# Complete finite-strain isotropic thermo-elasticity 

Paul Bouteiller

## To cite this version:

Paul Bouteiller. Complete finite-strain isotropic thermo-elasticity. European Journal of Mechanics A/Solids, 2023, 100, pp.105017. 10.1016/j.euromechsol.2023.105017 . hal-04109554

HAL Id: hal-04109554

## https://hal.science/hal-04109554

Submitted on 30 May 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

6

## 7

# Complete finite-strain isotropic thermo-elasticity 

Paul Bouteiller ${ }^{\text {a,b }}$<br>${ }^{a}$ CEA, DAM, DIF, F-91297 Arpajon, France<br>${ }^{b}$ Université Paris-Saclay, CEA, Laboratoire Matière en Conditions Extrêmes, F-91680 Bruyères-le-Châtel, France


#### Abstract

\section*{5 Abstract}

This paper deals with finite strain isotropic thermo-elasticity without any specific Ansatz regarding the Helmholtz free energy. On the theoretical side, an Eulerian setting of isotropic thermo-elasticity is developed, based on the objective left Cauchy-Green tensor along with the Cauchy stress. The construction of the elastic model relies on a particular invariants choice of the strain measure. These invariants are built so that a succession of elementary experiments, in which the invariants evolve independently, ensures the complete identification of the Helmholtz free energy and thus of the thermo-elastic constitutive law. Expressions idealizing these experimental tests are proposed. A wide range of hyperelastic models are found to be a special case of the model proposed herein.


Keywords: Finite strain thermo-elasticity, experimental identification

## Introduction

Elasticity is probably one of the most extensively discussed topics in the analytical mechanics literature over the last few centuries. Since Robert Hooke's linear relationship linking the current Cauchy stress $\boldsymbol{\sigma}$ to the linearized strain $\varepsilon$, many authors studied small strain but also non-linear elasticity, the latter beeing well suited to study materials, such as polymer, which undergo large deformations. The fundamental ground of this phenomenon, which can preceed more complexe non linear events, is now a very well assessed subject, both theoretically [3, 9, 25] and experimentally.

However, specific Ansatz are still commonly used concerning the Helmholtz free energy from which the elastic constitutive law is derived. Classical small-strain elasticity, linking the second Piola-Kirchoff tensor $\boldsymbol{\pi}$ to the right Green-Lagrange strain tensor $\boldsymbol{E}$ relies on a quadratic expansion of the Helmholtz free energy with respect to the small strain $\boldsymbol{E}$. As regards finite strain elasticity, where such series expansion are no longer valid, the construction is based on a choice of the Helmholtz free energy according to invariants of the deformation tensor. Two main categories are often distinguished, depending on the state variables

[^0]chosen. On the one hand, Neo-Hookean[20], Mooney-Rivlin [22], or Yeoh [36] models postulate a polynomial expansion of the free energy with respect to the fundamental invariants of a strain tensor. The coefficients involved in the polynomial expansion are fitted to reproduce experimental data. On the other hand, stretched based models, such as Ogden's which develop the Helmholtz free energy as a power series, retain the positive eigenvalues of the right Cauchy-Green tensor $\boldsymbol{C}$. Ogden models have been found to be very efficient for large strain elasticity of rubber like material [6]. Many articles have offered reviews of these different models and their respective pros and cons [7, 10, 19, 23].

The first objective of this paper is to present a choice of a triplet of kinematically significant invariants, allowing for the rigorous identification of the Helmholtz free energy, through a succession of simple elementary experiments and without a posteriori adjustment. These invariants are chosen so that they can evolve independently.

In Section 1, we will recall the main lines of construction of the finite strain isotropic thermo-elastic behavior. In Section 2, we will present the invariants of the left Cauchy-Green deformation tensor used for the construction of the elastic model and the resulting constitutive law. Then, we will study, Section 3 , the successive elementary experiments which allow the identification of the Helmholtz free energy. Physically realistic expressions for the experimental data required to build the model will then be proposed in Section 4.

## 1. Reminder of finite strain elasticity

This section briefly presents the fundamental assumption of finite strain isotropic thermo-elasticity within an Eulerian setting. The formulation of the general theory is given in the thermodynamic framework by introducing the Helmholtz free energy as a function of a finite strain measure, and by exploring both the first and the second law of thermodynamics.

### 1.1. Framework

Let $\mathscr{D}_{0}^{m} \in \mathbb{R}^{3}$ be a material domain corresponding to the reference configuration of a continuous medium. Particles labeled by $\boldsymbol{x}_{0} \in \mathscr{D}_{0}^{m}$ are transported in the current configuration to $\boldsymbol{x}_{t} \in \mathscr{D}_{t}^{m}$ by a one-to-one mapping $\varphi_{t}$. Let us note $\boldsymbol{F}$ the deformation gradient of this mapping.

Let us consider two observers (reference frames) $\mathscr{R}$ and $\tilde{\mathscr{R}}$. We denote $\boldsymbol{Q}_{0}$ the orthogonal transformation linking these two reference frames at the initial time and $\boldsymbol{Q}_{t}$ the orthogonal tensor linking these same two observers in the current configuration. A second-rank tensor, $\boldsymbol{Y}$ for the observer $\mathscr{R}$, equal to $\tilde{\boldsymbol{Y}}$ for the observer $\tilde{\mathscr{R}}$, is called invariant (or "objective Lagrangian") [21] if:

$$
\begin{equation*}
\tilde{\boldsymbol{Y}}=\boldsymbol{Q}_{0} \cdot \boldsymbol{Y} \cdot \boldsymbol{Q}_{0}^{\top} \quad \boldsymbol{Q}_{0} \in S O(3) ; \forall t . \tag{1}
\end{equation*}
$$

Similarly, we will call "objective" any second-rank tensor $\boldsymbol{X}$ satisfying the following equality:

$$
\begin{equation*}
\tilde{\boldsymbol{X}}=\boldsymbol{Q}_{t} \cdot \boldsymbol{X} \cdot \boldsymbol{Q}_{t}^{\top} \quad \boldsymbol{Q}_{t} \in S O(3) . \tag{2}
\end{equation*}
$$

The invariance or objectivity property of a tensor is intrinsic and comes either from its mathematical definition or from a physically motivated assumption. The first step in the construction of a constitutive law is the choice of a deformation measure, which cannot be simply $\boldsymbol{F}$, among two large families:

- On the one hand, strain tensors based on the product $\boldsymbol{C}=\boldsymbol{F}^{\top} \cdot \boldsymbol{F}$ or its related tensors of the form $\boldsymbol{Y}=\frac{1}{n}\left(\boldsymbol{U}^{n}-\mathbf{1}\right)$ with $\boldsymbol{U}=\sqrt{\boldsymbol{C}}$ (pure stretch). These deformation tensors are often called "right strain mesure", "Lagrangian" or "invariant" because they obey the relation (1). These strain measures are used to close the mechanical problem by relating the first or second Piola-Kirchoff stress tensor $\boldsymbol{P}$, $\boldsymbol{\pi}$ (which are also postulate invariant) to the displacement field $\boldsymbol{u}$.
- On the other hand, strain measure, called "left", "Eulerian" or "objective", and based on the left CauchyGreen deformation $\boldsymbol{B}=\boldsymbol{F} \cdot \boldsymbol{F}^{\top}$ (subsequently called "Finger tensor") are more rarely considered to build elasticity models. These deformation measurements effectively link the objective Cauchy stress $\sigma$ to the displacement field.


### 1.2. Elastic behavior

The finite-strain isotropic elastic behavior is based on the following three assertions:

1. Temperature and a strain measure are the only state variables.
2. The stress measure is a state function.
3. In all evolutions, the intrinsic dissipation is null.

The first assumption 1) forbids the presence of additional internal variables which could be the witness of phenomena such as plasticity, damage etc... But also anisotropy through a structural tensor [4, 37]. The second assumption 2 excludes viscous effects and a possible strain rate $\boldsymbol{D}$ ( $\boldsymbol{D}$ denotes the symmetric part of the spatial velocity gradient) dependency of the Cauchy stress. The last assumption 3 ensures reversibility in the sense that the entropy production other than thermal is zero. To satisfy these three requirements, most treatises devote the so-called "Lagrangian" approaches and relate the second Piola-Kirchoff tensor to the right Green-Lagrange deformation tensor. Eulerian formulations of elasticity are rarely considered (see nevertheless $[24,31])$, especially since the Cauchy stress cannot be written as the simple derivative of a potential with respect to a strain measure. implies that the Cauchy stress necessarily expands as a quadratic polynomial of the strain measure:

$$
\begin{equation*}
\boldsymbol{\sigma}=K_{0}^{B} \mathbf{1}+K_{1}^{B} \boldsymbol{B}+K_{2}^{B} \boldsymbol{B}^{2} \tag{3}
\end{equation*}
$$

The coefficients $K_{i}^{B}$ are scalar functions of the temperature, but also of a triplet of $\boldsymbol{B}$ invariants. Their expressions, as a function of the free energy, can be deduced from the nullity of the intrinsic dissipation (see [35] p.295) written on the current (spatial) configuration:

$$
\begin{equation*}
\Phi=-\rho\left(\dot{\psi}^{m}+\dot{T} s^{m}\right)+\boldsymbol{\sigma}: \boldsymbol{D}=0 \tag{4}
\end{equation*}
$$

For an isotropic material, the Helmholtz massic free energy is an isotropic function [4, 32] $\psi^{m}(T, \boldsymbol{B})=$ $\psi^{m}\left(T, \boldsymbol{Q}_{t} \cdot \boldsymbol{B} \cdot \boldsymbol{Q}_{t}^{\top}\right), \forall \boldsymbol{Q}_{t} \in S O(3)$ so that there exists $f_{\psi}^{B}$ such that:

$$
\begin{equation*}
\psi^{m}(T, \boldsymbol{B})=f_{\psi}^{B}\left(T, B_{\mathrm{I}}, B_{\mathrm{II}}, B_{\mathrm{III}}\right) \tag{5}
\end{equation*}
$$

where the fundamental invariants $\left(B_{\mathrm{I}}, B_{\mathrm{II}}, B_{\mathrm{III}}\right)$ denote the coefficients of the characteristic polynomial of $\boldsymbol{B}$ ( main invariant). Note that any other triplet of invariants bijectively related to ( $B_{\mathrm{I}}, B_{\mathrm{II}}, B_{\mathrm{III}}$ ), is legitimate. Thus, the "principal" invariants defined by $B_{i}=\frac{1}{i} \operatorname{Tr} \boldsymbol{B}^{i}$ are often used to construct elastic laws as their partial derivatives with respect to the strain measure are straightforward. In the following, we will retain invariants whose kinematic meaning is clear.

