximation (SOLA).*

Proof. We argue by viscous regularization, the proof being divided into 3 steps. We first introduce the viscous regularization we consider, and we get a uniform estimate in V_η for the resulting equations. Then we extract subsequences by weak compactness and Theorem 2.1. Then we pass to the limit when the viscous parameter ε goes to 0.

Step 1. Approximations and estimate. We argue by singular perturbations. Let $\varepsilon > 0$. According to standard results (see for instance in [6]), we know the existence of $\mathbf{u}_\varepsilon \in V_0$ such that for any $\mathbf{v} \in V_0$

$$(2.21) \quad \lambda \int_{\Omega} \mathbf{u}_\varepsilon \cdot \mathbf{v} + \int_{\Omega} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla \mathbf{v}^T + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} + \varepsilon \int_{\Omega} \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}.$$

Taking $\mathbf{v} = \mathbf{u}_\varepsilon$ as test, we get the following energy balance, since the transport term vanishes,

$$(2.22) \quad \lambda \int_{\Omega} |\mathbf{u}_\varepsilon|^2 + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}_\varepsilon|^2 + \varepsilon \int_{\Omega} |\nabla \mathbf{u}_\varepsilon|^2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_\varepsilon.$$

In particular, as $\mathbf{f} \in L^2(\Omega)^N$, the family $(\mathbf{u}_\varepsilon)_{\varepsilon > 0}$ is bounded in V_η .

Step 2. Extracting subsequences. From the bound in V_η , we deduce the existence of a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ and $\mathbf{u} \in V_\eta$ such that the sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$:

1. Weakly converges to \mathbf{u} in V_η ,

2. Weakly converges to \mathbf{u} in $W_0^p(\Omega)^N$ for all $p < p_\eta$,
3. Strongly converges to \mathbf{u} in $L^q(\Omega)^N$ for all $q < p_\eta^*$, where p_η^* is given by (2.13).

Item 1) is straightforward. Items 2) and 3) require additional comments. Indeed, let $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ which weakly converges to \mathbf{u} in V , as given by item 1, and let $1 < p < p_\eta$ be fixed. As the sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ by Theorem 2.1, we can extract a subsequence, still denoted $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$, which weakly converges to some $\mathbf{v} \in W_0^{1,p}(\Omega)$, and by the way in $\mathcal{D}'(\Omega)$. But as $\mathbf{u}_{\varepsilon_n} \rightarrow \mathbf{u}$ in V -weak, by (2.2) convergence in $\mathcal{D}'(\Omega)$ also holds. Therefore by the uniqueness of the limit, $\mathbf{u} = \mathbf{v}$. Let now $p < q < p_\eta$. From the sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ we can extract a sub-sequence which weakly converges to some \mathbf{v} in $W_0^{1,q}(\Omega)$. By the same argument as above, $\mathbf{u} = \mathbf{v}$ and by uniqueness of the limit, the whole sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ converges to \mathbf{u} in $W_0^{1,q}(\Omega)$ -weak, hence items 2) and 3).

Step 3. Passing to the limit in the equations. Let us take $\mathbf{v} \in \mathcal{V}$ as test in (2.27). It is easily checked that when $n \rightarrow \infty$,

$$(2.23) \quad \begin{aligned} \lambda \int_{\Omega} \mathbf{u}_{\varepsilon_n} \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} &\rightarrow \int_{\Omega} \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v}, \\ \int_{\Omega} \mathbf{u}_{\varepsilon_n} \otimes \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v}^T &\rightarrow \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}^T. \end{aligned}$$

It remains to deal with the term $\varepsilon \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v}$. As $\nabla \mathbf{v} \in L^\infty(\Omega)$,

$$\int_{\Omega} \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} \rightarrow \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v},$$

hence

$$\varepsilon \int_{\Omega} \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} \rightarrow 0.$$

In conclusion, \mathbf{u} satisfies (2.20) for any $\mathbf{v} \in \mathcal{V}$. Let $\mathbf{v} \in W_{\delta,\eta}$. There exists $q \in]\delta, p_\eta[$ such that $\mathbf{v} \in K_{\eta,(q/2)'}$. It follows the existence of a sequence $(\mathbf{v}_n)_{n \in \mathbb{N}}$ of vector fields in \mathcal{V} which converges to \mathbf{v} in both V_η and $W_0^{1,(q/2)'(\Omega)^N}$, which explains the reason of the special construction of $W_{\eta,\delta}$. Taking \mathbf{v}_n as test in (2.20), and passing to the limit when $n \rightarrow \infty$ yields \mathbf{u} satisfies (2.20) for any $\mathbf{v} \in W_{\eta,\delta}$. \square

Remark 2.7. *By De Rham Theorem, we know the existence of a pressure $p \in L_{loc}^2(\Omega)$ such that (2.4) holds in $\mathcal{D}'(\Omega)$.*

2.3 Strong convergence

We address in this section the question of the strong convergence of the approximations to Problem (2.20) constructed above. We first must check if a δ -weak solution satisfies the energy balance for a suitable choice of δ and η . The following holds.

