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Emile Deleage

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# WELL-POSEDNESS OF REYNOLDS AVERAGED EQUATIONS FOR COMPRESSIBLE FLUIDS WITH A VANISHING PRESSURE

ÉMILE DELÉAGE<sup>1,2</sup>

ABSTRACT. We show that the Reynolds averaged equations for compressible fluids (neglecting third order correlations) are well-posed in  $H^s$  when the pressure vanishes in dimensions  $d = 2$  and  $3$ . In order to do this, we show that the system is Friedrichs-symmetrizable. This model belongs to the class of non-conservative hyperbolic systems. Hence the usual symmetrisation method for conservation laws can not be used here.

## 1. INTRODUCTION AND MAIN RESULTS

We study the Reynolds averaged equations for compressible fluids, where third-order correlations are neglected. This system can be written in Eulerian coordinates as

$$\begin{aligned}
 (1.1a) \quad & \partial_t \rho + \operatorname{div}(\rho v) = 0, \\
 (1.1b) \quad & \partial_t v + (v \cdot \nabla v) + \frac{1}{\rho} (\nabla p + \operatorname{div}(\rho P)^T) = 0, \\
 (1.1c) \quad & \partial_t P + (v \cdot \nabla) P + \frac{\partial v}{\partial x} P + P \left( \frac{\partial v}{\partial x} \right)^T = 0.
 \end{aligned}$$

The variables are the averaged density  $\rho > 0$ , the averaged velocity  $v \in \mathbb{R}^d$  ( $d = 2$  or  $3$ ), and  $P$  is the Reynolds stress tensor,  $P \in S_d^{++}(\mathbb{R})$ . The function  $p$  is the pressure of the fluid and is a function of the density  $\rho$ , through an equation of state  $p = p(\rho)$ . The map  $\rho \mapsto p(\rho)$  is supposed to be of class  $C^1$  and non-decreasing. Typical pressure laws are of the form  $p(\rho) = a\rho^\gamma$ , with  $a > 0$  and  $\gamma > 0$  two constants.

The tensor  $P$  is a classical Reynolds tensor appearing in the Reynolds averaging of turbulent flows for barotropic compressible fluids (see [20], [25], [18]). It also appears in the description of free surface shear flows, where the averaging operator is the depth averaging (cf. Annexe C, A7 in [22] for a derivation of the model). In that latter case, the density  $\rho$  must be replaced by the water depth, often denoted  $h$ . The pressure is then given by  $p(h) = gh^2/2$  (cf. [22] for instance).

System (1.1) is hyperbolic whenever  $p'(\rho) \geq 0$  and  $P$  is definite positive, as it has been proved in [22]. Equation (1.1a) shows that the density  $\rho$  is conserved.

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<sup>1</sup>AIX MARSEILLE UNIV., CNRS, CENTRALE MARSEILLE, I2M UMR CNRS 7373, 13000 MARSEILLE, FRANCE

<sup>2</sup>UNIV. GRENOBLE ALPES, CNRS, INRAE, IRD, GRENOBLE INP, IGE, 38000 GRENOBLE, FRANCE

*E-mail address:* emile.deleage@univ-amu.fr.

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The conservation of momentum  $\rho v$  also holds: (1.1b) rewrites as

$$(1.2) \quad \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + pI_d + \rho P)^T = 0.$$

One can also deduce from (1.1) the conservation of energy:

$$(1.3) \quad \partial_t e + \operatorname{div}(e v + p v + \rho P v) = 0, \quad \text{with } e := \frac{1}{2} \rho |v|^2 + \rho \mathcal{E}(\rho) + \frac{1}{2} \operatorname{Tr}(\rho P).$$

The map  $\rho \mapsto \rho \mathcal{E}(\rho)$  is called the volumic internal energy, and is linked to the pressure via the relation  $p(\rho) = \rho^2 \mathcal{E}'(\rho)$ . The term  $\rho |v|^2/2$  is the volumic kinetic energy of the fluid, and the term  $\operatorname{Tr}(\rho P)/2$  is the energy associated to the tensor  $P$ .