According to the fundamental invariants ( $B_{\mathrm{I}}, B_{\mathrm{II}}, B_{\mathrm{III}}$ ), the isotropic finite strain thermo-elastic law reads, (see Appendix A but also [16, 31]):

$$
\begin{equation*}
\boldsymbol{\sigma}=\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}}\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} f_{\psi}^{B} \mathbf{1}+\left(\partial_{B_{\mathrm{I}}} f_{\psi}^{B}+B_{\mathrm{I}} \partial_{B_{\mathrm{II}}} f_{\psi}^{B}\right) \boldsymbol{B}-\partial_{B_{\mathrm{II}}} f_{\psi}^{B} \boldsymbol{B}^{2}\right) . \tag{6}
\end{equation*}
$$

### 1.4. Elastic model based on volumetric split

Most materials exhibit completely different volumetric and isochoric responses so that we make use of the isochoric-volumetric commutative split of the gradient $\boldsymbol{F}[13,30]$ :

$$
\begin{align*}
\boldsymbol{F} & =\left(J^{\frac{1}{3}} \mathbf{1}\right) \cdot\left(J^{\frac{-1}{3}} \boldsymbol{F}\right)=\left(J^{\frac{1}{3}} \mathbf{1}\right) \cdot \overline{\boldsymbol{F}},  \tag{7}\\
\Rightarrow \boldsymbol{B} & =\left(J^{\frac{2}{3}} \mathbf{1}\right) \cdot\left(J^{\frac{-2}{3}} \boldsymbol{B}\right)=\boldsymbol{B}^{s p h} \cdot \overline{\boldsymbol{B}} \tag{8}
\end{align*}
$$

$\boldsymbol{B}^{s p h}$ is a spherical tensor whose only invariant is the volume expansion $J$ :

$$
\begin{equation*}
J=\operatorname{det} \boldsymbol{F}=\sqrt{\operatorname{det} \boldsymbol{B}}=\sqrt{B_{\mathrm{III}}} \tag{9}
\end{equation*}
$$

The isochoric Finger tensor $\overline{\boldsymbol{B}}$ satisfies $\operatorname{det} \overline{\boldsymbol{B}}=1$ and is therefore represented by two invariants. We note, without any ambiguity, $\bar{B}_{\mathrm{I}}$ and $\bar{B}_{\mathrm{II}}$ the two fundamental invariants of the isochoric Finger tensor. These invariants are trivially related to those of $\boldsymbol{B}$ by:

$$
\begin{equation*}
\bar{B}_{\mathrm{I}}=J^{-\frac{2}{3}} B_{\mathrm{I}}=B_{\mathrm{III}}^{-\frac{1}{3}} B_{\mathrm{I}} \text { and } \bar{B}_{\mathrm{II}}=J^{-\frac{4}{3}} B_{\mathrm{II}}=B_{\mathrm{III}}^{-\frac{2}{3}} B_{\mathrm{II}} . \tag{10}
\end{equation*}
$$

One can find a function $\tilde{f}_{\psi}^{B}$ of the invariants ( $T, \bar{B}_{\mathrm{I}}, \bar{B}_{\mathrm{II}}, B_{\mathrm{III}}$ ) coinciding with the Helmholtz massic free energy:

$$
\begin{equation*}
\psi^{m}(T, \boldsymbol{B})=f_{\psi}^{B}\left(T, B_{\mathrm{I}}, B_{\mathrm{II}}, B_{\mathrm{III}}\right)=\tilde{f}_{\psi}^{B}\left(T, \bar{B}_{\mathrm{I}}, \bar{B}_{\mathrm{II}}, B_{\mathrm{III}}\right) . \tag{11}
\end{equation*}
$$

The partial derivatives of the new function $\tilde{f}_{\psi}^{B}$ are then obtained by simple algebraic calculations. The elastic law (6) is then transformed to give (see Appendix B):

$$
\begin{align*}
& \boldsymbol{\sigma}=\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}}\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B} \mathbf{1}+\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \boldsymbol{D e v} \overline{\boldsymbol{B}}-\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \boldsymbol{D e v}\left(\overline{\boldsymbol{B}}^{2}-\operatorname{Tr}(\overline{\boldsymbol{B}}) \overline{\boldsymbol{B}}\right)\right), \\
& \boldsymbol{\sigma}=\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}}\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B} \mathbf{1}+\frac{\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B}}{B_{\mathrm{III}}^{\frac{1}{3}}} \boldsymbol{\operatorname { e r }} \boldsymbol{v} \boldsymbol{B}-\frac{\partial_{\overline{\bar{B}}_{\mathrm{II}}} \tilde{f}_{\psi}^{B}}{B_{\mathrm{III}}^{3}} \boldsymbol{\operatorname { s e n }}\left(\boldsymbol{B}^{2}-\operatorname{Tr}(\boldsymbol{B}) \boldsymbol{B}\right)\right) . \tag{12}
\end{align*}
$$

$\boldsymbol{D e v} \boldsymbol{X}=\boldsymbol{X}-\left(\frac{1}{3} \operatorname{Tr} \boldsymbol{X}\right) \mathbf{1}$ refers to the deviatoric (trace-less) part of a second-order tensor.
(12) emphasizes the deep link between the isochoric strain and the deviatoric part of the stress.

## 2. Construction of the thermo-elastic model

The invariants used so far are natural, but inefficients for the identification of the Helmholtz free energy, as they evolve simultaneously in classical experimental tests. The corner-stone of the subsequent section is the choice of independent invariants associated with simple elementary experiments, allowing them to evolve one after the other.

### 2.1. State variables

First invariant of the isochoric Finger tensor. We introduce a first invariant $\gamma_{s}$ of the isochoric Finger tensor $\overline{\boldsymbol{B}}$, defined by:

$$
\begin{equation*}
\gamma_{s}=\sqrt{\bar{B}_{\mathrm{I}}-3}=\sqrt{\frac{B_{\mathrm{I}}}{B_{\mathrm{III}}^{\frac{1}{3}}}-3} \tag{13}
\end{equation*}
$$

${ }_{115} \gamma_{s}$ is bijectively related to the maximum distortion $\delta_{\max }($ see $[15])$ :

$$
\begin{equation*}
\gamma_{s}=\sqrt{3} \sqrt{\left(\delta_{\max }\right)^{\frac{2}{3}}-1} \text { with } \delta_{\max }=\max _{\boldsymbol{Q} \in S O(3)} \frac{1}{\left|\operatorname{det}\left(\boldsymbol{Q} \cdot \boldsymbol{u}_{t}, \boldsymbol{Q} \cdot \boldsymbol{u}_{t}^{\prime}, \boldsymbol{Q} \cdot \boldsymbol{u}_{t}^{\prime \prime}\right)\right|} \tag{14}
\end{equation*}
$$

where $\left(\boldsymbol{u}_{t}, \boldsymbol{u}_{t}^{\prime}, \boldsymbol{u}_{t}^{\prime \prime}\right)$ denotes an initially orthogonal material direction triplet (at $\left.t=0, \operatorname{det}\left(\boldsymbol{u}_{0}, \boldsymbol{u}_{0}^{\prime}, \boldsymbol{u}_{0}^{\prime \prime}\right)=1\right)$. The maximum distortion $\delta_{\max }$ reflects the maximum decrease of the solid angle formed by three initially

$$
\begin{equation*}
\boldsymbol{x}_{t}=\boldsymbol{\varphi}\left(\boldsymbol{x}_{0}, t\right)=J^{\frac{1}{3}}\left(\boldsymbol{x}_{0}+\gamma Y \boldsymbol{e}_{x}\right) \tag{15}
\end{equation*}
$$

simple algebra yield $\gamma_{s}=\gamma$, so that $\gamma_{s}$ equalized the motion magnitude.


Figure 1: Isochoric planar simple shear kinematics

Second invariant of the isochoric Finger tensor. We propose to construct a second invariant of $\overline{\boldsymbol{B}}$ which remains null in any isochoric simple shear test. Thus, one may clarify the dependence of the Helmholtz massic free energy with respect to this new invariant. The isochoric Finger tensor associated with the motion (15) reads:

$$
\begin{equation*}
\overline{\boldsymbol{B}}=\left(1+\gamma^{2}\right)\left(\boldsymbol{e}_{x} \otimes \boldsymbol{e}_{x}\right)+\gamma\left(\boldsymbol{e}_{x} \otimes \boldsymbol{e}_{y}+\boldsymbol{e}_{y} \otimes \boldsymbol{e}_{x}\right)+\boldsymbol{e}_{y} \otimes \boldsymbol{e}_{y}+\boldsymbol{e}_{z} \otimes \boldsymbol{e}_{z} . \tag{16}
\end{equation*}
$$

Set $\gamma_{\perp}=0$ and $e=1$ in (D.3) shows that the two fundamentals invariants of (16) stay equal $\bar{B}_{\mathrm{II}}=\bar{B}_{\mathrm{I}}$.
We define the invariant $\gamma_{t}$, identically zero in any spherical evolution and in a simple shear test, by:

$$
\begin{equation*}
\gamma_{t}=\sqrt{\bar{B}_{\mathrm{I}}-3}-\sqrt{\bar{B}_{\mathrm{II}}-3}=\gamma_{s}-\sqrt{\frac{B_{\mathrm{II}}}{J^{\frac{4}{3}}}-3}, \tag{17}
\end{equation*}
$$

$\gamma_{t}$ is always well defined (i.e. the terms under the roots are always positive) by concavity of the logarithm. A closed form of its variations' domain is given in Appendix C.

### 2.2. Constitutive law with the new invariants set

The list of reduced state variables retained is thus $\left(T, J, \gamma_{s}, \gamma_{t}\right)(9),(13),(17)$. The relations are inversed in:

$$
\begin{equation*}
J=B_{\mathrm{III}}^{\frac{1}{2}} ; \quad \gamma_{s}=\sqrt{\bar{B}_{\mathrm{I}}-3} ; \quad \gamma_{t}=\gamma_{s}-\sqrt{\bar{B}_{\mathrm{II}}-3} \Leftrightarrow B_{\mathrm{III}}=J^{2} ; \quad \bar{B}_{\mathrm{I}}=\gamma_{s}^{2}+3 ; \quad \bar{B}_{\mathrm{II}}=\left(\gamma_{s}-\gamma_{t}\right)^{2}+3 ; \tag{18}
\end{equation*}
$$

orthogonal material directions. Moreover, in an isochoric simple shear test, defined by the transformation field (see Figure 1): by applying the chain rule:

$$
\begin{equation*}
\boldsymbol{\sigma}=\rho_{0}\left(\partial_{J} \tilde{f}_{\psi} \mathbf{1}+\frac{1}{\gamma_{s} J^{\frac{5}{3}}}\left(\partial_{\gamma_{s}} \tilde{f}_{\psi}+\partial_{\gamma_{t}} \tilde{f}_{\psi}\right) \boldsymbol{D} \boldsymbol{e} \boldsymbol{v} \boldsymbol{B}+\frac{\partial_{\gamma_{t}} \tilde{f}_{\psi}}{\left(\gamma_{s}-\gamma_{t}\right) J^{\frac{7}{3}}} \boldsymbol{D} \boldsymbol{e} \boldsymbol{v}\left(\boldsymbol{B}^{2}-\operatorname{Tr}(\boldsymbol{B}) \boldsymbol{B}\right)\right) . \tag{19}
\end{equation*}
$$

This is the most general isotropic thermo-elastic law, formulated with respect to the variables $\left(T, J, \gamma_{s}, \gamma_{t}\right)$. Postulating that the free energy does not depend on the $\gamma_{t}$ invariant, i.e. $\partial_{\gamma_{t}} \tilde{f}_{\psi}=0$, reduces (19) to the elastic law used by $[24,26,29,31]$. It should be noted, however, that in these papers, the expression of the free energy is presented as a hypothesis and is not deduced from experimentally assessed stresses. These models, being limited to the triplet $\left(T, J, \gamma_{s}\right)$, choose not to distinguish energetically two different deformations which would share the same maximum distortion $\delta_{\max }$ and thus the same invariant $\gamma_{s}$.

## 3. Experimental identification

The thermodynamic construction presented herein follows the generic method presented by Guarrigues [16, 17].