Lemma 2.1. *Let $0 < \eta < \eta'_c$, where*

$$(2.24) \quad \begin{cases} \eta'_c = 1/9 & \text{when } N = 3, \\ \eta'_c = 1/3 & \text{when } N = 2. \end{cases}$$

Then there exists $\delta_0 \in]2, p_\eta^[$ such that for all $\delta \in]\delta_0, p_\eta^*[$, any δ -weak solution \mathbf{u} of Problem (2.20) satisfies the energy balance (2.5).*

Proof. The energy balance holds when we take \mathbf{u} itself as test vector field. This is the case when

$$(2.25) \quad W_{\eta,\delta} = V_\eta.$$

To ensure that (2.25) is satisfied, it is necessary that $\left(\frac{p_\eta^*}{2}\right)' < p_\eta$, which yields the bound (2.24) after elementary calculations. Moreover, as $p \rightarrow p'$ is non-increasing and continuous, there exists $\delta_0 \in]2, p_\eta^*[$, such that $\left(\frac{p_\eta^*}{2}\right)' < \left(\frac{\delta_0}{2}\right)' < p_\eta$. Therefore, for $\delta_0 < q < p_\eta^*$, $\left(\frac{q}{2}\right)' < p_\eta$ and by Theorem 2.1, $V_\eta \subset W_0^{(q/2)'(\Omega)^N}$. Therefore, by (2.2)

$$\|\mathbf{v}\|_{\eta,(q/2)'} \leq C\|\mathbf{v}\|_{1,2;\varrho,\eta},$$

which means $V_\eta \hookrightarrow K_{\eta,(q/2)'}$, hence $V_\eta = K_{\eta,(q/2)'}$ by (2.18), leading to (2.25). \square

Remark 2.8. *After some calculus, we find for optimal value of δ_0 :*

$$(2.26) \quad \begin{cases} \delta_0 = \frac{12}{5-3\eta} & \text{when } N = 3, \\ \delta_0 = \frac{4}{2-\eta} & \text{when } N = 2. \end{cases}$$

We are now able to prove the following convergence result.

Theorem 2.3. *Let $0 < \eta < \eta'_c$, $\delta \in]\delta_0, p_\eta[$, and let $(k_{\varepsilon_n})_{\varepsilon_n > 0}$ be a family in $L^1(\Omega)$ which converges in $L^1(\Omega)$ to some k in $L^1(\Omega)$. Let $\mathbf{u}_\varepsilon \in V_0$ such that for any $\mathbf{v} \in V_0$*

$$(2.27) \quad \lambda \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{v} - \int_\Omega \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla \mathbf{v}^T + \int_\Omega (\varrho^\eta + \varepsilon) \tilde{\nu}(k_{\varepsilon_n}) \nabla \mathbf{u}_\varepsilon : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

Then there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that the sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ strongly converges in V_η to a δ -weak solution $\mathbf{u} \in V_\eta$ of Problem (2.4) which in addition satisfies the energy balance (2.5). Moreover, the sequence $(\varrho^\eta \nu_{\text{turb}}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2)_{n \in \mathbb{N}}$ converges strongly in $L^1(\Omega)$ to $\varrho^\eta \nu_{\text{turb}}(k) |\nabla \mathbf{u}|^2$.

Proof. We follow the proof of Theorem 2.2, in order to ensure the existence of a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$, such that the sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ weakly converges in V_η to a δ -weak solution $\mathbf{u} \in V_\eta$ of Problem (2.4). The only addition to be made, is to check how to pass to the limit in the diffusion term. We first notice that the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ can be chosen such that the sequence $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ converges a.e. to k , by the inverse Lebesgue Theorem. Let $\mathbf{v} \in \mathcal{V}$, $\delta < q < p_\eta$. Then

$$(2.28) \quad \sqrt{\varrho^\eta} \nabla \mathbf{u}_{\varepsilon_n} \rightarrow \sqrt{\varrho^\eta} \nabla \mathbf{u} \quad \text{weakly in } L^2(\Omega)^{N \times N},$$

while

$$(2.29) \quad \sqrt{\varrho^\eta} \tilde{\nu}(k_{\varepsilon_n}) \nabla \mathbf{v} \rightarrow \sqrt{\varrho^\eta} \tilde{\nu}(k) \nabla \mathbf{v} \quad \text{a.e. in } \Omega,$$

because $\tilde{\nu}$ is a continuous function. Therefore, since

$$|\sqrt{\varrho^\eta} \tilde{\nu}(k_{\varepsilon_n}) \nabla \mathbf{v}| \leq \nu_M |\sqrt{\varrho^\eta} \tilde{\nu}(k_{\varepsilon_n}) \nabla \mathbf{v}| \in L^2(\Omega),$$

by (2.3), then the convergence in (2.29) also holds in $L^2(\Omega)^{N \times N}$, by Lebesgue Theorem. In conclusion

$$\int_\Omega \varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) \nabla \mathbf{u}_{\varepsilon_n} : \nabla \mathbf{v} \rightarrow \int_\Omega \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v},$$

as $n \rightarrow \infty$. The rest is as in the proof of Theorem 2.2, the energy balance being satisfied by Lemma (2.24).