In [9], it was shown that system (1.1) admits a variational formulation, as it is often the case in physics when the energy of a system is conserved. Define the Lagrangian density

$$(1.4) \quad \mathcal{L}(\rho, v, P) := \frac{1}{2} \rho |v|^2 - \rho \mathcal{E}(\rho) - \frac{1}{2} \operatorname{Tr}(\rho P),$$

and the corresponding action

$$(1.5) \quad \mathcal{A} := \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \mathcal{L} dx dt.$$

Then one can show that (1.1b) is the Euler-Lagrange equation given by the stationary action principle applied to the action (1.5), under the two constraints (1.1a) and (1.1c).

The tensor  $P$  admits an additional conservation law, sometimes called conservation of enstrophy, that is a consequence of (1.1c) and can be written

$$(1.6) \quad \partial_t \left( \frac{\det P}{\rho^2} \right) + v \cdot \nabla \left( \frac{\det P}{\rho^2} \right) = 0.$$

Note that equation (1.1c) implies that the symmetry of  $P$  is conserved by the evolution. Equation (1.6) then implies that, if  $P(t) \in S_d^{++}(\mathbb{R})$  for some instant  $t$ , then this property is true for all times.

It has been proved in [10] that system (1.1) does not admit any further conservation law. Thus, in dimension  $d = 2$  or  $3$ , system (1.1) is not conservative. Hence the usual symmetrisation method of Godunov (cf. [11]) and Lax and Friedrichs (cf. [8]) for hyperbolic systems of conservation laws (see for instance [24], pages 83-84) can not be used here.

However, one can show that system (1.1) is Friedrichs-symmetrizable when the pressure vanishes. More precisely, we state the following theorem :

**Theorem 1.** *Let  $d = 2$  or  $3$ . Suppose that  $\rho$  takes values in  $\mathbb{R}_+^*$  and  $P$  takes values in  $S_d^{++}(\mathbb{R})$ . Then the two following properties are equivalent:*

- (1) *System (1.1) is Friedrichs-symmetrizable*
- (2) *The pressure  $p$  is constant:  $p'(\rho) = 0$ , or the tensor  $P$  is a scalar matrix, i.e. there exists  $\lambda = \lambda(t, x) \in \mathbb{R}$  such that  $P = \lambda I_d$ .*

Let us make few comments about this theorem:

- The tensor  $P = \lambda I_d$  is a solution of system (1.1) only for trivial velocities; hence, for applications of this result, the case  $p' = 0$  seems more interesting.

- This property holds in the variables  $(\rho, v, P)$ . It could be possible that system (1.1), written in different variables, appears to be symmetrizable even when  $p' \neq 0$ .
- When  $d = 1$ , system (1.1) is symmetrizable, even when  $p' \neq 0$ . In fact, one can prove that one-dimensional hyperbolic systems are always Friedrichs-symmetrizable (cf. [19]). In higher dimension  $d \geq 2$ , this does not hold anymore.

As a consequence of Theorem 1, we have the following result regarding the well-posedness of system (1.1):

**Proposition 1.** *Let  $d = 2$  or  $3$ ,  $s > 1 + d/2$  and  $\mathcal{U} := \mathbb{R}_+^* \times \mathbb{R}^d \times S_d^{++}(\mathbb{R})$ . Let  $\bar{Y} := (\bar{\rho}, \bar{u}, \bar{P}) \in \mathcal{U}$  and  $Y_0$  taking values in  $\mathcal{U}$  such that  $Y_0 - \bar{Y} \in H^s(\mathbb{R}^d)$ . We consider the Cauchy problem associated to (1.1) with initial data  $Y_0$ . There exists  $T > 0$  such that (1.1) with  $p' = 0$  has a unique classical solution  $Y(t)$  in  $\mathcal{C}^1([0, T] \times \mathbb{R}^d)$  with values in  $\mathcal{U}$  achieving the initial data  $Y(0) = Y_0$ . Furthermore,  $Y - \bar{Y}$  belongs to  $\mathcal{C}([0, T], H^s(\mathbb{R}^d)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^d))$ .*

Proposition 1 is a consequence of Theorem 1 and of the theory of well-posedness of Kato for quasilinear evolution equations (cf. [17]). For a detailed proof of this result, see for instance Sect. 10 in [2].