Let $\tilde{f}_{\psi}$ denote the free energy variation within the non-isothermal thermo-elastic process from a reference configuration, $E_{0}=\left(T_{0}, 1,0,0\right)$ for which the free energy is set to zero, to an arbitrary configuration given by the state variables $\left(T, J, \gamma_{s}, \gamma_{t}\right)$ :

$$
\begin{equation*}
\psi^{m}=\tilde{f}_{\psi}\left(T, J, \gamma_{s}, \gamma_{t}\right) \tag{20}
\end{equation*}
$$

To define the Helmholtz massic free energy at any point $\left(T, J, \gamma_{s}, \gamma_{t}\right)$ of the state space, we define a particular path

$$
E_{0}=\left(T_{0}, 1,0,0\right) \xrightarrow{\mathscr{P}^{(1)}} E_{1}=(T, 1,0,0) \xrightarrow{\mathscr{P}^{(2)}} E_{2}=(T, J, 0,0) \xrightarrow{\mathscr{P}^{(3)}} E_{3}=\left(T, J, \gamma_{s}, 0\right) \xrightarrow{\mathscr{P}^{(4)}} E_{t}=\left(T, J, \gamma_{s}, \gamma_{t}\right)
$$

such that $\tilde{f}_{\psi}$ is additively found, given the energy variation in every path:

$$
\begin{equation*}
\psi^{m}=g^{(1)}(T)+g^{(2)}(T, J)+g^{(3)}\left(T, J, \gamma_{s}\right)+g^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right) \tag{21}
\end{equation*}
$$

- $g^{(1)}(T)$ is the Helmholtz massic free energy variation within $\mathscr{P}^{(1)}$ (strain-locked heating):

$$
\begin{equation*}
\dot{T} \neq 0 ; \quad J=1 ; \quad \dot{J}=0 ; \quad \gamma_{s}=0 ; \quad \dot{\gamma_{s}}=0 ; \quad \gamma_{t}=0 ; \quad \dot{\gamma_{t}}=0 \tag{22}
\end{equation*}
$$

- $g^{(2)}(T, J)$ is the Helmholtz massic free energy variation within $\mathscr{P}^{(2)}$ (isothermal spherical motion):

$$
\begin{equation*}
\dot{T}=0 ; \quad \dot{J} \neq 0 ; \quad \gamma_{s}=0 ; \quad \dot{\gamma}_{s}=0 ; \quad \gamma_{t}=0 ; \quad \dot{\gamma}_{t}=0 \tag{23}
\end{equation*}
$$

- $g^{(3)}\left(T, J, \gamma_{s}\right)$ is the Helmholtz massic free energy variation within $\mathscr{P}^{(3)}$ (isochoric simple shear test):

$$
\begin{equation*}
\dot{T}=0 ; \quad \dot{J}=0 ; \quad \dot{\gamma}_{s} \neq 0 ; \quad \gamma_{t}=0 ; \quad \dot{\gamma}_{t}=0 \tag{24}
\end{equation*}
$$

- $g^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right)$ is the Helmholtz massic free energy variation within $\mathscr{P}^{(4)}$ (isothermal isochoric isotrace motion):

$$
\begin{equation*}
\dot{T}=0 ; \quad \dot{J}=0 ; \quad \dot{\gamma}_{s}=0 ; \quad \dot{\gamma}_{t} \neq 0 \tag{25}
\end{equation*}
$$

with the initial conditions:

$$
\begin{equation*}
g^{(1)}\left(T_{0}\right)=0 ; \quad g^{(2)}(T, 1)=0, \forall T ; \quad g^{(3)}(T, J, 0)=0, \forall(T, J) ; \quad g^{(4)}\left(T, J, \gamma_{s}, 0\right)=0, \forall\left(T, J, \gamma_{s}\right) ; \tag{26}
\end{equation*}
$$

which leads to cancel the following partial derivatives:

$$
\begin{align*}
& \partial_{T} g^{(2)}(T, 1)=0 ; \quad \partial_{T} g^{(3)}(T, J, 0)=\partial_{J} g^{(3)}(T, J, 0)=0 \quad \forall T, J \\
& \partial_{T} g^{(4)}\left(T, J, \gamma_{s}, 0\right)=\partial_{J} g^{(4)}\left(T, J, \gamma_{s}, 0\right)=\partial_{\gamma_{s}} g^{(4)}\left(T, J, \gamma_{s}, 0\right)=0 \quad \forall T, J, \gamma_{s} \tag{27}
\end{align*}
$$

The general form of the massic entropy is given by the Helmholtz relation:

$$
\begin{equation*}
s^{m}=-\partial_{T} \tilde{f}_{\psi}=-\partial_{T} g^{(1)}-\partial_{T} g^{(2)}-\partial_{T} g^{(3)}-\partial_{T} g^{(4)} \tag{28}
\end{equation*}
$$

The massic internal energy is deduced from the definition of the Helmholtz free energy:

$$
\begin{equation*}
e^{m}=\tilde{f}_{\psi}+T s^{m}=g^{(1)}+g^{(2)}+g^{(3)}+g^{(4)}-T\left(\partial_{T} g^{(1)}+\partial_{T} g^{(2)}+\partial_{T} g^{(3)} \partial_{T} g^{(4)}\right) \tag{29}
\end{equation*}
$$

### 3.1. Elementary evolutions

The identification of the four unknown functions $g^{(1)}(T), g^{(2)}(T, J), g^{(3)}\left(T, J, \gamma_{s}\right)$ and $g^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right)$ comes down to experimental measures in a few experiments carried under the following ideal experimental conditions:

1) body forces are negligible;
2) state variables $\left(T, J, \gamma_{s}, \gamma_{t}\right)$ are uniform across the specimen.
3) experimental data are evaluated in a quasi-static setting so that the kinetic energy can be neglected; $g^{(1)}(T)$ will be determined by a simple energy balance whereas direct integration of experimental stress data shall clarify the rest of the Helmholtz free energy.

### 3.1.1. Strain-locked pure heating

The first evolution $\mathscr{P}^{(1)}$ is a pure heating, strain locked, experiment: $\boldsymbol{B}=\mathbf{1}, J=1, \dot{J}=0, \gamma_{s}=0$, $\dot{\gamma}_{s}=0, \gamma_{t}=0, \dot{\gamma}_{t}=0$. We measure the algebraic massic heat $q_{e x p}^{m(1)}(T)$ supplied to the system to go from $T_{0}$ to $T$. The conservation of energy between the states $E_{0}$ and $E_{1}$ is written (quasi-static evolution: $\Delta e_{c}^{m}=0$, no deformation: $w^{\text {mext }}=0$ ):

$$
\begin{equation*}
e_{1}^{m}-\underbrace{e_{0}^{m}}_{0}=q_{e x p}^{m(1)}(T), \tag{30}
\end{equation*}
$$

which leads to the differential equation in $g^{(1)}$ from (29):

$$
\begin{align*}
& \left.g^{(1)}(T)-T\left(\partial_{T} g^{(1)}(T)+\partial_{T} g^{(2)}(T, 1)+\partial_{T} g^{(3)}(T, 1,0)\right)+\partial_{T} g^{(4)}(T, 1,0,0)\right)=q_{e x p}^{m(1)}(T), \\
\Leftrightarrow & g^{(1)}(T)-T \partial_{T} g^{(1)}(T)=q_{e x p}^{m(1)}(T) \quad \text { see }(27), \\
\Leftrightarrow & g^{(1)}(T)=-T \int_{T_{0}}^{T} \frac{q_{e x p}^{m(1)}(\tilde{T})}{\tilde{T}^{2}} \mathrm{~d} \tilde{T} . \tag{31}
\end{align*}
$$

$$
\begin{equation*}
\boldsymbol{\sigma}=\rho_{0}\left(\partial_{J} \tilde{f}_{\psi} \mathbf{1}+\frac{1}{\gamma_{s} J^{\frac{5}{3}}}\left(\partial_{\gamma_{s}} \tilde{f}_{\psi}+\partial_{\gamma_{t}} \tilde{f}_{\psi}\right) \boldsymbol{D} \boldsymbol{e} \boldsymbol{v} \boldsymbol{B}+\frac{\partial_{\gamma_{t}} \tilde{f}_{\psi}}{\left(\gamma_{s}-\gamma_{t}\right) J^{\frac{7}{3}}} \boldsymbol{D} \boldsymbol{e} \boldsymbol{v}\left(\boldsymbol{B}^{2}-\operatorname{Tr}(\boldsymbol{B}) \boldsymbol{B}\right)\right) \tag{36}
\end{equation*}
$$

190 Given the equalities $\gamma_{t}=0, \partial_{\gamma_{s}} g^{(4)}\left(T, J, \gamma_{s}, 0\right)=0$, simple calculations show that the tangential stress $\tau_{e x p}^{(3)}\left(T, J, \gamma_{s}\right)=\sigma_{12}^{(3)}$ within this evolution is directly related to $\partial_{\gamma_{s}} g^{(3)}$ by:

$$
\begin{equation*}
\tau_{e x p}^{(3)}=\frac{\rho_{0}}{J} \partial_{\gamma_{s}} g^{(3)}\left(T, J, \gamma_{s}\right) \tag{37}
\end{equation*}
$$

In this experiment, we measure the tangential stress $\tau_{\text {exp }}^{(3)}$ as a function of $\gamma$ and we deduce by a simple integration the value of the function $g^{(3)}$ :

$$
\begin{equation*}
g^{(3)}=\frac{J}{\rho_{0}} \int_{0}^{\gamma_{s}} \tau_{e x p}^{(3)}(T, J, \gamma) \mathrm{d} \gamma \tag{38}
\end{equation*}
$$

### 3.1.4. Iso- $\gamma_{s}$ iso- $T$ isochoric double shear test

$\mathscr{P}^{(4)}$ is an isothermal, isochoric, iso- $\gamma_{s}$ experiment. Several tests grants this triple condition. For the sake of simplicity, we introduce herein the "double shear test", eventhough this experiment does not scan the whole domain of variation of $\gamma_{t}$. In the Appendix D, we will present a wider two-parameter test family, which include the present "double shear experiment", and which browse the whole domain for the variable $\gamma_{t}$.

Isochoric double shear test. Consider the motion defined by the following transformation (see also Figure 2):


Figure 2: Double shear kinematic

$$
\begin{equation*}
\boldsymbol{x}_{t}=J^{\frac{1}{3}}\left(\boldsymbol{x}_{0}+\sqrt{\gamma_{s}^{2}-\gamma_{\perp}^{2}} Y \boldsymbol{e}_{x}+\gamma_{\perp} Z \boldsymbol{e}_{y}\right) \tag{39}
\end{equation*}
$$

where $\boldsymbol{x}_{0}=X \boldsymbol{e}_{x}+Y \boldsymbol{e}_{y}+Z \boldsymbol{e}_{z}$ is the reference position of a particle, $J$ is the volume dilatation prior from path $\mathscr{P}^{(2)}$ and constant in path $\mathscr{P}^{(3)}$ and $\mathscr{P}^{(4)}, \gamma_{s}$ is the slip prior from path $\mathscr{P}^{(3)}$ and constant in path $\mathscr{P}^{(4)}$. The components of the Finger tensor $\boldsymbol{B}=\boldsymbol{F} \cdot \boldsymbol{F}^{\top}$ in the orthonormal basis $\left\{\boldsymbol{e}_{x}, \boldsymbol{e}_{y}, \boldsymbol{e}_{z}\right\}$ for this isochoric motion are:

$$
\boldsymbol{F}=J^{\frac{1}{3}}\left(\begin{array}{ccc}
1 & \sqrt{\gamma_{s}^{2}-\gamma_{\perp}^{2}} & 0  \tag{40}\\
0 & 1 & \gamma_{\perp} \\
0 & 0 & 1
\end{array}\right) \Rightarrow \overline{\boldsymbol{B}}=\left(\begin{array}{ccc}
1+\gamma_{s}^{2}-\gamma_{\perp}^{2} & \sqrt{\gamma_{s}^{2}-\gamma_{\perp}^{2}} & 0 \\
\sqrt{\gamma_{s}^{2}-\gamma_{\perp}^{2}} & 1+\gamma_{\perp}^{2} & \gamma_{\perp} \\
0 & \gamma_{\perp} & 1
\end{array}\right) \Rightarrow \bar{B}_{\mathrm{I}}=\gamma_{s}^{2}+3
$$

This test continuously shifts the deformation from a simple shear test in the ( $\boldsymbol{e}_{x}, \boldsymbol{e}_{y}$ ) plane to the same state in the $\left(\boldsymbol{e}_{y}, \boldsymbol{e}_{z}\right)$ plane while keeping the first invariant $\gamma_{s}$ constant.

Stress tensor in the double shear test. Injecting the strain measure deduced from (39) in (36), one can see that the out of plane shear stress $\sigma_{13}^{(4)}$ is directly related to the derivative of the Helmholtz massic free energy with

The expression for the massic free energy $g^{(4)}$ is obtained once more by integrating the experimental data:

$$
\begin{equation*}
g^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right)=\frac{J}{\rho_{0}} \int_{0}^{\gamma_{t}} \frac{\gamma_{s}-\tau}{\sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}} \tau_{e x p}^{(4)}\left(T, J, \gamma_{s}, \tau\right) \mathrm{d} \tau \tag{42}
\end{equation*}
$$

When $\gamma_{\perp}$ evolves in the interval $\left[0, \frac{\gamma_{s}}{\sqrt{2}}\right]$, the invariant $\gamma_{t}$ takes its values in $\gamma_{t} \in\left[\gamma_{s}\left(1-\sqrt{1+\frac{\gamma_{s}^{2}}{4}}\right), 0\right]$ (see Appendix D).