It remains to prove the strong convergence. To do so we use the so called "energy method". We deduce from (2.28) and the same argument as above that

$$(2.30) \quad \sqrt{\varrho^\eta} \sqrt{\tilde{\nu}(k_{\varepsilon_n})} \nabla \mathbf{u}_{\varepsilon_n} \rightarrow \sqrt{\varrho^\eta} \sqrt{\tilde{\nu}(k)} \nabla \mathbf{u} \quad \text{weakly in } L^2(\Omega)^{N \times N}.$$

Therefore,

$$(2.31) \quad \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2.$$

Now, we write the energy balances satisfied by \mathbf{u} and $\mathbf{u}_{\varepsilon_n}$:

$$(2.32) \quad \begin{cases} \lambda \int_{\Omega} |\mathbf{u}_{\varepsilon_n}|^2 + \int_{\Omega} \varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2 + \varepsilon_n \int_{\Omega} |\nabla \mathbf{u}_{\varepsilon_n}|^2 = \langle \mathbf{f}, \mathbf{u}_{\varepsilon_n} \rangle \\ \lambda \int_{\Omega} |\mathbf{u}|^2 + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 = \langle \mathbf{f}, \mathbf{u} \rangle. \end{cases}$$

As $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ converges to \mathbf{u} strongly in $L^2(\Omega)^N$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\mathbf{u}_{\varepsilon_n}|^2 = \int_{\Omega} |\mathbf{u}|^2, \quad \lim_{n \rightarrow \infty} \langle \mathbf{f}, \mathbf{u}_{\varepsilon_n} \rangle = \langle \mathbf{f}, \mathbf{u} \rangle.$$

Therefore, by (2.32), we have

$$(2.33) \quad \lim_{n \rightarrow \infty} \left(\int_{\Omega} \varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2 + \varepsilon_n \int_{\Omega} |\nabla \mathbf{u}_{\varepsilon_n}|^2 \right) = \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2,$$

which combined with (2.31) leads to

$$(2.34) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \varrho^\eta \tilde{\nu}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2 = \int_{\Omega} \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2, \quad \lim_{n \rightarrow \infty} \varepsilon_n \int_{\Omega} |\nabla \mathbf{u}_{\varepsilon_n}|^2 = 0.$$

Therefore, combining (2.30) with (2.34) yields the strong convergence in $L^2(\Omega)^{N \times N}$ of $(\sqrt{\varrho^\eta} \sqrt{\tilde{\nu}(k_{\varepsilon_n})} \nabla \mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ to $\sqrt{\varrho^\eta} \sqrt{\tilde{\nu}(k)} \nabla \mathbf{u}$, hence the strong convergence in $L^1(\Omega)$ of the sequence $(\varrho^\eta \nu_{\text{turb}}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2)_{n \in \mathbb{N}}$ to $\varrho^\eta \nu_{\text{turb}}(k) |\nabla \mathbf{u}|^2$. Moreover, by the standard argument based on the continuity of $\tilde{\nu}$, (2.3) and the a.e. convergence of $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ to k , we get

$$(2.35) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \varrho^\eta |\nabla \mathbf{u}_{\varepsilon_n}|^2 = \int_{\Omega} \varrho^\eta |\nabla \mathbf{u}|^2,$$

which combined with (2.28) and the L^2 -strong convergence of $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ to \mathbf{u} , yields the strong convergence in V_η . \square

3 TKE with a fixed velocity

Let $\mathbf{u} \in V_\eta$, $k \in L^1(\Omega)$ be given, $0 < \eta < \eta'_c$, $0 < \beta < 1$, and let

$$(3.1) \quad \mathbb{D} = \mathbb{D}(\mathbf{u}, k) = \varrho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2 \in L^1(\Omega),$$

and consider the following TKE equation, satisfied by $k = k(x)$:

$$(3.2) \quad \begin{cases} \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) + \frac{k|k|^{1/2}}{\varrho} = \mathbb{D}(\mathbf{u}, k) & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

Remark 3.1. Equation (3.2) can be written as

$$\mathbf{u} \cdot \nabla k - \varrho^\beta \tilde{\mu}(k) \Delta k - \nabla(\varrho^\beta \tilde{\mu}(k)) \cdot \nabla k + \frac{k|k|^{1/2}}{\varrho} = \mathbb{D}(\mathbf{u}, k).$$

Therefore as $\varrho > 0$ inside Ω , the strong maximum principle method used in [17] can be adapted to this case, so that any C^2 -strong solution k to (3.2) is such that $k > 0$ in Ω .

Throughout this section, $\mathbb{D}(\mathbf{u}, k)$ is supposed to be bounded in L^1 uniformly in \mathbf{u} and k according to (2.5), which means that for any \mathbf{u} and k , we take

$$(3.3) \quad \|\mathbb{D}(\mathbf{u}, k)\|_{0,1;\Omega} \leq D = \frac{\|\mathbf{f}\|_{0,2;\Omega}^2}{2\lambda}.$$

Theorem 3.1. Assume $\mathbf{u} \in C^0(\Omega)$, and let $k \in C^2(\overline{\Omega})$ be any strong solution to (3.2). Then for any

$$(3.4) \quad \beta < \frac{1}{2N-1} = \beta_c, \quad q < q_\beta = N'(p_\beta - 1) = N' \left(\frac{1-\beta}{1+\beta} \right),$$

where $p_\beta = \frac{2}{1+\beta}$, we have

$$(3.5) \quad \|k\|_{1,q;\Omega} + \|\varrho^{-1} k^{3/2}\|_{0,1;\Omega} \leq C = C(D, \beta).$$