System (1.1) has been used over the last years in the modeling of turbulent flows, including for numerical simulations (see [23], [15], [10], [14]). However, the well-posedness of (1.1) in dimension  $d \geq 2$  is still uncertain today. Proposition 1 states an answer to this question in the case of a vanishing pressure. One could also obtain system (1.1) with  $p' = 0$  when modeling a fluid for which the pressure gradient  $\nabla p$  is negligible compared to the turbulence of the fluid  $\operatorname{div}(\rho P)$  in (1.1b). This amounts to consider the high Mach number limit. This limit was studied by various authors in the Euler and Navier-Stokes cases and can lead to a vanishing pressure gradient ([12], [21]). The hypothesis  $p' = 0$  can also be found in the literature in a model called “pressureless gas dynamics”. This system appears in different physical contexts. It is used in cosmology in order to model galaxies formation in the presence of gravitational instability (see [26]). These equations also arise as an hydrodynamic limit of kinetic equations, when considering a cold plasma (see [4]), or the granular gases equation (see [16]). Another field of applications is the study of collective behaviors, in which the pressureless Euler equations appear with additional non-local dissipative terms ([6], [7]). Eventually, the study of the pressureless equations is motivated by numerical analysis: some numerical solvers for Euler equations also use the case  $p' = 0$  as an intermediate step in their splitting method, this allows to decouple the convective subsystem from the acoustic one ([1], see also [13]).

The mathematical analysis of the pressureless Euler equations is constrained by the fact that the system is only weakly hyperbolic. As a consequence, phenomena like creation of vacuum or high concentrations (delta shocks) can occur (see for instance [26], [4], [5], [3]). System (1.1) with  $p' = 0$  can thus be seen as a hyperbolic version of the pressureless gas model. In this sense, the Reynolds tensor  $P$  brings more regularity to the model.

**Notations.** We denote  $\partial_i := \partial/\partial x_i$  the partial derivative of the variable  $x_i$ , for  $1 \leq i \leq d$ . If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a scalar function, we denote by  $\nabla f \in \mathbb{R}^d$  the gradient of  $f$ , i.e. the vector field of components  $\partial_i f$ ,  $1 \leq i \leq d$ .

If  $Z = (Z_1, \dots, Z_n) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a vector field, the divergence of  $Z$  is the scalar function defined by

$$\operatorname{div}(Z) := \partial_1 Z_1 + \dots + \partial_n Z_n.$$

We also denote by  $\partial Z / \partial x$  the Jacobian matrix of  $Z$ , i.e. the matrix of coefficients  $(\partial Z / \partial x)_{i,j} = \partial Z_i / \partial x_j$ , for  $1 \leq i, j \leq d$ .

If  $Z = (Z_i)_{1 \leq i \leq d}$  and  $Z' = (Z'_i)_{1 \leq i \leq d}$  are two vector fields, we denote  $Z \otimes Z'$  the second order tensor defined by  $Z \otimes Z' := Z(Z')^T$ , i.e. the matrix of coefficients  $(Z \otimes Z')_{i,j} = Z_i Z'_j$ , for  $1 \leq i, j \leq d$ .

If  $A : \mathbb{R}^d \rightarrow M_d(\mathbb{R})$  is a second order tensor, we defined the divergence of  $A$  as the line vector of  $\mathbb{R}^d$  whose  $i$ -th component is given by the divergence of the  $i$ -th column of  $A$ .

For any positive integer  $d$ , we denote  $I_d \in M_d(\mathbb{R})$  the identity matrix of size  $d$ . We denote  $S_d^{++}(\mathbb{R})$  the set of symmetric definite positive matrices, i.e. the symmetric matrices of size  $d$  with a positive spectrum.

## 2. PROOF OF THE THEOREM

We now give the proof of Theorem 1.