Note that this out-of-plane shear-stress $\sigma_{13}^{(4)}$ is purely "non-linear" in the sense that it arises from the quadratic term of the strain measure, therefore, it cannot be predicted by any "linearized" versions of the elastic law or by any model neglecting the contribution of the second strain invariant $\bar{B}_{\mathrm{II}}$.

The "double shear experiment" is not a standard laboratory test but can be performed using an efficient hexapod machine (Stewart machine [11, 34]).

### 3.1.5. Free thermal expansion $\left(\mathscr{P}^{(5)}\right)$

A direct measurement of the massic heat $q_{e x p}^{m(1)}$ in the first path $\mathscr{P}^{(1)}$ (strain-locked heating) is rather difficult. This experiment can be replaced by a free-stress thermal expansion $\left(p_{e x p}^{(2)}\left(T, J_{e x p}^{(5)}(T)\right)=0, \forall T\right)$, followed by an isothermal spherical deformation which brings the volume expansion $J_{\text {exp }}^{(5)}(T)$ back to 1 .

Writing the conservation of energy for this evolution between the initial state ( $T_{0}, 1,0,0$ ) and the freely expanded state $\left(T, J_{e x p}^{(5)}, 0,0\right)$, it follows: $e_{5}^{m}=q_{e x p}^{m(5)}(T)$. Moreover, the massic internal energy in the freely expanded state $\left(T, J_{e x p}^{(5)}, 0,0\right)$ is also given by (29):

$$
\begin{gather*}
e_{5}^{m}=q_{e x p}^{m(1)}(T)-\frac{1}{\rho_{0}} \int_{1}^{J_{e x p}^{(5)}} p_{e x p}^{(2)}(T, \tilde{J})-T \partial_{T} p_{e x p}^{(2)}(T, \tilde{J}) \mathrm{d} \tilde{J},  \tag{43}\\
\Rightarrow q_{e x p}^{m(1)}(T)=q_{e x p}^{m(5)}(T)+\frac{1}{\rho_{0}} \int_{1}^{J_{e x p}^{(5)}} p_{e x p}^{(2)}(T, \tilde{J})-T \partial_{T} p_{e x p}^{(2)}(T, \tilde{J}) \mathrm{d} \tilde{J} . \tag{44}
\end{gather*}
$$

(44) proves that given the experimental pressure $p_{\text {exp }}^{2}(T, J)$ and the free-stress expansion massic heat $q_{e x p}^{m(5)}(T)$, one may determine $q_{e x p}^{m(1)}(T)$. The function (31) is then finally written:

$$
\begin{equation*}
\left.g^{(1)}(T)=-T \int_{T_{0}}^{T} \frac{q_{e x p}^{m(5)}(\tilde{T})}{\tilde{T}^{2}} \mathrm{~d} \tilde{T}-\frac{T}{\rho_{0}} \int_{T_{0}}^{T}\left(\int_{1}^{J_{e x p}^{(5)}(\tilde{T})} p_{e x p}^{(2)}(\tilde{T}, \tilde{J})-\tilde{T} \partial_{T} p_{e x p}^{(2)}(\tilde{T}, \tilde{J})\right) \mathrm{d} \tilde{J}\right) \mathrm{d} \tilde{T} \tag{45}
\end{equation*}
$$

### 3.2. Synthesis

The complete identification of the finite strain thermo-elastic model relies on four experimental data:

1) $q_{e x p}^{m(1)}(T)$ : massic heat exchanged in the first path $\mathscr{P}^{(1)}$ (strain-locked heating or cooling), with the condition:

$$
\begin{equation*}
q_{e x p}^{m(1)}\left(T_{0}\right)=0 . \tag{46}
\end{equation*}
$$

2) $p_{\text {exp }}^{(2)}(T, J)$ : pressure in the spherical evolution $\mathscr{P}^{(2)}$ (isothermal spherical deformation), with the condition:

$$
\begin{equation*}
p_{e x p}^{(2)}(T, 1)=p_{e x p}^{(1)}(T), \forall T . \tag{47}
\end{equation*}
$$

3) $\tau_{e x p}^{(3)}\left(T, J, \gamma_{s}\right)$ : in-plane shear stress in the isochoric planar shear test $\mathscr{P}^{(3)}$, along with the condition:

$$
\begin{equation*}
\tau_{e x p}^{(3)}(T, J, 0)=0, \forall T \forall J \tag{48}
\end{equation*}
$$

4) $\tau_{\text {exp }}^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right)$ : out-of-plane tangential stress in the fourth path $\mathscr{P}^{(4)}$, with the condition:

$$
\begin{equation*}
\tau_{e x p}^{(4)}\left(T, J, \gamma_{s}, 0\right)=0, \forall T \forall J \forall \gamma_{s} \tag{49}
\end{equation*}
$$

$g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)}$ which determine the state functions are expressed in terms of these measurements:

$$
\begin{align*}
g^{(1)}(T) & =-T \int_{T_{0}}^{T} \frac{q_{e x p}^{m(1)}(\tilde{T})}{\tilde{T}^{2}} \mathrm{~d} \tilde{T} \text { or }(45),  \tag{50}\\
g^{(2)}(T, J) & =-\frac{1}{\rho_{0}} \int_{1}^{J} p_{e x p}^{(2)}(T, \tilde{J}) \mathrm{d} \tilde{J},  \tag{51}\\
g^{(3)}\left(T, J, \gamma_{s}\right) & =\frac{J}{\rho_{0}} \int_{0}^{\gamma_{s}} \tau_{e x p}^{(3)}(T, J, \gamma) \mathrm{d} \gamma,  \tag{52}\\
g^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right) & =\frac{J}{\rho_{0}} \int_{0}^{\gamma_{t}} \frac{\gamma_{s}-\tau}{\sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}} \tau_{e x p}^{(4)}\left(T, J, \gamma_{s}, \tau\right) \mathrm{d} \tau . \tag{53}
\end{align*}
$$

The internal massic energy is assumed to be a state function, as to ensure (29) takes a simpler expression:

$$
\begin{equation*}
e^{m}=q_{e x p}^{m(1)}+g^{(2)}-T \partial_{T} g^{(2)}+g^{(3)}-T \partial_{T} g^{(3)}+g^{(4)}-T \partial_{T} g^{(4)} . \tag{54}
\end{equation*}
$$

For the numerical assessment of the model, it is not necessary to compute explicitly $g^{(1)}$ because this function does not appear neither in the constitutive law, nor in the internal energy.

### 3.3. Volumetric/isochoric splitting assumption

As already mentioned, hyperelastic materials exhibit radically different volume and shear behavior. This explains a well-accepted splitting of the Helmholtz free energy with respect to the isochoric/volumetric part of the strain tensor (see [24, 27]). In this section, we emphasize that this partition comes down to a very
natural assertion: the pressure remains unchanged in isochoric evolutions. With the previous results, the pressure in any state $\left(T, J, \gamma_{s}, \gamma_{t}\right)$ is:

$$
\begin{align*}
-\frac{\operatorname{Tr} \boldsymbol{\sigma}}{3}= & -\rho_{0} \partial_{J}\left(g^{(2)}(T, J)+g^{(3)}\left(T, J, \gamma_{s}\right)+g^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right)\right) \\
= & p_{e x p}^{(2)}(T, J)+\int_{0}^{\gamma_{s}} \tau_{e x p}^{(3)}\left(T, J, \gamma_{s}\right)-J \partial_{J} \tau_{e x p}^{(3)}\left(T, J, \gamma_{s}\right) \mathrm{d} \gamma  \tag{55}\\
& -\int_{0}^{\gamma_{t}} \frac{\gamma_{s}-\tau}{\sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}}\left(\tau_{e x p}^{(4)}+J \partial_{J} \tau_{e x p}^{(4)}\right)\left(T, J, \gamma_{s}, \tau\right) \mathrm{d} \tau
\end{align*}
$$

The pressure in any state is a function of the tangential stresses $\tau_{e x p}^{(3)}, \tau_{e x p}^{(4)}$ measured during the paths $\mathscr{P}^{(3)}, \mathscr{P}^{(4)}$. If one performs the $\mathscr{P}^{(3)}$ experiment from the initial state $\left(p_{\text {exp }}^{(2)}\left(T_{0}, 1\right)=0\right.$, no prior volume expansion), the Cauchy stress tensor generated is not deviatoric.

Hypothesis Optional Simplification. In isochoric deformation, the pressure does not vary. Both conditions $\partial_{\gamma_{t}} \operatorname{Tr} \boldsymbol{\sigma}=0, \partial_{\gamma_{s}} \operatorname{Tr} \boldsymbol{\sigma}=0$ and (55) lead to the differential equations:

$$
\begin{align*}
& \tau_{e x p}^{(4)}+J \partial_{J} \tau_{e x p}^{(4)}=0 \Rightarrow \tau_{e x p}^{(4)}\left(T, J, \gamma_{s}, \gamma_{t}\right)=\frac{\hat{\tau}_{e x p}^{(4)}\left(T, \gamma_{s}, \gamma_{t}\right)}{J} \\
& \tau_{e x p}^{(3)}+J \partial_{J} \tau_{e x p}^{(3)}=0 \Rightarrow \tau_{e x p}^{(3)}\left(T, J, \gamma_{s}\right)=\frac{\hat{\tau}_{e x p}^{(3)}\left(T, \gamma_{s}\right)}{J} \tag{56}
\end{align*}
$$

Inserting these expressions into the general formula (52)-(53), (56) obviously split the Helmholtz free energy into an isochoric/volumetric contributions:

$$
\begin{equation*}
\psi^{m}\left(T, J, \gamma_{s}, \gamma_{t}\right)=g^{(1)}(T)+g^{(2)}(T, J)+g^{(3)}\left(T, \gamma_{s}\right)+g^{(4)}\left(T, \gamma_{s}, \gamma_{t}\right) \tag{57}
\end{equation*}
$$

This assumption is certainly valid in a wide range of solicitation (e.g is consistent with the classical linearized elasticity), but must be somehow validated experimentally. Note that the complete model (50)-(53) can take into account the volumetric/deviatoric coupling which is relevant for high velocity impact [28]. Indeed, extreme strain-rate experiments can induced finite elastic deformations before plasticity occurs, so that pressure-dependent shear modulus are requiered (see for instance the SCG model $[2,33]$ ).

### 3.4. Time derivative of the massic internal energy

The massic internal energy takes a rather elegant form:

$$
\begin{align*}
e^{m}= & q_{e x p}^{m(5)}(T)-\frac{1}{\rho_{0}} \int_{J_{e x p}^{(5)}(T)}^{J}\left(p_{e x p}^{(2)}-T \partial_{T} p_{e x p}^{(2)}\right)(T, \tilde{J}) \mathrm{d} \tilde{J}+\frac{1}{\rho_{0}} \int_{0}^{\gamma_{s}}\left(\hat{\tau}_{e x p}^{(3)}-T \partial_{T} \hat{\tau}_{e x p}^{(3)}\right)\left(T, \tilde{\gamma}_{s}\right) \mathrm{d} \tilde{\gamma}_{s} \\
& +\frac{1}{\rho_{0}} \int_{0}^{\gamma_{t}} \frac{\gamma_{s}-\tau}{\sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}}\left(\hat{\tau}_{e x p}^{(4)}-T \partial_{T} \hat{\tau}_{e x p}^{(4)}\right)\left(T, \gamma_{s}, \tau\right) \mathrm{d} \tau . \tag{58}
\end{align*}
$$

The time derivative of the internal energy, essential for the numerical implementation, is obtained by evaluating the partial derivatives of the massic internal energy with respect to its four variables (see Appendix E for details):

$$
\begin{equation*}
\frac{\mathrm{d} e^{m}}{\mathrm{~d} t}=\frac{\partial e^{m}}{\partial T} \dot{T}+\frac{\partial e^{m}}{\partial J} \dot{J}+\frac{\partial e^{m}}{\partial \gamma_{s}} \dot{\gamma}_{s}+\frac{\partial e^{m}}{\partial \gamma_{t}} \dot{\gamma_{t}} . \tag{59}
\end{equation*}
$$

## 4. Idealizations of elementary paths

The expressions (50)-(53) yields the "exact" thermo-elastic behavior of an isotropic material and lies upon four experimental results. In this section, we will propose physically consistent expressions for the experimental stresses. We will see that some rather simple forms reduce our model to a few hyperelastic models classically found in the literature.