Proof. Step 1. A priori estimate over the sets $\{n \leq k < n+1\}$. Let $n \in \mathbb{N}^*$, $H_n : \mathbb{R} \rightarrow \mathbb{R}$ be the odd function given by

$$(3.6) \quad H_n : t \mapsto \begin{cases} 0 & \text{if } 0 \leq t < n, \\ t - n & \text{if } n \leq t < n+1, \\ 1 & \text{if } t \geq n+1. \end{cases}$$

Notice that H_n is continuous, piecewise C^1 , the derivative of which has a finite number of discontinuity and $H_n(0) = 0$. Therefore, Stampacchia's Theorem [23] applies, in particular, for any $\kappa \in W_0^{1,r}(\Omega)$, $H_n(\kappa) \in W_0^{1,r}(\Omega)$, $\nabla H_n(\kappa) = \nabla \kappa H_n'(\kappa)$ a.e and $\nabla \kappa = 0$ over the level sets $\{\kappa = C\}$ that are not of null measure.

Let k be a strong solution of (3.2). We take $H_n(k)$ as test function in (3.2) and integrate over Ω . Due to the boundary conditions, on one hand we have,

$$(3.7) \quad \int_{\Omega} H_n(k) \mathbf{u} \cdot \nabla k = 0,$$

(see in [15] for instance). On the other hand as H_n is an odd function that satisfies $H_n(t) \geq 0$ over \mathbb{R}^+ , then $H_n(k)k \geq 0$, which yields

$$(3.8) \quad \int_{\Omega} \frac{H_n(k)k|k|^{1/2}}{\varrho} \geq 0.$$

According to Stampacchia Theorem, Stokes formula yields

$$(3.9) \quad \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \cdot \nabla H_n(k) \leq \int_{\Omega} \mathbb{D}H_n(k) \leq D,$$

given that $|H_n| \leq 1$.

Let $B_n \subset \Omega$ be defined for any $n \in \mathbb{N}^*$ by (recall that $k > 0$ in Ω),

$$B_n = \{x \in \Omega / n \leq k(x) < n + 1\}.$$

We obtain by Stampacchia's Theorem

$$(3.10) \quad \nabla k \cdot \nabla H_n(k) = |\nabla k|^2 \mathbb{1}_{B_n} \quad \text{a.e. } \Omega.$$

Inequality (3.9) yields, by (3.10),

$$(3.11) \quad \int_{B_n} \varrho^\beta |\nabla k|^2 \leq \frac{1}{\nu_m} D.$$

The left-hand side of (3.9) can be transformed another way. Indeed, since $H'_n(k) = \mathbb{1}_{B_n}$ a.e. Ω , one has

$$(3.12) \quad \nabla k \cdot \nabla H_n(k) = |\nabla H_n(k)|^2 \mathbb{1}_{B_n} = |\nabla H_n(k)|^2 \quad \text{a.e. } \Omega,$$

which yields by (3.10)

$$(3.13) \quad |\nabla k|^2 \mathbb{1}_{B_n} = |\nabla H_n(k)|^2 \quad \text{a.e. } \Omega.$$

Thus, (3.11) becomes

$$(3.14) \quad \int_{\Omega} \varrho^\beta |\nabla H_n(k)|^2 \leq \frac{1}{\nu_m} D.$$

Applying Theorem 2.1, we deduce from (3.14) that for any $p < p_\beta = \frac{2}{1+\beta}$, there exists a constant $C = C(p, \beta, \nu_m, D) > 0$ such that

$$(3.15) \quad \int_{B_n} |\nabla k|^p = \int_{\Omega} |\nabla H_n(k)|^p \leq C.$$

Step 2. Ladder process. We deduce now the $W^{1,p}$ estimate (3.5) from (3.15) by adapting to this case the Boccardo-Gallouët technique [5]. To do so, Let $p < p_\beta$, $1 < q < p$. We get by Hölder's Inequality

$$(3.16) \quad \int_{B_n} |\nabla k|^q \leq \left(\int_{B_n} |\nabla k|^p \right)^{q/p} |B_n|^{1-(q/p)} \leq C^{q/p} |B_n|^{1-(q/p)}.$$

Let $n_0 \in \mathbb{N}^*$. On the one hand one has

$$(3.17) \quad \sum_{n=0}^{n_0-1} \int_{B_n} |\nabla k|^q \leq n_0 C^{q/p} |\Omega|^{1-(q/p)} \leq n_0 \max(C; |\Omega|).$$

On the other hand, let $r \geq 1$, the definition of B_n yields

$$(3.18) \quad \int_{B_n} |k|^r \leq n^r |B_n|.$$

We get by (3.16)

$$(3.19) \quad \sum_{n=n_0}^{+\infty} \int_{B_n} |\nabla k|^q \leq C^{q/p} \sum_{n=n_0}^{+\infty} |B_n|^{1-(q/p)} \leq C^{q/p} \sum_{n=n_0}^{+\infty} \frac{1}{n^{\frac{r(p-q)}{p}}} \left(\int_{B_n} |k|^r \right)^{\frac{p-q}{p}}.$$

Then the discrete Hölder Inequality yields

$$(3.20) \quad \sum_{n=n_0}^{+\infty} \int_{B_n} |\nabla k|^q \leq C^{q/p} \left[\sum_{n=n_0}^{+\infty} \int_{B_n} |k|^r \right]^{\frac{p-q}{p}} \left[\sum_{n=n_0}^{+\infty} \frac{1}{n^{\frac{r(p-q)}{q}}} \right]^{q/p}.$$