*Proof.* We write system (1.1) in matricial form:

$$\partial_t Y + A(Y, \nabla)Y = 0, \quad \text{with } Y := \begin{pmatrix} \rho \\ v \\ \tilde{P} \end{pmatrix} \in \mathbb{R}^{1+d+d(d+1)/2}$$

and, if  $\xi = (\xi_i)_{1 \leq i \leq d} \in \mathbb{R}^d$ ,

$$(2.1) \quad A(Y, \xi) := \begin{pmatrix} v \cdot \xi & \rho \xi^T & 0 \\ \frac{1}{\rho}(p'(\rho)I_d + P)\xi & (v \cdot \xi)I_d & C(\xi) \\ 0 & D(\xi) & (v \cdot \xi)I_{d(d+1)/2} \end{pmatrix}.$$

When  $d = 2$ , the symmetric matrix  $P = (P_{ij})_{1 \leq i, j \leq 2}$  can be identified as a vector  $\tilde{P} \in \mathbb{R}^3$  given by

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix}.$$

The matrices  $C(\xi)$  and  $D(\xi)$  are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \quad \text{and} \quad D(\xi) := \begin{pmatrix} 2P_{11}\xi_1 + 2P_{12}\xi_2 & 0 \\ P_{21}\xi_1 + P_{22}\xi_2 & P_{11}\xi_1 + P_{12}\xi_2 \\ 0 & 2P_{12}\xi_1 + 2P_{22}\xi_2 \end{pmatrix}.$$

When  $d = 3$ , the symmetric matrix  $P$  can be identified as a vector  $\tilde{P} \in \mathbb{R}^6$ :

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix}.$$

The matrices  $C(\xi)$  and  $D(\xi)$  are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}$$

and

$$D(\xi) := \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix},$$

where  $(P\xi)_i$  denotes the  $i$ -th component of the vector  $P\xi$ .

**We first show the implication** (2)  $\Rightarrow$  (1). Namely, if  $p' = 0$  or  $P = \lambda Id$ , then system (1.1) is Friedrichs-symmetrizable.

We thus suppose that  $p' = 0$  or  $P = \lambda Id$ .

Consider  $S = S(Y) \in S_n^{++}(\mathbb{R})$  (with  $n = (d+1)(d+2)/2$ ) defined as a block matrix, compatible with  $A$ :

$$(2.2) \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix}, \text{ with } S_2 \in M_d(\mathbb{R}) \text{ and } S_3 \in M_{d(d+1)/2}(\mathbb{R}).$$

Note that  $S$  is symmetric definite positive if and only if the matrices  $S_2$  and  $S_3$  are also symmetric definite positive.

We can now compute the product  $SA$  by block matrix multiplication. We obtain that

$$(2.3) \quad SA = \begin{pmatrix} v \cdot \xi & \rho \xi^T & 0 \\ \frac{1}{\rho} S_2(p'(\rho)Id + P)\xi & (v \cdot \xi)S_2 & S_2 C(\xi) \\ 0 & S_3 D(\xi) & (v \cdot \xi)S_3 \end{pmatrix}$$

Case  $d = 2$ . Let us choose

$$S_2 := \mu P^{-1} \text{ and } S_3 := \mu \begin{pmatrix} \frac{1}{2}q_{11}^2 & q_{12}q_{11} & \frac{1}{2}q_{12}^2 \\ q_{12}q_{11} & q_{11}q_{22} + q_{12}^2 & q_{12}q_{22} \\ \frac{1}{2}q_{12}^2 & q_{12}q_{22} & \frac{1}{2}q_{22}^2 \end{pmatrix},$$

where we denoted

$$(2.4) \quad P^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \in S_2^{++}(\mathbb{R}),$$

and

$$(2.5) \quad \mu := \begin{cases} \rho^2 & \text{when } p'(\rho) = 0, \\ \rho^2 \frac{\lambda}{p'(\rho) + \lambda} & \text{when } P = \lambda Id. \end{cases}$$

The matrix  $S_2$  is symmetric definite positive.