### 4.1. Pressure assumption

Hypothesis Experimental pressure $p_{\text {exp }}^{(2)}$. We assume the following form for the pressure $p_{\text {exp }}^{(2)}$ in any isothermal spherical deformation:

$$
\begin{equation*}
p_{e x p}^{(2)}(T, J)=p_{e x p}^{(1)}(T)-\kappa(T) \ln J \tag{60}
\end{equation*}
$$

where $p_{\text {exp }}^{(1)}(T)$ is the pressure from the path $\mathscr{P}^{(1)}$
$\kappa(T)$ is a temperature-dependent bulk modulus.
The pressure (60) is reasonable (note for instance $\lim _{J \rightarrow \infty} p_{\text {exp }}^{(2)}=-\infty, \lim _{J \rightarrow 0} p_{\text {exp }}^{(2)}=\infty$ ), but predicts a finite strain energy under infinite compression $J \rightarrow 0$. For further details about the volumetric part of the Helmholtz free energy from which the pressure is derived, we refer to [12] and references therein. A more reasonable approximation of the experimental stress would be:

$$
\begin{align*}
p_{e x p}^{(2)} & =p_{e x p}^{(1)}(T)-\frac{\kappa(T)}{2}\left(J-\frac{1}{J}\right),  \tag{61}\\
\Rightarrow g^{(2)}(T, J) & =-\frac{(J-1) p_{e x p}^{(1)}(T)}{\rho_{0}}+\frac{\kappa(T)}{2}\left(\frac{1}{2}\left(J^{2}-1\right)-\ln J\right) . \tag{62}
\end{align*}
$$

It's worth noticing that (62) coincides with that proposed by [31] which is a variant of [8].
For the sake of calculus simplicity, we will however keep the pressure given by (60) for the following, noting that the differences between (60) and (61) are very small for any volumic expansion up to $200 \%$. Both model comes down to the so called linearized elasticity when $J \approx 1$ as $\ln J \approx \frac{1}{2}\left(J-\frac{1}{J}\right) \approx \operatorname{Tr} \varepsilon$.
$g^{(2)}$ is obtained using both (60) and (51):

$$
\begin{equation*}
g^{(2)}(T, J)=-\frac{(J-1) p_{e x p}^{(1)}(T)}{\rho_{0}}+\frac{\kappa(T)}{\rho_{0}}(J \ln (J)+1-J) . \tag{63}
\end{equation*}
$$

If we replace the first path $\mathscr{P}^{(1)}$ by a free expansion $\mathscr{P}^{(5)}$ (in which one save $q_{e x p}^{m(5)}(T)$ and $J_{e x p}^{(5)}(T)$ ) followed by an isothermal spherical compression to bring $J$ back to $1, p_{\text {exp }}^{(1)}(T)$ is obtained by evaluating (60) in $\left(T, J_{\text {exp }}^{(5)}\right)$. Remembering that the stress is zero in a freely expanded test, it comes:

$$
\begin{equation*}
p_{\text {exp }}^{(1)}(T)=\kappa(T) \ln \left(J_{\text {exp }}^{(5)}(T)\right) . \tag{64}
\end{equation*}
$$

Therefore, by reinjecting this expression (64) into the general formula (60), we deduce the expression of $p_{\text {exp }}^{(2)}$ :

$$
\begin{equation*}
p_{e x p}^{(2)}(T, J)=-\kappa(T) \ln \frac{J}{J_{e x p}^{(5)}(T)} \tag{65}
\end{equation*}
$$

Hypothesis Free-stress volume expansion $J_{\text {exp }}^{(5)}(T)$. One can set arbitrary for $J_{\text {exp }}^{(5)}(T)$ :

$$
\begin{equation*}
J_{e x p}^{(5)}(T)=1+\beta\left(T-T_{0}\right), \tag{66}
\end{equation*}
$$

where $\beta$ refers to a free-stress expansion coefficient.
Inserting both the experimental pressure (64) and the free-stress volume expansion (66) in $(63), g^{(2)}(T, J)$ now yields:

$$
\begin{equation*}
g^{(2)}(T, J)=\frac{\kappa(T)}{\rho_{0}}\left(J \ln \left(\frac{J}{1+\beta\left(T-T_{0}\right)}\right)+1-J+\ln \left(1+\beta\left(T-T_{0}\right)\right)\right) . \tag{67}
\end{equation*}
$$

### 4.2. Massic heat $q_{e x p}^{m(1)}$

As a reminder, the first path $\mathscr{P}^{(1)}$ can be substituted by a free thermal expansion followed by isothermal compression. We then obtain:

$$
\begin{equation*}
q_{e x p}^{m(1)}(T)=q_{e x p}^{m(5)}(T)+\frac{1}{\rho_{0}} \int_{1}^{J_{e x p}^{(5)}} p_{e x p}^{(2)}(T, \tilde{J})-T \partial_{T} p_{e x p}^{(2)}(T, \tilde{J}) \mathrm{d} \tilde{J} \quad[\text { see (44) }] . \tag{68}
\end{equation*}
$$

The conservation of energy in a free expansion is then written according to (65):

$$
\begin{equation*}
q_{e x p}^{m(5)}(T)=q_{e x p}^{m(1)}(T)+\frac{1}{\rho_{0}}\left(T \kappa(T) \frac{\partial_{T} J_{e x p}^{(5)}}{J_{e x p}^{(5)}}\left(J_{e x p}^{(5)}-1\right)+\left(\kappa(T)-T \kappa^{\prime}(T)\right)\left(1+\ln \left(J_{e x p}^{(5)}\right)-J_{e x p}^{(5)}\right)\right) . \tag{69}
\end{equation*}
$$ example:

$$
\begin{equation*}
q_{e x p}^{m(5)}(T)=C_{p}\left(T-T_{0}\right), \tag{70}
\end{equation*}
$$

293

### 4.3. Calculation of $g^{(3)}$

Hypothesis In-plane shear stress $\hat{\tau}_{\text {exp }}^{(3)}$. We assume that a temperature dependent simple shear modulus $\mu(T)$ linearly links the shear stress with respect to the shear parameter $\gamma$ :

$$
\begin{equation*}
\hat{\tau}_{e x p}^{(3)}(T, J, \gamma)=\mu(T) \gamma . \tag{71}
\end{equation*}
$$

The shear stress (71) leads to a quadratic massic free energy with respect to $\gamma_{s}$ that is also independent of the volume expansion $J$ :

$$
\begin{equation*}
g^{(3)}\left(T, \gamma_{s}\right)=\frac{\mu(T) \gamma_{s}^{2}}{2 \rho_{0}} \tag{72}
\end{equation*}
$$

This Helmholtz massic free energy has for example been postulated in [29]. We now choose to enhance the model considering the contribution of the second invariant $\bar{B}_{\text {II }}$ through $\gamma_{t}$.

### 4.4. Calculation of $g^{(4)}$

The out-of-plane shear stress is a purely non-linear effect and its interpretation in terms of a "classical" shear or compressibility coefficient is therefore impossible.
Hypothesis Out-of-plane shear stress $\hat{\tau}_{\text {exp }}^{(4)}$. We postulate that the shear stress $\hat{\tau}_{\text {exp }}^{(4)}$ in the fourth path $\mathscr{P}^{(4)}$ is given by:

$$
\begin{equation*}
\hat{\tau}_{e x p}^{(4)}\left(T, \gamma_{s}, \tau\right)=\alpha(T) \sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}, \tag{73}
\end{equation*}
$$

$\alpha(T)$ refers to a non-linear shear modulus. We further recall that the invariant $\gamma_{t}$ is always negative during the double shear test so that the square root involved in (73) is well-defined. Using the expression (73) $g^{(4)}$ now reads:

$$
\begin{equation*}
g^{(4)}\left(T, \gamma_{s}, \gamma_{t}\right)=\frac{\alpha(T)}{\rho_{0}} \int_{0}^{\gamma_{t}}\left(\gamma_{s}-\tau\right) \mathrm{d} \tau=\frac{\alpha(T)}{2 \rho_{0}}\left(\gamma_{s}^{2}-\left(\gamma_{s}-\gamma_{t}\right)^{2}\right) . \tag{74}
\end{equation*}
$$

### 4.5. Idealized elastic law

With the idealizations of the experimental stresses (60), (71), (73) the $g^{(\bullet)}$ functions were explicitly calculated (63), (72), (74). The derivation of the Helmholtz free energy yields the current Cauchy stress:

$$
\begin{equation*}
\boldsymbol{\sigma}=\rho_{0}\left(\partial_{J} \tilde{f}_{\psi} \mathbf{1}+\frac{1}{\gamma_{s} J^{\frac{5}{3}}}\left(\mu(T) \gamma_{s}+\alpha(T)\left(\gamma_{s}-\gamma_{t}\right)\right) \boldsymbol{D} \boldsymbol{e} \boldsymbol{v} \boldsymbol{B}+\frac{\alpha(T)}{J^{\frac{7}{3}}} \boldsymbol{D} \boldsymbol{e} \boldsymbol{v}\left(\boldsymbol{B}^{2}-\operatorname{Tr}(\boldsymbol{B}) \boldsymbol{B}\right)\right) . \tag{75}
\end{equation*}
$$

The invariant $\gamma_{t}$ is upper bounded by $\gamma_{s}$ whereas the positivity of the second invariant $J_{2}^{B}$ implies the lower bound:

$$
\begin{equation*}
J_{2}^{B}=\sqrt{\frac{3}{2}}\|\boldsymbol{D e v} \boldsymbol{B}\| \geq 0 \Rightarrow \gamma_{t} \geq \gamma_{s}\left(1-\sqrt{\frac{1+\gamma_{s}^{2}}{4}}\right) . \tag{76}
\end{equation*}
$$

Finally $\lim _{\gamma_{s} \rightarrow 0} \frac{\gamma_{t}}{\gamma_{s}}=0$ so that (75) does not diverge when $\gamma_{s} \rightarrow 0$. As we will see in the following section, the expression (75) is algebraically equivalent to a standard Mooney-Rivlin model.

## 5. Analogy with the Mooney-Rivlin model

If we retain the idealizations $(71),(73)$, the dependence of the massic free energy with respect to the isochoric invariants is written:

$$
\begin{align*}
g^{(3)}\left(T, \gamma_{s}\right)+g^{(4)}\left(T, \gamma_{s}, \gamma_{t}\right) & =\frac{1}{2 \rho_{0}}\left(\mu(T) \gamma_{s}^{2}+\alpha(T)\left(\gamma_{s}^{2}-\left(\gamma_{s}-\gamma_{t}\right)^{2}\right)\right) \\
& =\frac{1}{2 \rho_{0}}\left(\mu(T)\left(\bar{B}_{\mathrm{I}}-3\right)+\alpha(T)\left(\left(\bar{B}_{\mathrm{I}}-3\right)-\left(\bar{B}_{\mathrm{II}}-3\right)\right)\right) . \tag{77}
\end{align*}
$$

The right and left Cauchy-Green strain tensors $\boldsymbol{C}$ and $\boldsymbol{B}$ share their invariants $B_{i}=C_{i}$, as $\boldsymbol{C}$ and $\boldsymbol{B}$ are related through the rotation tensor $\boldsymbol{R}$ arising from the polar decomposition of the transformation gradient $\boldsymbol{F}=\boldsymbol{R} \cdot \boldsymbol{U}=\boldsymbol{V} \cdot \boldsymbol{R}$. Therefore the free energy (77) coincides exactly with that of a Mooney-Rivlin model [22]:

$$
\begin{equation*}
\psi_{M-R}^{m}=g^{1}(T)+g^{2}(T, J)+\frac{C_{01}}{\rho_{0}}\left(\bar{B}_{\mathrm{I}}-3\right)+\frac{C_{10}}{\rho_{0}}\left(\bar{B}_{\mathrm{II}}-3\right), \tag{78}
\end{equation*}
$$

with the parameters $C_{01}=\frac{1}{2}(\mu(T)+\alpha(T)), C_{10}=-\frac{\alpha(T)}{2}$.
If we assume that the shear stress along the $\mathscr{P}^{(4)}$ path is identically zero, the massic free energy becomes independent of the $\gamma_{t}$ invariant and we then observe that the Helmholtz massic free energy evolves linearly with $\bar{B}_{\mathrm{I}}-3$, which corresponds to the Neo-Hookean model $\psi_{N-H}^{m}=g^{1}(T)+g^{2}(T, J)+\frac{\mu}{2 \rho_{0}}\left(\bar{B}_{\mathrm{I}}-3\right)$. Consequently our thermodynamic approach includes these two models which result from the experimental idealizations (71) (73).