Inequality (3.20) brings a condition on $r \geq 1$. Indeed, the right-hand side of (3.20) is finite if and only if the Riemann series converges, which means

$$(3.21) \quad r > \frac{q}{p-q}.$$

Let the two constants $\lambda_1(n_0)$ and $\lambda_2(n_0)$ be defined by

$$\begin{cases} \lambda_1(n_0) = n_0 \max(C; |\Omega|) \\ \lambda_2(n_0) = C^{q/p} \left[\sum_{n=n_0}^{+\infty} \frac{1}{n^{\frac{r(p-q)}{q}}} \right]^{q/p}, \end{cases}$$

with r satisfying (3.21). Consequently, (3.17) and (3.20) ensure the existence of two constants $\lambda_1(n_0)$ and $\lambda_2(n_0)$ such that $\lambda_2(n_0) \xrightarrow{n_0 \rightarrow +\infty} 0$ and

$$(3.22) \quad \int_{\Omega} |\nabla k|^q \leq \lambda_1(n_0) + \lambda_2(n_0) \left(\int_{B_n} |k|^r \right)^{\frac{p-q}{p}} = \lambda_1(n_0) + \lambda_2(n_0) \|k\|_{0,r;\Omega}^{\frac{r(p-q)}{p}}.$$

In order to obtain an inequality involving the norm $\|k\|_{1,q;\Omega}$, the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ must stand, which brings another condition on r :

$$(3.23) \quad 1 \leq r \leq q^* = \frac{Nq}{N-q}.$$

For both conditions (3.21) and (3.23) to be satisfied, one needs to have

$$(3.24) \quad \frac{q}{p-q} < \frac{Nq}{N-q} \quad \text{hence} \quad q < \frac{N(p-1)}{N-1}.$$

As this must be satisfied for any $p < p_\beta$, we get the following bound for q ,

$$(3.25) \quad q < N'(p_\beta - 1) = q_\beta.$$

Condition (3.25) makes sense if and only if β is such that

$$(3.26) \quad 1 < q_\beta < p_\beta \quad \text{hence} \quad 2 - \frac{1}{N} < p_\beta < N.$$

Since $0 \leq \beta < 1$, the condition $p_\beta < N$ is always satisfied. However, $p_\beta > 2 - \frac{1}{N}$ is equivalent to

$$(3.27) \quad \beta < \frac{1}{2N-1} =: \beta_c.$$

Conditions (3.27) and (3.25) then yield the bounds

$$(3.28) \quad \begin{aligned} \beta_c &= \frac{1}{3} \quad \text{and} \quad q_\beta = 2 \left(\frac{2}{1+\beta} - 1 \right) \quad \text{if} \quad N = 2 \\ \beta_c &= \frac{1}{5} \quad \text{and} \quad q_\beta = \frac{3}{2} \left(\frac{2}{1+\beta} - 1 \right) \quad \text{if} \quad N = 3. \end{aligned}$$

Therefore, if $\beta < \beta_c$ and $q < q_\beta$, considering r satisfying (3.21) and (3.23) for all $p < p_\beta$, $q < q_\beta$, $\beta < \beta_c$, Inequality (3.22) becomes

$$(3.29) \quad \|k\|_{1,q;\Omega}^q \lesssim \lambda_1(n_0) + c\lambda_2(n_0) \|k\|_{1,q;\Omega}^{\frac{r(p-q)}{p}}.$$

As when $0 < \beta < \beta_c$, we verify $\gamma = \frac{q_\beta^*(p_\beta - q_\beta)}{p_\beta q_\beta} < 1$, then we deduce from (3.29) that for $q < q_\beta$ and $0 < \beta < \beta_c$,

$$(3.30) \quad \|k\|_{1,q;\Omega} \leq C = C(\beta, \nu_m, D),$$

a constant that blows up when $q \rightarrow q_\beta$ or $\beta \rightarrow \beta_c$.

Step 3. $L^{3/2}(\Omega, \rho^{-1})$ estimate. Let $G_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be the odd function given by

$$(3.31) \quad G_\varepsilon : \begin{cases} G_\varepsilon(x) = \frac{x}{\varepsilon} & \text{if } 0 \leq x < \varepsilon, \\ G_\varepsilon(x) = 1 & \text{if } 1 \leq x. \end{cases}$$

We take $G_\varepsilon(k)$ as test function in (3.2). By the same argument as above,

$$\int_{\Omega} \mathbf{u} \cdot \nabla k G_\varepsilon(k) = 0, \quad \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \cdot \nabla G_\varepsilon(k) = \int_{\Omega} G'_\varepsilon(k) \varrho^\beta \tilde{\mu}(k) |\nabla k|^2 \geq 0,$$

hence, since $0 \leq G_\varepsilon(k) \leq 1$,

$$\int_{\Omega} \frac{G_\varepsilon(k) k^{\frac{3}{2}}}{\rho} \leq \|\mathbb{D}\|_{0,1;\Omega}.$$

As $k > 0$ and $\rho > 0$ in Ω , $\frac{G_\varepsilon(k) k^{\frac{3}{2}}}{\rho} \rightarrow \frac{k^{\frac{3}{2}}}{\rho}$ a.e. in Ω . Then by Fatou's Lemma,

$$(3.32) \quad 0 \leq \int_{\Omega} \frac{k^{\frac{3}{2}}}{\rho} \leq \|\mathbb{D}\|_{0,1;\Omega}.$$