We see that the principal minors of  $\mu^{-1}S_3$  are given by  $M_1 = \frac{1}{2}q_{11}^2 > 0$ ,  $M_2 = \frac{1}{2}q_{11}^2(q_{11}q_{22} - q_{12}^2) > 0$ , and  $M_3 = \frac{1}{4}(q_{11}q_{22} - q_{12}^2)^3 > 0$  (recall that  $P$  is positive definite). Hence by Sylvester's criterion,  $S_3$  is definite positive, and  $S$ , defined by (2.2), is a positive definite matrix.

We compute that

$$(2.6) \quad \frac{1}{\rho} S_2 (p'(\rho) Id + P) \xi = \rho \xi,$$

$$S_2 C(\xi) = \mu \begin{pmatrix} q_{11}\xi_1 & q_{11}\xi_2 + q_{12}\xi_1 & q_{12}\xi_2 \\ q_{12}\xi_1 & q_{12}\xi_2 + q_{22}\xi_1 & q_{22}\xi_2 \end{pmatrix},$$

and, since  $q_{i1}P_{1j} + q_{i2}P_{2j} = \delta_{ij}$  by (2.4),

$$S_3 D(\xi) = \mu \begin{pmatrix} q_{11}\xi_1 & q_{12}\xi_1 \\ q_{12}\xi_1 + q_{11}\xi_2 & q_{22}\xi_1 + q_{12}\xi_2 \\ q_{12}\xi_2 & q_{22}\xi_2 \end{pmatrix} = [S_2 C(\xi)]^T.$$

Hence for any  $Y \in \mathcal{U}$ , and for any  $\xi \in \mathbb{R}^2$ , the matrix  $S(Y)$  is symmetric definite positive and (2.3) shows that the matrix  $S(Y)A(Y, \xi)$  is symmetric. As a consequence, (1.1) is Friedrichs-symmetrizable when  $d = 2$ .

*Case  $d = 3$ .* Define again  $S$  by equation (2.2). Choose  $S_2 = \mu P^{-1}$  with  $\mu$  as in (2.5), such that  $S_2$  is symmetric definite positive and (2.6) holds again. Define  $S_3$  by

$$S_3 := \mu \begin{pmatrix} \frac{1}{2}q_{11}^2 & q_{11}q_{12} & q_{11}q_{13} & \frac{1}{2}q_{12}^2 & q_{12}q_{13} & \frac{1}{2}q_{13}^2 \\ q_{11}q_{12} & q_{11}q_{22} + q_{12}^2 & q_{11}q_{23} + q_{12}q_{13} & q_{12}q_{22} & q_{12}q_{23} + q_{22}q_{13} & q_{13}q_{23} \\ q_{11}q_{13} & q_{11}q_{23} + q_{13}q_{12} & q_{11}q_{33} + q_{13}^2 & q_{12}q_{23} & q_{12}q_{33} + q_{23}q_{13} & q_{13}q_{33} \\ \frac{1}{2}q_{12}^2 & q_{12}q_{22} & q_{12}q_{23} & \frac{1}{2}q_{22}^2 & q_{22}q_{23} & \frac{1}{2}q_{23}^2 \\ q_{12}q_{13} & q_{12}q_{23} + q_{13}q_{22} & q_{12}q_{33} + q_{13}q_{23} & q_{22}q_{23} & q_{22}q_{33} + q_{23}^2 & q_{23}q_{33} \\ \frac{1}{2}q_{13}^2 & q_{13}q_{23} & q_{13}q_{33} & \frac{1}{2}q_{23}^2 & q_{23}q_{33} & \frac{1}{2}q_{33}^2 \end{pmatrix},$$

where we denoted again

$$P^{-1} := \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}.$$

We see that  $S_3$  is symmetric. Furthermore, the principal minors of  $\mu^{-1}S_3$  are given by  $M_1 = q_{11}^2/2 > 0$ ,  $M_2 = q_{11}^2(q_{11}q_{22} - q_{12}^2)/2 > 0$ ,  $M_3 = q_{11}^3 \det(P^{-1})/2 > 0$ ,  $M_4 = q_{11}(q_{11}q_{22} - q_{12}^2)^2 \det(P^{-1})/4 > 0$ ,  $M_5 = (q_{11}q_{22} - q_{12}^2)^2 \det(P^{-1})^2/4 > 0$  and  $M_6 = \det(P^{-1})^4/8 > 0$ . Hence, by Sylvester's criterion,  $S_3$  is definite positive and  $S$  is symmetric definite positive.