More generally, if one looks at the in-plane shear stress curve $\tau_{\text {exp }}^{(3)}$ as a function of the slip parameter $\gamma$, it is certain that a polynomial expansion fitting experimental data can be found. By symmetry, this polynomial expansion must necessarily contain only even powers:

$$
\begin{equation*}
\hat{\tau}_{e x p}^{(3)}(T, \gamma)=\sum_{i} \mu_{i} \gamma^{2 i+1} \tag{79}
\end{equation*}
$$

(79) then generates a generalized Neo-Hookean model (power series of the free energy with respect to the " $\bar{B}_{\mathrm{I}}-3$ " invariant).

Moreover, if we postulate by the same reasoning that the out-of-plane shear stress $\hat{\tau}_{\text {exp }}^{(4)}$ develops in the form:

$$
\begin{equation*}
\hat{\tau}_{e x p}^{(3)}(T, \gamma)=\sum_{i} \mu_{i} \gamma^{2 i+1} \text { and } \hat{\tau}_{e x p}^{(4)}\left(T, \gamma_{s}, \tau\right)=\sum_{i} \alpha_{i} \sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}\left(\gamma_{s}-\tau\right)^{2 i} . \tag{80}
\end{equation*}
$$

Then one found the following Helmholtz massic free energy:

$$
\begin{align*}
& \psi^{m}=\frac{1}{\rho_{0}} \sum_{i} \frac{\mu_{i}(T)}{2 i+2}\left(\bar{B}_{\mathrm{I}}-3\right)^{i}+\frac{\alpha_{i}(T)}{2 i+2}\left[\left(\bar{B}_{\mathrm{I}}-3\right)^{i}-\left(\bar{B}_{\mathrm{II}}-3\right)^{i}\right]+g^{(1)}(T)+g^{(2)}(T, J), \\
& \psi^{m}=\frac{1}{\rho_{0}} \sum_{i} C_{i 0}\left(\bar{B}_{\mathrm{I}}-3\right)^{i}+C_{0 i}\left(\bar{B}_{\mathrm{II}}-3\right)^{i}+g^{(1)}(T)+g^{(2)}(T, J) \quad C_{i j}=0 \text { if } i \neq j \tag{81}
\end{align*}
$$

That is, a generalized Mooney-Rivlin material [22] .

In the general case, the integral expressions given in section 3.2 yield the exact finite strain isotropic thermo-elastic constitutive law.

## 6. Conclusion

We have developed a complete finite-strain isotropic thermo-elastic model without any assumption concerning the form of the Helmholtz free energy. The identification of the latter relies on the successive
realization of four elementary experiments, and more precisely the measurement of: a massic heat, a pressure, and two shear stresses. Experiments must be carried many times in order to scan all the accessible values of the state variables e.g. the double shear test $\mathscr{P}^{(4)}$ must be performed for a whole collection of temperature $T$, volume expansion $J$, and slip $\gamma_{s}$.

Some idealizations of the experimental stresses have been proposed. They can be replaced by any realistic curves, or even be given by the upscaling results of a large scale molecular dynamic simulation.

The well known Mooney-Rivlin model, and consequently the Neo-Hookean, are special cases of our methodology. They are derived from an assumption concerning the shear stress along the $\mathscr{P}^{(3)}$ and $\mathscr{P}^{(4)}$ paths.

The present work could admit several natural extensions among which:

- The description of an anisotropic medium by the introduction of structural tensors $\left\{\boldsymbol{N}_{t}^{\bullet}\right\}$ representing the actual directions of anisotropy. Enhancing the Helmholtz free energy with these new unknowns necessarily yields cross-invariant effects, reflecting the current orientation of the deformation with respect to the anisotropic directions.
- A finite strain plasticity model with the addition of an objective plastic strain tensor $\boldsymbol{B}^{p}$ in the arguments' list of the Helmholtz free energy. Taking this tensor into account adds 6 state variables. Three of them are specific to the plastic strain $\boldsymbol{B}^{p}$ (2 if one further assumes that the plastic evolution is isochoric), three others orient the principal directions orthogonal trihedron of $\boldsymbol{B}$ and $\boldsymbol{B}^{p}$.
- The study of large-amplitude shock waves and more particularly shock tails. Indeed, after a large plastic phase which is out of the scope of this study, the elastic release wave on the shock tail can be considered as large strain. A proper coupling between both, an equation of state driving the pressure, and the deviatoric elastic law presented herein is a very interesting perspective.


## Acknowledgement

The author would like to thank his colleagues from the CEA for their fruitful remarks but also Pr. Jean Guarrigues whose rigorous work gave rise to this article.

## Appendix A. Justification of the finite strain isotropic elastic law

The definition of the Finger tensor is $\boldsymbol{B}=\boldsymbol{F} \cdot \boldsymbol{F}^{\top}$, hence its time derivative reads:

$$
\begin{align*}
\dot{\boldsymbol{B}} & =\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{\top}+\boldsymbol{F} \cdot \dot{\boldsymbol{F}}^{\top}=\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right) \cdot\left(\boldsymbol{F} \cdot \boldsymbol{F}^{\top}\right)+\left(\boldsymbol{F} \cdot \boldsymbol{F}^{\top}\right) \cdot\left(\boldsymbol{F}^{-\top} \cdot \dot{\boldsymbol{F}}^{\top}\right),  \tag{A.1}\\
\dot{\boldsymbol{B}} & =(\boldsymbol{D}+\boldsymbol{W}) \cdot \boldsymbol{B}+\boldsymbol{B} \cdot(\boldsymbol{D}-\boldsymbol{W}) .
\end{align*}
$$

where $\boldsymbol{D}$ (resp $\boldsymbol{W}$ ) denotes the symmetric (resp skew-symmetric) part of the spatial velocity gradient.
$\forall n \in \mathbb{Z}$ :

$$
\begin{equation*}
\boldsymbol{B}^{n}: \dot{\boldsymbol{B}}=\boldsymbol{B}^{n}:((\boldsymbol{D}+\boldsymbol{W}) \cdot \boldsymbol{B}+\boldsymbol{B} \cdot(\boldsymbol{D}-\boldsymbol{W}))=2 \boldsymbol{B}^{n+1}: \boldsymbol{D} \tag{A.2}
\end{equation*}
$$

$$
\frac{\mathrm{d} B_{\mathrm{I}}}{\mathrm{~d} t}=\operatorname{Tr} \frac{\mathrm{d} \boldsymbol{B}}{\mathrm{~d} t}=\mathbf{1}: \dot{\boldsymbol{B}} ; \quad \frac{\mathrm{d} B_{\mathrm{II}}}{\mathrm{~d} t}=\left(B_{\mathrm{I}} \mathbf{1}-\boldsymbol{B}^{\top}\right): \frac{\mathrm{d} \boldsymbol{B}}{\mathrm{~d} t} ; \quad \frac{\mathrm{d} B_{\mathrm{III}}}{\mathrm{~d} t}=\left(B_{\mathrm{II}} \mathbf{1}-B_{\mathrm{I}} \boldsymbol{B}^{\top}+\boldsymbol{B}^{2 \top}\right): \frac{\mathrm{d} \boldsymbol{B}}{\mathrm{~d} t}=B_{\mathrm{III}} \boldsymbol{B}^{-\top}: \frac{\mathrm{d} \boldsymbol{B}}{\mathrm{~d} t} ;
$$

$$
\begin{equation*}
\dot{B}_{\mathrm{I}}=2 \boldsymbol{B}: \boldsymbol{D} ; \quad \dot{B}_{\mathrm{II}}=2\left(B_{\mathrm{I}} \boldsymbol{B}-\boldsymbol{B}^{2}\right): \boldsymbol{D} ; \quad \dot{B}_{\mathrm{III}}=B_{\mathrm{III}} \mathbf{1}: \boldsymbol{D} \tag{A.3}
\end{equation*}
$$

The nullity of intrinsic dissipation (4), is written using the coincident isotropic function (5):

$$
\begin{equation*}
-\rho\left(\partial_{T} f_{\psi}^{B} \dot{T}+\partial_{B_{\mathrm{I}}} f_{\psi}^{B} \dot{B}_{\mathrm{I}}+\partial_{B_{\mathrm{II}}} f_{\psi}^{B} \dot{B}_{\mathrm{II}}+\partial_{B_{\mathrm{III}}} f_{\psi}^{B} \dot{B}_{\mathrm{III}}+\dot{T} s^{m}\right)+\boldsymbol{\sigma}: \boldsymbol{D}=0 \tag{A.4}
\end{equation*}
$$ not depend on $\boldsymbol{D}$, the nullity of (A.4) $\forall T \forall \boldsymbol{D}$ yields the elastic law (6).

## Appendix B. Thermo-elastic law with respect to isochoric invariants

The partial derivatives of the new function $\tilde{f}_{\psi}^{B}$ defined by (11) are simple algebraic calculations:

$$
\begin{align*}
\partial_{B_{\mathrm{I}}} f_{\psi}^{B} & =\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \partial_{B_{\mathrm{I}}} \bar{B}_{\mathrm{I}}=B_{\mathrm{III}}^{-\frac{1}{3}} \partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B}, \\
\partial_{B_{\mathrm{II}}} f_{\psi}^{B} & =\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \partial_{B_{\mathrm{II}}} \bar{B}_{\mathrm{II}}=B_{\mathrm{III}}^{-\frac{2}{3}} \partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B}, \\
\partial_{B_{\mathrm{III}}} f_{\psi}^{B} & =\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \partial_{B_{\mathrm{III}}} \bar{B}_{\mathrm{I}}+\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \partial_{B_{\mathrm{III}}} \bar{B}_{\mathrm{II}}+\partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B},  \tag{B.1}\\
\partial_{B_{\mathrm{III}}} f_{\psi}^{B} & =-\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \frac{\bar{B}_{\mathrm{II}}}{3 B_{\mathrm{III}}}-\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \frac{2 \bar{B}_{\mathrm{II}}}{3 B_{\mathrm{III}}}+\partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B} .
\end{align*}
$$

And the elastic law (6) now reads:

$$
\begin{align*}
\boldsymbol{\sigma}=\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}} & \left(\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B}-\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \frac{\bar{B}_{\mathrm{I}}}{3}-\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \frac{2 \bar{B}_{\mathrm{II}}}{3}\right) \mathbf{1}\right. \\
& \left.+\left(B_{\mathrm{III}}^{-\frac{1}{3}} \partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B}+B_{\mathrm{III}}^{\frac{1}{3}} \bar{B}_{\mathrm{I}} B_{\mathrm{III}}^{-\frac{2}{3}} \partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B}\right) \boldsymbol{B}-B_{\mathrm{III}}^{-\frac{2}{3}} \partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \boldsymbol{B}^{2}\right),  \tag{B.2}\\
\boldsymbol{\sigma}=\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}} & \left(\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B}-\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \bar{B}_{\mathrm{I}}\right.\right. \\
3 & \left.\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \frac{2 \bar{B}_{\mathrm{II}}}{3}\right) \mathbf{1} \\
& \left.+\left(\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B}+\bar{B}_{\mathrm{I}} \partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B}\right) \overline{\boldsymbol{B}}-\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \overline{\boldsymbol{B}}^{2}\right) .
\end{align*}
$$