Notice that this last estimate does not require any restriction about β . \square

Remark 3.2. Let T_n be the truncation at height n . Then by taking $T_n(k)$ as test in (3.2), we see that $T_n(k) \in W_0^{1,p}(\Omega)$, uniformly in n , namely for any $p < p_\beta$,

$$(3.33) \quad \|T_n(k)\|_{1,p;\Omega} \leq C(D, \nu_m, \beta).$$

Remark 3.3. We get the same estimate if instead of (3.2) we consider its following regularization, where $\varepsilon > 0$,

$$(3.34) \quad \begin{cases} \mathbf{u} \cdot \nabla k - \operatorname{div}(\varrho^\beta \tilde{\mu}(k) \nabla k) - \varepsilon \Delta k + \frac{k^{3/2}}{\varrho + \varepsilon} = \mathbb{D}(\mathbf{u}, k) & \text{in } \Omega, \\ k \geq 0 & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

for a given $\mathbf{u} \in V_0$, and considering renormalized solutions $k \in \bigcap_{r < N'} W_0^{1,r}(\Omega)$ such that

for any $n \in \mathbb{N}$, $T_n(k) \in H_0^1(\Omega)$ (see [7, 15]). For these solutions, tests can be taken in $W_0^{1,\infty}(\Omega)$, as well as function of the form $h(k)\varphi$, for $q \in W_0^{1,\infty}(\Omega)$ and $h : \mathbb{R} \rightarrow \mathbb{R}$, continuous, bounded, C^1 -piecewise such that $h' \in L^\infty(\mathbb{R})$ with a finite number of discontinuities. This is what justifies that the whole procedure above to get estimates also applies to this regularized equation.

In the following we set

$$(3.35) \quad \mathbb{K}_\beta = \bigcap_{1 \leq q < q_\beta} W_0^{1,q}(\Omega).$$

4 Main result

According to the previous sections, the assumptions about the data are the following:

$$(4.1) \quad \begin{cases} \tilde{\nu}, \tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R} \text{ are continuous and } \forall k \in \mathbb{R}, & 0 < \nu_m \leq \tilde{\nu}(k), \tilde{\mu}(k) \leq \nu_M, \\ \mathbf{f} \in L^2(\Omega)^N, & \lambda > 0 \\ 0 < \eta < \eta'_c = \frac{1}{6N-9}, & \beta < \beta_c = \frac{1}{2N-1}. \end{cases}$$

Definition 4.1. When (4.1) holds, we say that $(\mathbf{u}, k) \in V_\eta \times \mathbb{K}_\beta$ is a weak solution to the initial problem (1.4) if $k \geq 0$ a.e in Ω , and if for all $(\mathbf{v}, \varphi) \in V_\eta \times \mathcal{D}(\Omega)$,

$$(4.2) \quad \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} - \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}^T + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

$$(4.3) \quad - \int_{\Omega} k \mathbf{u} \cdot \nabla \varphi + \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \nabla \varphi + \int_{\Omega} \frac{k^{3/2} \varphi}{\varrho} = \int_{\Omega} \mathbb{D}(\mathbf{u}, k) \varphi,$$

where $\mathbb{D}(\mathbf{u}, k) = \rho^\eta \tilde{\nu}(k) |\nabla \mathbf{u}|^2$.

Remark 4.1. As shown in section 2.3, the assumption about η makes sure that all the integrals in (4.2) are well defined. Moreover, by (2.5) and (3.3), $\mathbb{D}(\mathbf{u}, k) \in L^1(\Omega)$. Therefore, by (3.5), it is guaranteed that all the integrals in (4.3) are well defined, except the transport term about which we must take care. However, $k \in L^{3/2}(\Omega)$ and as $V_\eta \subset H_{loc}^1(\Omega)^N$, then for a given $\varphi \in \mathcal{D}(\Omega)$, we have at least $\mathbf{u} \in L^3(\text{supp}(\varphi))$. Therefore, $k \mathbf{u} \in L^1(\text{supp}(\varphi))$, which makes well defined the integral $\int_{\Omega} k \mathbf{u} \cdot \nabla \varphi$ since $\nabla \varphi \in L^\infty(\Omega)$.

Definition 4.2. We say that $(\mathbf{u}, k) \in V_0 \times (\cap_{q < N'} W_0^{1,q}(\Omega))$ is a weak solution to the perturbed Problem (1.5) if $k \geq 0$ a.e in Ω , and if for all $(\mathbf{v}, \varphi) \in V_0 \times \mathcal{D}(\Omega)$,

$$(4.4) \quad \lambda \int_{\Omega} \mathbf{u} \cdot \mathbf{v} - \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v}^T + \int_{\Omega} \varrho^\eta \tilde{\nu}(k) \nabla \mathbf{u} : \nabla \mathbf{v} + \varepsilon \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v},$$

$$(4.5) \quad - \int_{\Omega} k \mathbf{u} \cdot \nabla \varphi + \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \nabla \varphi + \varepsilon \int_{\Omega} \nabla k \cdot \nabla \varphi + \int_{\Omega} \frac{k^{3/2} \varphi}{\varrho + \varepsilon} = \int_{\Omega} \mathbb{D}(\mathbf{u}, k) \varphi.$$

By [15], we already know:

Theorem 4.1. Let $\varepsilon > 0$. Assume that $\tilde{\nu}, \tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, bounded and nonnegative, $\mathbf{f} \in V'_0$, $\lambda \geq 0$. Let $0 \leq \eta, \beta \leq 1$. Then Problem (1.5) has a weak solution $(\mathbf{u}_\varepsilon, k_\varepsilon) \in V_0 \times (\cap_{q < N'} W_0^{1,q}(\Omega))$.