We also check that

$$S_3 D(\xi) = [S_2 C(\xi)]^T.$$

Hence equation (2.3) shows that  $SA$  is symmetric, and, consequently, (1.1) is Friedrichs-symmetrizable when  $d = 3$ .

**We now show that (1)  $\Rightarrow$  (2).** Suppose that system (1.1) is Friedrichs-symmetrizable, i.e. there is a matrix  $S$  such that the product  $SA$  is symmetric.

We write  $S$  as a block matrix, compatible with  $A$ :

$$S := \begin{pmatrix} S_1 & \alpha & \beta \\ \alpha^T & S_2 & \gamma \\ \beta^T & \gamma^T & S_3 \end{pmatrix}.$$

We compute the product  $SA$  by block multiplication. Since  $SA$  is symmetric, for any block of  $SA$ , denoted  $(SA)_{i,j}$  ( $1 \leq i, j \leq 3$ ), we must have  $(SA)_{i,j} = (SA)_{j,i}^T$ . For  $i = j = 3$ , we obtain the constraint

$$\gamma^T C(\xi) + (v \cdot \xi)S_3 = [\gamma^T C(\xi) + (v \cdot \xi)S_3]^T.$$

Since  $S_3$  is symmetric, we deduce that the product  $\gamma^T C(\xi)$  is symmetric, for any  $\xi \in \mathbb{R}^d$ . By computing explicitly the product, we obtain that the only possibility is that  $\gamma = 0$ .

For  $i = 1$  and  $j = 3$ , we obtain the constraint

$$[\alpha C(\xi) + (v \cdot \xi)\beta]^T = (v \cdot \xi)\beta^T + \frac{1}{\rho}\gamma^T(p'(\rho)Id + P)\xi.$$

Since  $\gamma = 0$ , we obtain that  $\alpha C(\xi) = 0$ , for any  $\xi \in \mathbb{R}^d$ . By computing explicitly the product, we also obtain that  $\alpha = 0$ . Hence  $S$  has to be of the form

$$S = \begin{pmatrix} S_1 & 0 & \beta \\ 0 & S_2 & 0 \\ \beta^T & 0 & S_3 \end{pmatrix}.$$

For  $i = 2$  and  $j = 3$ , we obtain the constraint

$$(2.7) \quad \rho\beta^T\xi^T + S_3D(\xi) = [S_2C(\xi)]^T$$

Solving the linear system (2.7) for  $S_2, S_3$  and  $\beta$  gives after some computations that there exists two constants  $\lambda_1, \lambda_2$  such that

$$(2.8) \quad S_2 = \lambda_1 P^{-1} \quad \text{and} \quad \beta = \begin{cases} \lambda_2(q_{11}, 2q_{12}, q_{22}) & \text{when } d = 2, \\ \lambda_2(q_{11}, 2q_{12}, 2q_{13}, q_{22}, 2q_{23}, q_{33}) & \text{when } d = 3. \end{cases}$$

Note that it follows from these computations that  $\beta D(\xi) = 2\lambda_2\xi^T$ .

For  $i = 1$  and  $j = 2$  we obtain the constraint

$$(2.9) \quad \frac{1}{\rho}S_2(p'(\rho)Id + P)\xi = [S_1\rho\xi^T + \beta D(\xi)]^T = (S_1\rho + 2\lambda_2)\xi.$$

Equation (2.9) implies that  $S_2$  is proportional to  $(p'(\rho)Id + P)^{-1}$ . Since  $S_2$  is also proportional to  $P^{-1}$  by (2.8), we obtain that the two matrices  $P$  and  $P + p'(\rho)Id$  are proportional (recall that  $S_2$  is invertible). Hence  $p' = 0$  or  $P = \lambda Id$ .  $\square$

#### DECLARATIONS

- Conflict Of Interest statement: there is no competing interest.
- Data availability statement: there is no associated data.

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