This formula is more elegantly written using the following relations, which hold for any tensor of order 2 :

$$
\begin{align*}
X_{\mathrm{II}}=\frac{1}{2}\left(X_{\mathrm{I}}^{2}-\operatorname{Tr} \boldsymbol{X}^{2}\right) \Rightarrow-\frac{2}{3} X_{\mathrm{II}} \mathbf{1}+X_{\mathrm{I}} \boldsymbol{X}-\boldsymbol{X}^{2} & =-\frac{1}{3}\left(X_{\mathrm{I}}^{2}-\operatorname{Tr} \boldsymbol{X}^{2}\right) \mathbf{1}+X_{\mathrm{I}} \boldsymbol{X}-\boldsymbol{X}^{2}  \tag{B.3}\\
& =-\operatorname{Dev}\left(\boldsymbol{X}^{2}-\operatorname{Tr}(\boldsymbol{X}) \boldsymbol{X}\right) .
\end{align*}
$$

$$
\begin{align*}
\boldsymbol{\sigma} & =\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}}\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B} \mathbf{1}+\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B} \boldsymbol{D e v} \overline{\boldsymbol{B}}-\partial_{\bar{B}_{\mathrm{II}}} \tilde{f}_{\psi}^{B} \boldsymbol{D e v}\left(\overline{\boldsymbol{B}}^{2}-\operatorname{Tr}(\overline{\boldsymbol{B}}) \overline{\boldsymbol{B}}\right)\right), \\
\boldsymbol{\sigma} & =\frac{2 \rho_{0}}{\sqrt{B_{\mathrm{III}}}}\left(B_{\mathrm{III}} \partial_{B_{\mathrm{III}}} \tilde{f}_{\psi}^{B} \mathbf{1}+\frac{\partial_{\bar{B}_{\mathrm{I}}} \tilde{f}_{\psi}^{B}}{B_{\mathrm{III}}^{\frac{1}{3}}} \boldsymbol{D e v} \boldsymbol{B}-\frac{\partial_{\overline{\bar{B}}_{\mathrm{II}}} \tilde{f}_{\psi}^{B}}{B_{\mathrm{III}}^{3}} \boldsymbol{D e v}\left(\boldsymbol{B}^{2}-\operatorname{Tr}(\boldsymbol{B}) \boldsymbol{B}\right)\right) . \tag{B.4}
\end{align*}
$$

## Appendix C. Evolution domain of $\gamma_{t}$

Let $\left(\bar{\lambda}_{i}\right)_{i \in \llbracket 1,3 \rrbracket}$ be the principal stretches of the isochoric Finger tensor $\overline{\boldsymbol{B}}$. We further assume that the invariant $\gamma_{s}$ remains fixed. The following system is necessarily verified by the triplet $\left(\bar{\lambda}_{i}\right)_{i \in \llbracket 1,3 \rrbracket}$ :

$$
\begin{align*}
\bar{\lambda}_{1}^{2}+\bar{\lambda}_{2}^{2}+\bar{\lambda}_{3}^{2} & =\gamma_{s}^{2}+3=\bar{B}_{\mathrm{I}}  \tag{C.1}\\
\bar{\lambda}_{1}^{2} \bar{\lambda}_{2}^{2}+\bar{\lambda}_{2}^{2} \bar{\lambda}_{3}^{2}+\bar{\lambda}_{1}^{2} \bar{\lambda}_{3}^{2} & =\bar{B}_{\mathrm{II}}  \tag{C.2}\\
\bar{\lambda}_{1} \bar{\lambda}_{2} \bar{\lambda}_{3} & =1 . \tag{C.3}
\end{align*}
$$

By combining the equations (C.1) and (C.3), we deduce:

$$
\begin{equation*}
\bar{\lambda}_{2}^{4}+\left(\bar{\lambda}_{1}^{2}-\bar{B}_{\mathrm{I}}\right) \bar{\lambda}_{2}^{2}+\frac{1}{\bar{\lambda}_{1}^{2}}=0 \tag{C.4}
\end{equation*}
$$

388
(C.4) has a positive or zero discriminant:

$$
\begin{equation*}
\Delta=\left(\bar{\lambda}_{1}^{2}-\bar{B}_{\mathrm{I}}\right)^{2}-\frac{4}{\bar{\lambda}_{1}^{2}} \geq 0 \Rightarrow \bar{\lambda}_{2 \pm}^{2}=\frac{\bar{B}_{\mathrm{I}}-\bar{\lambda}_{1}^{2} \pm \sqrt{\left(\bar{\lambda}_{1}^{2}-\bar{B}_{\mathrm{I}}-\frac{2}{\bar{\lambda}_{1}}\right)\left(\bar{\lambda}_{1}^{2}-\bar{B}_{\mathrm{I}}+\frac{2}{\bar{\lambda}_{1}}\right)}}{2} \tag{C.5}
\end{equation*}
$$

The positivity of the term under the square root implies that (C.4) has real solutions if and only if the eigenvalue $\bar{\lambda}_{1}$ evolves in the interval:

$$
\begin{equation*}
\bar{\lambda} \in\left[X_{2}, X_{0}\right] \text { with } X_{k}=3 \sqrt{\tilde{\alpha}} \cos \left(\frac{1}{3} \arccos \left(-(\tilde{\alpha})^{-\frac{3}{2}}\right)+\frac{2 k \pi}{3}\right) \text { with } \tilde{\alpha}=\frac{\bar{B}_{\mathrm{I}}}{3} \geq 0 . \tag{C.6}
\end{equation*}
$$

${ }^{391}$ We can then express $\gamma_{t}$ as a function of the unique variable $\bar{\lambda}_{1}$ :

$$
\begin{equation*}
\bar{B}_{\mathrm{II}}\left(\bar{\lambda}_{1}\right)=\bar{\lambda}_{1}^{2} \bar{\lambda}_{2 \pm}^{2}+\frac{1}{\bar{\lambda}_{1}^{2}}+\frac{1}{\bar{\lambda}_{2 \pm}^{2}}=\bar{\lambda}_{1}^{2} \bar{B}_{\mathrm{I}}-\bar{\lambda}_{1}^{4}+\frac{1}{\bar{\lambda}_{1}^{2}} \Rightarrow \gamma_{t}\left(\bar{\lambda}_{1}\right)=\gamma_{s}-\sqrt{\bar{B}_{\mathrm{II}}\left(\bar{\lambda}_{1}\right)-3} \tag{C.7}
\end{equation*}
$$

Fixing $\gamma_{s}$ successively, we plot the evolution domain of $\gamma_{t}$ (see Figure C.3a) and compare it to the set that

$$
\begin{equation*}
\gamma_{t} \in\left[\gamma_{s}\left(1-\sqrt{1+\frac{\gamma_{s}^{2}}{4}}\right), 0\right] \tag{C.8}
\end{equation*}
$$

Some values of the shear parameter $\gamma_{t}$ are not accessible through the double shear test (see Figure C.3b). Therefore, we present in the following Appendix D a more general motion which scan the whole domain of variation for $\gamma_{t}$.


Figure C.3: Value of the second invariant $\gamma_{t}$ for various $\gamma_{s}$

## Appendix D. Double shear traction

We define as "isochoric double shear-traction test" the motion given by the following gradient:

$$
\boldsymbol{F}=J^{\frac{1}{3}}\left(\begin{array}{ccc}
\frac{1}{\sqrt{e}} & \gamma & 0  \tag{D.1}\\
0 & \frac{1}{\sqrt{e}} & \gamma_{\perp} \\
0 & 0 & e
\end{array}\right) \Rightarrow \overline{\boldsymbol{B}}=\left(\begin{array}{ccc}
\frac{1}{e}+\gamma^{2} & \frac{\gamma}{\sqrt{e}} & 0 \\
& \frac{1}{e}+\gamma_{\perp}^{2} & \gamma_{\perp} e \\
\text { Sym } & & e^{2}
\end{array}\right)
$$

The "double shear test" mentioned in Section 3.1.4 corresponds to $e=1$. We also denote as "simple tensileshear test" the case $\gamma_{\perp}=0$.

The square of the Finger isochoric strain tensor is equal to:

$$
\overline{\boldsymbol{B}}^{2}=\left(\begin{array}{ccc}
\left(\frac{1}{e}+\gamma^{2}\right)^{2}+\frac{\gamma^{2}}{e} & \frac{\gamma}{\sqrt{e}}\left(\frac{2}{e}+\gamma^{2}+\gamma_{\perp}^{2}\right) & \gamma \gamma_{\perp} \sqrt{e}  \tag{D.2}\\
& \frac{\gamma^{2}}{e}+\left(\gamma_{\perp} e\right)^{2}+\left(\frac{1}{e}+\gamma_{\perp}^{2}\right)^{2} & \gamma_{\perp} e\left(\frac{1}{e}+\gamma_{\perp}^{2}+e^{2}\right) \\
\text { Sym } & & \left(\gamma_{\perp} e\right)^{2}+e^{4}
\end{array}\right)
$$

It's worth noticing a non-zero $\bar{B}_{13}^{2}$ deformation which induces a purely non-linear stress. This out-of plane shear stress is relevant for identifying the dependence of the free energy on the second invariant $\gamma_{t}$.

A simple computation of the characteristic polynomial of the isochoric Finger tensor shows that the fundamental invariants are equal to:

$$
\begin{equation*}
\bar{B}_{\mathrm{I}}=\gamma^{2}+\gamma_{\perp}^{2}+e^{2}+\frac{2}{e} \text { and } \bar{B}_{\mathrm{II}}=\gamma^{2}\left(\gamma_{\perp}^{2}+e^{2}\right)+2 e+\frac{1}{e^{2}}+\frac{\gamma_{\perp}^{2}}{e^{2}} \tag{D.3}
\end{equation*}
$$

The conservation of the parameter $\gamma_{s}$ is ensured if at each instant, the following equality holds:

$$
\begin{equation*}
\bar{B}_{\mathrm{I}}=\gamma^{2}+\gamma_{\perp}^{2}+e^{2}+\frac{2}{e}=\gamma_{s}^{2}+3 \text { with } e(0)=1, \gamma(0)=\gamma_{s}, \gamma_{\perp}(0)=0 \tag{D.4}
\end{equation*}
$$

The motion has two independent parameters: a shear $\gamma_{\perp}$ and a traction magnitude $e$. The value of $\gamma$ is adjusted so as to guarantee at each instant the equality (D.4).