We prove the following result:

Theorem 4.2. Assume that (4.1) holds. Then the initial Problem (1.4) has a weak solution obtained by approximation (SOLA). More precisely, let $(\mathbf{u}_\varepsilon, k_\varepsilon)$ be a weak solution to the perturbed problem (1.5). Then there exists $(\varepsilon_n)_{n \in \mathbb{N}}$ that converges to 0 and $(\mathbf{u}, k) \in V_\eta \times \mathbb{K}_\beta$ a weak solution to the initial problem such that the sequence $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ strongly converges to \mathbf{u} in V_η and the sequence $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ weakly converges to k in $W_0^{1,q}(\Omega)$ for all $1 \leq q < q_\beta$.

Proof. The proof is the synthesis of the sections 2 and 3. Passing to the limit in the fluid equation was already done in the proof of Theorem 2.2. Theorem 2.3 ensures the strong convergence of the source term in $L^1(\Omega)$. We already know estimates for the TKE from Theorem 3.1 and Remark 3.3. Therefore, we summarize in a first step the sequence

extraction process, and we proceed in a second step to pass to the limit in the TKE equation.

Step 1. Extracting subsequences - Due to (4.1), particularly the choices of η and β , we know from step 1 of the proof of Theorem 2.2 and Theorem 3.1, that there exists $(\varepsilon_n)_{n \in \mathbb{N}}$ that converges to 0, such that the family $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ is bounded in V_η , and the family $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ is bounded in $W_0^{1,q}(\Omega)$ for any $1 < q < q_\beta$ (recall that q_β is given by (3.4)). Arguing as Step 2 of the proof of Theorem 2.2, based on a standard compactness argument combined to an uniqueness argument and Lebesgue inverse Theorem, we deduce that there exists $(\varepsilon_n)_{n \in \mathbb{N}}$, $\mathbf{u} \in V_\eta$ and $k \in \mathbb{K}_\beta$ such that

1. $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ converges weakly to \mathbf{u} in V_η ,
2. $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ converges weakly to \mathbf{u} in $W_0^p(\Omega)^N$ for all $1 < p < p_\eta$,
3. $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ converges strongly to \mathbf{u} in $L^q(\Omega)^N$ for all $q < p_\eta^*$,
4. $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ converges weakly to k in $W_0^q(\Omega)$ for all $1 < q < q_\beta$,
5. $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ converges strongly to k in $L^r(\Omega)$ for all $1 < r < q_\beta^*$,
6. $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ converges a.e. to k .

Moreover, by Theorem 2.3 and from the proof of Theorem 2.2, we can add to the list that $(\mathbf{u}_{\varepsilon_n})_{n \in \mathbb{N}}$ converges strongly to \mathbf{u} in V_η , and that (\mathbf{u}, k) satisfy (4.2). We also know that the sequence $(\varrho^\eta \nu_{\text{turb}}(k_{\varepsilon_n}) |\nabla \mathbf{u}_{\varepsilon_n}|^2)_{n \in \mathbb{N}}$ converges strongly in $L^1(\Omega)$ to $\varrho^\eta \nu_{\text{turb}}(k) |\nabla \mathbf{u}|^2$. Finally, as $V_\eta \subset H_{loc}^1(\Omega)$, by the same argument as in the proof of 2.2 based on the uniqueness of the limit, $(\varepsilon_n)_{n \in \mathbb{N}}$ can be chosen so that for any $\omega \subset\subset \Omega$, $(\mathbf{u}_{\varepsilon_n}|_\omega)_{n \in \mathbb{N}}$ converges strongly to $\mathbf{u}|_\omega$ in $L^p(\omega)^N$, for all $p < \infty$ when $N = 2$, and $p < 6$ when $N = 3$.

Step 2. Passing to the limit in the TKE equation - Let $\varphi \in \mathcal{D}(\Omega)$, and let $\omega \subset\subset \Omega$ denotes the support of φ . From the above, we already know that when $n \rightarrow \infty$,

$$\int_{\Omega} \mathbb{D}(\mathbf{u}_{\varepsilon_n}, k_{\varepsilon_n}) \varphi \rightarrow \int_{\Omega} \mathbb{D}(\mathbf{u}, k) \varphi.$$

By a usual argument [6, 15], we also have

$$\int_{\Omega} \varrho^\beta \tilde{\mu}(k_{\varepsilon_n}) \nabla k_{\varepsilon_n} \nabla \varphi \rightarrow \int_{\Omega} \varrho^\beta \tilde{\mu}(k) \nabla k \nabla \varphi.$$

Moreover, let $1 < q < q_\beta$ be fixed. By the weak convergence of $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ to k in $W_0^{1,q}(\Omega)$ to k and $\mathcal{D}(\Omega) \subset W_0^{1,q'}(\Omega)$,

$$\int_{\Omega} \nabla k_{\varepsilon_n} \nabla \varphi \rightarrow \int_{\Omega} \nabla k \nabla \varphi.$$

hence

$$\varepsilon \int_{\Omega} \nabla k_{\varepsilon_n} \nabla \varphi \rightarrow 0.$$

Whether we are in dimension 2 or in dimension 3, we easily check that $3/2 < q_\beta^*$. Therefore, as $\varrho|_\omega \geq C_\omega > 0$,

$$\frac{k_{\varepsilon_n}^{3/2}}{\varrho + \varepsilon_n} \rightarrow \frac{k^{3/2}}{\varrho} \quad \text{in } L^1(\omega),$$

which leads to

$$\int_{\Omega} \frac{k_{\varepsilon_n}^{3/2} \varphi}{\varrho + \varepsilon_n} \rightarrow \int_{\Omega} \frac{k^{3/2} \varphi}{\varrho}.$$

Finally, by the same argument using $\mathbf{u}_{\varepsilon_n} \rightarrow \mathbf{u}$ in $L^3(\omega)$,

$$\int_{\Omega} k_{\varepsilon_n} \mathbf{u}_{\varepsilon_n} \cdot \nabla \varphi \rightarrow \int_{\Omega} k \mathbf{u} \cdot \nabla \varphi.$$

As we were able to pass to the limit in the TKE equations, we have shown that (\mathbf{u}, k) also satisfies (4.3), which concludes the proof. \square

References

- [1] Christine Bernardi, Tomás Chacón Rebollo, Frédéric Hecht, and Roger Lewandowski. Automatic insertion of a turbulence model in the finite element discretization of the Navier-Stokes equations. *Math. Models Methods Appl. Sci.*, 19(7):1139–1183, 2009.
- [2] L. Berselli and R. Lewandowski. On the reynolds time-averaged equations and the long-time behavior of leray-hopf weak solutions, with applications to ensemble averages. *Nonlinearity*, 32(11):4579–4608, 2019.
- [3] Luigi C. Berselli and Dominic Breit. On the existence of weak solutions for the steady Baldwin-Lomax model and generalizations. *J. Math. Anal. Appl.*, 501(1):Paper No. 124633, 28, 2021.
- [4] Luigi C. Berselli, Alex Kaltenbach, M. Růžička, and Roger Lewandowski. On the existence of weak solutions for an unsteady rotational Smagorinsky model. *Pure Appl. Funct. Anal.*, 8(1):83–102, 2023.
- [5] Lucio Boccardo and Thierry Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.
- [6] T. Chacón-Rebollo and R. Lewandowski. *Mathematical and Numerical Foundations of Turbulence Models and Applications*. Modeling and Simulation in Science, Engineering and Technology. Springer New York, 2014.
- [7] Gianni Dal Maso, François Murat, Luigi Orsina, and Alain Prignet. Definition and existence of renormalized solutions of elliptic equations with general measure data. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(5):481–486, 1997.
- [8] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.
- [9] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, and L. Tartar. On a turbulent system with unbounded eddy viscosities. *Nonlinear Anal.*, 52(4):1051–1068, 2003.
- [10] Kiera Kean, William Layton, and Michael Schneier. On the Prandtl-Kolmogorov 1-equation model of turbulence. *Philos. Trans. Roy. Soc. A*, 380(2226):Paper No. 20210054, 15, 2022.
- [11] A. Kufner. *Weighted Sobolev spaces*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.

- [12] J. Lederer and R. Lewandowski. A RANS 3D model with unbounded eddy viscosities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 24(3):413–441, 2007.
- [13] F. Legeais and R. Lewandowski. Solutions obtained by approximations of a degenerated smagorinsky’s model. 2023.
- [14] R. Lewandowski. *Analyse mathématique et océanographie*, volume 39 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1997.
- [15] R. Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Anal.*, 28(2):393–417, 1997.
- [16] Roger Lewandowski. On a one dimensional turbulent boundary layer model. *Pure Appl. Funct. Anal.*, 5(5):1115–1129, 2020.
- [17] Roger Lewandowski and Bijan Mohammadi. Existence and positivity results for the ϕ - θ and a modified k - ϵ two-equation turbulence models. *Math. Models Methods Appl. Sci.*, 3(2):195–215, 1993.
- [18] B. Mohammadi and O. Pironneau. *Analysis of the k-epsilon turbulence model*. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.
- [19] A. S. Monin and A. M. Obukhov. Basic laws of turbulent mixing in the surface layer of the atmosphere. *Tr. Akad. Nauk. SSSR Geophys. Inst.*, 24(151):163–187, 1954.
- [20] S.-B. Pope. *Turbulent flows*. Cambridge University Press, Cambridge, 2000.
- [21] Jacques Rappaz and Jonathan Rochat. On non-linear Stokes problems with viscosity depending on the distance to the wall. *C. R. Math. Acad. Sci. Paris*, 354(5):499–502, 2016.
- [22] Jacques Rappaz and Jonathan Rochat. On some weighted Stokes problems: applications on Smagorinsky models. In *Contributions to partial differential equations and applications*, volume 47 of *Comput. Methods Appl. Sci.*, pages 395–410. Springer, Cham, 2019.
- [23] G. Stampacchia. *Équations elliptiques du second ordre à coefficients discontinus*. Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l’Université de Montréal, Montreal, Que., 1966.