We then deduce, by simple calculations, the value of the invariant $\gamma_{t}$ :

$$
\begin{equation*}
\gamma_{t}\left(e, \gamma_{\perp}\right)=\gamma_{s}-\sqrt{\left(\gamma_{\perp}^{2}+e^{2}\right)\left(\bar{B}_{\mathrm{I}}-\gamma_{\perp}^{2}-e^{2}-\frac{2}{e}\right)+\frac{1}{e^{2}}+2 e+\frac{\gamma_{\perp}^{2}}{e}-3} \tag{D.5}
\end{equation*}
$$

with the necessary conditions of existence:

$$
\begin{equation*}
\bar{B}_{\mathrm{I}}-\gamma_{\perp}^{2}-e^{2}-\frac{2}{e} \geq 0 \text { and } e>0 . \tag{D.6}
\end{equation*}
$$

The existence conditions (D.6) define a sub-domain of $\mathbb{R}^{2}$. At $\gamma_{\perp} \leq \gamma_{s}$ fixed, the range of variation of $e$ is analytic and simply given by $e \in\left[e_{2}, e_{0}\right]$ with:

$$
\begin{equation*}
e_{k}=2 \sqrt{\tilde{\alpha}} \cos \left(\frac{1}{3} \arccos \left(-(\tilde{\alpha})^{-\frac{3}{2}}\right)+\frac{2 k \pi}{3}\right) \text { with } \tilde{\alpha}=\frac{\bar{B}_{\mathrm{I}}-\gamma_{\perp}^{2}}{3} \geq 0 . \tag{D.7}
\end{equation*}
$$

Hence, the values of the parameter $\gamma_{t}$ that can be scanned through the simple shear tensile test ((D.9) with $\left.\gamma_{\perp}=0\right)$ are:

$$
\begin{align*}
& \gamma_{t}(e, 0)=\gamma_{s}-\sqrt{e^{2}\left(\bar{B}_{\mathrm{I}}-e^{2}-\frac{2}{e}\right)+\frac{1}{e^{2}}+2 e} \quad e \in\left[e_{2}(0), e_{0}(0)\right], e_{k}(0) \text { given by (D.7). }  \tag{D.8}\\
& \gamma_{t}(e, 0)=\gamma_{s}-\sqrt{e^{2} \bar{B}_{\mathrm{I}}-e^{4}+\frac{1}{e^{2}}}
\end{align*}
$$

Whereas for the double-shear test ((D.9) with $e=1$ ):

$$
\begin{equation*}
\gamma_{t}\left(1, \gamma_{\perp}\right)=\gamma_{s}-\sqrt{\left(\gamma_{\perp}^{2}+1\right)\left(\bar{B}_{\mathrm{I}}-\gamma_{\perp}^{2}-3\right)+\gamma_{\perp}^{2}}=\gamma_{s}-\sqrt{\gamma_{s}^{2}+\gamma_{\perp}^{2}\left(\gamma_{s}^{2}-\gamma_{\perp}^{2}\right)} . \tag{D.9}
\end{equation*}
$$

We represent in figure D. 4 the values of $\gamma_{t}$ covered by the double shear tensile test with $\gamma_{s}=3$. The red line corresponds to the single tensile-shear test $\left(\gamma_{\perp}=0\right)$, while the orange line corresponds to the double shear test $(e=1)$. The widest $\gamma_{t}$ interval is scanned when $\gamma_{\perp}=0$ (see D.4), i.e., during a single tensile-shear test.

We further represent the $\gamma_{t}$ that can be scanned by both the tensile-shear (Figure (D.5b)) and double shear (Figure D.5a) tests for various $\gamma_{s}$. We note that the double-shear test only explores negative values of $\gamma_{t}$ while the tensile-shear test scans the whole domain (see Figure D. 6 or more simply the equality between (C.7) and (D.8) as $e=\bar{\lambda}_{1}$ ).

The identification of the constitutive law with respect to the second invariant $\gamma_{t}$ Section 3.1.4 was built upon the double shear test. The reason is quite simple: the single shear tensile test, although scanning the whole domain for $\gamma_{t}$, does not allow to simply identify the dependence on $\gamma_{t}$ because the non-linear stress $\sigma_{13}$ vanishes. Therefore, measuring either $\sigma_{12}$ or $\sigma_{13}$ will couple $g^{(4)}$ to the other functions. The most efficient test, but also the hardest to perform experimentally, shall set a small shear strain $\gamma_{\perp}$ and then perform a tensile test by varying $e$. Hence $\sigma_{13}$ would be non-zero and one could characterize $\tilde{f}_{\psi}$ for $\gamma_{t} \geq 0$.


Figure D.4: Value of $\gamma_{t}, \gamma_{s}=3$ fixed, for different values of $e$ and $\gamma_{\perp}$


Figure D.5: $\gamma_{t}$ for various $\gamma_{s}$

## Appendix E. Generic partial derivatives of the internal energy

The partial derivative with respect to temperature is certainly the most cumbersome:

$$
\begin{align*}
\frac{\partial e^{m}}{\partial T}= & \frac{\mathrm{d} q_{e x p}^{m(5)}}{\mathrm{d} T}-\frac{1}{\rho_{0}} \frac{\mathrm{~d} J_{e x p}^{(5)}(T)}{\mathrm{d} T} T \partial_{T} p_{e x p}^{(2)}\left(T, J_{e x p}^{(5)}(T)\right) \\
& \left.+\frac{1}{\rho_{0}} \int_{1}^{J} T \partial_{T}^{2} p_{e x p}^{(2)}(T, \tilde{J})\right) \mathrm{d} \tilde{J}-\frac{1}{\rho_{0}} \int_{0}^{\gamma_{s}} T \partial_{T}^{2} \hat{\tau}_{e x p}^{(3)}\left(T, \tilde{\gamma}_{s}\right) \mathrm{d} \tilde{\gamma}_{s}  \tag{E.1}\\
& -\frac{1}{\rho_{0}} \int_{0}^{\gamma_{t}} \frac{\gamma_{s}-\tau}{\sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}} T \partial_{T}^{2} \hat{\tau}_{e x p}^{(4)}\left(T, \gamma_{s}, \tau\right) \mathrm{d} \tau .
\end{align*}
$$

The partial derivative with respect to the volume expansion $J$ thanks to the volumetric/isochoric split:

$$
\begin{equation*}
\frac{\partial e^{m}}{\partial J}=-\frac{1}{\rho_{0}}\left(p_{e x p}^{(2)}(T, J)-T \partial_{T} p_{e x p}^{(2)}(T, J)\right) \tag{E.2}
\end{equation*}
$$



Figure D.6: Upper and lower bound of $\gamma_{t}$

434
The partial derivative with respect to $\gamma_{s}$ and $\gamma_{t}$ are written as:
$\frac{\partial e^{m}}{\partial \gamma_{s}}=\frac{1}{\rho_{0}}\left(\hat{\tau}_{e x p}^{(3)}\left(T, \gamma_{s}\right)-T \partial_{T} \hat{\tau}_{e x p}^{(3)}\left(T, \gamma_{s}\right)\right)+\frac{1}{\rho_{0}} \int_{0}^{\gamma_{t}} \partial_{\gamma_{s}}\left(\frac{\gamma_{s}-\tau}{\sqrt{\left(\gamma_{s}-\tau\right)^{2}-\gamma_{s}^{2}}}\left(\hat{\tau}_{e x p}^{(4)}-T \partial_{T} \hat{\tau}_{e x p}^{(4)}\right)\left(T, \gamma_{s}, \tau\right)\right) \mathrm{d} \tau$,
$\frac{\partial e^{m}}{\partial \gamma_{t}}=\frac{1}{\rho_{0}} \frac{\gamma_{s}-\gamma_{t}}{\sqrt{\left(\gamma_{s}-\gamma_{t}\right)^{2}-\gamma_{s}^{2}}}\left(\hat{\tau}_{e x p}^{(4)}-T \partial_{T} \hat{\tau}_{e x p}^{(4)}\right)\left(T, \gamma_{s}, \gamma_{t}\right)$.

## References

[1] J. Akella, G. Smith, R. Grover, Y. Wu, and S. Martin. Static eos of uranium to 100 gpa pressure. High Pressure Research, 2(5-6):295-302, 1990.
[2] B. Banerjee. An evaluation of plastic flow stress models for the simulation of high-temperature and high-strain-rate deformation of metals. arXiv preprint cond-mat/0512466, 2005.
[3] M. A. Biot. Thermoelasticity and irreversible thermodynamics. Journal of Applied Physics, 27(3):240-253, 1956.
[4] J. Boehler. On irreducible representations for isotropic scalar functions. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 57(6):323-327, 1977.
[5] J.-P. Boehler. A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 59(4): 157-167, 1979.
[6] J. Cassels, N. Hitchin, et al. Nonlinear elasticity: theory and applications. 2001.
[7] G. Chagnon, M. Rebouah, and D. Favier. Hyperelastic energy densities for soft biological tissues: a review. Journal of Elasticity, 120(2):129-160, 2015.
[8] P. G. Ciarlet. Mathematical elasticity: Three-dimensional elasticity. SIAM, 2021.
[9] B. D. Coleman and W. Noll. The thermodynamics of elastic materials with heat conduction and viscosity. In The foundations of mechanics and thermodynamics, pages 145-156. Springer, 1974.
[10] H. Dal, Y. Badienia, K. Açikgöz, and F. A. Denlï. A comparative study on hyperelastic constitutive models on rubber: State of the art after 2006. Constitutive Models for Rubber XI, pages 239-244, 2019.
[11] M. Dalemat. Une experimentation reussie pour l'identification de la reponse mecanique sans loi de comportement: Approche data-driven appliquee aux membranes elastomeres. PhD thesis, ecole centrale de Nantes, 2019.
[12] S. Doll and K. Schweizerhof. On the development of volumetric strain energy functions. J. Appl. Mech., 67(1):17-21, 2000.
[13] P. Flory. Thermodynamic relations for high elastic materials. Transactions of the Faraday Society, 57:829-838, 1961.
[14] J. Garrigues. Algèbre et analyse tensorielles pour l'étude des milieux continus. 2016.
[15] J. Garrigues. Cinématique des milieux continus, 2016.
[16] J. Garrigues. Comportement élastique, 2017.
[17] J. Garrigues. Comportements inélastiques, 2018.
[18] C. G'sell, S. Boni, and S. Shrivastava. Application of the plane simple shear test for determination of the plastic behaviour of solid polymers at large strains. Journal of Materials Science, 18(3):903-918, 1983.
[19] H. B. Khaniki, M. H. Ghayesh, R. Chin, and M. Amabili. A review on the nonlinear dynamics of hyperelastic structures. Nonlinear Dynamics, pages 1-32, 2022.
[20] B. Kim, S. B. Lee, J. Lee, S. Cho, H. Park, S. Yeom, and S. H. Park. A comparison among neo-hookean model, mooneyrivlin model, and ogden model for chloroprene rubber. International Journal of Precision Engineering and Manufacturing, 13(5):759-764, 2012.
[21] S. Korobeynikov. Objective tensor rates and applications in formulation of hyperelastic relations. Journal of Elasticity, 93(2):105-140, 2008.
[22] N. Kumar and V. V. Rao. Hyperelastic mooney-rivlin model: determination and physical interpretation of material constants. Parameters, 2(10):01, 2016.
[23] S. K. Melly, L. Liu, Y. Liu, and J. Leng. A review on material models for isotropic hyperelasticity. International Journal of Mechanical System Dynamics, 1(1):71-88, 2021.
[24] C. Miehe. Entropic thermoelasticity at finite strains. aspects of the formulation and numerical implementation. Computer Methods in Applied Mechanics and Engineering, 120(3-4):243-269, 1995.
[25] R. W. Ogden. Non-linear elastic deformations. Courier Corporation, 1997.
[26] J. Pamin, B. Wcisło, and K. Kowalczyk-Gajewska. Gradient-enhanced large strain thermoplasticity with automatic linearization and localization simulations. Journal of Mechanics of Materials and Structures, 12(1):123-146, 2016.
[27] C. Sansour. On the physical assumptions underlying the volumetric-isochoric split and the case of anisotropy. European Journal of Mechanics-A/Solids, 27(1):28-39, 2008.
[28] M. Scheidler. On the coupling of pressure and deviatoric stress in hyperelastic materials. In Proceedings of the 13th Army Symposium on Solid Mechanics, pages 539-550, 1993.
[29] J. Simo and C. Miehe. Associative coupled thermoplasticity at finite strains: Formulation, numerical analysis and implementation. Computer Methods in Applied Mechanics and Engineering, 98(1):41-104, 1992.
[30] J. Simo, R. L. Taylor, and K. Pister. Variational and projection methods for the volume constraint in finite deformation elasto-plasticity. Computer Methods in Applied Mechanics and Engineering, 51(1-3):177-208, 1985.
[31] J. C. Simo and T. J. Hughes. Computational inelasticity, volume 7. Springer Science \& Business Media, 2006.
[32] G. Smith. On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors. International Journal of Engineering Science, 9(10):899-916, 1971.
[33] D. Steinberg, S. Cochran, and M. Guinan. A constitutive model for metals applicable at high-strain rate. Journal of Applied Physics, 51(3):1498-1504, 1980.
[34] D. Stewart. A platform with six degrees of freedom. Proceedings of the institution of mechanical engineers, 180(1):
[36] O. H. Yeoh. Some forms of the strain energy function for rubber. Rubber Chemistry and Technology, 66(5):754-771, 1993.
[37] Q.-S. Zheng and A. Spencer. Tensors which characterize anisotropies. International Journal of Engineering Science, 31 (5):679-693, 1993.


[^0]:    Email address: paul.bouteiller@cea.fr (Paul Bouteiller)

